Adaptive Controller Design and Disturbance Attenuation for SISO Linear Systems with Noisy Output Measurements and Partly Measured Disturbances*

Sheng Zeng† Zigang Pan‡

Abstract

In this paper, we present robust adaptive controller design for SISO linear systems with noisy output measurements and partly measured disturbances. Using the worst-case analysis approach, we formulate the robust adaptive control problem as a nonlinear H^∞-optimal control problem under imperfect state measurements, and solve it using game theory. The design paradigm is the same as (Pan and Başar 2000) with the only difference being the treatment of the measured disturbances. The same result as (Pan and Başar 2000) is achieved. In addition, when the relative degrees from the measured disturbances to the output are no less than that from the control input, the controller designed achieves the *zero* disturbance attenuation level with respect to the measured disturbance inputs. The asymptotic tracking objective is achieved even if the measured disturbance is only uniformly bounded, without requiring it to be of finite energy. This strong robustness property is then illustrated by numerical examples.

Keywords Nonlinear H^∞ control; cost-to-come function; integrator backstepping; adaptive control; measured disturbances.

1 Introduction

The design of adaptive controllers has been an important research topic since 1970*s*. The classic adaptive control design is based on the certainty equivalence principle (Goodwin and Sin 1984, Goodwin and Mayne 1987), which has been shown to be effective for linear sytstems with or without stochastic disturbance inputs (Morse 1980, Kumar 1985, Narendra and Annaswamy 1989, Guo and Chen 1991, Ren and Kumar 1992, Naik *et al.* 1992, Guo 1996, Hsu *et al.* 1999). Using this approach, the controller is designed as if the unknown parameters are known, in implementation these unknown parameters are substituted by their on-line estimates, which are generated through a variety of identifiers, as long as the estimates satisfy certain properties independent of the controller. This approach leads to structurally simple adaptive controllers. Yet, early designs based on this

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†Department of Electrical and Computer Engineering and Computer Science, University of Cincinnati, Cincinnati, OH 45221-0030. Tel: 513-556-1036; Email: zengs@ececs.uc.edu.

‡Department of Electrical and Computer Engineering and Computer Science, 814 Rhodes Hall, Mail Loc. 0030, University of Cincinnati, Cincinnati, OH 45221-0030. Tel: 513-556-1808; Fax: 513-556-7326; Email: zpan@ececs.uc.edu. All correspondences should be addressed to this author.

approach has been shown to be nonrobust (Ioannou and Kokotović 1983, Rohrs *et al.* 1985) when the system is subject to exogenous disturbance inputs and unmodeled dynamics. This approach has also failed to generalize to nonlinear systems with severe nonlinearities. This has motivated the study of robust adaptive control in 1980s and 1990s, and nonlinear adaptive control in 1990s.

Nonlinear adaptive control attracted a lot of research attention in 1990s after the celebrated characterization of feedback linearizable or partially feedback linearizable systems (Isidori 1995). A breakthrough is achieved when the integrator backstepping methodolgy was introduced (Kanellakopoulos *et al.* 1991) to systematically design adaptive controllers for parametric strict-feedback and parametric pure-feedback nonlinear systems. This has led to an period of intense research into the topic of nonlinear adaptive control when a large volume of results flourished, see the book (Krstić *et al.* 1995) for a complete list of references. More recently, this approach has been generalized to systems with unknown sign of the high frequency gain (Ye 2001). This nonlinear design methodology has also been applied to linear systems (Krstić *et al.* 1994) to compare performance with the certainty equivalence approach. As to be expected, a systematically designed nonlinear adatpive control law leads to better closed-loop performance than that for the certainty equivalence based design when the system is free of disturbance. Yet this approach has also been shown to to nonrobust (Başar *et al.* 1996) when the system is subject to exogenous disturbance input.

Robust adaptive control has been an important research topic in late 1980s and early 1990s. This research leads to various modification techniques in adaptive control design in order to render the closed-loop systems robust (Datta and Ioannou 1994, Ioannou and Sun 1996). Despite their successes, they fell short of directly addressing the disturbance attenuation property of the closed-loop system.

The objectives of robust adaptive control are to improve transient response, to accomondate unmodeled dynamics, and to reject exogenous disturbance inputs. These objectives are the same as those that motivate the study of the H^∞-optimal control problem, where these objectives are fulfilled by studying the disturbance attenuation property of the system. The game-theoretic approach to H^∞-optimal control (Başar and Bernhard 1995) offers the most promising tool to generalize the results to nonlinear systems (van der Schaft 1991, 1992, Isidori and Astolfi 1992, Didinsky *et al.* 1993, Didinsky 1994, Marino *et al.* 1994, Isidori and Kang 1995). These observations and results motivated the worst-case analysis approach to adaptive control, where the adaptive control problem is formulated as a nonlinear H^∞-optimal control problem under imperfect state measurements. The unknown parameter vector is viewed as part of the expanded state vector, and the measures of transient response, disturbance attenuation, and asymptotic tracking are all incorporated into a single game-theoretic cost function. The cost-to-come function analysis is applied to obtain the state estimator for the expanded state vector, which results in an on-line parameter identifier and a state estimator for the original system. This step converts the nonlinear H^∞-optimal control problem under imperfect state measurements into one under full information measurements. This full information measurement problem is then solved for a suboptimal solution using the integrator backstepping methodology. This design paradigm has been applied to worst-case parameter identification problems (Didinsky *et al.* 1995, Pan and Başar 1996), which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems (Didinsky and Başar 1997, Pan and Başar 1998, Tezcan and Başar 1999, Pan and Başar 2000, Arslan and Başar 2001), which has led to new classes of parametrized robust adaptive controllers for linear and nonlinear systems.

In this paper, we further generalize the worst-case analysis based approach to linear systems

with partly mearsured disturbance inputs. We assume that the linear system admits a known upper bound for its dynamic order, a known relative degree, a known sign of high-frequency gain with a nonvanishing bound away from 0, a strictly minimum phase transfer function from the control input to the output, and some other assumptions, which are the same as (Pan and Başar 2000). The adaptive control design follows the same paradigm as discussed above, which leads to two classes of parametrized controllers in closed form with the following robustness properties. The close-loop system admits a guaranteed disturbance attenuation level with respect to the exogneous disturbance input, where the ultimate attenuation lower bound is equal to the noise intensity in the measurement channel. The closed-loop system is totally stable (Chen 1984) with respect to the disturbance input and the initial condition. Furthemore, it achieves asymptotic tracking of the reference trajectory for all uniformly bounded disturbances inputs that are of bounded energy. These results are the same as those of (Pan and Başar 2000). In addition, with proper scaling, the controller achieves any positive attenuation level with respect to the measured disturbance. When the output has relative degrees for the measured disturbances that are greater than or equal to that for the control input, the controller designed achieves *zero* disturbance attenuation level with respect to the measured disturbances. This does not mean that we achieve decoupling from the measured disturbances, but means that the measured disturbances does not have any effect on the squared $\mathcal{L}_2$ norm of the tracking error. Therefore, when the unmeasured disturbance is $\mathcal{L}_2 \cap \mathcal{L}_\infty$, the tracking error asymptotically converges to zero for any measured disturbance that is uniformly bounded.

The balance of the paper is organized as follows. In Section 2, we present the fomulation of the adaptive control problem and discuss the general solution methodology. In Section 3, we obtain parameter identifier and state estimator using the cost-to-come function analysis. Then we derive the adaptive control law in Section 4, and present the main result on the robustness of the closed-loop system. The theoretical results are illustrated by two numerical examples in Section 5. The paper ends with some concluding remarks in Section 6, and an appendix which contains some important lemmas that are need for the derivation in the main body of the paper.

2 Problem Formulation

We consider the adaptive control problem for single-input-single-output (SISO) linear systems. We make the following assumption on the unknown system.

Assumption 1 *The linear system is known to be at most n dimensional.*

By adding addtional dyanmics if necessary, we consider the following true system dynamics:

$$\dot{x} = \grave{A}x + \grave{B}u + \grave{D}_1\grave{w} + \grave{D}_2\breve{w} \tag{1a}$$

$$y = \grave{C}x + \grave{E}\grave{w} \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state vector; $u \in \mathbb{R}$ is the scalar control input; $y \in \mathbb{R}$ is the scalar system output; $\grave{w} \in \mathbb{R}^{\grave{q}}$ is the unmeasured disturbance input vector; $\breve{w} \in \mathbb{R}^s$ is the measured disturbance input vector; the state x has initial condition $x(0)$; and all input and output signals y, u, $\breve{w}$, and $\grave{w}$ are continuous in the extended $\mathcal{L}_2$ spaces, i.e. $\mathcal{L}_{2e}$; the matrices $\grave{A}$, $\grave{B}$, $\grave{C}$, $\grave{D}_1$, $\grave{D}_2$, and $\grave{E}$ are of the appropriate dimensions, generally unknown or partially unknown. The transfer function from u to y is $H(s) = \grave{C}(sI - \grave{A})^{-1}\grave{B}$.

