Order Reduction of n for Robust Adaptive Control Design of SISO Linear Systems*

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Abstract

In this paper, we propose an order-reduction design methodology to simplify the adaptive controller obtained in [1] by n integrators. We study the same class of linear systems as [1], make the same assumptions, and have the same formulation and approach to the problem. The main difference between our design methodology and that of [1] lies in the step 0 of the control design step. In this paper, we skip step 0 and immediately start the integrator backstepping procedure without stabilizing the filtered dynamics of the output. This relieves us from generating the reference trajectory for the filtered dynamics of the output and thus reducing the controller order by n. The trade-off for this order reduction is that the worst-case estimate for the expanded state vector has to be chosen as a suboptimal choice, rather than the optimal choice. Exactly the same robustness properties can be established for the reduced-order controllers as those of [1]. There is no definite performance comparison that can be made theoretically between the reduced-order controller and the full-order controller of [1]. Based on a few simulation examples, we observe that the reduced-order controller does not perform better than the full-order controller. Two of these examples are presented in the paper.

Key words: adaptive control; nonlinear H^∞ control; cost-to-come function; integrator backstepping.

1 Introduction

Adaptive control has been an important research topic in control theory. Its early development since 1970s has been dominated by certainty equivalence principle [2, 3], which decouples the parameter estimation design from the control design, by making use of some standard parameter estimators and supplying the estimates to control law as if they were true parameters. The certainty equivalence based design simplifies the controller structure considerably and leads to many successful applications [4, 5, 6] for linear systems. On the other hand, early designs using this approach are shown to be nonrobust when the system has unmodeled dynamics and deterministic exogenous disturbance inputs [7], [8]. Furthermore, it is unsuccessful to generalize this approach

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to the nonlinear systems with severe nonlinearity. All of these drawbacks motivate the study of nonlinear adaptive control design in 1990s and robust adaptive control design in 1980s and 1990s.

One of the major research focus for nonlinear adaptive control design is in (partially) feedback linearizable systems which are geometrically characterized in [9]. The introduction of integrator backstepping methodology [10] provides a systematic design tool to obtain the adaptive control law for the class of parametric strict- (or pure-) feedback nonlinear systems. This method admits great design flexibility evident in the selection of the value function and the virtual control laws. See the book [11] for a complete list of references. The adaptive control design based on this method achieves better system performance for linear systems if the system has no disturbance input, in comparison with the certainty equivalence based adaptive controller design [11]. However, it is shown that such design may be nonrobust if system is subject to exogenous disturbance input [12].

The main objectives of robust adaptive control are to improve transient performance, accommodate unmodeled dynamics, and tolerate exogenous disturbance inputs, which are consistent with the objectives of H^∞-optimal control [13]. One key feature of H^∞-optimal control is that the disturbance attenuation property of the closed-loop system can be precisely described. Therefore, we formulate the robust adaptive control problem as a nonlinear H^∞-optimal control problem under imperfect state measurements, through which, the objectives of robust adaptive control can be then achieved. The game-theoretic approach to H^∞-optimal control problem offers the most promising approach to address nonlinear H^∞-optimal control problems [14] with many success [15, 16, 17, 18, 19, 20]. These motivate the worst-case analysis based approach to robust adaptive control design [21, 22, 23, 24, 25, 1, 26], where the measures of disturbance attenuation, asymptotic tracking, and transient performance are all incorporated into a single soft-constrained game theoretic cost function. In this approach, the unknown parameters are treated as part of the expanded state vector. An application of the cost-to-come function methodology [16, 27] to the nonlinear H^∞-optimal control problem yields a finite dimensional estimator for the expanded system, and converts the H^∞-optimal control problem with imperfect state measurements into one with full-information measurements. Then, the integrator backstepping methodology is applied to solve this full information measurement problem. The above design paradigm has been successfully applied to the identification problem [21, 22] and robust adaptive control problems [24, 25, 1, 26], which indicates that the resulted identifiers and adaptive controllers have strong robustness. Encouraged by these successes, we continue to research further on this topic.

Motivated by the result of [1], we studied reduced-order adaptive controller design in [28]. The key to the order reduction is step 0 in the controller design step of [1]. In [28], instead of generating reference trajectory for the entire state vector of the filtered dynamics of the measured output as is done in [1], we generate only the reference trajectory for a particular linear combination of the state vector of the filtered dynamics. Thus, the controller order is reduced by $n-1$ or $n-2$ depending on the eigen structure of a feedback matrix, as compared with the full-order controller design proposed in [1]. It is proved that the closed-loop system, after order reduction, achieves the same strong robustness properties as [1]. Simulation resluts demonstrate a significant improvement in transient performance of the closed-loop system with the reduced-order controller.

In this paper, we continue to study the reduced-order adaptive controller design , in comparison with the full-order controller achieved in [1]. We study the same class of linear systems as [1], make the same assumptions, and have the same problem formulation and solution approach to the problem. We assume the system under consideration has known upper bound for its dynamic order, is observable, admits a transfer function that is strictly minimum phase with known relative degree. The true system may be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and the uncontrollable modes on the imaginary axis are uncontrollable from the disturbance input. Based on these assumptions, the unknown system is transformed into the design model, where it is linear in all of the unknown quantities. We assume that the measurement channel is noisy to avoid singularity in estimation design. We also assume that the unknown parameter vector belongs to a convex compact set characterized by a known smooth nonnegative radially unbounded and strictly convex function $P(\theta)$. Furthermore, for any parameter vector which belongs to the convex compact set, the corresponding high frequency gain is never zero. The robust adaptive control problem is then formulated as a nonlinear H^∞-optimal control problem under imperfect state measurements, where the objectives of asymptotic tracking and disturbance attenuation are incorporated into a single soft-constrained game theoretic cost function. We adopt a game theoretic solution to this problem by separating it into estimation design and controller design steps. In estimation design, we apply the cost-to-come function methodology to obtain the finite dimensional estimator. Then, we apply the soft projection algorithm to relieve the persistency of excitation assumption. This estimation step is exactly the same as [1]. Therefore, we summarize the result of the estimation design in [1] for ease of reference. The main difference between this paper and [1] starts in the controller design step. As [1], we still apply the integrator backstepping methodology to derive the control law. But, the controller design begins from step 1, without first stabilizing the filtered dynamics of the output as step 0 does in [1]. This relieves us from generating the reference trajectory for state of the filtered dynamics of the output to track. Therefore, the controller structure can be simplified by n integrators, where n is the upper bound of the order of the unknown system. The rest of the backstepping design procedure is similar to that of [1]. The lack of step 0 results in the lack of one nonpositive drift term in the closed-loop value function, which may degrade the performance of the reduced-order controller. In addition, in order to guarantee the boundness of the closed-loop signals, the worst-case estimate for the expanded state will be chosen suboptimally, rather than optimally. Exactly the same robustness results are established for the reduced-order controller as the full-order counterpart of [1]. It is shown that, whenever the disturbance input is bounded and the reference trajectory and its derivatives up to rth order, where r being the relative degree of the transfer function of the true system, are bounded. Then, all signals in the closed-loop system are bounded. It is also shown that the reduced-order controller achieves the desired disturbance attenuation level, whose ultimate lower bound is the noise intensity in the measurement channel. When the disturbance input is bounded and of finite energy, and the reference trajectory and its derivatives up to rth order are bounded, then we have asymptotic tracking. Two examples are included to compare the system performance of the reduced-order controller and that of the full-order controller. This completes the preview of this paper.

The organization of this paper is as follows. In Section 2, we list the notations to be used in the paper. We present the problem formulation of the robust adaptive control problem in Section 3. Then, we present the summary of estimation design obtained in [1] in Section 4 for the convenience of readers. In Section 5, the controller design is discussed in detail, where we made use of the integrator backstepping methodology. In Section 6, we present the main robustness results in terms of a theorem. Then, two examples are included in Section 7 to compare the system performance of the reduced-order controller with that of the full-order controller [1]. The paper ends with some concluding remarks in Section 8 and two appendices.

2 Notations

We denote the real line by $\mathbb{R}$, the set of natural numbers by $\mathbb{N}$. We say that a function f belongs to $\mathcal{C}$ if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is continuously (partial) differentiable up to kth order. For a vector or matrix A, A' denotes its transpose. For any $z \in \mathbb{R}^n$ and any $n \times n$-dimensional symmetric matrix M, $n \in \mathbb{N}$, $|z|_M^2$ denotes $z'Mz$ and $|z|^2$ denotes $z'z$. For any $b \in \mathbb{R}$, $\mathrm{sgn}(b) = \begin{cases} -1 & b < 0 \\ 0 & b = 0 \\ 1 & b > 0 \end{cases}$. For any matrix M, the vector $\overrightarrow{M}$ is formed by stacking up its column vectors. For any symmetric matrix M, the vector $\overleftarrow{M}$ is formed by stacking up the column vectors of the lower triangular part of M. For $n \times n$ -dimensional symmetric matrices M_1 and M_2, where $n \in \mathbb{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. Denote $e_{n,i}$ to be $[\mathbf{0}_{1\times(i-1)}\ 1\ \mathbf{0}_{1\times(n-i)}]'$, for $i = 1, \ldots, n$ and $n \in \mathbb{N}$. For any matrix M, $\|M\|_p$ denotes its p-induced norm, $1 \leq p \leq \infty$. $\mathcal{L}_2$ denotes the set of square integrable functions, and $\mathcal{L}_\infty$ denotes the set of bounded functions.

3 Problem Formulation

The reduced-order control problem is a further work based on the results of [1]. The linear system under consideration satisfies the following assumption.

Assumption 1 *The linear system is known to be at most n dimensional, where $n \in \mathbb{N}$.* $\diamond$

The true system dynamics are given by

$$\begin{aligned} \dot{\grave{x}} &= \grave{A}\grave{x} + \grave{B}u + \grave{D}\grave{w}, \qquad \grave{x}(0) = \grave{x}_0 && \text{(1a)} \\ y &= \grave{C}\grave{x} + \grave{E}\grave{w} && \text{(1b)} \end{aligned}$$

where $\grave{x}$ is the $\grave{n}$-dimensional state vector; $\grave{n} \in \mathbb{N}$; u is the scalar control input; y is the scalar system output; $\grave{w}$ is the $\grave{q}$-dimensional disturbance input, $\grave{q} \in \mathbb{N}$; all signals in the system are assumed to be continuous, i.e., in the space $\mathcal{C}$; and the matrices $\grave{A}$, $\grave{B}$, $\grave{C}$, $\grave{D}$, and $\grave{E}$ are generally unknown or partially unknown. The true system (1) satisfies the following assumption.

Assumption 2 *The pair ($\grave{A}$, $\grave{C}$) is observable. The transfer function $H(s) = \grave{C}(sI_{\grave{n}} - \grave{A})^{-1}\grave{B}$ is known to have relative degree $r \in \mathbb{N}$, and is strictly minimum phase. The uncontrollable part (with respect to the control input u) of the unknown system is stable in the sense of Lyapunov. Any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}$ on the jw-axis are uncontrollable from the disturbance $\grave{w}$.* $\diamond$

By Assumption 1, $\grave{n} \leq n$. In the case of $\grave{n} < n$, there is a method, outlined in [1], to add dynamics to (1) such that the system has the same input and output mapping as (1), admits order n, and satisfies Assumption 2. Hence, without loss of generality, we assume $\grave{n} = n$.

By Assumption 2, there always exist a state transformation and a disturbance transformation

$$x = \grave{T}\grave{x}; \quad w = \grave{M}\grave{w}$$

such that the system can be written as

$$\begin{aligned} \dot{x} &= Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Bu + Dw; \qquad x(0) = x_0 && \text{(2a)} \\ y &= Cx + Ew && \text{(2b)} \end{aligned}$$

where $\grave{T}$ is an unknown real invertible matrix; $\grave{M}$ is an unknown real $q \times \grave{q}$-dimensional matrix, $q \in \mathbb{N}$; $\theta \in \mathbb{R}^\sigma$ is the vector of unknown parameters of the system, $\sigma \in \mathbb{N}$; and the matrices A, $\bar{A}_{211}$, $\bar{A}_{212}$, B, D, C, and E are known and admit the following structure

$$A = \begin{bmatrix} a_{11} & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots & & \vdots \\ a_{r-2\,1} & \dots & \dots & \dots & 1 & 0 & \dots & 0 \\ a_{r-1\,1} & \dots & \dots & \dots & a_{r-1\,r-1} & 1 & \dots & 0 \\ a_{r1} & \dots & \dots & \dots & a_{r\,r-1} & a_{rr} & \dots & a_{rn} \\ \vdots & & & & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & \dots & \dots & a_{n\,r-1} & a_{nr} & \dots & a_{nn} \end{bmatrix}; \quad \bar{A}_{212} = \begin{bmatrix} \mathbf{0}_{(r-1)\times\sigma} \\ \bar{A}_{212\,0} \\ \bar{A}_{212\,r} \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & b_{p0} & \dots & b_{pn-r} \end{bmatrix}'; \quad C = \begin{bmatrix} 1 & \mathbf{0}_{1\times(n-1)} \end{bmatrix}$$

$\bar{A}_{212\,0}$ is a row vector; b_{pi}, $i = 1, \cdots, n-r$, are constants. Denote the elements of x by $[x_1 \ \dots \ x_n]'$. The equation (2) is called the design model which satisfies Assumptions 3—5 described below.

Assumption 3 *The matrices E is such that $EE'{>}0$. Define $\zeta = (EE')^{-1/2}$ and $L = DE'$.* $\diamond$

Because of the structures of A, $\bar{A}_{212}$, and B, the high frequency gain of the transfer function $\grave{H}(s)$, b_0, is equal to $b_{p0} + \bar{A}_{212\,0}\theta$. The assumption on the parameter θ is given below

Assumption 4 *The sign of the high-frequency gain b_0 is known. There exists a known smooth nonnegative radially unbounded strictly convex function $P : \mathbb{R}^\sigma \to \mathbb{R}$, such that θ belongs to the set $\Theta := \{\bar{\theta} \in \mathbb{R}^\sigma : P(\bar{\theta}) \le 1\}$. Furthermore, $\forall \bar{\theta} \in \Theta$, we have $\mathrm{sgn}(b_0)(b_{p0} + \bar{A}_{212\,0}\bar{\theta}) > 0$.* $\diamond$

The following assumption is made on the reference trajectory y_d that x_1 is to track.

Assumption 5 *The reference trajectory y_d is r times continuously differentiable. The signal y_d and the first r derivatives of y_d are available for feedback. Denote $Y_d := [y_d \ y_d^{(1)} \ \dots \ y_d^{(r)}]'$ and $Y_{d0} = [y_d(0) \ y_d^{(1)}(0) \ \cdots \ y_d^{(r-1)}(0)]'$.* $\diamond$

For the system (1), the control law is generated by

$$u(t) = \mu(y_{[0,t]}, Y_{d[0,t]}) \tag{3}$$

Furthermore, it must satisfy the following condition. For any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)})$ $\in \grave{\mathcal{W}} := \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathbb{R}^r \times \mathcal{C}$, which consists of the initial condition of the system, the true value of the parameters, the waveform of the disturbance input, the initial condition of the reference trajectory, and the waveform of the rth order derivative of the reference trajectory, there must be a unique solution $\grave{x}_{[0,\infty)}$ for the closed-loop system, which results in a continuous control function $u_{[0,\infty)}$. The class of these admissible controllers is denoted by $\mathcal{M}_u$.

The objective of the control design is to make the system output Cx to track the reference trajectory y_d asymptotically while attenuating the effect of the uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)})$, where the exogenous input $\grave{w}$ can be taken to be any open-loop time function, as in the case of H^∞-optimal control problems. For such uncertainty, we have $(x_0, \theta, w_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)})$ belongs to the set $\mathcal{W} = \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathbb{R}^r \times \mathcal{C}$. Then, a precise definition of the objective is further given below as that in [1].

Definition 1 *A controller $\mu \in \mathcal{M}_u$ is said to achieve* disturbance attenuation level γ *if there exist a nonnegative function $l(t, \theta, x, y_{[0,t]}, Y_{d[0,t]})$ and a constant $l_0 \geq 0$, such that, for all $t_f \geq 0$, the following inequality holds:*

$$\sup_{(x_0,\theta,\grave{w}_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} \leq 0 \tag{4}$$

where

$$\begin{aligned} J_{\gamma t_f} := \int_0^{t_f} ((x_1(\tau) - y_d(\tau))^2 + l(\tau, \theta, x(\tau), y_{[0,\tau]}, Y_{d[0,\tau]}) - \gamma^2 |w(\tau)|^2) d\tau \\ -\gamma^2 |[\theta' - \check{\theta}_0' \; x_0' - \check{x}_0']'|^2_{\bar{Q}_0} - l_0 \end{aligned} \tag{5}$$

where $\check{\theta}_0 \in \Theta$ is the initial guess of the unknown parameter vector θ, $\check{x}_0$ is the initial guess of the unknown initial state x_0, the $(n+\sigma)\times(n+\sigma)$-dimensional matrix $\bar{Q}_0 > 0$ is the weighting matrix on the initial estimation error, quantifying the level of confidence in the estimate $[\check{\theta}_0' \; \check{x}_0']'$, and $\bar{Q}_0^{-1}$ admits the structure $\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi_0' \end{bmatrix}$, where $Q_0 > 0$ and $\Pi_0 > 0$ are $\sigma \times \sigma$- and $n \times n$-dimensional, respectively. $\diamond$

The definition above aims to guarantee that, for any $t_f \geq 0$, the squared $\mathcal{L}_2$ norm of the output tracking error $x_1 - y_d$ on $[0, t_f]$ is bounded by γ^2 times the squared $\mathcal{L}_2$ norm of the transformed disturbance input $w_{[0,t]}$ plus a constant that depends only on the initial condition of the system.

The problem formulated above can be brought into the framework of H^∞-optimal control for affine-quadratic nonlinear systems with imperfect state measurements as in [1], the system dynamics (2) is expanded by adding the simple dynamics of θ: $\dot{\theta} = 0$. Let ξ denote the expanded state vector $\xi = [\theta' \; x']'$, which satisfies the following dynamics:

$$\dot{\xi} = \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ y\bar{A}_{211} + u\bar{A}_{212} & A \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0}_{\sigma\times 1} \\ B \end{bmatrix} u + \begin{bmatrix} \mathbf{0}_{\sigma\times q} \\ D \end{bmatrix} w =: \bar{A}(y,u)\xi + \bar{B}u + \bar{D}w; \tag{6a}$$

$$y = \begin{bmatrix} \mathbf{0}_{1\times\sigma} & C \end{bmatrix} \xi + Ew =: \bar{C}\xi + Ew \tag{6b}$$

The worst-case optimization of the cost function (4) can be carried out in two steps with the following inequality:

$$\begin{aligned} & \sup_{(x_0,\theta,\grave{w}_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} \\ = & \sup_{Y_{d0}\in\mathbb{R}^r, y_{[0,\infty)}\in\mathcal{C}, y^{(r)}_{d[0,\infty)}\in\mathcal{C}} \; \sup_{(x_0,\theta,\grave{w}_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\grave{\mathcal{W}}|y_{[0,\infty)},y^{(r)}_{d[0,\infty)},Y_{d0}} J_{\gamma t_f} \\ \leq & \sup_{Y_{d0}\in\mathbb{R}^r, y_{[0,\infty)}\in\mathcal{C}, y^{(r)}_{d[0,\infty)}\in\mathcal{C}} \; \sup_{(x_0,\theta,w_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},y^{(r)}_{d[0,\infty)},Y_{d0}} J_{\gamma t_f} \end{aligned} \tag{7}$$

The design procedure starts with the inner supremization, which can be interpreted as the evaluation of the worst-case performance with a known output waveform. As a function of the output, the control input waveform is independent of the actual disturbance input waveform, and can be viewed as an open-loop time function. This step is actually the estimation design step discussed next in Section 3. The outer supremization can be interpreted as the computation of the worst-case measurement waveform against a given control law, which is crucial for the determination of achievability of the objective (4). This step is the control design step carried out in Section 4. The design function $l(t, \theta, x, y_{[0,t]}, Y_{d[0,t]})$ is selected based on the same considerations as [1]: the existence of a solution to the problem; the ease of analysis of stability and robustness of the resulting closed-loop system.

This completes the formulation of the robust adaptive control problem. Next, we turn to the estimation and control design in the next two sections.

4 Estimation Design

In order to set up an appropriate basis for the discussion of control design in next section, we begin by reviewing and summarizing the key results of the estimation design obtained in [1].

Given Y_{d0}, the measurement waveform $y_{[0,\infty)}$, and the reference trajectory $y^{(r)}_{d[0,\infty)}$, then the control waveform $u_{[0,\infty)}$ is also known. The cost function we consider in this step is

$$\begin{aligned} J_{i\gamma t_f} &= \int_0^{t_f} (|x_1(\tau) - y_d(\tau)|^2 + |\xi(\tau) - \hat{\xi}(\tau)|^2_{\bar{Q}(\tau, y_{[0,\tau]}, Y_{d[0,\tau]})} - \gamma^2 |w(\tau)|^2) d\tau \\ &\quad -\gamma^2 |[\theta' - \check{\theta}_0' \quad x_0' - \check{x}_0']'|^2_{\bar{Q}_0} \end{aligned} \tag{8}$$

where $\hat{\xi}$ is the worst-case estimate for the expanded state ξ to be designed later; $\bar{Q}$ is the nonnegative-definite weighting function which exhibits a special structure given by

$$\begin{aligned} \bar{Q}(t, y_{[0,t]}, Y_{d[0,t]}) &= (\bar{\Sigma}(t))^{-1} \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \Delta \end{bmatrix} (\bar{\Sigma}(t))^{-1} + \begin{bmatrix} \epsilon(t)(\Phi(t))' C' (\gamma^2\zeta^2 - 1) C\Phi(t) & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \mathbf{0}_{n\times n} \end{bmatrix} \\ &= \begin{bmatrix} -(\Phi(t))' \\ I_n \end{bmatrix} \gamma^4 \Pi^{-1} \Delta \Pi^{-1} \begin{bmatrix} -(\Phi(t))' \\ I_n \end{bmatrix}' + \begin{bmatrix} \epsilon(t)(\Phi(t))' C' (\gamma^2\zeta^2 - 1) C\Phi(t) & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \mathbf{0}_{n\times n} \end{bmatrix} \end{aligned}$$

where $\bar{\Sigma}$ is the worst-case covariance matrix defined later in (14f); Φ and Π are defined in (14e) and (14a), respectively, $\Delta = \gamma^{-2}\beta_\Delta \Pi + \Delta_1$ with the constant $\beta_\Delta \geq 0$ and the $n \times n$-dimensional constant matrix $\Delta_1 > 0$; ϵ is a scalar function defined by

$$\epsilon(t) = \mathrm{Tr}((\Sigma(t))^{-1})/K_c, \quad \forall t \in [0, t_f] \tag{9}$$

or simply defined by

$$\epsilon(t) = 1, \qquad \forall t \in [0, t_f] \tag{10}$$

In (9), K_c is a constant that is greater than or equal to $\gamma^2 \mathrm{Tr}(Q_0)$, corresponding to the preselected maximum level of the quantity $\mathrm{Tr}((\Sigma(t))^{-1})$; and the matrix Σ will play the role of worst-case covariance matrix of the parameter estimation error. In the other case, with the choice of $\epsilon \equiv 1$, we observe, from (14b) and (14c), Σ and s_Σ remains constant matrices on $[0, \infty)$, which simplifies the controller structure by $\frac{\sigma(\sigma+1)}{2} + 1$ integrators. In the following, we will considers $\bar{Q}$ as a function $\bar{Q} : \mathbb{R}^{n\times\sigma} \times \mathbb{R} \to \mathbb{R}^{(n+\sigma)\times(n+\sigma)}$, $\bar{Q}(\Phi, s_\Sigma)$.