Assumption 2 *The pair $(\grave{A}, \grave{C})$ is observable. The transfer function $H(s)$ is known to have relative degree r, and is strictly minimum phase. Moreover, the uncontrollable part (with respect to u) of the unknown system is stable in the sense of Lyapunov. Any uncontrollable mode correponding to the eigenvalues of the system matrix on the $j\omega$-axis are uncontrollable from $\grave{w}$ and $\check{w}$.*

The above assumption implies that it is not necessary that the true system is controllable, but the uncontrollable part should be stable in the sense of Lyapunov, i.e. all eigenvalues of uncontrollable part have zero or negative real part and those with zero part have Jordan block of order 1, and uncontrollable from $\grave{w}$ and $\check{w}$.

As discussed in (Pan and Başar 2000), there exists a state diffeomorphism $x = \grave{T}\grave{x}$ and a disturbance transformation $w = \grave{M}\grave{w}$, such that system (1) can be transformed into the following form in the x cooridinates

$$\dot{x} = Ax + (y\bar{A}_{21} + u\bar{A}_{22})\theta + Dw + \sum_{j=1}^{s} \check{w}_j \bar{A}_{23j}\theta + \check{D}\check{w} \tag{2a}$$

$$y = Cx + Ew \tag{2b}$$

where θ is the σ-dimensional vector of unknown parameters of the systems; the matrics A, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{A}_{231}$, $\cdots$, $\bar{A}_{23s}$, D, $\check{D}$, C, and E are of appropriate dimensions, known and have the following structure

$$A = \begin{bmatrix} a_{11} & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots & & \vdots \\ a_{r-2,1} & \cdots & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ a_{r-1,1} & \cdots & \cdots & \cdots & a_{r-1,r-1} & 1 & \cdots & 0 \\ a_{r1} & \cdots & \cdots & \cdots & a_{r,r-1} & a_{rr} & \cdots & a_{rn} \\ \vdots & & & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{n,r-1} & a_{nr} & \cdots & a_{nn} \end{bmatrix}; \quad \bar{A}_{22} = \begin{bmatrix} \mathbf{0}_{(r-1)\times\sigma} \\ \bar{A}_{220} \\ \bar{A}_{22r} \end{bmatrix};$$

$$\bar{A}_{220} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix};$$

Then the uncertainty quadruple $(x(0), \theta, \grave{w}_{[0,\infty)}, \check{w}_{[0,\infty)})$ comprises the initial state, the true values of unknown parameters, unmeasured disturbances and the measured disturbances. We will design adaptive controller using system (2), which is called the design model.

Assumption 3 *The matrics D and E are such that $EE' > 0$. Define $\zeta := 1/(EE')^{\frac{1}{2}}$ and $L := DE'$.*

Because the structure of A, $\bar{A}_{22}$, the first element of the parameter vector θ, denoted by b_0 is equal to the high frequency gain of the transfer function $H(s)$. We partition parameter vector θ into

$$\theta = \begin{bmatrix} b_0 & \theta_s' \end{bmatrix}'$$

where θ_s is a $(\sigma - 1)$-dimensional vector.

Assumption 4 *The sign of the high-frequency gain b_0 is known, and without loss of generality, assume $b_0 > 0$. There exists a known smooth nonnegative radially-unbounded strictly convex function $P(\theta)$, such that the true value θ belongs to the closed convex set $\Theta_e := \{\theta : P(\theta) \leq 1\}$. Moreover, for any $\theta \in \Theta_e$, we have $b_0 > 0$.*

Assumption 4 delineated the *a priori* convex compact set where the parameter vector θ lies in, which will gurantee the stability of the closed-loop system and the boundedness of the estimate of θ.

The control law to system (2) is generated by the following control law

$$u(t) = \mu(t, y_{[0,t]}, \check{w}_{[0,t]}) \tag{3}$$

where $\mu : [0, \infty) \times \mathcal{L}_{2e} \times \mathcal{L}_{2e} \to \mathbb{R}$. We denote the class of these admissible controllers by $\mathcal{M}$.

The control objective is to design a robust adaptive controller for system (1), such that the output $y(t)$ tracks a desired reference signal $y_d(t)$ while rejecting the uncertainty quadruple, while all signals in the closed-loop system are uniformly bounded.

We make the following assumption about the reference signal, y_d.

Assumption 5 *The reference trajectory, y_d, is r times continuously differentiable.*

Define vector $Y_d = [y_d^{(0)}, \cdots, y_d^{(r)}]' \in \mathbb{R}^{r+1}$, where $y_d^{(0)} = y_d$, and $y_d^{(i)}$ is the ith order time derivative of y_d, $i = 1, \cdots, r$. The desired trajectory vector Y_d is uniformly bounded, i.e., for some $c_d > 0$

$$|Y_d(t)| \leq c_d, \qquad \forall t \in [0, \infty)$$

where $|\cdot|$ denote the Euclidean norm.

For design purposes, instead of attenuating the effect of $\dot{w}$ and $\check{w}$, we design the adaptive controller to attenuate the effect of w and $\check{w}$. This is done to allow our design paradigm to be carried out. This will result in a guaranteed attenuation level with respect to $\dot{w}$ and $\check{w}$ as well, see (Pan and Başar 2000) for a discussion. We take the uncertainty quadruple $(x(0), \theta, w_{[0,\infty)}, \check{w}_{[0,\infty)})$ to belong to the set $\mathcal{W} = \mathbb{R}^n \times \Theta_e \times \mathcal{L}_{2e} \times \mathcal{L}_{2e}$.

We make the control objective precise as follows.

Definition 1 *A controller μ is said to achieve disturbance attenuation level γ if there exist nonnegative functions $l(t, \theta, x, y_{[0,t]}, \check{w}_{[0,t]})$ and $l_0(\check{x}_0, \check{\theta}_0)$ such that*

$$\sup_{(x(0),\theta,w_{[0,\infty)},\check{w}_{[0,\infty)})\in\mathcal{W}} J^t \leq 0; \qquad \forall t \geq 0 \tag{4}$$

where

$$J^t := \int_0^t ((x_1 - y_d)^2 + l(\tau, \theta, x(\tau), y_{[0,\tau]}, \check{w}_{[0,\tau]}) - \gamma^2|w|^2 - \gamma^2|\check{w}|^2)d\tau - \gamma^2|\theta - \check{\theta}_0|^2_{Q_0}$$
$$-\gamma^2|x(0) - \check{x}_0|^2_{\Pi_0^{-1}} - l_0(\check{x}_0, \check{\theta}_0) \tag{5}$$

$\check{\theta}_0 \in \Theta_e$ is the initial guess of the unknown parameter vector; the $\sigma \times \sigma$ dimensional matrix $Q_0 > 0$ is the quadratic weighting on the error between the true value of θ and the initial guess $\check{\theta}_0$, quantifying the level of confidence in the estimate $\check{\theta}_0$; $\check{x}_0$ is the initial guess of the unknown initial state $x(0)$; and $\Pi_0^{-1} > 0$ is the the weighting on the initial state estimation error, quantifying the level of confidence in the estimate $\check{x}_0$; and $|z|_Q$ denotes z^TQz for any vector z and any symmetric matrix Q.

Clearly, when the inequality (4) is achieved, we have

$$\frac{\|x_1 - y_d\|_2^2}{\|w\|_2^2 + \|\check{w}\|_2^2 + |\theta - \check{\theta}_0|_{Q_0}^2 + |x(0) - \check{x}_0|_{\Pi_0^{-1}}^2 + C_0} \leq \gamma^2 \tag{6}$$

where $\|\cdot\|_2$ denote $\mathcal{L}_2$ norm, and C_0 is a constant. When $\|w\|_2$ and $\|\check{w}\|_2$ are finite, $\|x_1 - y_d\|_2$ is also finite, which implies $\lim_{t\to\infty} |x_1 - y_d| = 0$, under additional assumptions.

The following notation will be used thoughtout this paper. The vector $z := [z_1, \cdots, z_r]'$ denotes some transformed state vector; $\check{x}$ denotes the estimate of the current state of the system; $\tilde{x}$ denotes the state estimation error $x - \check{x}$; $\check{\theta}$ denotes the estimate of the parameter vector θ; $\tilde{\theta}$ denotes the estimation error $\theta - \check{\theta}$; any function symbol with an "over bar" will denote a function defined in the terms of the transformed state variables, such as $\bar{f}(z)$ being the identical function as $f(x)$; for any matrix M, the vector $\vec{M}$ is formed by stacking up its column vectors; $e_{j,i}$ denotes a j-dimensional column vector, all of its elements are 0, except its ith row is 1, such as $e_{4,2} = [0, 1, 0, 0]'$.