On the basis of Assumption 4, a projection-based design of the estimator is carried out.

For the purpose of keeping $\check{\theta}$ within a vicinity of Θ such that $\check{b}_0$ is always positive, a smooth projection algorithm is introduced in [1]. Define

$$\rho := \inf\{P(\bar{\theta}) \mid \bar{\theta} \in R^\sigma \text{ and } b_{p0} + \bar{A}_{212\,0}\bar{\theta} = 0\} \tag{11}$$

then, $\rho > 1$. Fix any $\rho_o \in (1, \rho)$. The design will try to guarantee that the estimate $\check{\theta}$ lies in the open set

$$\Theta_o := \{\bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) < \rho_o\}$$

then, we have that the estimate $\check{b}_0 := b_{p0} + \bar{A}_{212\,0}\check{\theta}$ is bounded away from 0.

Define the smooth projection function

$$\begin{aligned} P_r(\check{\theta}) &:= \begin{cases} \exp((1 - P(\check{\theta}))^{-1})\,(\rho_o - P(\check{\theta}))^{-3} \left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'; & \forall\check{\theta} \in \Theta_o\backslash\Theta \\ \mathbf{0}_{\sigma\times 1}; & \forall\check{\theta} \in \Theta \end{cases} && (12) \\ &=: \; p_r(\check{\theta}) \left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)' && (13) \end{aligned}$$

then, it is obvious that $P_r(\check{\theta})$ and $p_r(\check{\theta})$ are smooth on the set Θ_o, and $(\theta - \check{\theta})' P_r(\check{\theta}) \le 0$, $\forall\check{\theta} \in \Theta_o$.

Then the identifier dynamics are summarized as follows, which is the result of [1].

$$\begin{aligned}
&(A - \zeta^2 LC + \beta_\Delta/2 I_n)\Pi + \Pi\,(A - \zeta^2 LC + \beta_\Delta/2 I_n)' - \Pi C'\,(\zeta^2 - \gamma^{-2})C\Pi \\
&\qquad\qquad + DD' - \zeta^2 LL' + \gamma^2\Delta_1 = \mathbf{0} && \text{(14a)} \\
\dot{\Sigma} &= -(1-\epsilon)\Sigma\Phi' C'\,(\gamma^2\zeta^2 - 1)C\Phi\Sigma; \qquad \Sigma(0) = \gamma^{-2}Q_0^{-1} && \text{(14b)} \\
\dot{s}_\Sigma &= (\gamma^2\zeta^2 - 1)\,(1-\epsilon)C\Phi\Phi' C'; \qquad s_\Sigma(0) = \gamma^2\mathrm{Tr}(Q_0) && \text{(14c)} \\
A_f &= A - \zeta^2 LC - \Pi C' C\,(\zeta^2 - \gamma^{-2}) && \text{(14d)} \\
\dot{\Phi} &= A_f\Phi + y\bar{A}_{211} + u\bar{A}_{212}; \qquad \Phi(0) = \Phi_0 && \text{(14e)} \\
\bar{\Sigma}(t) &= \begin{bmatrix} \Sigma(t) & \Sigma(t)\,(\Phi(t))' \\ \Phi(t)\Sigma(t) & \gamma^{-2}\Pi + \Phi(t)\Sigma(t)\,(\Phi(t))' \end{bmatrix}; \\
(\bar{\Sigma}(t))^{-1} &= \begin{bmatrix} (\Sigma(t))^{-1} + \gamma^2(\Phi(t))'\Pi^{-1}\Phi(t) & -\gamma^2(\Phi(t))'\Pi^{-1} \\ -\gamma^2\Pi^{-1}\Phi(t) & \gamma^2\Pi^{-1} \end{bmatrix} && \text{(14f)} \\
\dot{\check{\theta}} &= -\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'\,(y_d - C\check{x}) - \begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + \gamma^2\zeta^2\Sigma\Phi' C'\,(y - C\check{x}); \\
&\quad \check{\theta}(0) = \check{\theta}_0 && \text{(14g)} \\
\dot{\check{x}} &= -\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'\,(y_d - C\check{x}) - \begin{bmatrix} \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c \\
&\quad + Bu + (y\bar{A}_{211} + u\bar{A}_{212})\check{\theta} + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi' C' + L)\,(y - C\check{x}); \; \check{x}(0) = \check{x}_0 && \text{(14h)} \\
W(\xi, \check{\xi}, \bar{\Sigma}) &= |\xi - \check{\xi}|^2_{\bar{\Sigma}^{-1}} = |\theta - \check{\theta}|^2_{\Sigma^{-1}} + \gamma^2|x - \check{x} - \Phi\,(\theta - \check{\theta})|^2_{\Pi^{-1}}; \\
&\quad W : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \to \mathbb{R} && \text{(14i)}
\end{aligned}$$

$$\dot{W}(\xi, \check{\xi}, \hat{\xi}, \bar{\Sigma}, y_d, u, w) = -|x_1 - y_d|^2 - \gamma^4|x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C\Phi}$$

$$+\gamma^2|w|^2 + |C\check{x} - y_d|^2 + |\xi_c|^2_{\bar{Q}(\Phi, s_\Sigma)} - \gamma^2\zeta^2|y - C\check{x}|^2 + 2(\theta - \check{\theta})' P_r(\check{\theta})$$
$$-\gamma^2|w - w_*(\xi, \check{\xi}, \bar{\Sigma}, w)|^2 \tag{14j}$$

where s_Σ is a scalar signal defined as $s_\Sigma(t) := \mathrm{Tr}((\Sigma(t))^{-1})$, which is introduced to avoid the computation of $(\Sigma(t))^{-1}$ on line; Φ is a filtered signal of y and u; $\check{\theta}$ is an estimate for θ; $\check{x}$ is an estimate for x; W is the value function for the estimation design step, which is defined when $\Sigma > 0$; $\xi_c = \hat{\xi} - \check{\xi}$ and $\check{\xi} = [\check{\theta}' \ \check{x}']'$; $w_* : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \times \mathbb{R}^q \to \mathbb{R}^q$ denotes the worst-case disturbance for the estimation step, given by

$$w_*(\xi, \check{\xi}, \bar{\Sigma}, w) \quad = \quad \zeta^2 E' (y - \bar{C}\xi) + \gamma^{-2} (I_q - \zeta^2 E'E) \bar{D}' \bar{\Sigma}^{-1} (\xi - \check{\xi}) \tag{15}$$

(14j) holds as long as $\Sigma > 0$ and $\check{\theta} \in \Theta_o$.

Lemma 1 of [1] guarantees that the matrix Σ is bounded.

Lemma 1 *Consider the dynamic equation (14b) for the covariance matrix Σ. Let $\gamma \geq \zeta^{-1}$, $Q_0 > 0$, (9) hold, and $K_c \geq \gamma^2 \mathrm{Tr}(Q_0)$. Then, the matrix Σ is uniformly upper and lower bounded as follows: whenever it exists on $[0, t_f]$ and Φ is continuous on $[0, t_f]$.*

$$K_c^{-1} I_\sigma \leq \Sigma(t) \leq \Sigma(0) = \gamma^{-2} Q_0^{-1}; \qquad \gamma^2 \mathrm{Tr}(Q_0) \leq \mathrm{Tr}((\Sigma(t))^{-1}) \leq K_c; \qquad \forall t \in [0, t_f] \tag{16}$$

We list Assumptions 7 and 8 in [1] below.

Assumption 6 *If the matrix $A - \zeta^2 LC$ is Hurwitz, then choose the desired disturbance attenuation level $\gamma \geq \zeta^{-1}$. In case $\gamma = \zeta^{-1}$, choose $\beta_\Delta \geq 0$ such that $A - \zeta^2 LC + \beta_\Delta/2 I_n$ is Hurwitz. If the matrix $A - \zeta^2 LC$ is not Hurwitz, then choose the desired disturbance attenuation level $\gamma > \zeta^{-1}$.* $\diamond$

Assumption 7 *The initial weighting matrix Π_0 is chosen as the unique positive-definite solution to the algebraic Riccati equation:*

$$(A - \zeta^2 LC + \beta_\Delta/2 I_n)\Pi + \Pi(A - \zeta^2 LC + \beta_\Delta/2 I_n)' - \Pi C'(\zeta^2 - \gamma^{-2}) C \Pi$$
$$+DD' - \zeta^2 LL' + \gamma^2 \Delta_1 = \mathbf{0} \tag{17}$$

$\diamond$

Assumption 6 makes the observations that the quantity ζ^{-1} is the ultimate lower bound on the achievable performance level for the adaptive system using the proposed method. Assumption 7 will initialize the Π dynamics at its steady state value for the purpose of simplifying the identifier. Under Assumption 7, the matrix A_f is Hurwitz.

Assuming that all signals exist, $\Sigma(t) > 0$ and $\check{\theta}(t) \in \Theta_o$, $\forall t \in [0, t_f]$, the cost function can equivalently be written as;

$$\begin{aligned} J_{i\gamma t_f} \quad = \quad & -|\theta - \check{\theta}(t_f)|^2_{(\Sigma(t_f))^{-1}} - \gamma^2 |x(t_f) - \check{x}(t_f) - \Phi(t_f)(\theta - \check{\theta}(t_f))|^2_{\Pi^{-1}} \\ & + \int_0^{t_f} (|C\check{x}(\tau) - y_d(\tau)|^2 + |\xi_c(\tau)|^2_{\bar{Q}(\Phi(\tau), s_\Sigma(\tau))} - \gamma^2 \zeta^2 |y(\tau) - C\check{x}(\tau)|^2 \\ & + 2(\theta - \check{\theta}(\tau))' P_r(\check{\theta}(\tau)) - \gamma^2 |w(\tau) - w_*(\xi(\tau), \check{\xi}(\tau), \bar{\Sigma}(\tau), w(\tau))|^2) d\tau \end{aligned} \tag{18}$$

To further simplify the estimator dynamics, we may generate Φ by the following $3n$-dimensional prefiltering system for y and u.

$$\begin{aligned} \dot{\eta} &= A_f\eta + p_n y; \quad \eta(0) = \eta_0 && \text{(19a)} \\ \dot{\lambda} &= A_f\lambda + p_n u; \quad \lambda(0) = \lambda_0 && \text{(19b)} \\ \dot{\lambda}_o &= A_f\lambda_o; \qquad \lambda_o(0) = p_n && \text{(19c)} \end{aligned}$$

where p_n is an n-dimensional vector such that (A_f, p_n) is controllable. Then, Φ is given by an algebraic relation in terms of η and λ.

$$\begin{aligned} \Phi &= \begin{bmatrix} A_f^{n-1}\eta & \ldots & A_f\eta & \eta \end{bmatrix} M_f^{-1}\bar{A}_{211} + \begin{bmatrix} A_f^{n-1}\lambda & \ldots & A_f\lambda & \lambda \end{bmatrix} M_f^{-1}\bar{A}_{212} \\ &+ \begin{bmatrix} A_f^{n-1}\lambda_o & \cdots A_f\lambda_o & \lambda_o \end{bmatrix} M_f^{-1}\Phi_{o0} \end{aligned} \tag{20}$$

where the matrix

$$M_f = \begin{bmatrix} A_f^{n-1}p_n & \ldots & A_f p_n & p_n \end{bmatrix} \tag{21}$$

and $\eta_0 \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{R}^n$, and $\Phi_{o0} \in \mathbb{R}^{n\times\sigma}$ are such that (20) holds at $t = 0$. This completes the summary of the identification design of [1]. Based on the result of this step, we now turn to the controller design step in the next section.

5 Control Design

We describe in this section the reduced-order controller design for the uncertain system. In view of the equivalent form (18) of the cost function obtained in the previous section, we can rewrite the inequality (7) to be

$$\begin{aligned} & \sup_{(x_0,\theta,\dot{w}_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\dot{\mathcal{W}}} J_{\gamma t_f} \\ \leq & \sup_{Y_{d0}\in\mathbb{R}^r, y_{[0,\infty)}\in\mathcal{C}, y^{(r)}_{d[0,\infty)}\in\mathcal{C}} \; \sup_{(x_0,\theta,w_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},y^{(r)}_{d[0,\infty)},Y_{d0}} J_{\gamma t_f} \\ \leq & \sup_{Y_{d0}\in\mathbb{R}^r, y_{[0,\infty)}\in\mathcal{C}, y^{(r)}_{d[0,\infty)}\in\mathcal{C}} \int_0^{t_f} (|C\check{x}(\tau) - y_d(\tau)|^2 + |\xi_c(\tau)|^2_{\bar{Q}(\Phi(\tau),s_\Sigma(\tau))} + \check{l}(t, y_{[0,t]}, Y_{d[0,t]}) \\ & \qquad - \gamma^2\zeta^2|y(\tau) - C\check{x}(\tau)|^2)d\tau - l_0 \end{aligned} \tag{22}$$

where $\check{l} = l - |\xi - \hat{\xi}|_{\bar{Q}} + (\theta - \check{\theta})' P_r(\check{\theta})$, the last inequality holds if all closed-loop signals exist, $\Sigma(t) > 0$, and $\check{\theta}(t) \in \Theta_o$, on $[0, t_f]$. The control design aims to guarantee that the supremum is less than or equal to zero for all measurement waveforms. The control design problem is an H^∞-optimal control problem with full information measurement. A new variable v is introduced, instead of considering y as the maximizing variable, to reveal the suitable dynamic structure for backstepping design. The variable v is formulated as

$$v := \zeta\,(y - C\check{x}) \tag{23}$$

In terms of v , we have

$$\begin{aligned}\dot{\check{\theta}} &= -\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'\left(y_d - C\check{x}\right) - \begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + \gamma^2\Sigma\Phi' C'\zeta v &\text{(24a)}\\ \dot{\check{x}} &= -\Phi\Sigma P_r(\check{\theta}) + A\check{x} + Bu - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'\left(y_d - C\check{x}\right) - \begin{bmatrix} \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c \\ &\quad + \bar{A}_{211}\check{\theta}C\check{x} + \bar{A}_{212}\check{\theta}u + (\zeta^{-2}\bar{A}_{211}\check{\theta} + \Pi C' + \gamma^2\Phi\Sigma\Phi' C' + L)\zeta v &\text{(24b)}\end{aligned}$$

Express $\check{x} = \begin{bmatrix} \check{x}_1 & \cdots & \check{x}_n \end{bmatrix}'$ and $\Phi = \begin{bmatrix} \Phi_1' & \cdots & \Phi_n' \end{bmatrix}'$, where $\check{x}_i$, $i = 1, \ldots, n$, are scalars and Φ_i, $i = 1, \ldots, n$ are row vectors. To begin with the backstepping design, we list the relevant dynamics below, which are expressed in the form of various functions on the independent variables.

$$\begin{aligned}\dot{s}_\Sigma &= (\gamma^2\zeta^2 - 1)\left(1 - \epsilon\right)\Phi_1\Phi_1' &\text{(25a)}\\ \dot{\Sigma} &= -(1-\epsilon)\Sigma\Phi_1'\left(\gamma^2\zeta^2 - 1\right)\Phi_1\Sigma &\text{(25b)}\\ \dot{\check{\theta}} &= \delta(y_d, \check{x}_1, \Phi_1, \check{\theta}, \overleftarrow{\Sigma}) + \varphi(\overrightarrow{\Phi}, \overleftarrow{\Sigma})\bar{Q}\xi_c + \kappa(\Phi_1, \overleftarrow{\Sigma})v &\text{(25c)}\\ \dot{\check{x}}_i &= f_i(y_d, \check{x}_1, \cdots, \check{x}_i, \check{\theta}, \Phi_1, \Phi_i, \overleftarrow{\Sigma}) + \check{x}_{i+1} + \varrho_i(\overrightarrow{\Phi}, \overleftarrow{\Sigma})\bar{Q}\xi_c \\ &\quad + h_i(\check{\theta}, \Phi_1, \Phi_i, \overleftarrow{\Sigma})v; \qquad i = 1, \cdots, r-1 &\text{(25d)}\\ \dot{\check{x}}_r &= f_r(y_d, \check{x}_1, \cdots, \check{x}_r, \check{\theta}, \Phi_1, \Phi_r, \overleftarrow{\Sigma}) + a_{r\,r+1}\check{x}_{r+1} + \cdots + a_{rn}\check{x}_n + (b_{p0} + \bar{A}_{212\,0}\check{\theta})u \\ &\quad + \varrho_r(\overrightarrow{\Phi}, \overleftarrow{\Sigma})\bar{Q}\xi_c + h_r(\check{\theta}, \Phi_1, \Phi_r, \overleftarrow{\Sigma})v &\text{(25e)}\\ \dot{\Phi}_i &= \psi_i(\check{x}_1, \Phi_1, \ldots, \Phi_i) + \Phi_{i+1} + e_{n,i}'\bar{A}_{211}\zeta^{-1}v; \qquad i = 1, \cdots, r-1 &\text{(25f)}\end{aligned}$$

where the nonlinear functions f_i, $i = 1, \cdots, r$, and δ are smooth as long as $\check{\theta} \in \Theta_o$; the nonlinear functions φ, κ, h_i, ϱ_i, $i = 1, \cdots, r$ are smooth; and the functions ψ_i, $i = 1, \cdots, r$, are linear. Note that ϵ is a smooth function of s_Σ for either definitions of (9) and (10).

Next, we will apply the integrator backstepping methodology to design the control law u. In view of a non-negative definite weighting of ξ_c in the cost function (22), the design for ξ_c will be carried out last. Therefore, we, for convenience, set ξ_c to be zero first in this backstepping procedure, which does not influence the result of backstepping design.

The design procedure is proceeded under two cases: $r = 1$ and $r > 1$. First, we consider the case $r = 1$.

Step 1: At this step, we define the transformed variable $z_1 = \check{x}_1 - y_d$. For notational consistency, we choose $\alpha_0 = y_d$ and $V_0 = 0$. Then in the application of Lemma 3 in [1], we make the following substitution

$$X_{1o} := \begin{bmatrix} y_d & \check{\theta}' & \overleftarrow{\Sigma}' & s_\Sigma \end{bmatrix}' \to x_o; \quad X_{1a} := \check{x}_1 \to x_a; \quad v \to w; \quad u \to u;$$

$$X_{1d} := \begin{bmatrix} y_d^{(1)} & \Phi_1 & \check{x}_2 & \ldots & \check{x}_n & \Phi_2 & \ldots & \Phi_n \end{bmatrix}' \to x_d; \quad \mathcal{D}_{1a} := \mathbb{R} \to D_a$$

$$\mathcal{D}_{1o} := \{X_{1o} : \check{\theta} \in \Theta_o\} \to D_o; \quad ; \mathcal{D}_{1d} := \mathbb{R}^{1+n\sigma+n-1} \to D_d; \quad \mathcal{D}_w := \mathbb{R} \to D_w; \quad \gamma \to \gamma$$

$$\mathcal{D}_{1o} \times \mathcal{D}_{1d} \to D_1; \quad \text{Any number in } \mathbb{N} \to k; \quad V_0 \to V_o; \quad 0 \to \sigma_o; \quad \alpha_0 \to \alpha_o; \quad 0 \to l_o;$$

$$F_{1o} := \begin{bmatrix} y_d^{(1)} \\ \delta \\ \overleftarrow{-(1-\epsilon)\Sigma\Phi_1'\left(\gamma^2\zeta^2 - 1\right)\Phi_1\Sigma} \\ (\gamma^2\zeta^2 - 1)(1-\epsilon)\Phi_1\Phi_1' \end{bmatrix} \to f_o;$$

$$H_{1o} := \begin{bmatrix} 0 & \kappa' & \overleftarrow{\mathbf{0}_{\sigma\times\sigma}} & 0 \end{bmatrix}' \to h_o; \; F_{1a} := f_1 + a_{12}\check{x}_2 + \cdots + a_{1n}\check{x}_n \to f_a$$

$$G_{1a} := (b_{p0} + \bar{A}_{212\,0}\check{\theta}) \to g_a; \quad H_{1a} := h_1 \to h_a$$

Choose a constant $\gamma_1 > 0$ and a smooth design function $\beta_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}$ such that

$$\beta_1(X_{1o}, X_{1a}, X_{1d}) \geq c_{\beta_1} > 0, \quad \forall (X_{1o}, X_{1a}, X_{1d}) \in \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d}$$

where c_{β_1} is a constant. Again, we make the following substitution by Lemma 3 in [1]:

$$\sqrt{\gamma_1} \to \delta; \quad (1+\beta_1)z_1 \to \phi; \quad V_1 \to V; \quad \mu \to \alpha$$

Then, $V_1 = \gamma_1 z_1^2$, $V_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \to \mathbb{R}$, and $\mu : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}$ are smooth and such that

$$\begin{aligned} \dot{V}_1(X_{1o}, X_{1a}, X_{1d}, u, v)|_{u=\mu(X_{1o},X_{1a},X_{1d})} &= -z_1^2 - \beta_1 z_1^2 + \gamma^2 v^2 - \gamma^2 (v - \nu_1(X_{1o}, X_{1a}, X_{1d}))^2; \\ &\quad \forall X_{1o} \in \mathcal{D}_{1o}, \quad \forall X_{1a} \in \mathcal{D}_{1a}, \quad \forall X_{1d} \in \mathcal{D}_{1d}, \quad \forall v \in \mathcal{D}_w \end{aligned}$$

where $\nu_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}$ is smooth and can be easily obtained by Lemma 3 in [1].

Hence, we complete the design of the control function

$$u = \mu(X_{1o}, X_{1a}, X_{1d}) \tag{26}$$

We denote the value function $V = V_1$.

This completes the backstepping design for the case $r = 1$.

Next, we consider the backstepping design for the case $r > 1$. By repeated application of Lemma 3 in [1], we may derive the control law.

Step 1: Define the transformed variable $z_1 = \check{x}_1 - y_d$. Similar as the case $r = 1$, we first choose $\alpha_0 = y_d$, and $V_0 = 0$. By Lemma 3 in [1], we make the following substitution:

$$\begin{aligned} &X_{1o} := \begin{bmatrix} y_d & \check{\theta}' & \overleftarrow{\Sigma}' & s_\Sigma \end{bmatrix}' \to x_o; \quad X_{1a} := \check{x}_1 \to x_a; \quad X_{1d} := \begin{bmatrix} y_d^{(1)} & \Phi_1 \end{bmatrix}' \to x_d \\ &v \to w; \quad \check{x}_2 \to u; \quad \mathcal{D}_{1o} := \{X_{1o} : \check{\theta} \in \Theta_o\} \to D_o; \quad \mathcal{D}_{1a} := \mathbb{R} \to D_a \\ &\mathcal{D}_{1d} := \mathbb{R}^{1+\sigma} \to D_d; \quad \mathcal{D}_w := \mathbb{R} \to D_w; \quad \mathcal{D}_{1o} \times \mathcal{D}_{1d} \to D_1; \quad \gamma \to \gamma \\ &\text{Any number in } \mathbb{N} \to k; \quad V_0 \to V_o; \quad 0 \to \sigma_o; \quad \alpha_0 \to \alpha_o; \quad 0 \to l_o; \\ &F_{1o} := \begin{bmatrix} y_d^{(1)} \\ \delta \\ \overleftarrow{-(1-\epsilon)\Sigma\Phi_1'(\gamma^2\zeta^2 - 1)\Phi_1\Sigma} \\ (\gamma^2\zeta^2 - 1)(1-\epsilon)\Phi_1\Phi_1' \end{bmatrix} \to f_o \\ &H_{1o} := \begin{bmatrix} 0 & \kappa' & \overleftarrow{\mathbf{0}_{\sigma\times\sigma}} & 0 \end{bmatrix}' \to h_o; \quad F_{1a} := f_1 \to f_a \\ &G_{1a} := 1 \to g_a; \quad H_{1a} := h_1 \to h_a \end{aligned}$$

Choose a constant $\gamma_1 > 0$ and a smooth design function $\beta_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}$ such that

$$\beta_1(X_{1o}, X_{1a}, X_{1d}) \geq c_{\beta_1} > 0, \quad \forall (X_{1o}, X_{1a}, X_{1d}) \in \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d}$$

where c_{β_1} is a constant. Again, we make the following substitution by Lemma 3 in [1]:

$$\sqrt{\gamma_1} \to \delta; \quad (1+\beta_1)z_1 \to \phi; \quad V_1 \to V; \quad \alpha_1 \to \alpha$$

Then, $V_1 = \gamma_1 z_1^2$, $V_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \to \mathbb{R}$, and $\alpha_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}$ are smooth and such that

$$\begin{aligned} \dot{V}_1(X_{1o}, X_{1a}, X_{1d}, \check{x}_2, v)|_{\check{x}_2=\alpha_1(X_{1o},X_{1a},X_{1d})} &= -z_1^2 - \beta_1 z_1^2 + \gamma^2 v^2 - \gamma^2 (v - \nu_1(X_{1o}, X_{1a}, X_{1d}))^2; \\ &\quad \forall X_{1o} \in \mathcal{D}_{1o}, \quad \forall X_{1a} \in \mathcal{D}_{1a}, \quad \forall X_{1d} \in \mathcal{D}_{1d}, \quad \forall v \in \mathcal{D}_w \end{aligned}$$

where $\nu_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}$ is smooth and appropriately defined in Lemma 3 in [1].

If $\check{x}_2$ had been the actual control input, then we would have used the following virtual control law $\check{x}_2 = \alpha_1(X_{1o}, X_{1a}, X_{1d})$ to guarantee the dissipation inequality with supply rate

$$-z_1^2 - \beta_1 z_1^2 + \gamma^2 v^2$$

This completes the first step of the backstepping design.

Step i, $1 < i < r$: We use Lemma 3 in [1] to finish the design at this step. Assume we have completed $i-1$ steps of the backstepping procedure, and obtained

$$\begin{aligned}
X_{jo} &:= \begin{bmatrix} y_d & \check{\theta}' & \overleftarrow{\Sigma}' & s_\Sigma & \check{x}_1 & y_d^{(1)} & \Phi_1 & \ldots & \check{x}_{j-1} & y_d^{(j-1)} & \Phi_{j-1} \end{bmatrix}'; \\
& \quad\; j = 1, \ldots, i-1 && \text{(27a)} \\
\mathcal{D}_{jo} &:= \{X_{jo} : \ \check{\theta} \in \Theta_o\}; \quad j = 1, \ldots, i-1; && \text{(27b)} \\
X_{ja} &:= \check{x}_j; \quad \mathcal{D}_{ja} := \mathbb{R}; \quad j = 1, \ldots, i-1 && \text{(27c)} \\
X_{jd} &:= \begin{bmatrix} y_d^{(j)} & \Phi_j \end{bmatrix}'; \quad \mathcal{D}_{jd} := \mathbb{R}^{1+\sigma}; \quad j = 1, \ldots, i-1 && \text{(27d)} \\
\alpha_j &: \quad \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}; \quad j = 1, \ldots, i-1 && \text{(27e)} \\
\beta_j &: \quad \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}; \quad \beta_j(X_{jo}, X_{ja}, X_{jd}) \geq c_{\beta_j} > 0; \quad c_{\beta_j} \in \mathbb{R}; \\
& \quad\; j = 1, \ldots, i-1 && \text{(27f)} \\
\nu_{i-1} &: \quad \mathcal{D}_{i-1\,o} \times \mathcal{D}_{i-1\,a} \times \mathcal{D}_{i-1\,d} \to \mathbb{R}; && \text{(27g)} \\
z_j &= \check{x}_j - \alpha_{j-1}(X_{jo}); \quad j = 1, \ldots, i-1 && \text{(27h)} \\
\gamma_j &> 0; \quad \gamma_j \in \mathbb{R}; \quad j = 1, \ldots, i-1 && \text{(27i)} \\
V_{i-1} &= \sum_{j=1}^{i-1} \gamma_j z_j^2; \quad V_{i-1} : \mathcal{D}_{i-1\,o} \times \mathcal{D}_{i-1\,a} \to \mathbb{R}; && \text{(27j)}
\end{aligned}$$

$$\begin{aligned}
& \dot{V}_{i-1}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d}, \check{x}_i, v)|_{\check{x}_i = \alpha_{i-1}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d})} = -z_1^2 - \sum_{j=1}^{i-1} \beta_j z_j^2 + \gamma^2 v^2 \\
& \qquad\qquad\qquad -\gamma^2 (v - \nu_{i-1}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d}))^2; \\
& \qquad \forall (X_{i-1\,o}, X_{i-1\,d}) \in \mathcal{D}_{i-1\,o} \times \mathcal{D}_{i-1\,d}, \quad \forall X_{i-1\,a} \in \mathcal{D}_{i-1\,a}, \quad \forall v \in \mathcal{D}_w && \text{(27k)} \\
\dot{X}_{i-1\,o} &= F_{i-1\,o}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d}) + H_{i-1\,o}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d}) v; && \text{(27l)} \\
\dot{X}_{i-1\,a} &= F_{i-1\,a}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d}) + G_{i-1\,a}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d}) \check{x}_i \\
& \quad + H_{i-1\,a}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d}) v; && \text{(27m)}
\end{aligned}$$

where the nonlinear function α_j, β_j, $j = 1, \cdots, i-1$, $F_{i-1\,o}$, $H_{i-1\,o}$, $F_{i-1\,a}$, $G_{i-1\,a}$, $H_{i-1\,a}$ and ν_{i-1} are smooth on their domain of definition, which is $\mathcal{D}_{i-1\,o} \times \mathcal{D}_{i-1\,a} \times \mathcal{D}_{i-1\,d}$; V_{i-1} is a smooth function on its domain of definition.

At step i, define

$$z_i = \check{x}_i - \alpha_{i-1}(X_{i-1\,o}, X_{i-1\,a}, X_{i-1\,d})$$

Again, we apply Lemma 3 in [1] to make the following substitutions

$$X_{io} := \begin{bmatrix} X'_{i-1\,o} & X_{i-1\,a} & X'_{i-1\,d} \end{bmatrix}' \to x_o; \quad X_{ia} := \check{x}_i \to x_a; \quad X_{id} = \begin{bmatrix} y_d^{(i)} & \Phi_i \end{bmatrix}' \to x_d;$$

$$\mathcal{D}_{io} := \mathcal{D}_{i-1\,o} \times \mathcal{D}_{i-1\,a} \times \mathcal{D}_{i-1\,d} = \{X_{io} : \breve{\theta} \in \Theta_o\} \to D_o; \quad \mathcal{D}_{ia} := \mathbb{R} \to D_a$$
$$v \to w; \quad \breve{x}_{i+1} \to u; \quad \mathcal{D}_{id} := \mathbb{R}^{1+\sigma} \to D_d; \quad \mathcal{D}_w \to D_w \quad \mathcal{D}_{io} \times \mathcal{D}_{id} \to D_1; \quad V_{i-1} \to V_o$$
$$\nu_{i-1} \to \sigma_o; \quad \alpha_{i-1} \to \alpha_o; \quad z_1^2 + \sum_{j=1}^{i-1} \beta_j z_j^2 \to l_o; \quad \text{Any number in } \mathbb{N} \to k; \quad \gamma \to \gamma$$
$$F_{io} := \begin{bmatrix} F'_{i-1\,o} & F_{i-1\,a} + G_{i-1\,a}\breve{x}_i & y_d^{(i)} & \psi_{i-1} + \Phi_i \end{bmatrix}' \to f_o; \quad F_{ia} := f_i \to f_a;$$
$$H_{io} := \begin{bmatrix} H'_{i-1\,o} & H'_{i-1\,a} & 0 & e'_{n,i-1}\bar{A}_{211}\zeta^{-1} \end{bmatrix}' \to h_o; \; G_{ia} := 1 \to g_a; \quad H_{ia} := h_i \to h_a;$$

Note that

$$X_{io} := \begin{bmatrix} y_d & \breve{\theta}' & \overleftarrow{\Sigma}' & s_\Sigma & \breve{x}_1 & y_d^{(1)} & \Phi_1 & \ldots & \breve{x}_{i-1} & y_d^{(i-1)} & \Phi_{i-1} \end{bmatrix}'$$

Choose a constant $\gamma_i > 0$ and a smooth design function $\beta_i : \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id} \to \mathbb{R}$ such that

$$\beta_i(X_{io}, X_{ia}, X_{id}) \geq c_{\beta_i} > 0, \quad \forall (X_{io}, X_{ia}, X_{id}) \in \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id}$$

where c_{β_i} is a constant. Again, we make the following substitution, by Lemma 3:

$$\sqrt{\gamma_i} \to \delta; \quad \beta_i z_i \to \phi; \quad V_i \to V; \quad \alpha_i \to \alpha$$

Then, $V_i = V_{i-1} + \gamma_i z_i^2$, $V_i : \mathcal{D}_{io} \times \mathcal{D}_{ia} \to \mathbb{R}$, and $\alpha_i : \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id} \to \mathbb{R}$ are smooth and such that

$$\begin{aligned} \dot{V}_i(X_{io}, X_{ia}, X_{id}, \breve{x}_{i+1}, v)|_{\breve{x}_{i+1}=\alpha_i(X_{io},X_{ia},X_{id})} &= -z_1^2 - \sum_{j=1}^{i} \beta_j z_j^2 + \gamma^2 v^2 - \gamma^2 (v - \nu_i(X_{io}, X_{ia}, X_{id}))^2; \\ &\forall (X_{io}, X_{id}) \in \mathcal{D}_{io} \times \mathcal{D}_{id}, \quad \forall X_{ia} \in \mathcal{D}_{ia}, \quad \forall v \in \mathcal{D}_w \end{aligned}$$

where $\nu_i : \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id} \to \mathbb{R}$ is smooth and properly defined.

If $\breve{x}_{i+1}$ had been the actual control input, then we would have used the following virtual control law $\breve{x}_{i+1} = \alpha_i(X_{io}, X_{ia}, X_{id})$ to guarantee the dissipation inequality with supply rate

$$-z_1^2 - \sum_{j=1}^{i} \beta_j z_j^2 + \gamma^2 v^2$$

This completes this step of backstepping design.

Step r: First define the transformed variable

$$z_r = \breve{x}_r - \alpha_{r-1}(X_{r-1\,o}, X_{r-1\,a}, X_{r-1\,d}) \tag{28}$$

Similar as step i, we apply Lemma 3 in [1] to make the following substitution

$$X_{ro} := \begin{bmatrix} X'_{r-1\,o} & X_{r-1\,a} & X'_{r-1\,d} \end{bmatrix}' \to x_o; \quad X_{ra} := \breve{x}_r \to x_a; \quad v \to w; \quad u \to u;$$
$$X_{rd} = \begin{bmatrix} y_d^{(r)} & \Phi_r & \breve{x}_{r+1} & \ldots & \breve{x}_n & \Phi_{r+1} & \ldots & \Phi_n \end{bmatrix}' \to x_d; \quad \mathcal{D}_{ra} := \mathbb{R} \to D_a$$
$$\mathcal{D}_{ro} := \mathcal{D}_{r-1\,o} \times \mathcal{D}_{r-1\,a} \times \mathcal{D}_{r-1\,d} = \{X_{ro} : \breve{\theta} \in \Theta_o\} \to D_o; \quad \mathcal{D}_w :\to D_w$$

$$\mathcal{D}_{rd} := \mathbb{R}^{1+(n-r+1)\sigma+n-r} \to D_d; \quad \mathcal{D}_{ro} \times \mathcal{D}_{rd} \to D_1; \quad V_{r-1} = \sum_{i=1}^{r-1} \gamma_i z_i^2 \to V_o$$

$$\nu_{r-1} \to \sigma_o; \quad \alpha_{r-1} \to \alpha_o; \quad z_1^2 + \sum_{j=1}^{r-1} \beta_j z_j^2 \to l_o; \quad \text{Any number in } \mathbb{N} \to k; \quad \gamma \to \gamma$$

$$F_{ro} := \left[\begin{array}{cccc} F'_{r-1\,o} & F_{r-1\,a} + G_{r-1\,a}\check{x}_r & y_d^{(r)} & \psi_{r-1} + \Phi_r \end{array} \right]' \to f_o;$$

$$H_{ro} := \left[\begin{array}{cccc} H'_{r-1\,o} & H'_{r-1\,a} & 0 & e'_{n,r-1}\bar{A}_{211}\zeta^{-1} \end{array} \right]' \to h_o; \; F_{ra} := f_r + a_{r\,r+1}\check{x}_{r+1} + \cdots a_{rn}\check{x}_n \to f_a$$

$$G_{ra} := b_{p0} + \bar{A}_{212\,0}\check{\theta} \to g_a; \quad H_{ra} := h_r \to h_a;$$

Note that

$$X_{ro} := \left[\begin{array}{cccccccccc} y_d & \check{\theta}' & \overleftarrow{\Sigma}' & s_\Sigma & \check{x}_1 & y_d^{(1)} & \Phi_1 & \ldots & \check{x}_{r-1} & y_d^{(r-1)} & \Phi_{r-1} \end{array} \right]'$$

Choose a constant $\gamma_r > 0$ and a smooth design function $\beta_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$ such that

$$\beta_r(X_{ro}, X_{ra}, X_{rd}) \geq c_{\beta_r} > 0, \quad \forall (X_{ro}, X_{ra}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd}$$

where c_{β_r} is a constant. Again, we make the following substitution, by Lemma 3 in [1]:

$$\sqrt{\gamma_r} \to \delta; \quad \beta_r z_r \to \phi; \quad V_r \to V; \quad \mu \to \alpha$$

Then, $V_r = V_{r-1} + \gamma_r z_r^2$, $V_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \to \mathbb{R}$, and $\mu : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$ are smooth and such that

$$\begin{aligned} \dot{V}_r(X_{ro}, X_{ra}, X_{rd}, u, v)|_{u=\mu(X_{ro}, X_{ra}, X_{rd})} &= -z_1^2 - \sum_{j=1}^{r} \beta_j z_j^2 + \gamma^2 v^2 - \gamma^2 (v - \nu_r(X_{ro}, X_{ra}, X_{rd}))^2; \\ &\quad \forall (X_{ro}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{rd}, \quad \forall X_{ra} \in \mathcal{D}_{ra}, \quad \forall v \in \mathcal{D}_w \end{aligned}$$

where $\nu_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$ is smooth and appropriately defined.

Hence, we finally finish the design of control function

$$u = \mu(X_{ro}, X_{ra}, X_{rd}) \tag{29}$$

We note that the value function $V = V_r$.

This completes the backstepping design procedure.

In summary, for both of two cases: $r = 1$ and $r > 1$, we have obtained X_{jo}, X_{ja}, X_{jd}, $\mathcal{D}_{jo}$, $\mathcal{D}_{ja}$, $\mathcal{D}_{jd}$, $\gamma_j \in \mathbb{R}$, $\alpha_0 : \mathcal{D}_{1o} \to \mathbb{R}$, $\beta_j : \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}$, $j = 1, \cdots, r$, $\alpha_j : \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}$, $j = 1, \cdots, r-1$, $\nu_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$, $V : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \to \mathbb{R}$, and $\mu : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$ such that $X_{j+1\,o} = [X'_{jo}\ X_{ja}\ X'_{jd}]'$, $\mathcal{D}_{j+1\,o} = \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd}$, $j = 1, \ldots, r-1$, α_j, $j = 0, \cdots, r-1$, ν_r, V, and μ are smooth, $\gamma_j > 0$, $\beta_j(X_{jo}, X_{ja}, X_{jd}) \geq c_{\beta_j} > 0$, $\forall (X_{jo}, X_{ja}, X_{jd}) \in \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd}$, $X_{ja} = \check{x}_j$, $X_{jo} = \left[\begin{array}{ccccccccccc} y_d & \check{\theta}' & \overleftarrow{\Sigma}' & s_\Sigma & \check{x}_1 & y_d^{(1)} & \Phi_1 & \ldots & \check{x}_{j-1} & y_d^{(j-1)} & \Phi_{j-1} \end{array} \right]'$, $\mathcal{D}_{jo} := \{X_{jo} \mid \check{\theta} \in \Theta_o\}$, $j = 1, \cdots, r$, $X_{rd} = \left[\begin{array}{ccccccc} y_d^{(r)} & \Phi_r & \check{x}_{r+1} & \ldots & \check{x}_n & \Phi_{r+1} & \ldots & \Phi_n \end{array} \right]'$, $\mathcal{D}_{rd} = \{X_{rd}\}$, and

$$V(X_{ro}, X_{ra}) = \sum_{j=1}^{r} \gamma_j \left(\check{x}_j - \alpha_{j-1}(X_{jo})\right)^2$$

$$
\begin{aligned}
\dot V(X_{ro}, X_{ra}, X_{rd}, u, v)|_{u=\mu(X_{ro},X_{ra},X_{rd})} \quad &= \quad -(\check x_1 - \alpha_0(X_{1o}))^2 - \sum_{j=1}^{r} \beta_j\,(\check x_j - \alpha_{j-1}(X_{jo}))^2 \\
&\quad\quad +\gamma^2 v^2 - \gamma^2 (v - \nu_r(X_{ro}, X_{ra}, X_{rd}))^2 \\
&\quad\quad \forall (X_{ro}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{rd}, \quad \forall X_{ra} \in \mathcal{D}_{ra}, \quad \forall v \in \mathbb{R}
\end{aligned}
$$

In the following, we will always set $u = \mu(X_{ro}, X_{ra}, X_{rd})$.

Next, we consider the design of ξ_c. The derivative of V, when ξ_c is not necessarily 0, is given by

$$
\begin{aligned}
\dot V(X_{ro}, X_{ra}, X_{rd}, \hat\xi, v) = & -(\check x_1 - \alpha_0(X_{1o}))^2 - \sum_{j=1}^{r} \beta_j\,(\check x_j - \alpha_{j-1}(X_{jo}))^2 - \gamma^2 (v - \nu_r(X_{ro}, X_{ra}, X_{rd}))^2 \\
& +\gamma^2 v^2 + (\varsigma_r(X_{ro}, X_{ra}, X_{rd}))' \bar Q \xi_c \\
& \forall (X_{ro}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{rd}, \quad \forall X_{ra} \in \mathcal{D}_{ra}, \quad \forall v \in \mathbb{R}, \quad \forall \xi_c \in \mathbb{R}^{\sigma+n}
\end{aligned}
$$

where $\varsigma_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}^{\sigma+n}$ is smooth and appropriately defined. The closed-loop system admits the state vector

$$
X := \begin{bmatrix} \theta' & x' & X'_{ro} & X_{ra} & \Phi_r & \check x_{r+1} & \cdots & \check x_n & \Phi_{r+1} & \cdots & \Phi_n \end{bmatrix}'
$$

which belongs to the set $\mathcal{D} := \{X : \ \Sigma > 0, s_\Sigma > 0, \check\theta \in \Theta_o\}$. Based on the value functions of the estimation design and control design, we obtain the following value function for the closed-loop adaptive nonlinear system:

$$
U = V + W = |\theta - \check\theta|^2_{\Sigma^{-1}} + \gamma^2 |x - \check x - \Phi\,(\theta - \check\theta)|^2_{\Pi^{-1}} + \sum_{j=1}^{r} \gamma_j\,(\check x_j - \alpha_{j-1})^2 \tag{30}
$$

where $U : \mathcal{D} \to \mathbb{R}$ is smooth. The derivative of U along the solution of the closed-loop dynamics is given by

$$
\begin{aligned}
\dot U(X, y_d^{(r)}, \hat\xi, w) \quad = \quad & -|x_1 - y_d|^2 - \gamma^4 |x - \hat x - \Phi\,(\theta - \hat\theta)|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat\theta|^2_{\Phi' C' C \Phi} \\
& +2(\theta - \check\theta)' P_r(\check\theta) + |\xi_c|^2_{\bar Q(\Phi, s_\Sigma)} + (\varsigma_r(X_{ro}, X_{ra}, X_{rd}))' \bar Q(\Phi, s_\Sigma)\xi_c - \sum_{j=1}^{r} \beta_j z_j^2 \\
& +\gamma^2 |w|^2 - \gamma^2 |w - w_*(\xi, \check\xi, \bar\Sigma, w)|^2 - \gamma^2 |v - \nu_r(X_{ro}, X_{ra}, X_{rd})|^2 \\
= \quad & -|x_1 - y_d|^2 - \gamma^4 |x - \hat x - \Phi\,(\theta - \hat\theta)|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat\theta|^2_{\Phi' C' C \Phi} \\
& +|\xi_c + \frac{1}{2}\varsigma_r(X_{ro}, X_{ra}, X_{rd})|^2_{\bar Q(\Phi, s_\Sigma)} - \frac{1}{4}|\varsigma_r(X_{ro}, X_{ra}, X_{rd})|^2_{\bar Q(\Phi, s_\Sigma)} \\
& +2(\theta - \check\theta)' P_r(\check\theta) - \sum_{j=1}^{r} \beta_j z_j^2 + \gamma^2 |w|^2 - \gamma^2 |w - w_{opt}(X, y_d^{(r)})|^2 \\
& \forall X \in \mathcal{D}, \quad \forall y_d^{(r)} \in \mathbb{R}, \quad \forall \hat\xi \in \mathbb{R}^{\sigma+n}, \quad \forall w \in \mathbb{R}^q
\end{aligned} \tag{31}
$$

where the worst-case disturbance with respect to the value function U is

$$
w_{opt}(X, y_d^{(r)}) = \zeta E' \nu_r + \gamma^{-2}\,(I_q - \zeta^2 E' E) \bar D' \bar\Sigma^{-1}\,(\xi - \check\xi) + \zeta^2 E' C\,(\check x - x) \tag{32}
$$

and w_{opt}:$\mathcal{D} \times \mathbb{R} \to \mathbb{R}^q$.