Let ξ denote the expanded state vector $\xi = [\theta', x']'$. Note that

$\dot{\theta} = 0$, we have the following expanded dynamics for system (2)

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ y\bar{A}_{21} + u\bar{A}_{22} + \sum_{i=1}^{s} \bar{A}_{23i}\check{w} & A \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0}_{\sigma\times q} \\ D \end{bmatrix} w + \begin{bmatrix} \mathbf{0}_{\sigma\times s} \\ \check{D} \end{bmatrix} \check{w} && \text{(7a)} \\
&=: \bar{A}\xi + \bar{D}w + \bar{\check{D}}\check{w} \\
y &= \begin{bmatrix} \mathbf{0}_{1\times\sigma} & C \end{bmatrix} \xi + Ew =: \bar{C}\xi + Ew && \text{(7b)}
\end{aligned}$$

The worst-case optimization of the cost function (5) can be carried out in two steps as depicted in the following equations.

$$\sup_{(x(0),\theta,w_{[0,\infty)},\check{w}_{[0,\infty)})\in\mathcal{W}} J^t = \sup_{y_{[0,\infty)},\check{w}_{[0,\infty)}} \sup_{(x(0),\theta,w_{[0,\infty)})|y_{[0,\infty)},\check{w}_{[0,\infty)}} J^t \tag{8}$$

The right-hand supremum operator will be carried out first. It is the identification design step, which will be presented in Section 3. Succinctly stated, in this step, we will calculate the maximam cost that is consistant with the given measurement waveform.

The left-hand supremum operator will be carried out second. It is the controller design step, which will be dicussed in Section 4. In this step we use a backstepping method to design the control input u, and prove that all state varibles of the closed-loop system are uniformly bounded in time for any uniformly bounded disturbance input waveforms.

This completes the formulation of the robust adaptive control problem. Next, we turn to the identification design step in the next section.

3 Design of A Worst-case Indentifier

In this section, we present the identification design for the adaptive control problem formulated. In this step, the measurement waveform $y_{[0,\infty)}$ and $\check{w}_{[0,\infty)}$ are assumed to be known. Since the control input is a causal function of y and $\check{w}$, then it is known. This calculation uses the cost-to-come function methodology. The relevant cost function for this step is, after ignoring terms which are

constant in this optimization

$$J_{i\gamma}^t = \int_0^t \left(|x_1 - y_d|^2 + |\xi - \hat{\xi}|^2_{\bar{\bar{Q}}(\tau, y_{[0,\tau]}, \check{w}_{[0,\tau]})} - \gamma^2 |w|^2 \right) d\tau - \gamma^2 |\theta - \check{\theta}_0|^2_{Q_0} - \gamma^2 |x(0) - \check{x}_0|^2_{\Pi_0^{-1}} \tag{9}$$

where the first positive definite term inside the integral is required by the objective of the adaptive control design (5); the second nonnegative definite term inside the integral is required for robustness considerations of the complete adaptive system and is part of the function l; $\hat{\xi}$ is the worst-case estimate for the expanded state ξ, yet to be determined; $\bar{\bar{Q}}$ is a matrix-valued weighting function to be defined shortly; and the last two terms are required by the objective (5).

By the cost-to-come function analysis of (Pan and Başar 2000), we have

$$\begin{aligned} \dot{\bar{\Sigma}} &= (\bar{A} - \zeta^2 \bar{L}\bar{C})\bar{\Sigma} + \bar{\Sigma}(\bar{A} - \zeta^2 \bar{L}\bar{C})' + \frac{1}{\gamma^2}\bar{D}\bar{D}' - \frac{1}{\gamma^2}\zeta^2 \bar{L}\bar{L}' \\ &\quad -\bar{\Sigma}(\gamma^2\zeta^2 \bar{C}'\bar{C} - \bar{C}'\bar{C} - \bar{\bar{Q}})\bar{\Sigma}; \quad \bar{\Sigma}(0) = \frac{1}{\gamma^2}\begin{bmatrix} Q_0 & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \Pi_0^{-1} \end{bmatrix}^{-1} \end{aligned} \tag{10a}$$

$$\begin{aligned} \dot{\check{\xi}} &= (\bar{A} + \bar{\Sigma}(\bar{C}'\bar{C} + \bar{\bar{Q}}))\check{\xi} + \zeta^2(\gamma^2 \bar{\Sigma}\bar{C}' + \bar{L})(y - \bar{C}\check{\xi}) + \bar{\bar{D}}\check{w} \\ &\quad -\bar{\Sigma}(\bar{C}' y_d + \bar{\bar{Q}}\hat{\xi}); \quad \check{\xi}(0) = \begin{bmatrix} \check{\theta}_0' & \check{x}_0' \end{bmatrix}' \end{aligned} \tag{10b}$$

where $\bar{L}$ is defined as $\bar{L} = \begin{bmatrix} \mathbf{0}_{1\times\sigma} & L' \end{bmatrix}'$.

Then, the cost function (9) can be equivalently written as

$$\begin{aligned} J_{i\gamma}^t = &- |\xi(t) - \check{\xi}(t)|^2_{\bar{\Sigma}^{-1}(t)} + \int_0^t (|\bar{C}\check{\xi} - y_d|^2 + |\hat{\xi} - \check{\xi}|^2_{\bar{\bar{Q}}} - \gamma^2\zeta^2 |y - \bar{C}\check{\xi}|^2 \\ &- \gamma^2 |w - w^*(\tau, u_{[0,\tau]}, y_{[0,\tau]}, \check{w}_{[0,\tau]}, \xi_{[0,\tau]}, \hat{\xi}_{[0,\tau]}, \check{\xi}_{[0,\tau]})|^2) d\tau \end{aligned} \tag{11}$$

where w^* is the worst-case disturbance, given by

$$w^*(\tau, u_{[0,\tau]}, y_{[0,\tau]}, \check{w}_{[0,\tau]}, \xi_{[0,\tau]}, \hat{\xi}_{[0,\tau]}, \check{\xi}_{[0,\tau]}) = \zeta^2 E'(y - \bar{C}\xi) + \frac{1}{\gamma^2}(I - \zeta^2 E'E)\bar{D}'\bar{\Sigma}^{-1}(\xi - \check{\xi}) \tag{12}$$

Define $\xi_c := \hat{\xi} - \check{\xi}$, which will be determined later to improve the performance of the adaptive system.

The following derivation for the identifier closely resembles that in (Pan and Başar 2000). Partition $\bar{\Sigma}$ as

$$\bar{\Sigma} = \begin{bmatrix} \Sigma & \bar{\Sigma}_{12} \\ \bar{\Sigma}_{21} & \bar{\Sigma}_{22} \end{bmatrix} \tag{13}$$

and introduce $\Phi := \bar{\Sigma}_{21}\Sigma^{-1}$ and $\Pi := \gamma^2(\bar{\Sigma}_{22} - \bar{\Sigma}_{21}\Sigma^{-1}\bar{\Sigma}_{12})$.

For the boundedness of Σ, we make the following assumption on the weighting matrix $\bar{\bar{Q}}$,

Assumption 6 *The weighting matrix $\bar{\bar{Q}}$ in (9) is given by*

$$\bar{\bar{Q}} = \bar{\Sigma}^{-1}\begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \Delta \end{bmatrix}\bar{\Sigma}^{-1} + \begin{bmatrix} \epsilon\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \mathbf{0}_{n\times n} \end{bmatrix}$$

where Δ is an $n \times n$ positive-definite matrix, and ϵ is a scalar function defined by

$$\epsilon(\tau) := \frac{\mathrm{Tr}(\Sigma^{-1}(\tau))}{K_c}, \qquad K_c \geq \gamma^2 \mathrm{Tr}(Q_0), \qquad \tau \geq 0 \tag{14}$$

Then, we have the following differential equation of Σ, Φ,
and Π

$$\dot{\Sigma} = -(1-\epsilon)\Sigma\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi\Sigma; \ \Sigma(0) = \frac{1}{\gamma^2}Q_0^{-1} \tag{15a}$$

$$\dot{\Phi} = (A - \zeta^2 LC - \frac{1}{\gamma^2}\Pi C'(\gamma^2\zeta^2 - 1)C)\Phi + y\bar{A}_{21} + u\bar{A}_{22} + \sum_{i=1}^{s} \bar{A}_{23i}\breve{w}_i; \ \Phi(0) = \mathbf{0}_{n\times\sigma} \tag{15b}$$

$$\begin{aligned}\dot{\Pi} &= (A - \zeta^2 LC)\Pi + \Pi(A - \zeta^2 LC)' - \Pi C'(\zeta^2 - \frac{1}{\gamma^2})C\Pi + DD' \\ &\quad -\zeta^2 LL' + \gamma^2\Delta; \qquad \Pi(0) = \Pi_0\end{aligned} \tag{15c}$$

The matrix Σ will play the role of worst-case covariance matrix of the parameter estimation error. Assumption 6 guarantees that Σ is uniformly bounded from above and uniformly bounded from below away from 0 as depicted in the following Lemma, whose proof is given in (Pan and Başar 2000).

Lemma 1 *Consider the dynamic equation (15a) for the covariance matrix Σ. Let Assumption 6 hold and $\gamma \geq \zeta^{-1}$. Then, the matrix Σ is uniformly upper and lower bounded as follows:*

$$\frac{1}{K_c}I_{\sigma\times\sigma} \leq \Sigma(\tau) \leq \Sigma(0) = \frac{1}{\gamma^2}Q_0^{-1}; \qquad \gamma^2\mathrm{Tr}(Q_0) \leq \mathrm{Tr}(\Sigma^{-1}(\tau)) \leq K_c; \qquad \forall\tau \in [0,t]$$

To simpify the on-line calculation, we define $s(\tau) := 1/\mathrm{Tr}(\Sigma^{-1}(\tau))$, and its time derivative is given by

$$\dot{s} = -s^2(\gamma^2\zeta^2 - 1)(1-\epsilon)C\Phi\Phi' C'; \qquad s(0) = \frac{1}{\gamma^2\mathrm{Tr}(Q_0)} \tag{16}$$

Then, $\epsilon(\tau) = K_c^{-1}s^{-1}(\tau)$, which does not require the inversion of Σ.

From Assumption 6 and (15a), we note that $\gamma \geq \zeta^{-1}$. This means the quantity ζ^{-1} is the ultimate lower bound on the achievable performance level for the adaptive system, using the design method proposed in this paper.

Assumption 7 *If the matrix $A - \zeta^2 LC$ is Hurwitz, then the desired disturbance attenuation level $\gamma \geq \zeta^{-1}$. If the matrix $A - \zeta^2 LC$ is not Hurwitz, then the desired disturbance attenuation level $\gamma > \zeta^{-1}$.*

Assumption 8 *the initial weighting matrix Π_0 in (15c) is chosen as the unique positive definite solution to the algebraic Riccati equation:*

$$(A - \zeta^2 LC)\Pi + \Pi(A - \zeta^2 LC)' - \Pi C'(\zeta^2 - \frac{1}{\gamma^2})C\Pi + DD' - \zeta^2 LL' + \gamma^2\Delta = 0 \tag{17}$$

Then, we note that the unique positive-definite solution of (15c) is time-invariant and equal to the initial value Π_0.

To guarantee the boundness of estimated parameters without persistently exciting signals, we introduce soft projection design on the parameter estimate, which is based on the *a priori* information on the bounds of the true value of the parameter vector θ.

Define

$$\rho := \min_{\theta_s} P(0, \theta_s); \qquad \Theta_o := \{\theta : P(\theta) < \frac{1+\rho}{2}\}$$

By Assumption 4, we have $1 < \rho < \infty$, and the convexity of P implies the following inequality

$$\frac{\partial P}{\partial \theta}(\check{\theta})(\theta - \check{\theta}) < 0$$

Add the following term to the right-hand side of the dyanmics (10b) when $P(\check{\theta}) > 1$:

$$-\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\left(\frac{1+\rho}{2} - P(\check{\theta})\right)^3}\bar{\Sigma}\begin{bmatrix} \frac{\partial P}{\partial \theta}(\check{\theta}) & \mathbf{0}_{1\times n} \end{bmatrix}' \tag{18}$$

we have

$$\begin{aligned}\dot{\check{\xi}} &= -(1-\mathbf{1}_{\Theta_e}(\check{\theta}))\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\left(\frac{1+\rho}{2} - P(\check{\theta})\right)^3}\bar{\Sigma}\begin{bmatrix} \frac{\partial P}{\partial \theta}(\check{\theta}) & \mathbf{0}_{1\times n} \end{bmatrix}' + \bar{A}\check{\xi} - \bar{\Sigma}\bar{C}'(y_d - \bar{C}\check{\xi}) \\ &\quad -\bar{\Sigma}\bar{\bar{Q}}\xi_c + (\gamma^2\bar{\Sigma}\bar{C}' + \bar{L})\zeta^2(y - \bar{C}\check{\xi}) + \bar{\bar{D}}\check{w}; \qquad \check{\xi}(0) = \begin{bmatrix} \check{\theta}_0' & \check{x}_0' \end{bmatrix}'\end{aligned} \tag{19}$$

where the function $\mathbf{1}_{\Theta_e}(\check{\theta})$ is the indicator function of the set Θ_e, i.e.,

$$\mathbf{1}_{\Theta_e}(\check{\theta}) = \begin{cases} 1 & \check{\theta} \in \Theta_e \\ 0 & \check{\theta} \notin \Theta_e \end{cases}$$

It is easy to verify that the following nonlinear function

$$P_r(\check{\theta}) := (1-\mathbf{1}_{\Theta_e}(\check{\theta}))\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\left(\frac{1+\rho}{2} - P(\check{\theta})\right)^3}\left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \tag{20}$$

is smooth on the set Θ_o .

Partition $\check{\xi}$ into $(\check{\theta}', \check{x}')'$ to obtain the dynamics for these individual estimates. We sumarize the equations below. Partition $\check{\xi}$ into $(\check{\theta}', \check{x}')'$ to obtain the dynamics for these individual estimates. We sumarize the equations below.

$$(A - \zeta^2 LC)\Pi + \Pi(A - \zeta^2 LC) - \Pi C'(\zeta^2 - \frac{1}{\gamma^2})C\Pi + DD' - \zeta^2 LL' + \gamma^2\Delta = 0 \tag{21a}$$

$$\dot{\Sigma} = -(1-\epsilon)\Sigma\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi\Sigma; \quad \Sigma(0) = \frac{1}{\gamma^2}Q_0^{-1} \tag{21b}$$

$$\dot{s} = -s^2(\gamma^2\zeta^2 - 1)(1-\epsilon)C\Phi\Phi'C'; \quad s(0) = \frac{1}{\gamma^2 \mathrm{Tr}(Q_0)} \tag{21c}$$

$$\epsilon = \frac{1}{K_c s} \tag{21d}$$

$$A_f = A - \zeta^2 LC - \Pi C'C(\zeta^2 - \frac{1}{\gamma^2}) \tag{21e}$$

$$\dot{\Phi} = A_f\Phi + y\bar{A}_{21} + u\bar{A}_{22} + \sum_{i=1}^{s} \check{w}_i \bar{A}_{23i}; \quad \Phi(0) = \mathbf{0}_{n\times\sigma} \tag{21f}$$

$$\begin{aligned}\dot{\check{\theta}} &= -\Sigma P_r(\check{\theta}) - \Sigma\Phi'C'(y_d - C\check{x}) - \left[\begin{array}{cc} \Sigma & \Sigma\Phi' \end{array}\right]\bar{\bar{Q}}\xi_c \\ &\quad +\gamma^2\zeta^2\Sigma\Phi'C'(y - C\check{x}); \quad \check{\theta}(0) = \check{\theta}_0\end{aligned} \tag{21g}$$

$$\begin{aligned}\dot{\check{x}} &= -\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\frac{1}{\gamma^2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) - \left[\begin{array}{cc} \Phi\Sigma & \frac{1}{\gamma^2}\Pi + \Phi\Sigma\Phi' \end{array}\right]\bar{\bar{Q}}\xi_c \\ &\quad +(y\bar{A}_{21} + u\bar{A}_{22} + \sum_{j=1}^{s}\check{w}_j\bar{A}_{23j})\check{\theta} + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi'C' + L)(y - C\check{x}) \\ &\quad +\check{D}\check{w}; \qquad \check{x}(0) = \check{x}_0\end{aligned} \tag{21h}$$

For the controller structure simplification, the dynamics for Φ can be implimented with $(s+2)n$ integrators instead of the σn integrators. First, we observe the matrix A_f has the same structure as the matrix A. Then we introduce the matrix

$$M_f := \left[\begin{array}{ccccc} A_f^{n-1}p_n & \cdots & A_f p_n & p_n \end{array}\right] \tag{22}$$

where p_n is a n-dimensional vector such that the pair (A_f, p_n) is controllable, which implies that M_f is invertible. Then the following $(s+2)n$-dimensional prefiltering system for y, u, and $\check{w}$ generates the Φ online:

$$\dot{\eta} = A_f\eta + p_n y; \quad \eta(0) = 0 \tag{23a}$$

$$\dot{\lambda} = A_f\lambda + p_n u; \quad \lambda(0) = 0 \tag{23b}$$

$$\dot{\eta}^{\check{w}_1} = A_f\eta^{\check{w}_1} + p_n\check{w}_1; \quad \eta^{\check{w}_1}(0) = 0 \tag{23c}$$

$$\vdots$$

$$\dot{\eta}^{\check{w}_s} = A_f\eta^{\check{w}_s} + p_n\check{w}_s; \quad \eta^{\check{w}_s}(0) = 0 \tag{23d}$$

$$\begin{aligned}\Phi &= \left[\begin{array}{cccc} A_f^{n-1}\eta & \cdots & A_f\eta & \eta \end{array}\right]M_f^{-1}\bar{A}_{21} + \left[\begin{array}{cccc} A_f^{n-1}\lambda & \cdots & A_f\lambda & \lambda \end{array}\right]M_f^{-1}\bar{A}_{22} \\ &\quad +\sum_{j=1}^{s}\left[\begin{array}{cccc} A_f^{n-1}\eta^{\check{w}_j} & \cdots & A_f\eta^{\check{w}_j} & \eta^{\check{w}_j} \end{array}\right]M_f^{-1}\bar{A}_{23j}\end{aligned} \tag{23e}$$