The design of ξ_c is based on two considerations: (1). to guarantee $[0\ \gamma^{-2}\Pi]\bar{Q}\xi_c$ is bounded, which is applied later in the proof of Theorem 1; (2). to guarantee the terms $|\xi_c+\frac{1}{2}\varsigma_r|^2_{\bar{Q}}-\frac{1}{4}|\varsigma_r|^2_{\bar{Q}}+|\xi_c|^2_{\bar{Q}}$ is non-positive. Then, we choose ξ_c to be

$$\xi_c := \mu_\xi(X_{ro}, X_{ra}, X_{rd}) = -\frac{1}{2}\varepsilon_0(X_{ro}, X_{ra}, X_{rd})\varsigma_r(X_{ro}, X_{ra}, X_{rd}) \tag{33}$$

where

$$\varepsilon_0(X_{ro}, X_{ra}, X_{rd}) := \frac{|k_c|}{\sqrt{k_c^2 + |[\mathbf{0}\ \ \gamma^{-2}\Pi]\,\bar{Q}(\Phi, s_\Sigma)\,\varsigma_r(X_{ro}, X_{ra}, X_{rd})|^2 + 1}} \tag{34}$$

and $k_c \in \mathbb{R}$ is a constant. Clearly, μ_ξ: $\mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}^{\sigma+n}$ is a smooth function. We observe that $\xi_c \equiv 0$ when $k_c = 0$. This special case will result in simpler control system structure.

Till now, we have finished the design for the entire closed-loop system, which involves state X. We shall turn to study the robustness and tracking properties of the closed-loop system in the next section.

6 Main Result

In this section, we investigate the robustness and tracking properties of the closed-loop system with the reduced-order controller. The main achievements are stated in the form of a theorem.

The closed-loop system dynamics can be expressed as

$$\dot{X} = F(X, y_d^{(r)}) + G(X)w = F(X, y_d^{(r)}) + G(X)\grave{M}\grave{w}; \quad X(0) = X_0 \tag{35}$$

where the function F and G are smooth mappings of $\mathcal{D} \times \mathbb{R}$ and $\mathcal{D}$, respectively; and the initial state X_0 satisfies

$$\begin{aligned} X_0 \in \mathcal{D}_0 \quad := \quad & \{X_0 \in \mathcal{D} \ \mid \ \theta \in \Theta,\, \check{\theta}_0 \in \Theta,\ \Sigma(0) = \gamma^{-2}Q_0^{-1} > 0,\ \mathrm{Tr}((\Sigma(0))^{-1}) \le K_c, \\ & s_\Sigma(0) = \gamma^2\mathrm{Tr}(Q_0)\} \end{aligned}$$

As described in [1], the value function U satisfies the Hamilton-Jacobi-Isaacs equation

$$\begin{aligned} \frac{\partial U}{\partial X}(X)F(X, y_d^{(r)}) + \frac{1}{4\gamma^2}\frac{\partial U}{\partial X}(X)G(X)(G(X))'\left(\frac{\partial U}{\partial X}(X)\right)' + Q(X, y_d^{(r)}) = 0; \\ \forall X \in \mathcal{D}, \forall y_d^{(r)} \in \mathbb{R} \end{aligned} \tag{36}$$

where the function $Q : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is smooth and given by

$$\begin{aligned} Q(X, y_d^{(r)}) \quad = \quad & |x_1 - y_d|^2 + \gamma^4|x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C\Phi} \\ & -2(\theta - \check{\theta})'P_r(\check{\theta}) - (\varepsilon_0^2 - 2\varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/4 + \sum_{j=1}^{r}\beta_j z_j^2 \end{aligned}$$

The stability of the closed-loop system can not be deduced directly from the value function U, which is not a positive-definite function for the closed-loop system. The following theorem will state the stability properties of the closed-loop system, it shows that the closed-loop system admits strong robustness properties.

Theorem 1 *Consider the robust adaptive control problem formulated in Section 2, with Assumptions 1 − 7 holding. Then, the robust adaptive controller μ defined by (26) or (29), with ξ_c given by (33), achieves the following strong robustness properties for the closed-loop system.*

1. *Given $c_w \geq 0$ and $c_d \geq 0$, there exists a constant $c_c \geq 0$ and a compact set $\Theta_c \subset \Theta_o$, such that, for any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$ with*

$$|x_0| \leq c_w; \quad |\grave{w}(t)| \leq c_w; \quad |Y_d(t)| \leq c_d; \quad \forall t \in [0, \infty)$$

all closed-loop state variables x, $\check{x}$, $\check{\theta}$, Σ, s_Σ, and Φ are bounded as follows, $\forall t \in [0,\infty)$

$$|x(t)| \leq c_c, \quad |\check{x}(t)| \leq c_c, \quad \check{\theta}(t) \in \Theta_c, \quad , |\overrightarrow{\Phi(t)}| \leq c_c$$

$$K_c^{-1} I_\sigma \leq \Sigma(t) \leq \gamma^{-2} Q_0^{-1}, \qquad \gamma^2 \mathrm{Tr}(Q_0) \leq s_\Sigma(t) \leq K_c$$

therefore, there is a compact set $S \subset \mathcal{D}$ such that $X(t) \in S$, $\forall t \in [0,\infty)$. Hence, there exists a constant $c_u \geq 0$ such that $|u(t)| \leq c_u$, $|\hat{\xi}(t)| \leq c_u$, $|\eta(t)| \leq c_u$, $|\lambda(t)| \leq c_u$, and $|\lambda_o(t)| \leq c_u$, $\forall t \in [0,\infty)$.

2. *The controller μ belongs to $\mathcal{M}_u$ and achieves disturbance attenuation level γ for any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$.*

3. *For any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$ with $\grave{w}_{[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $Y_{d[0,\infty)} \in \mathcal{L}_\infty$, the output of the system x_1 asymptotically tracks the reference trajectory y_d, i.e.,*

$$\lim_{t\to\infty} (x_1(t) - y_d(t)) = 0$$

Proof For the first statement, fix any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$ such that $|x_0| \leq c_w$, $|\grave{w}(t)| \leq c_w$, $|Y_d(t)| \leq c_d$, $\forall t \in [0,\infty)$, for some $c_w \geq 0$ and $c_d \geq 0$. Consider the maximal length interval $[0, T_f)$ such that the differential equation (35) for the closed-loop system admits a solution that lies in $\mathcal{D}$ on this interval. It will be shown that the maximal length of the interval T_f is always ∞.

Under the definition of $\epsilon(t) = \mathrm{Tr}((\Sigma(t))^{-1})/K_c$, Lemma 1 in [1] guarantees that the covariance matrix Σ and then the signal s_Σ are boundeded as desired. If $\epsilon(t)$ is simply defined to be 1, Σ and s_Σ are constants, and therefore bounded as desired.

Define $\eta_c := \tilde{x} - \Phi\tilde{\theta}$. Introduce the new vector $X_e := \begin{bmatrix} \check{\theta}' & \eta_c' & z_1 & \cdots & z_r \end{bmatrix}'$, where $X_e : [0, T_f) \to \mathcal{D}_e := \{X_e : \check{\theta} \in \Theta_o\}$. Then, the value function U can be explained as $U = \bar{U}(t, X_e(t))$, where $\bar{U} : [0, T_f) \times \mathbb{R}^{n+\sigma+r} \to \mathbb{R}$. The derivative of $\bar{U}$ satisfies the following inequality:

$$\begin{aligned}
\dot{\bar{U}} &\leq -|\xi - \hat{\xi}|^2_{\bar{Q}} + (\varepsilon_0^2 - 2\varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/4 + 2(\theta - \check{\theta})' P_r(\check{\theta}) - \sum_{j=1}^{r} \beta_j z_j^2 + \gamma^2 \|\grave{M}\|_2^2 c_w^2 \\
&= -|\xi - \hat{\xi}|^2_{\bar{Q}} - |\xi_c|^2_{\bar{Q}} + (\varepsilon_0^2 - 2\varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/4 + |\xi_c|^2_{\bar{Q}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - \sum_{j=1}^{r} \beta_j z_j^2 + \gamma^2 \|\grave{M}\|_2^2 c_w^2 \\
&= -\frac{1}{2}|\xi - \check{\xi}|^2_{\bar{Q}} - \frac{1}{2}|\xi - 2\hat{\xi} + \check{\xi}|^2_{\bar{Q}} + (\varepsilon_0^2 - \varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/2 + 2(\theta - \check{\theta})' P_r(\check{\theta}) - \sum_{j=1}^{r} \beta_j z_j^2 + \gamma^2 \|\grave{M}\|_2^2 c_w^2
\end{aligned}$$

Since $0 \leq \varepsilon_0 < 1$, $(\varepsilon_0^2 - \varepsilon_0)|\varsigma_r|_{\bar{Q}}^2/2 \leq 0$. Then, we have

$$\dot{\bar{U}} \quad \leq \quad -\gamma^4/2|\eta_c|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - \sum_{j=1}^{r} c_{\beta_j} z_j^2 + \gamma^2 \|\grave{M}\|_2^2 c_w^2 \tag{37}$$

The right part of the inequality (37) tends to be negative infinity as X_e approaches to the boundary of $\mathcal{D}_e$. Then there exists a compact set $\Omega_1(c_w) \subset \mathcal{D}_e$ such that, $\forall t \in [0, T_f)$, if $X_e(t) \in \mathbb{R}^{n+\sigma+r} \setminus \Omega_1(c_w)$ then $X_e(t) \in \mathcal{D}_e \setminus \Omega_1(c_w)$ and $\dot{U} < 0$. Define

$$\begin{aligned} U_M(X_e) \quad &:= \quad K_c|\theta - \check{\theta}|^2 + \gamma^2|\eta_c|^2_{\Pi^{-1}} + \sum_{j=1}^{r} \gamma_j z_j^2 \\ U_m(X_e) \quad &:= \quad \gamma^2|\theta - \check{\theta}|^2_{Q_0} + \gamma^2|\eta_c|^2_{\Pi^{-1}} + \sum_{j=1}^{r} \gamma_j z_j^2 \end{aligned}$$

Then, we have $U_m(X_e) \leq \bar{U}(t, X_e) \leq U_M(X_e)$, $\forall t \in [0, T_f)$, $\forall X_e \in \mathbb{R}^{n+\sigma+r}$. Define sets $S_{1\alpha} := \{X_e \in \mathbb{R}^{n+\sigma+r} \mid U_m(X_e) \leq \alpha\}$, $\alpha \in \mathbb{R}$. Then, they are compact or empty sets, since $U_m(X_e)$ is positive definite, continuous and radially unbounded. According to Lemma 9 in [1], we make the following institutions:

$$\begin{aligned} &n + \sigma + r \to n;\ \mathbb{R}^{n+\sigma+r} \to D;\ [0, T_f) \to [t_0, t_1);\ \Omega_1 \to K;\ X_e \to \xi; \\ &\bar{U} \to V;\ U_M \to W_2;\ U_m \to W_1;\ S_{1\alpha} \to S_{1\alpha};\ c_1 \to \eta;\ S_{1c_1} \to S_{1\eta} \end{aligned}$$

Then, there is a compact set $S_{1c_1} \subset \mathbb{R}^{n+\sigma+r}$ such that $X_e(t) \in S_{1c_1}$, $\forall t \in [0, T_f)$, for some $c_1 \in \mathbb{R}$. Therefore, we can conclude that the elements in X_e: $\tilde{\theta}$, η_c, and $z_1, \cdots, z_r$ are bounded.

Considering the state η_c, whose derivative (see Appendix A for derivation detail) is given by

$$\dot{\eta}_c = A_f \eta_c + \gamma^{-2} \Pi C' (y_d - y) + Dw - (\zeta^2 L + \Pi C' (\zeta^2 - \gamma^{-2}))Ew - \frac{1}{2}[0 \;\; \gamma^{-2}\Pi]\bar{Q}\varepsilon_0\varsigma_r$$

then we express η_c in two parts

$$\begin{aligned} \eta_c \quad &= \quad \eta_{cy} + \eta_{cy_d} &(38a) \\ \dot{\eta}_{cy} \quad &= \quad A_f \eta_{cy} - \gamma^{-2}\Pi C' y &(38b) \\ \dot{\eta}_{cy_d} \quad &= \quad A_f \eta_{cy_d} + \gamma^{-2}\Pi C' y_d - (\zeta^2 L + \Pi C' (\zeta^2 - \gamma^{-2}))Ew + Dw - \frac{1}{2}[0 \;\; \gamma^{-2}\Pi]\bar{Q}\varepsilon_0\varsigma_r &(38c) \end{aligned}$$

Note that A_f is Hurwitz, y_d and w are bounded, $\frac{1}{2}[0 \;\; \gamma^{-2}\Pi]\bar{Q}\varepsilon_0\varsigma_r$ is bounded via the design of ξ_c. Therefore, η_{cy_d} is bounded, so is η_{cy} by the boundedness of η_c and η_{cy_d}.

By the dynamics (38b) of η_{cy} and Lemma 4, there is a particular linear combination of the components of η_{cy}, denoted by η_L, which is bounded, strictly minimum phase and has relative degree 1 with respect to the input y. By Lemma 5, the η_{cy} dynamics with input y and output η_L may serve as a reference system mentioned in Lemma 11 in [1]. Applying Lemma 14 in [1], we know that η_L is strictly minimum phase and has relative degree $r + 1$ with respect to the input u; and the composite system with input u and output η_L may also serve as a reference system.

To analyze the boundedness of η, we observe that the relative degree for each of the elements of η is at least 1 with respect to the input y, and is the output of a stable linear system. By Lemma 11 in [1], we have that η is bounded, where the reference system has output η_L and input y.

Next consider the boundedness of Φ, which can be decomposed into y dependent part and u dependent part as follows

$$
\begin{aligned}
\Phi &= \Phi_y + \Phi_u && \text{(39a)}\\
\Phi_y &= \left[\begin{array}{cccc} A_f^{n-1}\eta & \dots & A_f\eta & \eta \end{array}\right] M_f^{-1}\bar{A}_{211} =: \left[\begin{array}{c} \eta' T_1 \\ \vdots \\ \eta' T_n \end{array}\right] && \text{(39b)}\\
\dot{\Phi}_u &= A_f\Phi_u + u\bar{A}_{212}; \qquad \Phi_u(0) = \Phi_{u0} && \text{(39c)}
\end{aligned}
$$

where Φ_{u0} is the matrix such that $\Phi_y(0) + \Phi_{u0} = \Phi_0$. Clearly, Φ_y is bounded since η is bounded. Then, we can conclude the boundedness of Φ if Φ_u is proved to be bounded. Express Φ_u in terms of its row vectors $\left[\begin{array}{ccc} \Phi'_{u1} & \cdots & \Phi'_{un} \end{array}\right]'$. Using a similar argument as that in [1], we consider the boundedness of Φ_{u1} in two steps. Define

$$
\begin{aligned}
\dot{\lambda}_b &= A_f\lambda_b + e_{n,r}u; \quad \lambda_b(0) = \mathbf{0}_{n\times 1} && \text{(40a)}\\
\dot{\Phi}_{u_s} &= A_f\Phi_{u_s} + u\left[\begin{array}{c} \mathbf{0}_{r\times\sigma} \\ \bar{A}_{212\,r} \end{array}\right]; \quad \Phi_{u_s}(0) = \Phi_{u0} && \text{(40b)}
\end{aligned}
$$

Then, we have

$$\Phi_u = \Phi_{u_s} + \lambda_b\bar{A}_{212\,0}$$

Express λ_b in terms of its elements $\left[\begin{array}{ccc} \lambda_{b1} & \dots & \lambda_{bn} \end{array}\right]'$; and Φ_{u_s} in terms of its row vectors $\left[\begin{array}{ccc} \Phi'_{u_s1} & \dots & \Phi'_{u_sn} \end{array}\right]'$

The relative degree for each of the elements of Φ_{u_s1} is at least $r+1$ with respect to the input u, and is the output of a stable linear system. By lemma 11 in [1], this yields that Φ_{u_s1} is bounded, where the reference system has output η_L and input u and $\grave{w}$.

By the boundedness of $\tilde{x}-\Phi\tilde{\theta}$, η, Φ_{u_s1}, and $\tilde{\theta}$, we have that the signal $\tilde{x}_1 - \lambda_{b1}\bar{A}_{212\,0}\tilde{\theta}$ is bounded, so is $\check{x}_1$ in view of the boundedness of z_1 and y_d.

Notice the signal $\tilde{x}_1 - \lambda_{b1}\bar{A}_{212\,0}\tilde{\theta} = (x_1 - \lambda_{b1}b_0) - (\check{x}_1 - \lambda_{b1}(b_{p0} + \bar{A}_{212\,0}\check{\theta}))$. We analyze the item $x_1 - \lambda_{b1}b_0$ first, which can be generated by

$$
\begin{aligned}
\dot{x} - b_0\dot{\lambda}_b &= A_f(x - b_0\lambda_b) + \left[\begin{array}{c} \mathbf{0}_{r\times 1} \\ \bar{A}_{212r}\theta \end{array}\right] u + (\zeta^2 L + \Pi C'(\zeta^2 - \gamma^{-2}))(y - E\grave{M}\grave{w}) \\
&\quad + \left[\begin{array}{cccc} \mathbf{0}_{1\times r} & b_{p1} & \cdots & b_{p\,n-r} \end{array}\right]' u + \bar{A}_{211}\theta y + D\grave{M}\grave{w} && \text{(41a)}\\
x_1 - b_0\lambda_{b1} &= C(x - b_0\lambda_b) && \text{(41b)}
\end{aligned}
$$

Recalling the way to decompose Φ, we similarly decompose the signal $x_1 - \lambda_{b1}b_0$ into y dependent part and u dependent part:

$$
\begin{aligned}
\dot{x}_u &= A_f x_u + \left[\begin{array}{c} \mathbf{0}_{r\times 1} \\ \bar{A}_{212r}\theta \end{array}\right] u + \left[\begin{array}{cccc} \mathbf{0}_{1\times r} & b_{p1} & \cdots & b_{p\,n-r} \end{array}\right]' u; \;\; x_u(0) = x_0; && \text{(42a)}\\
x_{u1} &= Cx_u && \text{(42b)}
\end{aligned}
$$

$$\begin{aligned}
\dot{x}_y &= A_f x_y + (\zeta^2 L + \Pi C' (\zeta^2 - \gamma^{-2}))(y - E\grave{M}\grave{w}) + \bar{A}_{211}\theta y + D\grave{M}\grave{w}; \\
&\quad x_y(0) = \mathbf{0}_{n\times 1} && (42c) \\
x_{y1} &= C x_y && (42d) \\
x_1 - b_0\lambda_{b1} &= x_{u1} + x_{y1} && (42e)
\end{aligned}$$

The signal x_{u1} has relative degree at least $r+1$ with respect to u. It is bounded by Lemma 11 in [1], where the reference system has input u and output η_L. The signal x_{y1} has relative degree at least 1 with respect to y. It is bounded by Lemma 11 in [1], where the reference system has input y and output η_L. Then, $x_1 - b_0\lambda_{b1}$ is bounded on $[0, T_f)$. It can be further concluded that $\check{x}_1 - \lambda_{b1}(b_{p0} + \bar{A}_{212\,0}\check{\theta})$ is bounded.

Since $\check{x}_1$ is bounded, and $\check{b}_0$ is bounded away from 0 by the fact that $\check{\theta}(t) \in \Theta_o$, $\forall t \in [0, T_f)$, we have the boundedness of the signal λ_{b1}. This further implies the boundedness of the signals Φ_{u1} and Φ_1 and the boundedness of x_1 and y because of the boundedness of $x_1 - b_0\lambda_{b1}$ and $\grave{w}$.