Associated with the above identifier, introduce the value function

$$\begin{aligned}W(t, \xi(t), \check{\xi}(t), \bar{\Sigma}(t)) &= |\xi(t) - \check{\xi}(t)|^2_{\bar{\Sigma}^{-1}(t)} \\ &= |\theta - \check{\theta}(t)|^2_{\Sigma^{-1}(t)} + \gamma^2|x(t) - \check{x}(t) - \Phi(t)(\theta - \check{\theta}(t))|^2_{\Pi^{-1}}\end{aligned} \tag{24}$$

whose time derivative is given by

$$\dot{W} = -|x_1 - y_d|^2 - \gamma^4|x - \hat{x} - \Phi(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi'C'C\Phi} + \gamma^2|w|^2$$

$$+|C\check{x}-y_d|^2+|\xi_c|^2_{\bar{\tilde{Q}}}-\gamma^2\zeta^2|y-C\check{x}|^2-\gamma^2|w-w^*(t,u_{[0,t]},y_{[0,t]},\check{w}_{[0,t]},\xi_{[0,t]},\hat{\xi}_{[0,t]},\check{\xi}_{[0,t]})|^2$$
$$+2(\theta-\check{\theta})'P_r(\check{\theta}) \tag{25}$$

We note that the last term in $\dot{W}$ is nonpositive, zero on the set Θ_e and approaches $-\infty$ as $\check{\theta}$ approaches the boundary of the set Θ_o, which guarantees the boundness of $\check{\theta}$.

Then the cost function (9) can be equivalently written as:

$$\begin{aligned}
J^t_{i\gamma} &= J^t_{i\gamma}+W(0)-W(t)+\int_0^t \dot{W}d\tau \\
&= -|\theta-\check{\theta}(t)|^2_{\Sigma^{-1}(t)}-\gamma^2|x(t)-\check{x}(t)-\Phi(t)(\theta-\check{\theta}(t))|^2_{\Pi^{-1}}+\int_0^t(|C\check{x}-y_d|^2+|\xi_c|^2_{\bar{\tilde{Q}}} \\
&\quad -\gamma^2\zeta^2|y-C\check{x}|^2-\gamma^2|w-w^*(\tau,u_{[0,\tau]},y_{[0,\tau]},\check{w}_{[0,\tau]},\xi_{[0,\tau]},\hat{\xi}_{[0,\tau]},\check{\xi}_{[0,\tau]})|^2 \\
&\quad +2(\theta-\check{\theta})'P_r(\check{\theta}))d\tau
\end{aligned} \tag{26}$$

This completes the identification design step.

4 Control Design

In this section, we describe the controller design for the uncertain system under consideration. Note that, we ignored some terms in the cost function (5) in the identification step, since they are constant when y and $\check{w}$ are given. In the control design step, we will include such terms. Then, based on the cost function (5) in the Section 2, the controller design is to guarantee that the following supremum is less than or equal to zero for all measurement waveforms,

$$\begin{aligned}
&\sup_{(x(0),\theta,w_{[0,\infty)}\check{w}_{[0,\infty)})\in\mathcal{W}} J^t \\
&= \sup_{(x(0),\theta,w_{[0,\infty)},\check{w}_{[0,\infty)})\in\mathcal{W}} \left\{J^t_{i\gamma}+\int_0^t\left(l_1(\tau,y_{[0,\tau]},\check{w}_{[0,\tau]})-\gamma^2|\check{w}|^2\right)d\tau-l_0(\check{x}_0,\check{\theta}_0)\right\} \\
&= \sup_{y_{[0,\infty)},\check{w}_{[0,\infty)}}\ \sup_{(x(0),\theta,w_{[0,\infty)})|y_{[0,\infty)},\check{w}_{[0,\infty)}} \left\{J^t_{i\gamma}+\int_0^t\left(l_1(\tau,y_{[0,\tau]},\check{w}_{[0,\tau]})-\gamma^2|\check{w}|^2\right)d\tau-l_0(\check{x}_0,\check{\theta}_0)\right\} \\
&= \sup_{y_{[0,\infty)},\check{w}_{[0,\infty)}} \Big\{\int_0^t\Big(|C\check{x}-y_d|^2+|\xi_c|^2_{\bar{\tilde{Q}}}+2(\theta-\check{\theta})'P_r(\check{\theta})-\gamma^2\zeta^2|y-C\check{x}|^2 \\
&\qquad\qquad +l_1(\tau,y_{[0,\tau]},\check{w}_{[0,\tau]})-\gamma^2|\check{w}|^2\Big)d\tau-l_0(\check{x}_0,\check{\theta}_0)\Big\}
\end{aligned} \tag{27}$$

where function $l_1(\tau,y_{[0,\tau]},\check{w}_{[0,\tau]})$ is part of the weighting function $l(\tau,\theta,x,y_{[0,\tau]},\check{w}_{[0,\tau]})$ to be designed, which is a constant in the identifier design step and is therefore neglected.

By equation (27), we observe that the cost function is expressed in term of the states of the estimator we derived, whose dynamics are driven by the measurement y, input u, measured disturbance $\check{w}$, and the worst-case estimate for the expanded state vector $\hat{\xi}$, which are signals we either measure or can construct. This is then a nonlinear H^∞-optimal control problem under full information measurements. Instead of considering y and $\check{w}$ as the maximizing variable, we can equivalently deal with the transformed variable:

$$v:=\begin{bmatrix}\zeta(y-C\check{x})\\ \check{w}\end{bmatrix}$$

Then, we have

$$\begin{aligned}
\dot{\eta} &= A_f\eta + p_n C\check{x} + p_n(e'_{s+1,1}v/\zeta) && (28a)\\
\dot{\check{\theta}} &= -\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'(y_d - C\check{x}) - \left[\begin{array}{cc}\Sigma & \Sigma\Phi'\end{array}\right]\bar{\bar{Q}}\xi_c + \gamma^2\Sigma\Phi' C'\zeta e'_{s+1,1}v && (28b)\\
\dot{\check{x}} &= A\check{x} - (\frac{1}{\gamma^2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) + \bar{A}_{21}\check{\theta}C\check{x} - \Phi\Sigma P_r(\check{\theta}) - \left[\begin{array}{cc}\Phi\Sigma & \frac{1}{\gamma^2}\Pi + \Phi\Sigma\Phi'\end{array}\right]\bar{\bar{Q}}\xi_c\\
&\quad + \bar{A}_{22}\check{\theta}u + \left((\zeta^{-2}\bar{A}_{21}\check{\theta} + \Pi C' + \gamma^2\Phi\Sigma\Phi' C' + L)\zeta e'_{s+1,1} + \sum_{j=1}^{s}\bar{A}_{23j}\check{\theta}e'_{s+1,j+1}\right.\\
&\quad \left. + \left[\begin{array}{cc}\mathbf{0}_{n\times 1} & \check{D}\end{array}\right]\right)v && (28c)\\
\dot{W} &= -|x_1 - y_d|^2 - \gamma^4|x - \hat{x} - \Phi(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C\Phi}\\
&\quad + |C\check{x} - y_d|^2 + |\xi_c|^2_{\bar{\bar{Q}}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) + \gamma^2|\check{w}|^2 + \gamma^2|w|^2 - \gamma^2|v|^2\\
&\quad - \gamma^2|w - w^*(t, u_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, \xi_{[0,t]}, \hat{\xi}_{[0,t]}, \check{\xi}_{[0,t]})|^2 && (28d)
\end{aligned}$$

The variables to be designed at this stage include u and ξ_c. The design for ξ_c will be carried out last. Note that the structure of A in the dynamics is in strict-feedback form, we will use the backstepping methodology (Krstić *et al.* 1995) to design the control input u, which will guarantee the global uniform boundedness of the closed-loop system states and the asymptotic convergence of tracking error.