To show the existence of a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$, we similarly define the function as in [1]

$$\Upsilon = U + P(\check{\theta})\,(\rho_o - P(\check{\theta}))^{-1}$$

where Υ can be explained as $\Upsilon(X(t)) := \bar{\Upsilon}(t, X_e(t))$, where $\bar{\Upsilon} : [0, T_f) \times \mathcal{D}_e \to \mathbb{R}$ is $\mathcal{C}_1$. The time derivative of Υ along the solution of the dynamics (35) is given by

$$\begin{aligned}
\dot{\Upsilon} &= \dot{U} + \rho_o\,(\rho_o - P(\check{\theta}))^{-2}\frac{\partial P}{\partial \theta}(\check{\theta})\dot{\check{\theta}} \\
&\le -|x_1 - y_d|^2 - \gamma^4|x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C \Phi} + 2(\theta - \check{\theta})' P_r(\check{\theta}) \\
&\quad - \sum_{j=1}^{r} \beta_j z_j^2 + \gamma^2\|\grave{M}\|_2^2 c_w^2 + (\varepsilon_0^2 - 2\varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/4 + \rho_o\,(\rho_o - P(\check{\theta}))^{-2}\,\Big(-\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma P_r(\check{\theta}) \\
&\quad - \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\Phi' C'\,(y_d - \check{x}_1) + \frac{1}{2}\frac{\partial P}{\partial \theta}(\check{\theta})[\Sigma\ \ \Sigma\Phi']\bar{Q}\varepsilon_0\varsigma_r + \frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2\zeta^2\Sigma\Phi' C'\,(\tilde{x}_1 + E\grave{M}\grave{w})\Big) \\
&\le -\gamma^4|x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - |\xi_c|^2_{\bar{Q}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - \sum_{j=1}^{r}\beta_j z_j^2 + (\varepsilon_0^2 - \varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/2 \\
&\quad + \gamma^2\|\grave{M}\|_2^2 c_w^2 + \rho_o\,(\rho_o - P(\check{\theta}))^{-2}\,\Big(-\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma P_r(\check{\theta}) - \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\Phi' C'\,(y_d - \check{x}_1) \\
&\quad + \frac{1}{2}\frac{\partial P}{\partial \theta}(\check{\theta})[\Sigma\ \ \Sigma\Phi']\bar{Q}\varepsilon_0\varsigma_r + \frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2\zeta^2\Sigma\Phi' C'\,(\tilde{x}_1 + E\grave{M}\grave{w})\Big) \qquad (43)
\end{aligned}$$

Express $\varsigma_r = [\varsigma_{r1}'\ \varsigma_{r2}']'$,where ς_{r_1} and ς_{r_2} are σ- and n-dimensional, respectively. Then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial P}{\partial \theta}(\check{\theta})[\Sigma\ \ \Sigma\Phi']\bar{Q}\varepsilon_0\varsigma_r = \frac{1}{2}\frac{\partial P}{\partial \theta}(\check{\theta})\epsilon\Sigma\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi\varepsilon_0\varsigma_{r1} \\
&-|\xi_c|^2_{\bar{Q}} = -\gamma^4|\hat{x} - \check{x} - \Phi\,(\hat{\theta} - \check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \frac{1}{4}\epsilon(\gamma^2\zeta^2 - 1)|\varepsilon_0\varsigma_{r1}|^2_{\Phi' C' C\Phi}
\end{aligned}$$

Then, we have

$$-\gamma^4|x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - |\xi_c|^2_{\bar{Q}} + \frac{1}{2}\rho_o\,(\rho_o - P(\check{\theta}))^{-2}\frac{\partial P}{\partial \theta}(\check{\theta})[\Sigma\ \ \Sigma\Phi']\bar{Q}\varepsilon_0\varsigma_r$$

$$
\begin{aligned}
&= -\gamma^4/2|x-\check{x}-\Phi(\theta-\check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \gamma^4/2|x-2\hat{x}+\check{x}-\Phi(\theta-2\hat{\theta}+\check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} \\
&-\frac{1}{4}\epsilon(\gamma^2\zeta^2-1)\left|\varepsilon_0\varsigma_{r1}-\rho_o(\rho_o-P(\check{\theta}))^{-2}\Sigma\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2_{\Phi'C'C\Phi} \\
&+\frac{1}{4}\epsilon(\gamma^2\zeta^2-1)\rho_o^2(\rho_o-P(\check{\theta}))^{-4}\left|\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2_{\Sigma\Phi'C'C\Phi\Sigma}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\dot{\Upsilon} \;\le\; & -\gamma^4/2|x-\check{x}-\Phi(\theta-\check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta-\check{\theta})'P_r(\check{\theta}) - \sum_{j=1}^{r} c_{\beta_j} z_j^2 + \gamma^2\|\grave{M}\|_2^2 c_w^2 \\
& -\frac{\rho_o}{K_c}p_r(\check{\theta})(\rho_o-P(\check{\theta}))^{-2}\left|\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2 + \frac{\epsilon}{4}(\gamma^2\zeta^2-1)\rho_o^2(\rho_o-P(\check{\theta}))^{-4}\left|\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2_{\Sigma\Phi'C'C\Phi\Sigma} \\
& +\frac{\rho_o^2}{4}(\rho_o-P(\check{\theta}))^{-4}\left|\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2 + \left|-\Sigma\Phi'C'(y_d-\check{x}_1)+\gamma^2\zeta^2\Sigma\Phi'C'(\tilde{x}_1+E\grave{M}\grave{w})\right|^2 \\
\le\; & -\gamma^4/2|x-\check{x}-\Phi(\theta-\check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta-\check{\theta})'P_r(\check{\theta}) - \sum_{j=1}^{r} c_{\beta_j} z_j^2 \qquad (44) \\
& -\frac{\rho_o}{K_c}p_r(\check{\theta})(\rho_o-P(\check{\theta}))^{-2}\left|\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2 + (\rho_o-P(\check{\theta}))^{-4}\left|\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2 c + c \qquad (45)
\end{aligned}
$$

for some constant $c > 0$. This inequality follows from the boundedness of y_d, x_1, $\check{x}_1$, $C\Phi$, and $\grave{w}$. The right-hand-side of the inequality tends to be $-\infty$ as X_e approaches to the boundary of $\mathcal{D}_e$. Therefore, there exist a compact set $\Omega_2 \subset \mathcal{D}_e$, such that $\forall X_e(t) \in \mathcal{D}_e \setminus \Omega_2$ and $\dot{\Upsilon} < 0$. Note also that

$$
\begin{aligned}
U_m(X_e) + P(\check{\theta})(\rho_o-P(\check{\theta}))^{-1} \le \bar{\Upsilon}(t,X_e) \le U_M(X_e) + P(\check{\theta})(\rho_o-P(\check{\theta}))^{-1} \\
\forall (t,X_e) \in [0,T_f)\times\mathcal{D}_e
\end{aligned}
$$

Define sets $S_{2\bar{\alpha}} := \{X_e \in \mathcal{D}_e \;\;|\;\; U_m(X_e) + P(\check{\theta})(\rho_o-P(\check{\theta}))^{-1} \le \bar{\alpha}\}$, $\bar{\alpha} \in \mathbb{R}$. Then, they are compact or empty. By Lemma 9 in [1], we make the following substitutions:

$$
\begin{aligned}
& n+\sigma+r \to n;\; \mathcal{D}_e \to D;\; [0,T_f) \to [t_0,t_1);\; \Omega_2 \to K;\; X_e \to \xi; \\
& \bar{\Upsilon} \to V;\; U_M + P(\check{\theta})(\rho_o-P(\check{\theta}))^{-1} \to W_2; U_m + P(\check{\theta})(\rho_o-P(\check{\theta}))^{-1} \to W_1; \\
& S_{2\bar{\alpha}} \to S_{1\alpha};\; c_2 \to \eta;\; S_{2c_2} \to S_{1\eta}
\end{aligned}
$$

Then, there exists a compact set $S_{2c_2} \subset \mathcal{D}_e$ such that $X_e(t) \in S_{2c_2}$, $\forall t \in [0,T_f)$, for some $c_2 \in \mathbb{R}$. Therefore, there exists a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0,T_f)$.

Further proof of the first statement is dealt with in two different cases: $r = 1$ and $r \ge 2$.

Case 1: $r = 1$. We regard the true system as the reference system, where the input is u and the output is x_1. By Lemma 11 in [1], the whole state vector x is bounded on the time interval $[0,T_f)$, since it can be considered as stably filtered output signal of u and y, with relative degrees greater than or equal to 1. Φ_u is bounded by Lemma 11 in [1], since it is some stably filtered output signal of u. Then, Φ is bounded. Furthermore, $\check{x}$ is bounded by the boundedness of $\tilde{x} - \Phi\tilde{\theta}$.

Hence, we can conclude that there exists a compact set $\mathcal{S} \subset \mathcal{D}$ such that $X(t) \in \mathcal{S}$, $\forall t \in [0, T_f)$. Therefore, we can conclude that $T_f = +\infty$. This further implies that the control input u and $\hat{\xi}$ are bounded.

Case 2: $r \geq 2$. By Lemma 13 in [1], the true system (1) admits state space representation (78) of [1]. W.L.O.G., assume the true system (1) is given by the state space representation (78) of [1]. Denote the elements of $\bar{\grave{x}}_1$ by $[\bar{\grave{x}}_{11} \ \dots \ \bar{\grave{x}}_{1r}]'$. We use the mathmatical deduction to prove the uniform boundedness of the signals $\Phi_{u_s i}$, $\tilde{x}_i - \lambda_{bi}\bar{A}_{212\,0}\tilde{\theta}$, $\check{x}_i$, $x_i - b_0\lambda_{bi}$, λ_{bi}, Φ_{ui}, Φ_i, x_i, and $\bar{\grave{x}}_{1i}$, $i = 1, \cdots, r$, and deduce that $\bar{\grave{x}}_{1i} = \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{i\,i-1}x_{i-1} + x_i + \tilde{T}_{i3}\bar{\grave{x}}_3 + \tilde{T}_{i4}\bar{\grave{x}}_4$, $i = 1, \cdots, r$. where $\tilde{a}_{ij}$, $i = 1, \cdots, r$, $j = 1, \cdots, i-1$, are constants; and $\tilde{T}_{i3}$ and $\tilde{T}_{i4}$, $i = 1, \cdots, r$, are constant matrices.

1° $i = 1$. It has been proved previously that the signals $\Phi_{u_s 1}$, $\tilde{x}_1 - \lambda_{b1}\bar{A}_{212\,0}\tilde{\theta}$, $\check{x}_1$, $x_1 - b_0\lambda_{b1}$, λ_{b1}, Φ_{u1}, Φ_1, and x_1 are uniformly bounded. Comparing the design model (2) and the canonical form (78) in [1], we can easily obtain

$$\bar{\grave{x}}_{11} = x_1 + \tilde{T}_{13}\bar{\grave{x}}_3 + \tilde{T}_{14}\bar{\grave{x}}_4 \tag{46}$$

where $\tilde{T}_{13}$ and $\tilde{T}_{14}$ are some constant matrices. Then, $\bar{\grave{x}}_{11}$ is bounded.

2° Assume that the statement is true for $1 \leq i \leq k$, where $k \in \{1, 2, \cdots, r-1\}$.

3° $i = k+1$. According to **2**$^\circ$, the signals $\bar{\grave{x}}_{1j}$ $(j = 1, \cdots, k)$ are bounded. Then, we can apply Lemma 11 in [1] to conclude the boundedness of the output of the system (72) in [1] with relative degree $r_2 \geq r - k + 1$. By observing (40b), we find that the relative degree for each of the elements of $\Phi_{u_s\,k+1}$ is at least $r - k + 1$ with respect to the input u. By Lemma 11 in [1], then $\Phi_{u_s\,k+1}$ is bounded, where the reference system has output x_1 and input u and $\grave{w}$, in view of boundedness of $\bar{\grave{x}}_{1j}$, $j = 1, \cdots, k$. The boundedness of the signal $\tilde{x}_{k+1} - \lambda_{b\,k+1}\bar{A}_{212\,0}\tilde{\theta}$ can be easily obtained from the boundedness of $\tilde{x} - \Phi\tilde{\theta}$, η, $\Phi_{u_s\,k+1}$, and $\tilde{\theta}$. Furthermore, $\check{x}_{k+1}$ is bounded in view of the boundedness of z_{k+1} and $\alpha_k(X_{ko}, X_{ka}, X_{kd})$.

The signal $\tilde{x}_{k+1} - \lambda_{b\,k+1}\bar{A}_{212\,0}\tilde{\theta} = (x_{k+1} - \lambda_{b\,k+1}b_0) - (\check{x}_{k+1} - \lambda_{b\,k+1}(b_{p0} + \bar{A}_{212\,0}\check{\theta}))$. Now consider the term $x_{k+1} - \lambda_{b\,k+1}b_0$ first. By Equations (41a), (42a), and (42c), we obtain

$$\begin{aligned}
x_{k+1} - b_0\lambda_{b\,k+1} &= e'_{n,k+1}(x - b_0\lambda_b) \\
x_{u\,k+1} &= e'_{n,k+1}x_u \\
x_{y\,k+1} &= e'_{n,k+1}x_y \\
x_{k+1} - b_0\lambda_{b\,k+1} &= x_{u\,k+1} + x_{y\,k+1}
\end{aligned}$$

The signal $x_{u\,k+1}$ has relative degree at least $r - k + 1$ with respect to u, it is bounded by Lemma 11 in [1], where the reference system has input u and output x_1. The signal $x_{y\,k+1}$ has relative degree at least 1 with respect to y, its boundedness can be deduced by Lemma 11 in [1], where the reference system has input y and output η_L. Then, $x_{k+1} - b_0\lambda_{b\,k+1}$ is bounded. Therefore, $\check{x}_{k+1} - \lambda_{b\,k+1}(b_{p0} + \bar{A}_{212\,0}\check{\theta})$ is also bounded. By the boundedness of $\check{x}_{k+1}$ and $\check{b}_0$, we have the boundedness of the signal $\lambda_{b\,k+1}$. It further inplies the boundedness of the signals $\Phi_{u\,k+1}$, Φ_{k+1} and x_{k+1}.

By **2**$^\circ$, we have

$$x_k = \bar{\grave{x}}_{1k} - \tilde{a}_{k1}x_1 - \cdots - \tilde{a}_{k\,k-1}x_{k-1} - \tilde{T}_{k3}\bar{\grave{x}}_3 - \tilde{T}_{k4}\bar{\grave{x}}_4$$

Then, the dynamics of x_k can be written as

$$\dot{x}_k = \grave{a}_k\bar{\grave{x}}_{11} + \bar{\grave{x}}_{1\,k+1} + \bar{a}_{k1}x_1 + \cdots + \bar{a}_{kk}x_k + \bar{T}_{k3}\bar{\grave{x}}_3 + \bar{T}_{k4}\bar{\grave{x}}_4 + \bar{D}_k\grave{w} \tag{47}$$

where $\bar{a}_{kj}$, $j = 1, \cdots, k$, $\bar{T}_{k3}$, $\bar{T}_{k4}$, and $\bar{D}_k$ are some constant numbers or matrices which can be easily obtained and $\grave{a}_k$ is a constant defined in [1]. Referring the design model (2), the dynamics of x_k is

$$\dot{x}_k = (a_{k1} + e'_{n,k}\bar{A}_{211}\theta)x_1 + a_{k2}x_2 + \cdots + a_{kk}x_k + x_{k+1} + e'_{n,k}D\grave{M}\grave{w} + e'_{n,k}\bar{A}_{211}\theta E\grave{M}\grave{w} \tag{48}$$

Considering (47) and (48), along with (46), we finally obtain

$$\bar{\tilde{x}}_{1\,k+1} = \tilde{a}_{k+1\,1}x_1 + \cdots + \tilde{a}_{k+1\,k}x_k + x_{k+1} + \tilde{T}_{k+1\,3}\bar{\tilde{x}}_3 + \tilde{T}_{k+1\,4}\bar{\tilde{x}}_4 \tag{49}$$

where $\tilde{a}_{k+1\,j}$, $j = 1, \cdots, k$, $\tilde{T}_{k+1\,3}$, and $\tilde{T}_{k+1\,4}$ are some constant numbers or constant matrices. Then, $\bar{\tilde{x}}_{1\,k+1}$ is bounded.

Based on the above reasoning, we have proved that the signals $\Phi_{u_s i}$, $\tilde{x}_i - \lambda_{bi}\bar{A}_{212\,0}\tilde{\theta}$, $\breve{x}_i$, $x_i - b_0\lambda_{bi}$, λ_{bi}, Φ_{ui}, Φ_i, x_i, and $\bar{\tilde{x}}_{1i}$, $i = 1, \cdots, r$ are bounded, and the following equation

$$\bar{\tilde{x}}_{1i} = \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{i\,i-1}x_{i-1} + x_i + \tilde{T}_{i3}\bar{\tilde{x}}_3 + \tilde{T}_{i4}\bar{\tilde{x}}_4, \;\; i = 1, \cdots, r \tag{50}$$

are correct.

Then, the boundedness of x_j, $j = r+1, \cdots, n$ can be deduced by Lemma 11 in [1], since they can be viewed as stably filtered output signal of u and y, with relative degrees greater than or equal to 1. Similarly, $\Phi_{u\,r+1}$, $\cdots$, Φ_{un} are bounded by Lemma 11 in [1], since they are some stably filtered output signals of u. Therefore, the signals Φ_j, $j = r+1, \cdots, n$, are bounded. So is $\breve{x}_j$, $j = r+1, \cdots, n$, by the boundedness of $\tilde{x}_j - \Phi_j\tilde{\theta}$, $j = r+1, \cdots, n$.

Hence, we can conclude that there exists a compact set $\mathcal{S} \subset \mathcal{D}$ such that $X(t) \in \mathcal{S}$, $\forall t \in [0, T_f)$. Therefore, we can conclude that $T_f = +\infty$. This further implies that the control input u and $\hat{\xi}$ are bounded.

Thus, this completes the proof of statement 1.

Next, we prove the second statement. $\forall t_f \in [0, \infty)$, we know for any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$, there exist constants $c_w \geq 0$ and $c_d \geq 0$ such that $|\grave{w}(t)| \leq c_w$ and $|Y_d(t)| \leq c_d$, $\forall t \in [0, t_f]$. By the first statement and the causality of the closed-loop system, there exist a solution $X : [0, t_f] \to \mathcal{D}$ for the closed-loop system. Therefore, the closed-loop system admits a unique solution on $[0, \infty)$, which then implies that the proposed adaptive control law belongs to $\mathcal{M}_u$.

Now we choose l and l_0 as follows

$$\begin{aligned} l(t, \theta, x, y_{[0,t]}, Y_{d[0,t]}) \;&=\; \gamma^4 |x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C \Phi} \\ &\quad -2(\theta - \breve{\theta})' P_r(\breve{\theta}) - (\varepsilon_0^2 - 2\varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/4 + \sum_{j=1}^{r} \beta_j z_j^2 \\ l_0 \;&=\; V(X_{ro}(0), X_{ra}(0)) \end{aligned}$$

The function l is nonnegative along the solution of the closed-loop dynamics (35). Then, we obtain

$$J_{\gamma t_f} = J_{\gamma t_f} + \int_0^{t_f} \dot{U}(X(\tau), y_d^{(r)}(\tau), w(\tau))d\tau + U(X(0)) - U(X(t_f)) \leq -U(X(t_f)) \leq 0 \tag{51}$$

This shows that the controller μ, along with the choice of ξ_c, achieves the disturbance attenuation level γ as described in Definition 1. Thus, the second statement is established.

We continue to the proof of the third statement. For any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$ with $\grave{w}_{[0,\infty)} \in \mathcal{L}_2 \bigcap \mathcal{L}_\infty$ and $Y_{d[0,\infty)} \in \mathcal{L}_\infty$, we have

$$\int_0^\infty |x_1(t) - y_d(t)|^2 dt \leq U(X(0)) + \int_0^\infty \gamma^2 |\grave{M}\grave{w}(t)|^2 dt < +\infty$$

by inequality (31) and the second statement. It implies that $x_1 - y_d \in \mathcal{L}_2$ on the interval $[0, \infty)$. From the first statement, we have that $\dot{x}_1 - \dot{y}_d \in \mathcal{L}_\infty$ on the interval $[0, \infty)$. Therefore,

$$\lim_{t \to \infty} (x_1(t) - y_d(t)) = 0$$

This completes the proof of the theorem. $\square$

7 Examples

In this section, two examples are presented to illustrate the performance of the reduced-order controller designed in this paper.

Example 1

We consider the same system shown in Example 2 of [1]. Then the true system is given by

$$\begin{aligned} \begin{bmatrix} \dot{\grave{x}}_1 \\ \dot{\grave{x}}_2 \\ \dot{\grave{x}}_3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \theta & 0 \end{bmatrix} \begin{bmatrix} \grave{x}_1 \\ \grave{x}_2 \\ \grave{x}_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \grave{w} \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} 0 & 0.1 \end{bmatrix} \grave{w} \end{aligned}$$

where the initial condition for the state $\grave{x}$ and the true value of the parameter θ are set to be

$$\grave{x}_1(0) = 1; \quad \grave{x}_2(0) = 1; \quad \grave{x}_3(0) = 2; \quad \theta = -4$$

for simulation purposes.

By [1], the design model for the adaptive controller is obtained as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \left(y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right) \theta + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 & 0.1 \end{bmatrix} w \end{aligned}$$

via the state and disturbance transformation

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{bmatrix} \grave{x}; \quad w = \begin{bmatrix} 1 & 0 \\ 0 & -\theta \\ -\theta & \theta \\ 0 & 1 \end{bmatrix} \grave{w} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 4 & -4 \\ 0 & 1 \end{bmatrix} \grave{w}$$

The reference trajectory is generated by the following differential equation

$$\begin{aligned} \dot{x}_d &= -x_d + d \\ y_d &= x_d \end{aligned}$$

where d is the command signal. The initial state $x_d(0) = 0$.

For comparison purpose, the parameter selection is similar as that in [1]. The projection function $P(\theta)$ is chosen to be $P(\theta) = 0.01(\theta + 10)^2$. The initial states for the controller are set to be

$$\check{\theta}_0 = -1; \quad \check{x}_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}'; \; Q_0 = 1; \; \Phi_0 = \mathbf{0}_{3\times 1}; \; \eta_0 = \lambda_0 = \mathbf{0}_{3\times 1}$$

The desired disturbance attenuation level is $\gamma = 0.2$. The constant k_c in the expression of ξ_c is chosen to be 0. The other parameters are the same as [1] and are listed below

$$p_n = [0\ 0\ 1]'; \; \Delta_1 = I_3; \; \beta_\Delta = 0; \; K_c = 0.1 > \gamma^2 Q_0; \; \rho_o = 1.5; \; \gamma_1 = 0.5; \; \beta_1 = 0.5; \; \epsilon = K_c^{-1} s_\Sigma$$

We present the simulation results in Figure 1−4,

which shows the performance of the closed-loop system under reduced-order and full-order controllers for two different sets of inputs. The first set of simulations, Figures 1−2, is done by setting the reference trajectory and the arbitrary exogenous disturbance to zero. Figure 1 shows the system response with the reduced-order controller. Figure 2 shows the system response with the full-order controller. For both of the simulations, the tracking errors converge to zero and the parameter estimates converge to the true value, which is consistent to our theoretical findings. The control magnitudes are bounded by 7. The transient response for the reduced-order controller is worse, which may be due to the disappearance of the negative drift relating to η in the derivative of the closed-loop value function. However, from subfigures (b), we observe the convergence rates of the tracking error and the parameter estimation error are faster for the reduced-order controller than those for the full-order controller.

To illustrate the robustness of the controllers, we present the second set of simulations, Figures 3−4. The command signal and the disturbance input are set to be

$$\begin{aligned} d(t) &= 2.5\sin(2.8t); \\ \grave{w}(t) &= \begin{bmatrix} 0.3\sin(3t + \frac{\pi}{4}) \\ \text{Band-limited white noise with power 0.1, seed 1000 and sampling period 1 sec} \end{bmatrix} \end{aligned}$$

Based on Figures 3 and 4 , we observe that the desired attenuation level are achieved for both of the controllers based on subfigures (f). The parameter estimates asymptotically oscillate around the true value. The control magnitudes are bounded by 8 for both of the controllers. The transient response for the full-order performs better, but both of the controllers have comparable steady state, upper bounded by 0.15, which can be seen from subfigures (b).

Example 2

Consider a circuit shown in Figure 5. The resistance R is 1Ω. The capacitor C and the inductor L are linear and time invariant, $L = 1H$. v_i is a dependent voltage source. v_e is an unknown sinusoidal voltage source. v_{w1} is an unmeasured exogenous voltage disturbance; i_s is an unmeasured exogenous current disturbance. v_o is the voltage output. Our objective is to achieve the desired voltage output v_o by adjusting v_i.