Consider the dynamics of Φ,

$$\dot{\Phi} = A_f\Phi + y\bar{A}_{21} + u\bar{A}_{22} + \sum_{i=1}^{s}\check{w}_i\bar{A}_{23i}; \qquad \Phi(0) = \mathbf{0}_{n\times\sigma} \tag{29}$$

For ease of the ensuing study, we will separate Φ as the sum of several matrices,

$$\begin{aligned}
\Phi &= \Phi^u + \Phi^y + \Phi^{\check{w}} && (30a)\\
\Phi^y &= \left[\begin{array}{cccc}A_f^{n-1}\eta & \cdots & A_f\eta & \eta\end{array}\right]M_f^{-1}\bar{A}_{21} = \left[\begin{array}{c}\eta' T_1\\ \vdots\\ \eta' T_n\end{array}\right] && (30b)\\
\dot{\Phi}^u &= A_f\Phi^u + u\bar{A}_{22}; \qquad \Phi^u(0) = \mathbf{0}_{n\times\sigma} && (30c)\\
\dot{\Phi}^{\check{w}} &= A_f\Phi^{\check{w}} + \sum_{j=1}^{s}\check{w}_j\bar{A}_{23j}; \qquad \Phi^{\check{w}}(0) = \mathbf{0}_{n\times\sigma} && (30d)
\end{aligned}$$

where T_i, $i = 1, \cdots, n$ are $n \times \sigma$-dimensional constant matrices depending on A_f, M_f, and $\bar{A}_{21}$. Express Φ^u and $\Phi^{\check{w}}$ in terms of their row vectors, $\Phi^u = \left[\begin{array}{ccc}\Phi_1^{u\prime} & \cdots & \Phi_n^{u\prime}\end{array}\right]'$ and $\Phi^{\check{w}} = \left[\begin{array}{ccc}\Phi_1^{\check{w}\prime} & \cdots & \Phi_n^{\check{w}\prime}\end{array}\right]'$. Then, $C\Phi^y = \eta' T_1$, $C\Phi^u = \Phi_1^u$, and $C\Phi^{\check{w}} = \Phi_1^{\check{w}}$.

We summarized the dynamics for backstepping design in the following, where we have emphasized the dependence of various functions on the independent variables,

$$\dot{s} = -s^2(\gamma^2\zeta^2 - 1)(1 - \epsilon)(\eta' T_1 + \Phi_1^u + \Phi_1^{\check{w}})(\eta' T_1 + \Phi_1^u + \Phi_1^{\check{w}})' \tag{31a}$$

$$\epsilon = \frac{1}{K_c s} \tag{31b}$$

$$\dot{\Sigma} = -(1-\epsilon)\Sigma(\eta' T_1 + \Phi_1^u + \Phi_1^{\check{w}})'(\gamma^2\zeta^2 - 1)(\eta' T_1 + \Phi_1^u + \Phi_1^{\check{w}})\Sigma \tag{31c}$$

$$\dot{\check{\theta}} = \delta(y_d - \check{x}_1, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \check{\theta}, \vec{\Sigma}) + \varphi(\eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma})\bar{\bar{Q}}\xi_c + \kappa(\eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma})v \tag{31d}$$

$$\dot{\eta} = A_f\eta + p_n\check{x}_1 + p_n(e'_{s+1,1}v/\zeta) \tag{31e}$$

$$\begin{aligned}\dot{\check{x}}_1 = {} & \check{x}_2 + f_1(y_d - \check{x}_1, \check{x}_1, \check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma}) + \varrho_1(\eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma})\bar{\bar{Q}}\xi_c \\ & + h_1(\check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma})v\end{aligned} \tag{31f}$$

$$\begin{aligned}\dot{\check{x}}_2 = {} & \check{x}_3 + f_2(y_d - \check{x}_1, \check{x}_1, \check{x}_2, \check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_2^{\check{w}}, \Phi_1^u, \Phi_2^u, \vec{\Sigma}) + \varrho_2(\eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma})\bar{\bar{Q}}\xi_c \\ & + h_2(\check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_2^{\check{w}}, \Phi_1^u, \Phi_2^u, \vec{\Sigma})v\end{aligned} \tag{31g}$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{aligned}\dot{\check{x}}_{r-1} = {} & \check{x}_r + f_{r-1}(y_d - \check{x}_1, \check{x}_1, \check{x}_2, \cdots, \check{x}_{r-1}, \check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_{r-1}^{\check{w}}, \Phi_1^u, \Phi_{r-1}^u, \vec{\Sigma}) + \\ & \varrho_{r-1}(\eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma})\bar{\bar{Q}}\xi_c + h_{r-1}(\check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_{r-1}^{\check{w}}, \Phi_1^u, \Phi_{r-1}^u, \vec{\Sigma})v\end{aligned} \tag{31h}$$

$$\begin{aligned}\dot{\check{x}}_r = {} & a_{r,r+1}\check{x}_{r+1} + \cdots + a_{r,n}\check{x}_n + f_r(y_d - \check{x}_1, \check{x}_1, \check{x}_2, \cdots, \check{x}_r, \check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_r^{\check{w}}, \Phi_1^u, \Phi_r^u, \vec{\Sigma}) \\ & + \varrho_r(\eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma})\bar{\bar{Q}}\xi_c + h_r(\check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_r^{\check{w}}, \Phi_1^u, \Phi_r^u, \vec{\Sigma})v + \check{b}_0 u\end{aligned} \tag{31i}$$

$$\dot{\Phi}_1^u = \psi_1(\Phi_1^u) + \Phi_2^u \tag{31j}$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{\Phi}_{r-1}^u = \psi_{r-1}(\Phi_1^u, \cdots, \Phi_{r-1}^u) + \Phi_r^u \tag{31k}$$

$$\dot{\Phi}_1^{\check{w}} = \varpi_1(\Phi_1^{\check{w}}) + \Phi_2^{\check{w}} + \sum_{j=1}^{s} e'_{n,1}\bar{A}_{23j}e'_{s+1,j+1}v \tag{31l}$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{\Phi}_{r-1}^{\check{w}} = \varpi_{r-1}(\Phi_1^{\check{w}}, \cdots, \Phi_{r-1}^{\check{w}}) + \Phi_r^{\check{w}} + \sum_{j=1}^{s} e'_{n,r-1}\bar{A}_{23j}e'_{s+1,j+1}v \tag{31m}$$

where the nonlinear functions δ, $f_1, \cdots, f_r$ are smooth as long as $\check{\theta} \in \Theta_o$; the nonlinear functions φ, κ, $\varrho_1, \cdots, \varrho_r$, $h_1, \cdots, h_r$, $\psi_1, \cdots, \psi_{r-1}$, and ϖ_1, $\cdots$,ϖ_{r-1} are smooth. Here, we use Φ_1^u,$\cdots$,Φ_r^u, $\Phi_1^{\check{w}}$,$\cdots$,$\Phi_r^{\check{w}}$ as independent variables, instead of λ, $\eta_1^{\check{w}}$,$\cdots$,$\eta_s^{\check{w}}$, for the clarity of ensuing analysis.

We observe that the above dynamics is linear in ξ_c, which will be optimatized after backstepping design. Σ, s, $\Phi^{\check{w}}$, and $\check{\theta}$ will always be bounded by the design in Section 3, then they will not be stabilized in the control design. Φ^u is not necessary bounded, since the control input u appeared in their dynamics, it can not stabilzed in conjunction with $\check{x}$ using backstepping. Hence, we assume it is bounded, and prove later that it is indeed so under the derived control law.

The following backstepping design will achieve the γ level of disturbance attenuation with respect to the disturbance v.

Step 0: In this step, we try to stabilize η by virtual control law $\check{x}_1 = y_d$. Introduce variable η_d, as the desired trajectory of η, which satisfy the dynamics

$$\dot{\eta}_d = A_f\eta_d + p_n y_d; \qquad \eta_d(0) = \mathbf{0}_{n\times 1}$$

Define the error variable $\tilde{\eta} = \eta - \eta_d$. Then, $\tilde{\eta}$ satisfies the dynamics

$$\dot{\tilde{\eta}} = A_f\tilde{\eta} + p_n(e'_{s+1,1}v/\zeta) + p_n(\check{x}_1 - y_d) \tag{32}$$

By (Pan and Başar 2000), the following holds.

Lemma 2 *Given any Hurwitz matrix A_f, there exists a positive-definite matrix Y such that the following generalized algebraic Riccati equation admits a positive-definite solution Z:*

$$A_f' Z + Z A_f + \frac{1}{\gamma^2 \zeta^2} Z p_n p_n' Z + Y = 0 \tag{33}$$

Note that A_f in (32) is a Hurwitz matrix, then we define the following value function in terms of the positive definite matrix Z:

$$V_0(\tilde{\eta}) := |\tilde{\eta}|_Z^2 \tag{34}$$

Then its time derivative is given by

$$\dot{V}_0 = -|\tilde{\eta}|_Y^2 + \gamma^2 |v|^2 - \gamma^2 |v - \nu_0|^2 + 2\tilde{\eta}' Z p_n (\check{x}_1 - y_d)$$

where

$$\nu_0(\tilde{\eta}) = \frac{1}{\gamma^2 \zeta} e_{s+1,1} p_n' Z \tilde{\eta} \tag{35}$$

If $\check{x}_1$ is control input, then we may choose the control law

$$\check{x}_1 = y_d$$

and the design achieve attenuation level γ from the disturbance v to the output $Y^{1/2}(\eta - \eta_d)$. This complete the virtual control design for the η dynamics.

Step 1: Define the transformed variable

$$z_1 := \check{x}_1 - y_d \tag{36}$$

which is the deviation of $\check{x}_1$ from its desired trajectory y_d.