Take $\grave{x}_1 = i_1$, $\grave{x}_2 = v_c$ as the state variables, $u = v_i$ as the input, and $y = v_o$ as the output, $\grave{w} = [v_{w1}\ \ i_s]'$ as the exogenous disturbance. Then, the circuit admits the following state space representation

$$\dot{\grave{x}}_1 = -\frac{R}{L}\grave{x}_1 - \frac{1}{L}\grave{x}_2 + \frac{1}{L}v_e + \frac{1}{L}u$$

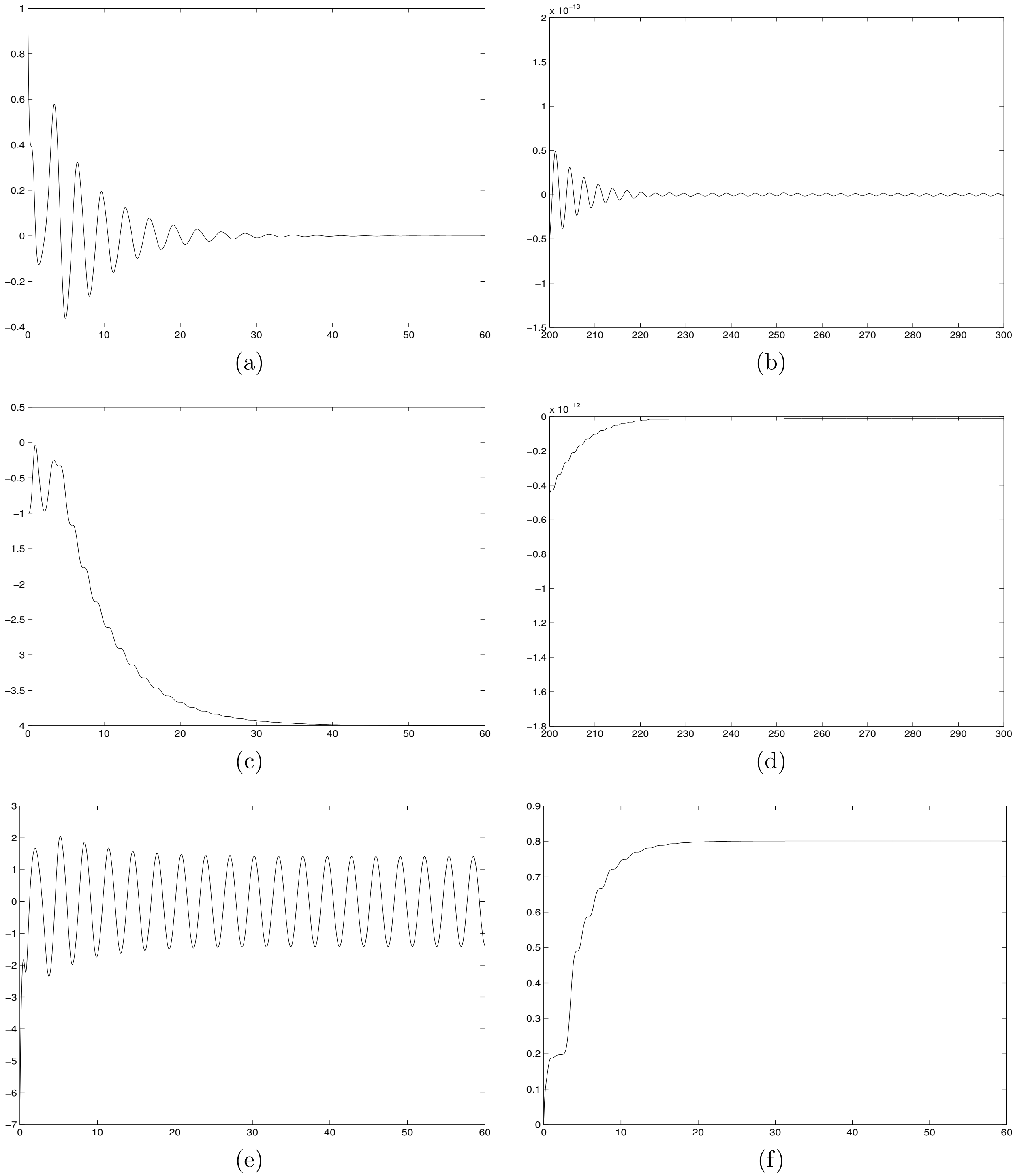

Figure 1: System response for Example 1 with reduced-order controller under $d(t) = 0$ and no arbitrarily varying disturbance.

(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; (d)Parameter estimation error (long term); (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

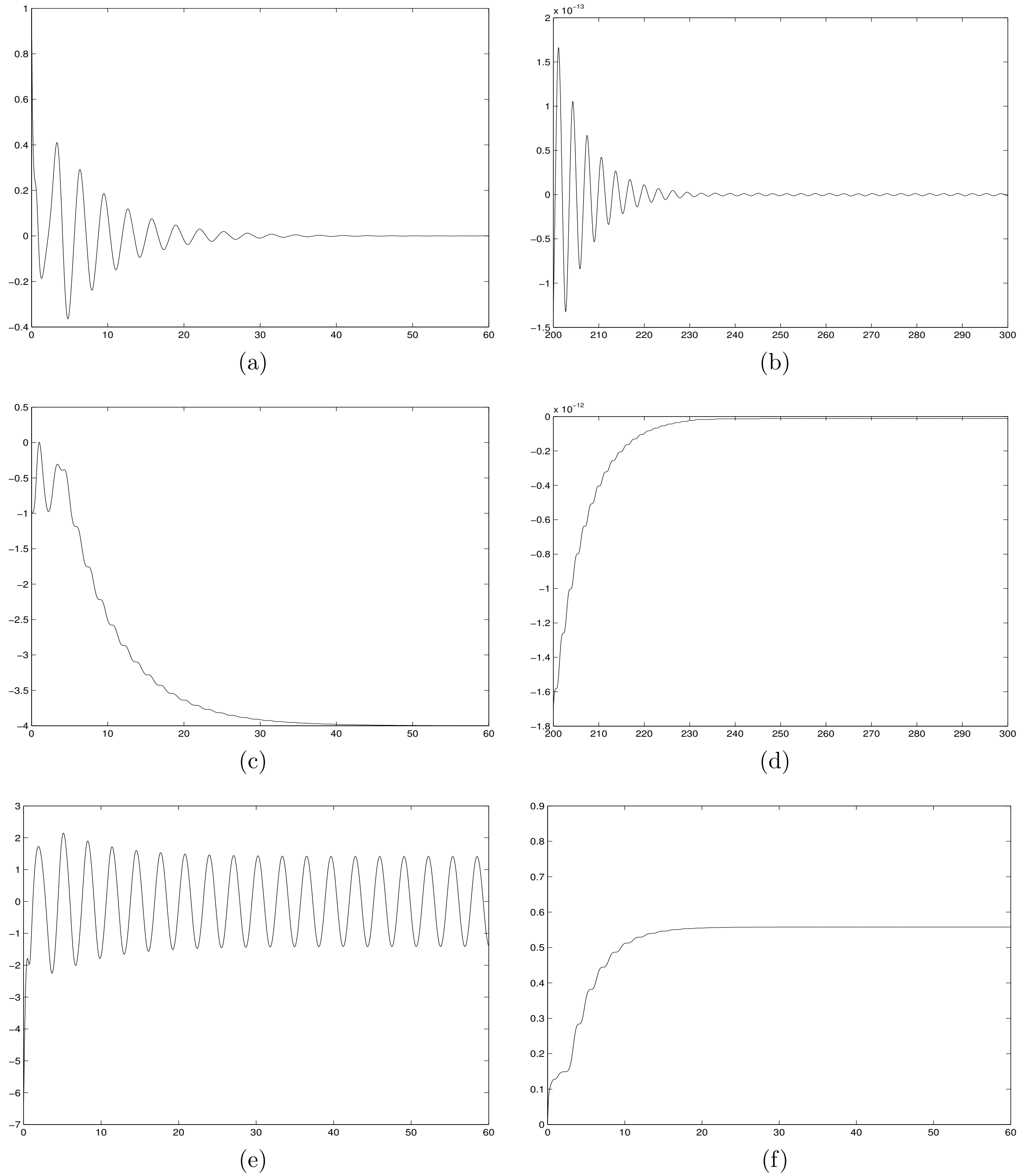

Figure 2: System response for Example 1 with full-order controller under $d(t) = 0$ and no arbitrarily varying disturbance.
(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; (d)Parameter estimation error (long term); (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

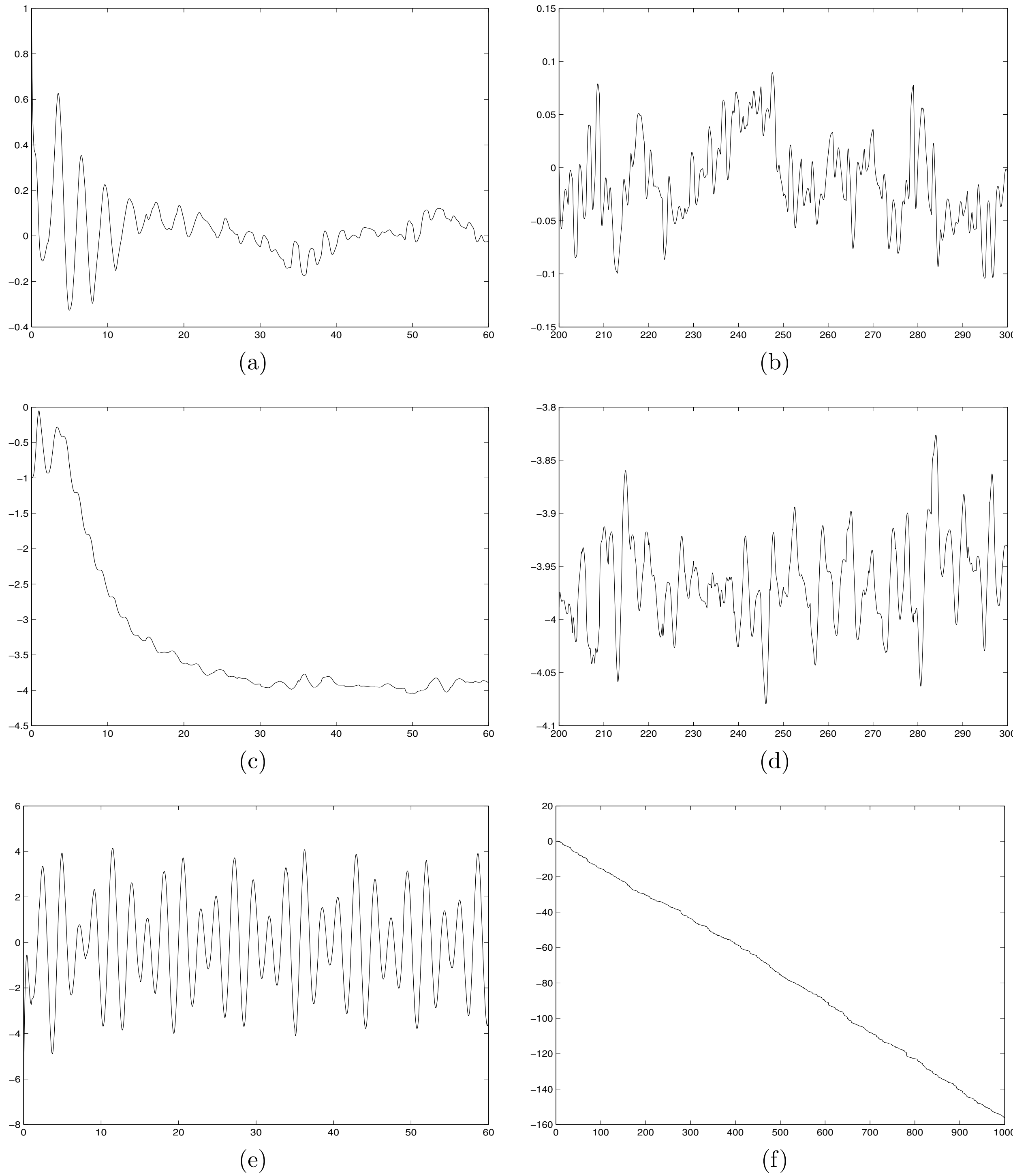

Figure 3: System response for Example 1 with reduced-order controller under $d(t) = 2.5\sin(2.8t)$ and arbitrarily varying disturbance $\dot{w}$.

(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; (d) Parameter estimate (Long term); (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

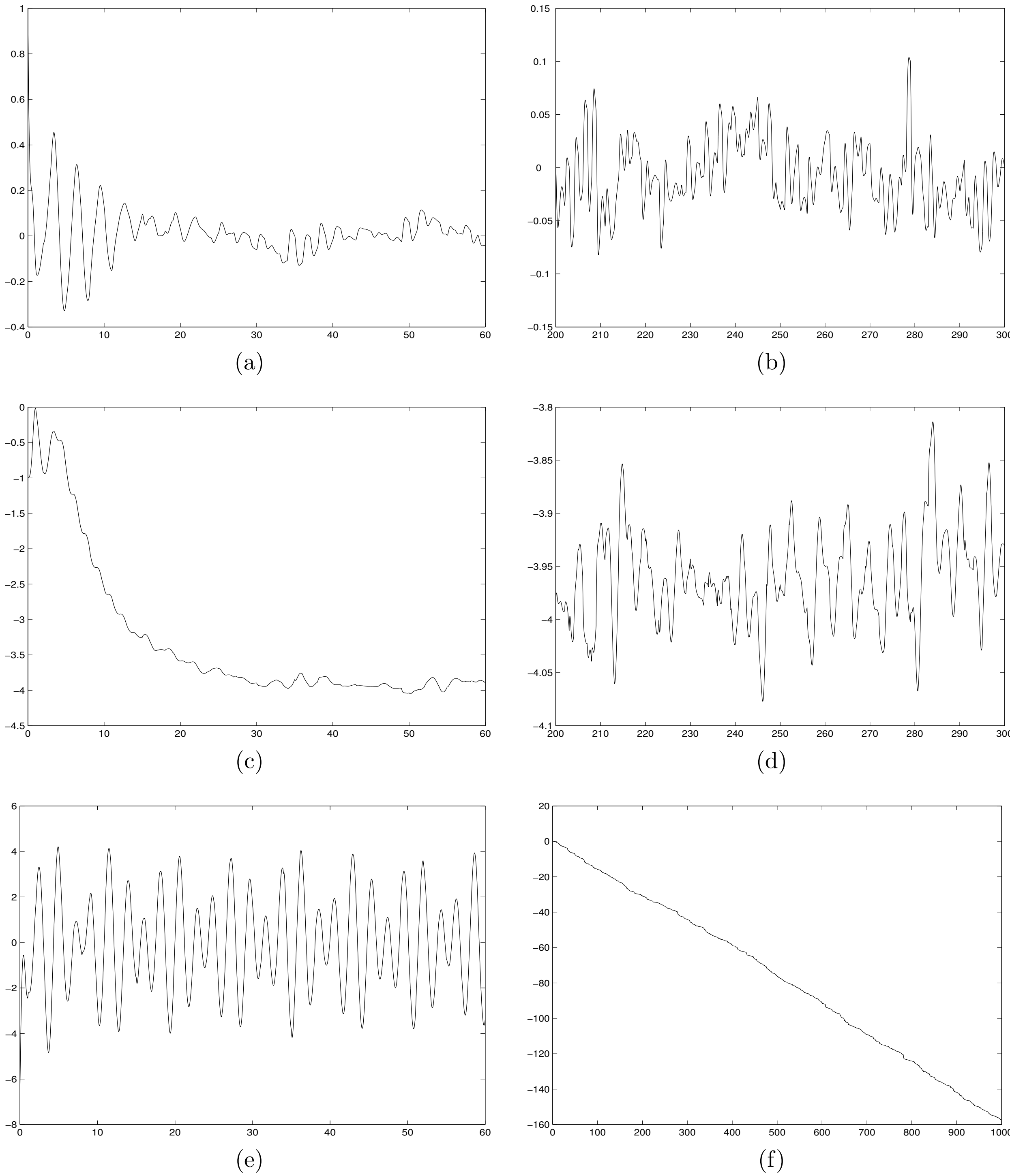

Figure 4: System response for Example 1 with full-order controller under $d(t) = 2.5\sin(2.8t)$ and arbitrarily varying disturbance.

(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; (d) Parameter estimate (Long term); (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

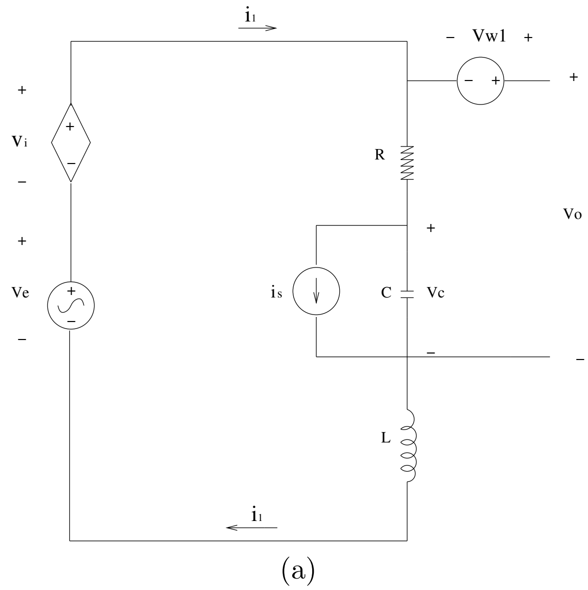

Figure 5: RLC circuit

$$
\begin{aligned}
\dot{\grave{x}}_2 &= \frac{1}{C}\grave{x}_1 - \frac{1}{C}\grave{w}_2 \\
y &= R\grave{x}_1 + \grave{x}_2 + \grave{w}_1
\end{aligned}
$$

Then sinusoidal component v_e can be modelled by a second-order linear system

$$
\begin{aligned}
\dot{\grave{x}}_3 &= \grave{x}_4 \\
\dot{\grave{x}}_4 &= \theta_f \grave{x}_3 \\
v_e &= \grave{x}_3
\end{aligned}
$$

The parameter θ_f determines the frequency of the sinusoid, which is assumed to be unknown. For simulation purposes, we will set $v_e(t) = \sqrt{2}sin(\sqrt{6}t + \frac{\pi}{4})$, which corresponds to the choice of

$$
\grave{x}_3(0) = 1; \quad \grave{x}_4(0) = \sqrt{6}; \qquad \theta_f = -6
$$

The initial condition for $\grave{x}_1$ and $\grave{x}_2$ are set to be $\grave{x}_1(0) = 1$ and $\grave{x}_2(0) = 0$

Then, the true system in state-space form is given by

$$
\dot{\grave{x}} = \begin{bmatrix} -R/L & -1/L & 1/L & 0 \\ 1/C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \theta_f & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} 1/L \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & -1/C \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \grave{w} \tag{52a}
$$

$$
y = \begin{bmatrix} R & 1 & 0 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} 1 & 0 \end{bmatrix} \grave{w} \tag{52b}
$$

It is easy to check that the true system satisfies the Assumptions 1 and 2.

The true values for the unknown parameters $\frac{1}{C}$ and θ_f are 6.5 and -6, respectively. We assume $\frac{1}{C}$ belongs to the interval $[0.5\ \ 6.5]$; θ_f belongs to the interval $[-6\ \ 0]$.

Since the system is observable, the overall system (52) can be transformed into the observer canonical form through the transformation

$$x = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1/C & 0 & 1 & 0 \\ -\theta_f & -\theta_f & 1/C & 1 \\ -\theta_f/C & 0 & 0 & 1/C \end{bmatrix} \grave{x}$$

Then, the observer canonical form of (52) is obtained as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1/C+\theta_f & 0 & 1 & 0 \\ \theta_f & 0 & 0 & 0 \\ \theta_f/C & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1/C \\ -\theta_f \\ -\theta_f/C \end{bmatrix} u + \begin{bmatrix} 0 & -1/C \\ 0 & 0 \\ 0 & \theta_f/C \\ 0 & 0 \end{bmatrix} \grave{w} && \text{(53a)} \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \end{bmatrix} \grave{w} && \text{(53b)} \end{aligned}$$

For the ease of simulation, we normalize the unknown parameters $\frac{1}{C}$ and θ_f. Define $\theta_1 = (\frac{1}{C} - 3.5)/3$, $\theta_2 = (\theta_f + 3)/3$, and $\theta_3 = \theta_1\theta_2$. Then, the new unknown parameters θ_1, θ_2, θ_3 all belong to the interval $[-1\ \ 1]$. With the new parameter vector $\theta = [\theta_1\ \ \theta_2\ \ \theta_3]'$, the equation (53) can be rewritten as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3\theta_1+3\theta_2-6.5 & 0 & 1 & 0 \\ 3\theta_2-3 & 0 & 0 & 1 \\ -9\theta_1+10.5\theta_2+9\theta_3-10.5 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3\theta_1+3.5 \\ -3\theta_2+3 \\ 9\theta_1-10.5\theta_2-9\theta_3+10.5 \end{bmatrix} u \\ &\quad + \begin{bmatrix} 0 & -3\theta_1-3.5 \\ 0 & 0 \\ 0 & -9\theta_1+10.5\theta_2+9\theta_3-10.5 \\ 0 & 0 \end{bmatrix} \grave{w} && \text{(54a)} \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \end{bmatrix} \grave{w} && \text{(54b)} \end{aligned}$$

Then, the design model for this adaptive controller is obtained as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ -6.5 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \\ -10.5 & 0 & 0 & 0 \end{bmatrix} x + \left(y \begin{bmatrix} 0 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & 3 & 0 \\ -9 & 10.5 & 9 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & -3 & 0 \\ 9 & -10.5 & -9 \end{bmatrix} \right) \theta + \begin{bmatrix} 1 \\ 3.5 \\ 3 \\ 10.5 \end{bmatrix} u \\ &\quad + \begin{bmatrix} 0 & 0.65 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 & 2.85 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \end{bmatrix} w \end{aligned}$$

where, we have introduced the disturbance transformation

$$w = \begin{bmatrix} 10 & 0 \\ 0 & -(3\theta_1 + 3.5)/0.65 \\ (3\theta_1 - 3\theta_2)/0.6 & 0 \\ -\theta_2/1.3 & (-9\theta_1 + 10.5\theta_2 + 9\theta_3 - 10.5)/3.9 \\ (9\theta_1 - 10.5\theta_2 - 9\theta_3)/2.85 & 0 \end{bmatrix} \grave{w} = \begin{bmatrix} 10 & 0 \\ 0 & -10 \\ 10 & 0 \\ 0.769 & -10 \\ 10 & 0 \end{bmatrix} \grave{w}$$

The reference trajectory is generated by

$$\begin{aligned} \dot{x}_d &= -x_d + d;\ x_d(0) = 0 \\ y_d &= x_d \end{aligned}$$

where d is the command input. We set the desired disturbance attenuation level $\gamma = 0.4 > \zeta^{-1} = 0.1$. The projection function $P(\theta)$ is selected to be

$$P(\theta) = \frac{1}{3}(\theta_1^2 + \theta_2^2 + \theta_3^2)$$

The initial states for the controller are selected to be

$$\check{x}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}';\quad \check{\theta}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}';\quad Q_0 = I_3;\ \Phi_0 = \mathbf{0}_{4\times 3};\ \eta_0 = \lambda_0 = \mathbf{0}_{4\times 1}$$

For the adaptive controller design, we select

$$p_n = [0\ 0\ 0\ 1]';\ K_c = 0.48 = \gamma^2 \mathrm{Tr}(Q_0);\ \Delta_1 = I_4;\ \beta_\Delta = 0;\ \rho_o = 2;\ \gamma_1 = 1;\ \beta_1 = 1;\ \epsilon = K_c^{-1} s_\Sigma$$

The simulation results are shown in Figures 6—9.

We still compare the system performances between the reduced-order controller and full-order controller through two sets of different simulations. First, we set the command input $d(t) = 2.7\sin(t)$ and the disturbance $\grave{w}(t)$ to be zero. Figures 6 and 7 show the system responses achieved with reduced-order controller and full-order controller, respectively. We observe that the simulation results for both of the controllers are very similar. The tracking errors asymptotically converge to zero, the parameter estimates also converge to their true values. The control inputs are bounded by 8. Both of the controllers achieve comparable transient and steady-state performance, which is different from the simulation results of Example 1. To analyze the reason, we may conclude that, with increase of dynamic order of the system, the advantage of simplified controller structure is more prominent, which compensates its negative effects.