Then the time derivative of z_1 is given by

$$\dot{z}_1 = \bar{f}_1(z_1, y_d, \check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma}) + \check{x}_2 - y_d^{(1)} + \varrho_1(\eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma}) \bar{\bar{Q}} \xi_c + h_1(\check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma}) v$$

where the function $\bar{f}_1$ is defined as

$$\bar{f}_1(z_1, y_d, \check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma}) = f_1(y_d - \check{x}_1, \check{x}_1, \check{\theta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma})$$

Introduce the value function for this step

$$\bar{V}_1 = V_0 + \frac{1}{2} z_1^2$$

whose derivative is given by

$$\begin{aligned} \dot{\bar{V}}_1 &= -|\tilde{\eta}|_Y^2 + \gamma^2 |v|^2 - \gamma^2 |v - \nu_0|^2 + 2\tilde{\eta} Z p_n z_1 + z_1 (\check{x}_2 - y_d^{(1)} + \bar{f}_1 + \varrho_1 \bar{\bar{Q}} \xi_c + h_1 v) \\ &= -z_1^2 - |\tilde{\eta}|_Y^2 - \bar{\beta}_1 z_1^2 + z_1 z_2 + \varsigma_1' \bar{\bar{Q}} \xi_c + \gamma^2 |v|^2 - \gamma^2 |v - \bar{\nu}_1|^2 \end{aligned}$$

where

$$
\begin{aligned}
z_2 &= \check{x}_2 - y_d^{(1)} - \bar{\alpha}_1 && (37a)\\
\bar{\nu}_1(z_1, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma}) &= \nu_0 + \frac{1}{2\gamma^2} h_1' z_1 && (37b)\\
\bar{\alpha}_1(z_1, y_d, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma}, s) &= -z_1 - \bar{\beta}_1 z_1 - 2p_n' Z\tilde{\eta} - \bar{f}_1 - h_1\nu_0 - \frac{1}{4\gamma^2} h_1 h_1' z_1 && (37c)\\
\bar{\beta}_1(z_1, y_d, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \Phi_1^u, \vec{\Sigma}, s) &\geq c_{\beta_1} > 0 && (37d)\\
\bar{\varsigma}_1(z_1, y_d, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \Phi^u, \vec{\Sigma}) &= \varrho_1' z_1 && (37e)
\end{aligned}
$$

c_{β_1} is any positive constant, and the nonlinear function $\bar{\beta}_1$ is to be chosen by the designer. Note that the function α_1 is smooth as long as $\check{\theta} \in \Theta_o$. If $\check{x}_2$ were the actual control, then we would choose the following control law:

$$\check{x}_2 = y_d^{(1)} + \bar{\alpha}_1$$

and set $\xi_c = 0$, to guarantee the dissipation inequlity with supply rate:

$$-|\check{x}_1 - y_d|^2 - |\tilde{\eta}|_Y^2 - \bar{\beta}_1 z_1^2 + \gamma^2 v^2$$

This complete the second step of backstepping design.

Step i, $1 < i < r$: We assume that we have completed $i-1$ steps of the backstepping procedure and obtained

$$
\begin{aligned}
z_j &= \check{x}_j - y_d^{(j-1)} - \bar{\alpha}_{j-1}(z_1, \cdots, z_{j-1}, y_d, \cdots, y_d^{(j-2)}, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \cdots, \Phi_{j-1}^{\check{w}}, \Phi_1^u, \cdots, \Phi_{j-1}^u,\\
&\quad \vec{\Sigma}, s); \qquad j = 2, \cdots, i && (38a)\\
\bar{V}_{i-1} &= |\tilde{\eta}|_Z^2 + \frac{1}{2}\sum_{j=1}^{i-1} z_j^2 && (38b)\\
\dot{\bar{V}}_{i-1} &= -z_1^2 - |\tilde{\eta}|_Y^2 - \sum_{j=1}^{i-1} \bar{\beta}_j z_j^2 + z_{i-1} z_i + \bar{\varsigma}_{i-1}' \bar{\bar{Q}} \xi_c + \gamma^2 |v|^2 - \gamma^2 |v - \bar{\nu}_{i-1}|^2 && (38c)\\
\bar{\beta}_j &= \bar{\beta}_j(z_1, \cdots, z_j, y_d, \cdots, y_d^{(j-1)}, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \cdots, \Phi_j^{\check{w}}, \Phi_1^u, \cdots, \Phi_j^u, \vec{\Sigma}, s) \geq c_{\beta_j} > 0,\\
&\quad j = 1, \cdots, i-1 && (38d)\\
\bar{\nu}_{i-1} &= \bar{\nu}_{i-1}(z_1, \cdots, z_{i-1}, y_d, \cdots, y_d^{(i-3)}, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \cdots, \Phi_{i-1}^{\check{w}}, \Phi_1^u, \cdots, \Phi_{i-1}^u, \vec{\Sigma}, s) && (38e)\\
\bar{\varsigma}_{i-1} &= \bar{\varsigma}_{i-1}(z_1, \cdots, z_{i-1}, y_d, \cdots, y_d^{(i-3)}, \check{\theta}, \tilde{\eta}, \eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma}, s) && (38f)
\end{aligned}
$$

where all the nonlinear functions are smooth as long as $\check{\theta} \in \Theta_o$.

At the current step i, the time derivative of z_i is given by

$$
\begin{aligned}
\dot{z}_i &= \check{x}_{i+1} - y_d^{(i)} + \bar{\chi}_{i1}(z_1, \cdots, z_i, y_d, \cdots, y_d^{(i-1)}, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \cdots, \Phi_i^{\check{w}}, \Phi_1^u, \cdots, \Phi_i^u, \vec{\Sigma}, s)\\
&\quad + 2\gamma^2 \bar{\chi}_{i2}(z_1, \cdots, z_{i-1}, y_d, \cdots, y_d^{(i-2)}, \check{\theta}, \tilde{\eta}, \eta, \Phi_1^{\check{w}}, \cdots, \Phi_i^{\check{w}}, \Phi_1^u, \cdots, \Phi_i^u, \vec{\Sigma}, s) v\\
&\quad + \bar{\chi}_{i3}(z_1, \cdots, z_{i-1}, y_d, \cdots, y_d^{(i-2)}, \check{\theta}, \tilde{\eta}, \eta, \vec{\Phi}^{\check{w}}, \vec{\Phi}^u, \vec{\Sigma}, s) \bar{\bar{Q}} \xi_c && (39a)\\
\chi_{i1} &= f_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \check{x}_j}(f_j + \check{x}_{j+1}) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d{}^{(j-1)}} y_d^{(j)} - \frac{\partial \alpha_{i-1}}{\partial \check{\theta}} \delta - \frac{\partial \alpha_{i-1}}{\partial \tilde{\eta}}(A_f \tilde{\eta} + p_n z_1)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial \alpha_{i-1}}{\partial \eta}(A_f\eta + p_n\check{x}_1) - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \Phi_j^{\check{w}}}(\Phi_{j+1}^{\check{w}} + \varpi_j)' - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \Phi_j^{u}}(\Phi_{j+1}^{u} + \psi_j)' \\
&-\frac{\partial \alpha_{i-1}}{\partial \vec{\Sigma}}(\epsilon-1)\overrightarrow{(\Sigma(\eta' T_1 + \Phi_1^{\check{w}} + \Phi_1^{u})'(\gamma^2\zeta^2 - 1)(\eta' T_1 + \Phi_1^{\check{w}} + \Phi_1^{u})\Sigma)} \\
&+\frac{\partial \alpha_{i-1}}{\partial s}(\gamma^2\zeta^2-1)(1-\epsilon)(\eta' T_1 + \Phi_1^{\check{w}} + \Phi_1^{u})(\eta' T_1 + \Phi_1^{\check{w}} + \Phi_1^{u})' s^2
\end{aligned} \tag{39b}
$$

$$
\begin{aligned}
\chi_{i2} &= \frac{1}{2\gamma^2}\left(h_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \check{x}_j}h_j - \frac{\partial \alpha_{i-1}}{\partial \check{\theta}}\kappa - \frac{\partial \alpha_{i-1}}{\partial \tilde{\eta}}\frac{p_n e'_{s+1,1}}{\zeta} - \frac{\partial \alpha_{i-1}}{\partial \eta}\frac{p_n e'_{s+1,1}}{\zeta}\right. \\
&\quad \left. - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \Phi_j^{\check{w}}}\sum_{k=1}^{s}\bar{A}'_{23k}e_{n,j}e'_{s+1,k+1}\right)
\end{aligned} \tag{39c}
$$

$$
\chi_{i3} = \varrho_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \check{x}_j}\varrho_j - \frac{\partial \alpha_{i-1}}{\partial \check{\theta}}\varphi \tag{39d}
$$