To illustrate the robustness of the closed-loop system, we make another set of simulations, whose results are shown in Figures 8—9, the command input and the exogenous disturbance input are set to be

$$\begin{aligned} d(t) &= 2.7\sin(t) \\ \grave{w}(t) &= \begin{bmatrix} 0.04\sin(2t + \frac{\pi}{3}) \\ 0.02\sin(5t + 1) \end{bmatrix} \end{aligned}$$

Figure 8 shows the closed-loop system performance with the reduced-order controller. Figure 9 shows the closed-loop system performance with full-order controller. We observe that, for both of

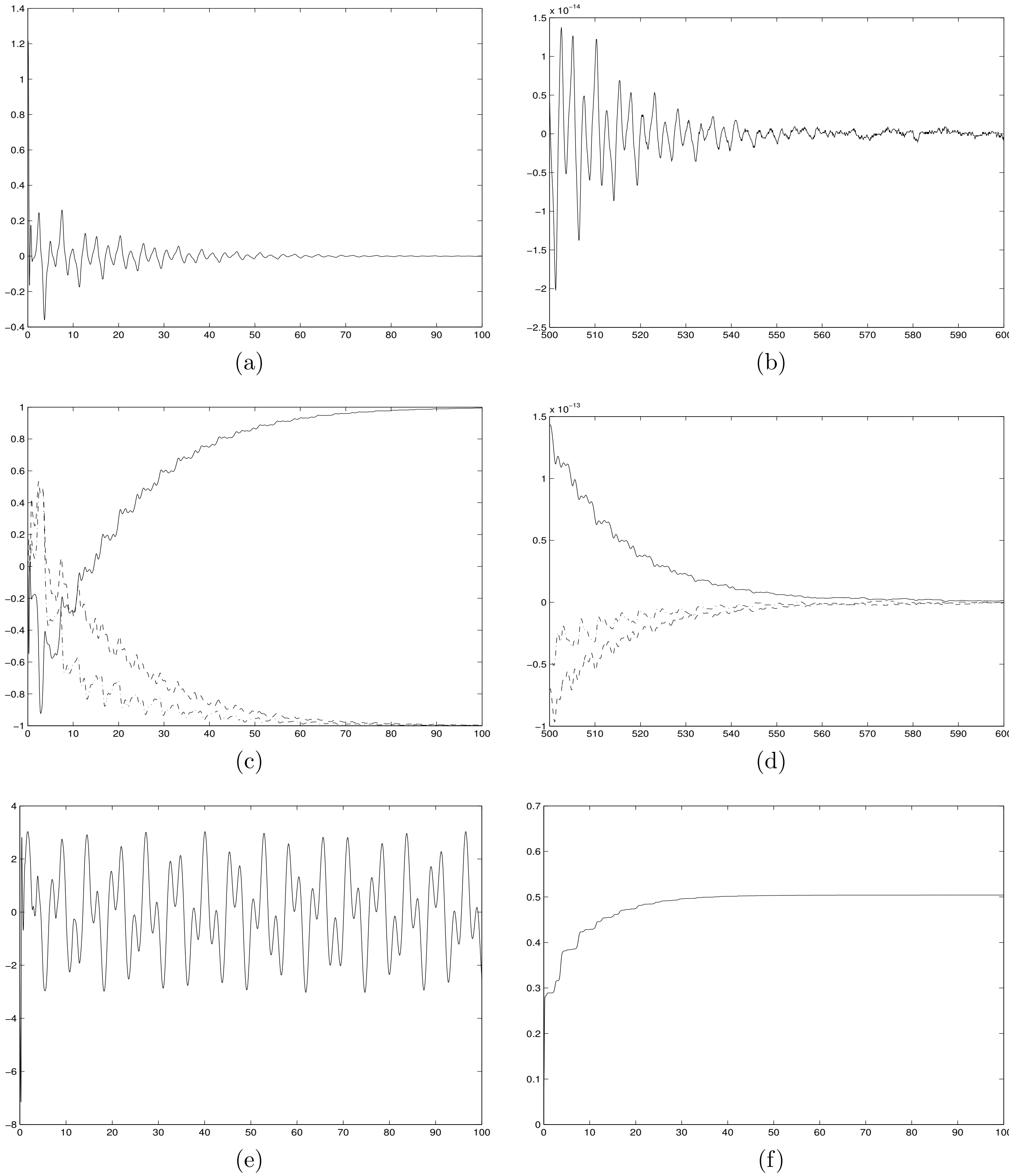

Figure 6: System response for Example 2 with reduced-order controller under $d(t) = 2.7\sin(t)$ and no arbitrarily varying disturbance $\grave{w}$.
(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; solid line for $\check{\theta}_1$; dash line for $\check{\theta}_2$; dash dot line for $\check{\theta}_3$; (d) Parameter estimation error (Long term); solid line for $\theta_1 - \check{\theta}_1$; dash line for $\theta_2 - \check{\theta}_2$; dash dot line for $\theta_3 - \check{\theta}_3$; (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

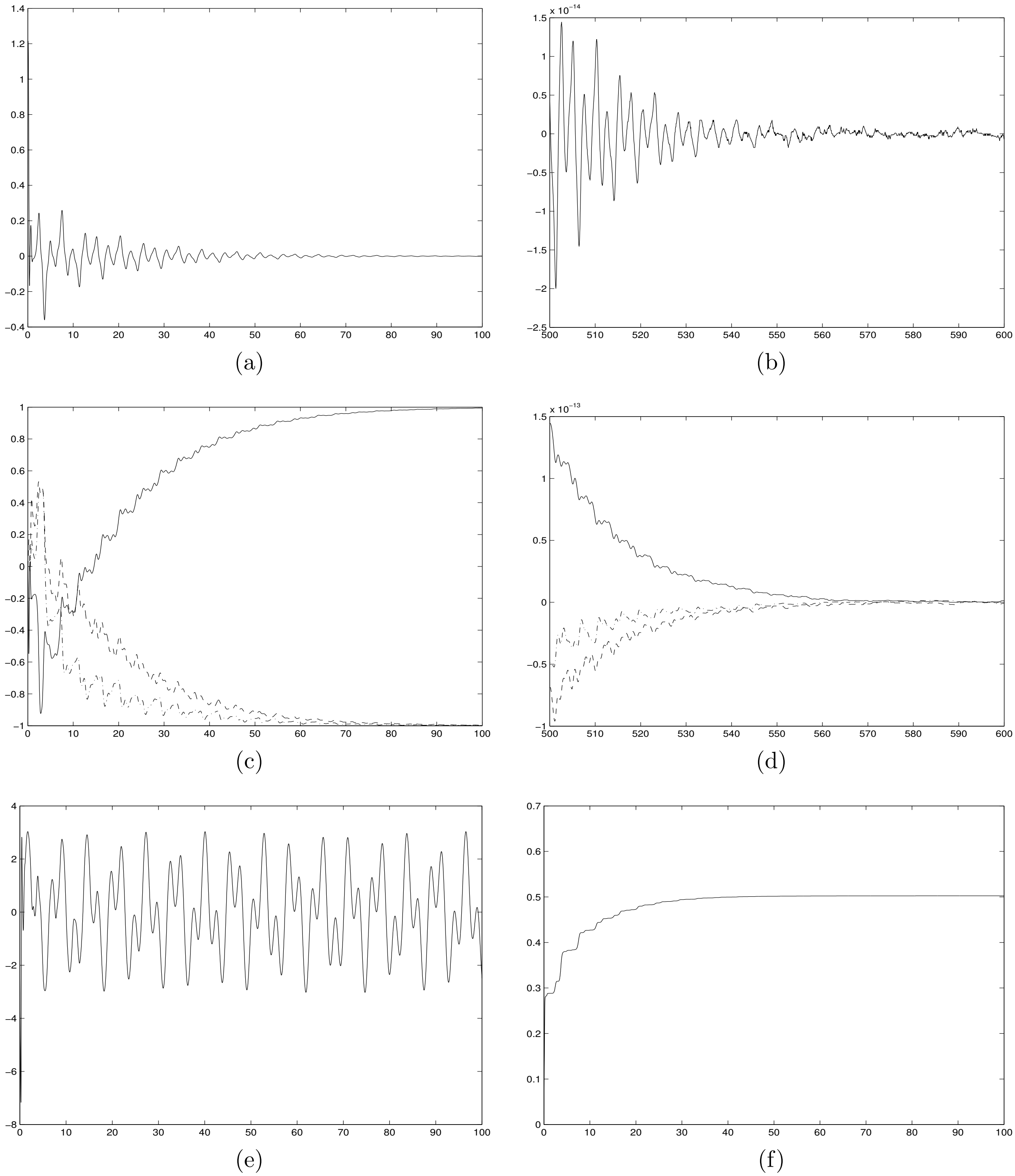

Figure 7: System response for Example 2 with full-order controller under $d(t) = 2.7\sin(t)$ and no arbitrarily varying disturbance.
(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; solid line for $\check{\theta}_1$; dash line for $\check{\theta}_2$; dash dot line for $\check{\theta}_3$; (d) Parameter estimation error (Long term); solid line for $\theta_1 - \check{\theta}_1$; dash line for $\theta_2 - \check{\theta}_2$; dash dot line for $\theta_3 - \check{\theta}_3$; (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

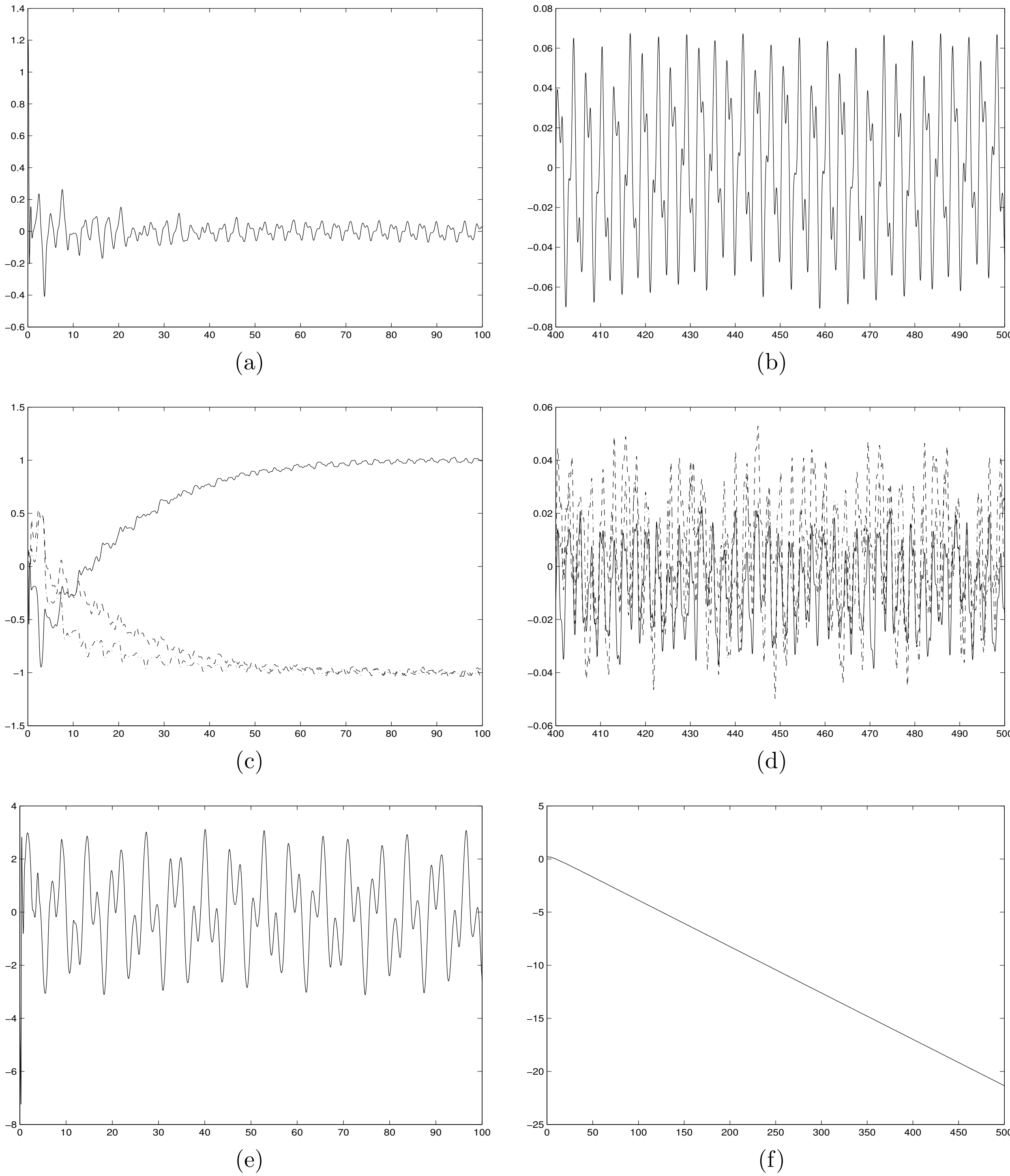

Figure 8: System response for Example 2 with reduced-order controller under $d(t) = 2.7sin(t)$ and arbitrarily varying disturbance $\grave{w}$.
(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; solid line for $\check{\theta}_1$; dash line for $\check{\theta}_2$; dash dot line for $\check{\theta}_3$; (d) Parameter estimation error (Long term); solid line for $\theta_1 - \check{\theta}_1$; dash line for $\theta_2 - \check{\theta}_2$; dash dot line for $\theta_3 - \check{\theta}_3$; (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

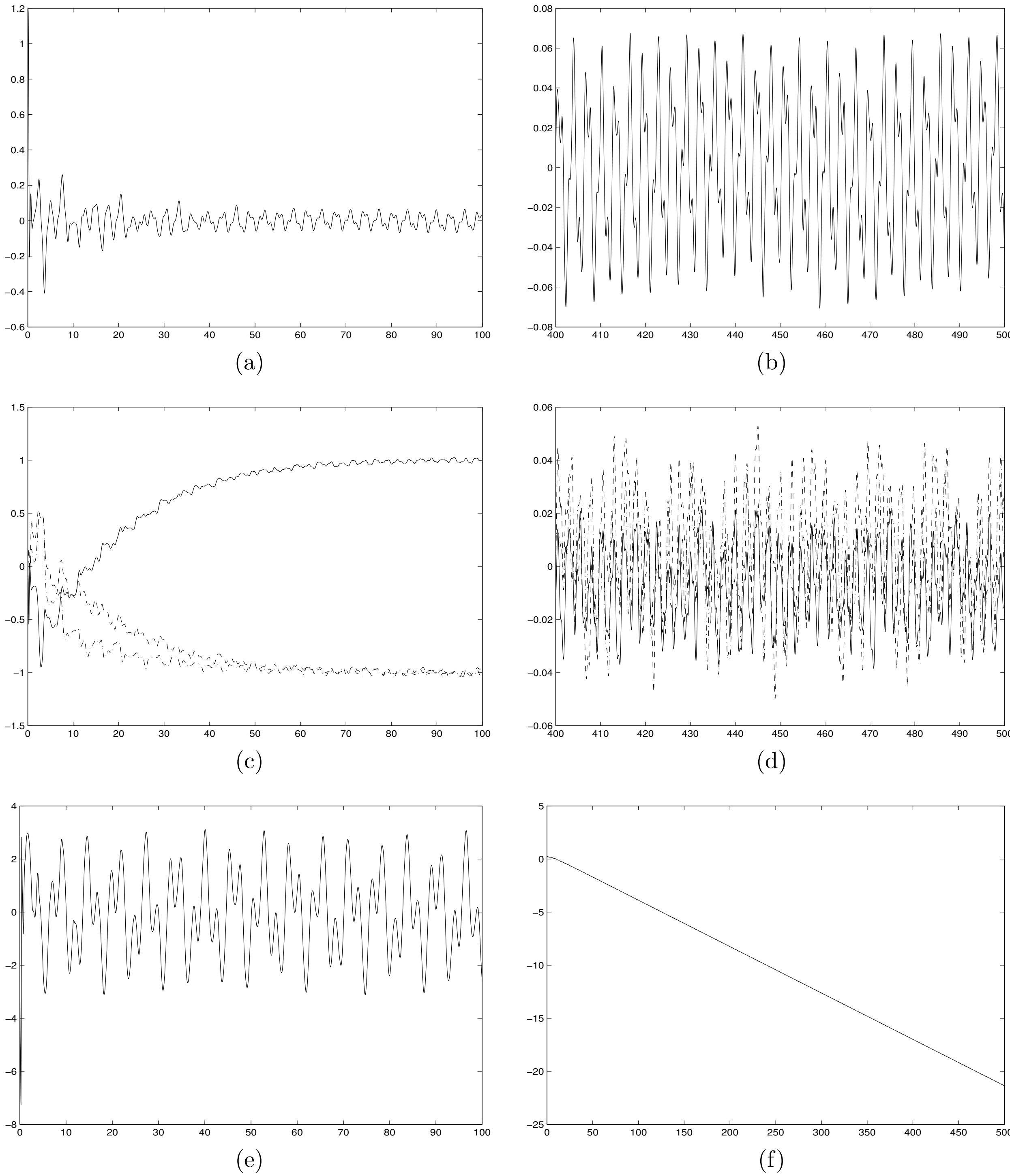

Figure 9: System response for Example 2 with full-order controller under $d(t) = 2.7\sin(t)$ and arbitrarily varying disturbance.
(a) Tracking error; (b) Tracking error (long term); (c) Parameter estimate; solid line for $\check{\theta}_1$; dash line for $\check{\theta}_2$; dash dot line for $\check{\theta}_3$; (d) Parameter estimation error (Long term); solid line for $\theta_1 - \check{\theta}_1$; dash line for $\theta_2 - \check{\theta}_2$; dash dot line for $\theta_3 - \check{\theta}_3$; (e) Control input; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

the controllers, the transient responses behave well, and the tracking errors reach steady state in about 70 seconds. The steady-state tracking errors are bounded by 0.08 from subfigures (b). The parameter estimates aslo oscillate around their true values in steady state. The control efforts are still bounded by 8. Subfigures (f) show that the desired disturbance attenuation levels are achieved. Both controllers have comparable system responses.

8 Conclusions

In this paper, we have studied the reduced-order adaptive control design for SISO linear systems with noisy output measurements. The main contribution of this paper is that the controller structure is simplified by n integrators without any additional assumptions or sacrifice in the strong robustness properties of the closed-loop system, in comparison with the full-order controller designed in [1]. We study the same class of systems as [1], make the same assumptions and have the same formulation and approach to the problem. The uncertain linear system under consideration is assumed to be observable, admits a strictly minimum phase transfer function with known relative degree, has a known upper bound for its dynamic order. The system is allowed to have uncontrollable part, which is assumed to be stable in the sense of Lyapunov, and the uncontrollable modes on the imaginary axis are uncontrollable from the disturbance $\grave{w}$. With these assumptions, the system is transformed into the design model which is linear in all of the unknown quantities and reveals the dependence of the unknown parameters. For the design model, we assume that the measurement channel is noisy, which yields a regular optimization problem for the estimation design step, and the true value of parameter vector belongs to a known convex compact set such that the sign of the high frequency gain is known. The robust adaptive control problem is then formulated as a nonlinear H^{∞}-optimal control problem under imperfect state measurements, where the unknown parameter vector is viewed as part of the expanded state vector, and the objectives of robust adaptive control are incorporated into the optimization of a single soft-constrained game theoretic cost function. A game theoretic approach is adopted to solve this problem. The adaptive control design is carried out in two steps, first step is the estimation design step, second step is the controller design step. The estimation design step is completely the same as [1]. In Section 4, we merely make a summary of the result of [1] for ease of reference. The main difference between this paper and [1] lies in the controller design step. At this step, although we still employ the backstepping methodology as [1], we start the controller design from step 1, without first stabilizing η dynamics as step 0 does in [1]. Then, there is no need to generate η_d dynamics for η to track, the dynamic order of the controller is thereby reduced by n. The step 0 of [1] may be skipped that does not affect the robustness of the closed-loop system because the dynamics of $\tilde{x} - \Phi\tilde{\theta}$ admits desired structure which may be substituted for the $\tilde{\eta}$ dynamics. On the other hand, the dynamics of $\tilde{x} - \Phi\tilde{\theta}$ have an undesirable feature that they depend also on $\hat{\xi}$. Then, the trade-off for the order-reduction is that $\hat{\xi}$ may not be set to the optimal choice. Also, the lack of step 0 leads to the disappearance of a nonpositive drift term related to η dynamics in the derivative of the closed-loop value function. All of these trade-offs may be responsible for the degradation of the closed-loop responses after order reduction, which can be observed in the examples presented in the paper. However, as mentioned above, the reduced-order controller is shown to guarantee exactly the same strong robustness properties as [1], namely, all closed-loop signals are bounded when the disturbance input is bounded and reference trajectory and its derivatives up to rth order are bounded; the controller achieves the desired disturbance attenuation level, whose ultimate lower bound is the noise intensity in the measurement channel;

the noiseless output asymptotically tracks the reference trajectory when the disturbance input is bounded and of finite energy and the reference trajectory and its derivatives up to rth order are bounded. Two examples are presented to further corroborate the theoretical findings.

This order-reduction scheme proposed in this paper may be applied to SISO linear systems with partly measured disturbances, and to SISO linear systems with repeated noisy measurements, these works are straightforward and easy to achieve, which will not be pursued in the near future. Another order-reduction scheme under consideration is to reduce the dynamic order of the controller by $n-1$ integrators. The main difference between this order-reduction scheme and the one described in this paper is that $n-1$ order-reduction scheme allows the optimal selection of $\hat{\xi}$.

A Derivation of dynamics for η_c

We provide a derivation for the dynamics of $\eta_c := x - \check{x} - \Phi(\theta - \check{\theta})$. The dynamics of x, $\check{x}$, Φ, and $\check{\theta}$ are as follows

$$
\begin{aligned}
\dot{x} &= Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Bu + Dw \\
\dot{\check{\theta}} &= -\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'(y_d - C\check{x}) - \begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + \gamma^2\zeta^2\Sigma\Phi' C'(y - C\check{x}) \\
\dot{\check{x}} &= -\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) - \begin{bmatrix} \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + Bu \\
&\quad + (y\bar{A}_{211} + u\bar{A}_{212})\check{\theta} + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi' C' + L)(y - C\check{x}) \\
\dot{\Phi} &= A_f\Phi + y\bar{A}_{211} + u\bar{A}_{212}
\end{aligned}
$$

Then the derivative of η_c can be written as

$$
\begin{aligned}
\dot{\eta}_c &= \dot{x} - \dot{\check{x}} - \dot{\Phi}(\theta - \check{\theta}) + \Phi\dot{\check{\theta}} \\
&= Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Bu + Dw - (-\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) \\
&\quad - \begin{bmatrix} \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + (y\bar{A}_{211} + u\bar{A}_{212})\check{\theta} + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi' C' + L)(y - C\check{x}) \\
&\quad + Bu) - (A_f\Phi + y\bar{A}_{211} + u\bar{A}_{212})(\theta - \check{\theta}) + \Phi(-\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'(y_d - C\check{x}) \\
&\quad - \begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + \gamma^2\zeta^2\Sigma\Phi' C'(y - C\check{x})) \\
&= A(x - \check{x}) + Dw + \gamma^{-2}\Pi C'(y_d - C\check{x}) + \begin{bmatrix} \mathbf{0} & \gamma^{-2}\Pi \end{bmatrix} \bar{Q}\xi_c - \zeta^2(\Pi C' + L)(y - C\check{x}) \\
&\quad - A_f\Phi(\theta - \check{\theta}) \\
&= \left(A_f + \zeta^2 LC + \Pi C' C(\zeta^2 - \gamma^{-2})\right)(x - \check{x}) + Dw + \gamma^{-2}\Pi C'(y_d - C\check{x}) + \begin{bmatrix} \mathbf{0} & \gamma^{-2}\Pi \end{bmatrix} \bar{Q}\xi_c \\
&\quad - \zeta^2(\Pi C' + L)(y - C\check{x}) - A_f\Phi(\theta - \check{\theta}) \\
&= A_f\eta_c + \left(\zeta^2 L + \Pi C'(\zeta^2 - \gamma^{-2})\right)(y - C\check{x} - Ew) + \gamma^{-2}\Pi C'(y_d - C\check{x}) + \begin{bmatrix} \mathbf{0} & \gamma^{-2}\Pi \end{bmatrix} \bar{Q}\xi_c \\
&\quad + Dw - \zeta^2(\Pi C' + L)(y - C\check{x}) \\
&= A_f\eta_c - (\zeta^2 L + \Pi C'(\zeta^2 - \gamma^{-2}))Ew + \gamma^{-2}\Pi C'(y_d - y) + Dw + \begin{bmatrix} \mathbf{0} & \gamma^{-2}\Pi \end{bmatrix} \bar{Q}\xi_c
\end{aligned}
$$

B Four useful lemmas

Lemma 2 *Let n_1, n_a, $q \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $D_a \subseteq \mathbb{R}^{n_a}$, $D_e \subseteq \mathbb{R}$, $D_d \subseteq \mathbb{R}^{n_1}$, $D_w \subseteq \mathbb{R}^q$, D_a, D_e, and D_d be nonempty and open, D_w contains a nonempty open subset of $\mathbb{R}^q$. Let f_a, g_a, h_a be* *mappings of $D_a \times D_d$ into $\mathbb{R}^{n_a}$, $\mathbb{R}^{n_a}$, and $\mathbb{R}^{n_a \times q}$, respectively. Let f_a, g_a, h_a be $\mathcal{C}_k$, $\gamma > 0$ and $\gamma \in \mathbb{R}$. Let $z_e : D_a \to D_e$ be $\mathcal{C}_{k+1}$, $\phi : D_a \times D_d \to \mathbb{R}$ be $\mathcal{C}_k$ and $\delta : D_e \to \mathbb{R}$ be $\mathcal{C}_{k+1}$. Assume that z_e and δ satisfy the following conditions:*

$$
\begin{array}{ll}
(i) & \dfrac{\partial z_e}{\partial x_a}(x_a) g_a(x_a, x_d) \neq 0, \quad \forall (x_a, x_d) \in D_a \times D_d. \\
(ii) & \delta(z_e(x_a)) > 0, \quad \forall x_a \in D_a. \\
(iii) & \pi_\delta(z_e(x_a)) = \delta(z_e(x_a)) + \dfrac{\partial \delta}{\partial z_e}(z_e(x_a)) z_e(x_a) > 0, \quad \forall x_a \in D_a.
\end{array}
$$

Let $V : D_a \to \mathbb{R}$ be defined by $V(x_a) = (\delta(z_e(x_a)) z_e(x_a))^2$, which is $\mathcal{C}_{k+1}$.

Consider the system dynamics

$$\dot{x}_a = f_a(x_a, x_d) + g_a(x_a, x_d) u + h_a(x_a, x_d) w \tag{56}$$

where $w(\cdot)$, $x_d(\cdot)$, and $u(\cdot)$ are continuous signals taking values in D_w, D_d, and $\mathbb{R}$, respectively.

Then there exists a $\mathcal{C}_k$ function $\alpha : D_a \times D_d \to \mathbb{R}$ defined by (59), such that the derivative of $V(x_a)$ along a solution of (56) with $u = \alpha(x_a, x_d)$ admits the following form

$$
\begin{aligned}
\dot{V}(x_a, x_d, u, w)|_{u=\alpha(x_a,x_d)} = -\phi(x_a, x_d) z_e(x_a) + \gamma^2 |w|^2 - \gamma^2 |w - \sigma(x_a, x_d)|^2 \\
\forall (x_a, x_d) \in D_a \times D_d, \quad \forall w \in D_w
\end{aligned} \tag{57}
$$

where $\sigma : D_a \times D_d \to \mathbb{R}^q$ is $\mathcal{C}_k$ and given by

$$\sigma(x_a, x_d) = \frac{1}{2\gamma^2} (h_a(x_a, x_d))' \left(\frac{\partial V}{\partial x_a}(x_a) \right)' \tag{58}$$

Proof $\forall x_a \in D_a$, $\forall x_d \in D_d$, $\forall w \in D_w$, $\forall u \in \mathbb{R}$, we have

$$
\begin{aligned}
\dot{V}(x_a, x_d, u, w) &= 2(\delta(z_e(x_a)))^2 \, z_e(x_a) \, \frac{\partial z_e}{\partial x_a}(x_a) \dot{x}_a + 2(z_e(x_a))^2 \delta(z_e(x_a)) \frac{\partial \delta}{\partial z_e}(z_e(x_a)) \frac{\partial z_e}{\partial x_a}(x_a) \dot{x}_a \\
&= 2 z_e(x_a) \, \delta(z_e(x_a)) \left(\delta(z_e(x_a)) \frac{\partial z_e}{\partial x_a}(x_a) \dot{x}_a + z_e(x_a) \frac{\partial \delta}{\partial z_e}(z_e(x_a)) \frac{\partial z_e}{\partial x_a}(x_a) \dot{x}_a \right) \\
&= 2 z_e(x_a) \delta(z_e(x_a)) \pi_\delta(z_e(x_a)) \frac{\partial z_e}{\partial x_a}(x_a) \left(f_a(x_a, x_d) + g_a(x_a, x_d) u + h_a(x_a, x_d) w \right) \\
&= z_e(x_a)(\chi_{11} + 2\gamma^2 \chi_{12} w + \chi_{13} u)
\end{aligned}
$$

where χ_{11}, χ_{12}, and χ_{13} are $\mathcal{C}_k$ functions on $D_a \times D_d$ and given by

$$
\begin{aligned}
\chi_{11}(x_a, x_d) &= 2\delta(z_e(x_a)) \pi_\delta(z_e(x_a)) \frac{\partial z_e}{\partial x_a}(x_a) f_a(x_a, x_d) \\
\chi_{12}(x_a, x_d) &= \gamma^{-2} \delta(z_e(x_a)) \pi_\delta(z_e(x_a)) \frac{\partial z_e}{\partial x_a}(x_a) h_a(x_a, x_d) \\
\chi_{13}(x_a, x_d) &= 2\delta(z_e(x_a)) \pi_\delta(z_e(x_a)) \frac{\partial z_e}{\partial x_a}(x_a) g_a(x_a, x_d)
\end{aligned}
$$

Define $\alpha : D_a \times D_d \to \mathbb{R}$ by

$$\alpha(x_a, x_d) = (\chi_{13}(x_a, x_d))^{-1} \left(-\chi_{11}(x_a, x_d) - \phi(x_a, x_d) - \gamma^2 \chi_{12}(x_a, x_d) (\chi_{12}(x_a, x_d))' z_e(x_a) \right) \tag{59}$$

Then, the derivative of V is given by

$$\dot{V}(x_a, x_d, u, w)|_{u=\alpha(x_a,x_d)} = -\phi(x_a, x_d) z_e(x_a) + \gamma^2 |w|^2 - \gamma^2 |w - \sigma(x_a, x_d)|^2$$

and (58) holds with

$$\sigma(x_a, x_d) = (\chi_{12}(x_a, x_d))' z_e(x_a) = \frac{1}{2\gamma^2} (h_a(x_a, x_d))' \left(\frac{\partial V}{\partial x_a}(x_a) \right)'$$

This completes the proof of the lemma. □

Lemma 3 *Let n_1, n_a, $q \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $D_a \subseteq \mathbb{R}^{n_a}$, $D_e \subseteq \mathbb{R}$, $D_d \subseteq \mathbb{R}^{n_1}$, D_a, D_e, and D_d be nonempty and open. Let f_a, g_a, h_a be mappings of $D_a \times D_d$ into $\mathbb{R}^{n_a}$, $\mathbb{R}^{n_a}$, and $\mathbb{R}^{n_a \times q}$, respectively. Let f_a, g_a, h_a be $\mathcal{C}_k$, $\gamma > 0$ and $\gamma \in \mathbb{R}$. Let $z_e : D_a \to D_e$ be $\mathcal{C}_{k+1}$, $\phi : D_a \times D_d \to \mathbb{R}$ be $\mathcal{C}_k$ and $\delta : D_e \to \mathbb{R}$ be $\mathcal{C}_{k+1}$. Assume that z_e and δ satisfy the following conditions.*

(i) $\quad \dfrac{\partial z_e}{\partial x_a}(x_a) g_a(x_a, x_d) \neq 0, \quad \forall (x_a, x_d) \in D_a \times D_d.$

(ii) $\quad \delta(z_e(x_a)) > 0, \quad \forall x_a \in D_a.$

(iii) $\quad \pi_\delta(z_e(x_a)) = \delta(z_e(x_a)) + \dfrac{\partial \delta}{\partial z_e}(z_e(x_a)) z_e(x_a) > 0, \quad \forall x_a \in D_a.$

Let $V : D_a \to \mathbb{R}$ be defined by $V(x_a) = (\delta(z_e(x_a)) z_e(x_a))^2$, which is $\mathcal{C}_{k+1}$. Then there exists a $\mathcal{C}_k$ function $\alpha : D_a \times D_d \to \mathbb{R}$ defined by (59), such that V satisfies the following Hamilton-Jacobi-Isaacs equation.

$$\begin{aligned} \frac{\partial V}{\partial x_a}(x_a)(f_a(x_a, x_d) + g_a(x_a, x_d)\alpha(x_a, x_d)) + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x_a}(x_a) h_a(x_a, x_d)(h_a(x_a, x_d))' (\frac{\partial V}{\partial x_a}(x_a))' \\ + \phi(x_a, x_d) z_e(x_a) = 0; \quad \forall (x_a, x_d) \in D_a \times D_d \end{aligned} \tag{60}$$

Proof The reasoning is based on Lemma 7 in [1].

Let $D_w = \mathbb{R}^q$. By Lemma 2, there exists a $\mathcal{C}_k$ function $\alpha : D_a \times D_d \to \mathbb{R}$ such that the derivative of $V(x_a(t))$ along the solution of the dynamics of (56) satisfying (57). We will apply Lemma 7 in [1] to show the desired result (60). We make the following substitutions.

$$\begin{gathered} n_a \to n,\ n_1 \to m,\ q \to q,\ D_a \to D,\ x_a \to x,\ D_d \to D_d,\ x_d \to x_d, \\ f_a(x_a, x_d) + g_a(x_a, x_d)\alpha(x_a, x_d)) \to f(x, x_d) \\ h_a(x_a, x_d) \to g(x, x_d),\ \phi(x_a, x_d) z_e(x_a) \to l(x, x_d), \\ \sigma \to \sigma,\ \gamma \to \gamma,\ V \to V,\ (57) \to (63) \\ D_a \times D_d \to D_1,\ D_w \to D_w,\ w \to w, \end{gathered}$$

Then, it is clear that V satisfies (60).

This completes the proof of the lemma. □

Lemma 4 *Consider the following dynamics*

$$\dot{x} = Ax + Bu \tag{61}$$

where x is an n-dimensional vector, $n \in \mathbb{N}$; A is an $n \times n$ real matrix; B is an $n \times 1$ real vector, $B \neq 0$; u is a scalar input. Then, there exists a linear combination of x, that is, $y = Cx$, such that the transfer function $H(s) = C(sI - A)^{-1}B$ is strictly minimum phase and has relative degree 1.

Proof Case 1: If the pair (A, B) is controllable. Then, there exists a nonsingular state transformation $x = T\bar{x}$, where

$$T = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & \dots & B \end{bmatrix} \begin{bmatrix} 1 & & & & \\ a_{n-1} & 1 & & \mathbf{0} & \\ \vdots & & \ddots & & \\ a_2 & & & \ddots & \\ a_1 & a_2 & \dots & a_{n-1} & 1 \end{bmatrix} \tag{62}$$

and a_i, $i = 0, 1, \dots, n-1$, are the coefficients of the polynominal $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$, such that the system (61) can be transformed to be

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{63}$$

where the matrices $\bar{A}$ and $\bar{B}$ admit the following structure

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}; \quad \bar{B} = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Take a $1 \times n$ real vector $\bar{C} = [b_0 \quad b_1 \quad \dots \quad b_{n-1}]$ with $b_{n-1} \neq 0$, such that all roots of the polynominal $b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1 s + b_0 = 0$ have negative real parts.

The system $(\bar{A}, \bar{B}, \bar{C})$ has transfer function as follows

$$\bar{C}(sI - \bar{A})^{-1}\bar{B} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

Clearly, the above transfer function is strictly minimum phase and has relative degree 1. We take $C = \bar{C}T^{-1}$ and obtain

$$H(s) = C(sI - A)^{-1}B = \bar{C}(sI - \bar{A})^{-1}\bar{B}$$

Therefore, the transfer function $H(s)$ is also strcitly minimum phase and has relative degree 1.

Case 2: If the pair (A, B) is not completely controllable. There exists a real invertible transformation $x = R\hat{x}$, such that the system (61) is transformed to be

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \tag{64}$$

where

$$\begin{aligned} \hat{x} &= \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}; \\ \hat{A} &= R^{-1}AR = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \mathbf{0} & \hat{A}_{22} \end{bmatrix}; \\ \hat{B} &= R^{-1}B = \begin{bmatrix} \hat{B}_1 \\ \mathbf{0} \end{bmatrix} \end{aligned}$$

$\hat{x}_1$ is n_1-dimensional, $n_1 \in \{1, 2, \cdots, n\}$; $\hat{A}_{11}$ is $n_1 \times n_1$ dimensional; $\hat{B}_1$ is $n_1 \times 1$ dimensional; and the pair $(\hat{A}_{11}, \hat{B}_1)$ is controllable.

According to Case 1, we may obtain a $1 \times n_1$ real vector $\hat{C}_1$ such that the transfer function $\hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1$ is strictly minimum phase and has relative degree 1.

Take $\hat{C} = [\hat{C}_1 \quad \hat{C}_2]$, where $\hat{C}_2$ is an arbitrary $1 \times (n - n_1)$ real vector. Then

$$\begin{aligned} \hat{C}(sI - \hat{A})^{-1}\hat{B} &= \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} (sI - \hat{A}_{11})^{-1} & (sI - \hat{A}_{11})^{-1}\hat{A}_{12}(sI - \hat{A}_{22})^{-1} \\ \mathbf{0} & (sI - \hat{A}_{22})^{-1} \end{bmatrix} \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \\ &= \hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1 \end{aligned}$$

Therefore, the transfer function is strictly minimum phase and has relative degree 1.

Take $C = \hat{C}R^{-1}$, we have

$$H(s) = C(sI - A)^{-1}B = \hat{C}(sI - \hat{A})^{-1}\hat{B}$$

Hence the transfer function $H(s)$ is strictly minimum phase and has relative degree 1.

This completes the proof of the lemma. □

Lemma 5 *Consider the following linear time-invariant system*

$$\dot{z} = Az + Bu + Dw \tag{65a}$$

$$y = Cz + Ew \tag{65b}$$

where z is an n-dimensional state vector, $n \in \mathbb{N}$; u and y are scalar input and output, respectively. w is a q-dimensional unknown disturbance input, $q \in \mathbb{N}$; The matrix A is Hurwitz. The transfer function $H(s) = C(sI - A)^{-1}B$ is strictly minimum phase and has relative degree $r \in \mathbb{N}$.

Then there exists a real invertible coordinate transformation matrix T such that the system (65) can be transformed into the form (70) in [1], and which satisfies the Assumptions 10 in [1] with index $r_1 = r$.

Proof First, we partition the system dynamics (65) into observable and unobservable parts through the state transformation $[x_o' \quad x_{\bar{o}}']' = T_o^{-1}z$ with a real invertible matrix T_o. Then the system (65) admits representation

$$\left[\begin{array}{c} \dot{x}_o \\ \hline \dot{x}_{\bar{o}} \end{array}\right] = \left[\begin{array}{c|c} A_o & \mathbf{0} \\ \hline A_{o\bar{o}} & A_{\bar{o}} \end{array}\right] \left[\begin{array}{c} x_o \\ \hline x_{\bar{o}} \end{array}\right] + \left[\begin{array}{c} B_o \\ \hline B_{\bar{o}} \end{array}\right] u + \left[\begin{array}{c} D_o \\ \hline D_{\bar{o}} \end{array}\right] w \tag{66a}$$

$$y = \left[\begin{array}{c|c} C_o & \mathbf{0} \end{array}\right] \left[\begin{array}{c} x_o \\ \hline x_{\bar{o}} \end{array}\right] + Ew \tag{66b}$$

where the state vector x_o is n_o-dimensional, $n_o \in \mathbb{N}$, and the partitioning of the matrices are compatible with the partitioning of $[x_o' \; x_{\bar{o}}']'$. The pair (A_o, C_o) is observable.

Now we only consider the observable part in (66). There exists a real invertible coordinate transformation $[x_{co}' \;\; x_{\bar{c}o}']' = T_c^{-1}x_o$ such that the observable part is further divided into the controllable and uncontrollable parts as follows

$$\left[\begin{array}{c} \dot{x}_{co} \\ \hline \dot{x}_{\bar{c}o} \end{array}\right] = \left[\begin{array}{c|c} A_{co} & A_{c\bar{c}o} \\ \hline \mathbf{0} & A_{\bar{c}o} \end{array}\right] \left[\begin{array}{c} x_{co} \\ \hline x_{\bar{c}o} \end{array}\right] + \left[\begin{array}{c} B_{co} \\ \hline \mathbf{0} \end{array}\right] u + \left[\begin{array}{c} D_{co} \\ \hline D_{\bar{c}o} \end{array}\right] w \tag{67}$$

$$y = \left[\begin{array}{c|c} C_{co} & C_{\bar{c}o} \end{array}\right] \left[\begin{array}{c} x_{co} \\ \hline x_{\bar{c}o} \end{array}\right] + Ew \tag{68}$$

where the state vector x_{co} is n_c-dimensional, $n_c \in \mathbb{N}$, and the partitioning of the matrices are compatible with that of $[x'_{co}, x'_{\bar{c}o}]'$; $A_{\bar{c}o}$ is Hurwitz. The triple (A_{co}, B_{co}, C_{co}) is observable and controllable. Clearly, the transfer function $H(s) = C_{co}(sI - A_{co})^{-1}B_{co}$. Let the transfer function be given by

$$H(s) = C_{co}(sI - A_{co})^{-1}B_{co} = \frac{b_0 s^{n_c - r} + b_1 s^{n_c-r-1} + \cdots + b_{n_c-r}}{s^{n_c} + a_1 s^{n_c-1} + \cdots + a_{n_c}}, \quad b_0 \neq 0 \tag{69}$$

By Lemma 12 in [1], there exists a real invertible matrix T_1 such that

$$\left[\begin{array}{c|c} T_1^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \left[\begin{array}{c|c} A_{co} & B_{co} \\ \hline C_{co} & 0 \end{array}\right] \left[\begin{array}{c|c} T_1 & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \mathbf{0} \\ \hline \hat{C}_1 & \mathbf{0} & 0 \end{array}\right] \tag{70}$$

where

$$\hat{A}_{11} = \left[\begin{array}{c|c} a_1 & \\ \vdots & I_{r-1} \\ a_{r-1} & \\ \hline a_r & \mathbf{0}_{1\times(r-1)} \end{array}\right]; \quad \hat{A}_{12} = \begin{cases} \begin{bmatrix} & \mathbf{0} \\ 1 & \end{bmatrix} & n_c \geq r+1 \\ [\ ] & n_c = r \end{cases};$$

$$\hat{A}_{21} = \left[\begin{array}{c|c} a_{r+1} & \\ \vdots & \mathbf{0}_{(n_c-r)\times(r-1)} \\ a_{n_c} & \end{array}\right]; \quad \hat{A}_{22} = \begin{cases} \left[\begin{array}{c|c} -b_1/b_0 & \\ \vdots & I_{n_c-r-1} \\ -b_{n_c-r-1}/b_0 & \\ \hline -b_{n_c-r}/b_0 & \mathbf{0}_{1\times(n_c-r-1)} \end{array}\right] & n_c \geq r+1 \\ [\ ] & n_c = r \end{cases}$$

$$\hat{B}_1 = \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & b_0 \end{bmatrix}'; \quad \hat{C}_1 = \begin{bmatrix} 1 & \mathbf{0}_{1\times(r-1)} \end{bmatrix}$$

where $\hat{A}_{22}$ is Hurwitz, because the transfer function $H(s)$ is strictly minimum phase.

Then the desired transformation matrix T is equal to

$$T = T_o \left[\begin{array}{c|c} T_c & \mathbf{0} \\ \hline \mathbf{0} & I_{n-n_o} \end{array}\right] \left[\begin{array}{cc|c} T_1 & \mathbf{0} & \\ \cline{1-2} 0 & I_{n_o-n_c} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & I_{n-n_o} \end{array}\right] \tag{71}$$

Then, the system can be transformed into the following form, via the transform matrix $[x'_1\, x'_2\, x'_{\bar{c}o}\, x'_{\bar{o}}]' = Tz$.

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \hline \dot{x}_{\bar{c}o} \\ \hline \dot{x}_{\bar{o}} \end{array}\right] = \left[\begin{array}{cc|c|c} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{12,3} & \mathbf{0} \\ \hat{A}_{21} & \hat{A}_{22} & & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & A_{\bar{c}o} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\hat{A}_{4,12}} & \hat{A}_{43} & A_{\bar{o}} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_{\bar{c}o} \\ \hline x_{\bar{o}} \end{array}\right] + \left[\begin{array}{c} \hat{B}_1 \\ \mathbf{0} \\ \hline \mathbf{0} \\ \hline B_{\bar{o}} \end{array}\right] u + \left[\begin{array}{c} \hat{D}_1 \\ \hat{D}_2 \\ \hline D_{\bar{c}o} \\ \hline D_{\bar{o}} \end{array}\right] w \tag{72a}$$

$$y = \left[\begin{array}{cc|c|c} \hat{C}_1 & \mathbf{0} & C_{\bar{c}o} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_{\bar{c}o} \\ \hline x_{\bar{o}} \end{array}\right] + Ew; \tag{72b}$$

where all matrices are constant matrices of appropriate dimension. Clearly, the system (72) admits the state space representation (70) in [1] and satisfies the Assumption 10 in [1] with index $r_1 = r$.

This completes the proof of the Lemma 5. □

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