Introduce the value function for this step as

$$
\bar{V}_i = \bar{V}_{i-1} + \frac{1}{2}z_i^2 \tag{40}
$$

whose derivative is

$$
\dot{\bar{V}}_i = -z_1^2 - |\tilde{\eta}|_Y^2 - \sum_{j=1}^{i}\bar{\beta}_j z_j^2 + z_i z_{i+1} + \bar{\varsigma}_i'\bar{\bar{Q}}\xi_c + \gamma^2|v|^2 - \gamma^2|v-\bar{\nu}_i|^2
$$

where we have introduced

$$
\begin{aligned}
z_{i+1} &= \check{x}_{i+1} - y_d^{(i)} - \bar{\alpha}_i(z_1,\cdots,z_i,y_d,\cdots,y_d^{(i-1)},\check{\theta},\tilde{\eta},\eta,\Phi_1^{\check{w}},\cdots,\Phi_i^{\check{w}},\Phi_1^{u},\cdots,\Phi_i^{u},\vec{\Sigma},s) && \text{(41a)}\\
\bar{\alpha}_i &= -\bar{\chi}_{i1} - 2\gamma^2\bar{\chi}_{i2}\bar{\nu}_{i-1} - \gamma^2\bar{\chi}_{i2}\bar{\chi}'_{i2}z_i - \bar{\beta}_i z_i - z_{i-1} && \text{(41b)}\\
\bar{\nu}_i &= \bar{\nu}_{i-1} + \bar{\chi}'_{i2}z_i && \text{(41c)}\\
\bar{\beta}_i &= \bar{\beta}_i(z_1,\cdots,z_i,y_d,\cdots,y_d^{(i-1)},\check{\theta},\tilde{\eta},\eta,\Phi_1^{\check{w}},\cdots,\Phi_i^{\check{w}},\Phi_1^{u},\cdots,\Phi_i^{u},\vec{\Sigma},s) \geq c_{\beta_i} > 0 && \text{(41d)}\\
\bar{\varsigma}_i &= \bar{\varsigma}_{i-1} + \bar{\chi}'_{i3}z_i && \text{(41e)}
\end{aligned}
$$

and all these nonlinear functions are smooth as long as $\check{\theta} \in \Theta_o$. The function $\bar{\beta}_i$ here is again a design function chosen by the designer. If $\check{x}_{i+1}$ is the actual control variable, we may choose the virtual control law as

$$
\check{x}_{i+1} = y_d^{(i)} + \bar{\alpha}_i \tag{42}
$$

and set $\xi_c = 0$, which then guarantee the dissipation inequality with a supply rate of

$$
-z_1^2 - |\tilde{\eta}|_Y^2 - \sum_{j=1}^{i}\bar{\beta}_j z_j^2 + \gamma^2|v|^2 \tag{43}
$$

This completes this step of backstepping design.

Step r: At this step the actual control appears in the derivative of z_r, which is given by

$$\dot{z}_r = a_{r,r+1}\check{x}_{r+1} + \cdots + a_{rn}\check{x}_n + \check{b}_0 u - y_d^{(r)} + \bar{\chi}_{r1} + 2\gamma^2\bar{\chi}_{r2}v + \bar{\chi}_{r3}\bar{\bar{Q}}\xi_c$$

where the functions χ_{r1}, χ_{r2}, and χ_{r3} are given as in (39), with i fixed at r. Partition $\chi_{r2} = \begin{bmatrix} \chi_{r2,1} & \chi_{r2,s} \end{bmatrix}$, where $\chi_{r2,1}$ is a scalar, and $\chi_{r2,s}$ is of appropriate dimensions.

Introduce the following value function for this step,

$$\bar{V}_r = \bar{V}_{r-1} + \frac{1}{2}z_r^2$$

Its derivative can be written as

$$\dot{\bar{V}}_r = -z_1^2 - |\tilde{\eta}|_Y^2 - \sum_{j=1}^{r}\bar{\beta}_j z_j^2 + \gamma^2|v|^2 - \gamma^2|v-\bar{\nu}_r|^2 + \bar{\varsigma}_r'\bar{\bar{Q}}\xi_c$$

with the control law defined by

$$\begin{aligned} u &= \bar{\mu}(z_1,\cdots,z_r,\check{x}_{r+1},\cdots,\check{x}_n,y_d,\cdots,y_d^{(r)},\check{\theta},\tilde{\eta},\eta,\Phi_1^{\check{w}},\cdots,\Phi_r^{\check{w}},\Phi_1^u,\cdots,\Phi_r^u,\check{w},\vec{\Sigma},s) \\ &:= -\frac{1}{\check{b}_0}(a_{r,r+1}\check{x}_{r+1} + \cdots + a_{rn}\check{x}_n - y_d^{(r)} - \bar{\alpha}_r) \end{aligned} \tag{44}$$

where $\bar{\beta}_r$ and $\bar{\varsigma}_r$ are as defined in (41d) and (41e) with i fixed at r, and

$$\bar{\alpha}_r = -\bar{\chi}_{r1} - 2\gamma^2\bar{\chi}_{r2,s}\check{w} - 2\gamma^2\bar{\chi}_{r2,1}e_{s+1,1}'\bar{\nu}_{r-1} - \gamma^2\bar{\chi}_{r2,1}^2 z_r - \bar{\beta}_r z_r - z_{r-1} \tag{45a}$$

$$\bar{\nu}_r = \bar{\nu}_{r-1} + e_{s+1,1}\bar{\chi}_{r2,1}z_r \tag{45b}$$

Clearly, the functions $\bar{\mu}$, $\bar{\chi}_{r1}$, $\bar{\chi}_{r2}$, $\bar{\chi}_{r3}$, $\bar{\nu}_r$, and $\bar{\varsigma}_r$ are smooth as long as $\check{\theta} \in \Theta_o$.

This complete the backstepping design procedure.

For the closed-loop adaptive nonlinear system, we have the following value function,

$$U := V_r + W = |\theta - \check{\theta}|_{\Sigma^{-1}}^2 + \gamma^2|x - \check{x} - \Phi(\theta-\check{\theta})|_{\Pi^{-1}}^2 + |\tilde{\eta}|_Z^2 + \frac{1}{2}\sum_{j=1}^{r}(\check{x}_j - y_d^{(j-1)} - \alpha_{j-1})^2 \tag{46}$$

where we have introduced $\alpha_0 = 0$ for notational consistency. The time derivative of this function is given by

$$\begin{aligned} \dot{U} &= -|x_1 - y_d|^2 - \gamma^4\left|x - \hat{x} - \Phi(\theta-\hat{\theta})\right|_{\Pi^{-1}\Delta\Pi^{-1}}^2 - \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \hat{\theta}\right|_{\Phi' C' C\Phi}^2 \\ &\quad +2(\theta-\check{\theta})'P_r(\check{\theta}) + |\xi_c|_{\bar{\bar{Q}}}^2 - \bar{\varsigma}_r'\bar{\bar{Q}}\xi_c - |\tilde{\eta}|_Y^2 - \sum_{j=1}^{r}\bar{\beta}_j z_j^2 + \gamma^2|w|^2 + \gamma^2|\check{w}|^2 \\ &\quad -\gamma^2\left|w - w^*(t,u_{[0,t]},y_{[0,t]},\check{w}_{[0,t]},\xi_{[0,t]},\hat{\xi}_{[0,t]},\check{\xi}_{[0,t]})\right|^2 - \gamma^2|v-\bar{\nu}_r|^2 \\ &= -|x_1 - y_d|^2 - \gamma^4\left|x - \hat{x} - \Phi(\theta-\hat{\theta})\right|_{\Pi^{-1}\Delta\Pi^{-1}}^2 - \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \hat{\theta}\right|_{\Phi' C' C\Phi}^2 \\ &\quad +2(\theta-\check{\theta})'P_r(\check{\theta}) + \left|\xi_c + \frac{1}{2}\bar{\varsigma}_r\right|_{\bar{\bar{Q}}}^2 - \frac{1}{4}|\bar{\varsigma}_r|_{\bar{\bar{Q}}}^2 - |\tilde{\eta}|_Y^2 - \sum_{j=1}^{r}\bar{\beta}_j z_j^2 + \gamma^2|w|^2 + \gamma^2|\check{w}|^2 \\ &\quad -\gamma^2\left|w - w^*(t,u_{[0,t]},y_{[0,t]},\check{w}_{[0,t]},\xi_{[0,t]},\hat{\xi}_{[0,t]},\check{\xi}_{[0,t]})\right|^2 - \gamma^2|v-\bar{\nu}_r|^2 \end{aligned} \tag{47}$$

Then the optimal choice for the variable ξ_c and $\hat{\xi}$ are

$$\xi_c^* = -\frac{1}{2}\bar{\varsigma}_r; \Longleftrightarrow \hat{\xi}^* = \check{\xi} - \frac{1}{2}\bar{\varsigma}_r \tag{48}$$

which yields that the closed-loop system is dissipative with storage function U and supply rate

$$-|x_1 - y_d|^2 + \gamma^2|w|^2 + \gamma^2|\check{w}|^2 \tag{49}$$

Furthermore, the worst case disturbance with respect to the value function U is given by

$$w_{opt} = \zeta E' e'_{s+1,1}\bar{\nu}_r + \frac{1}{\gamma^2}(I - \zeta^2 E'E)\bar{D}'\bar{\Sigma}^{-1}(\xi - \check{\xi}) + \zeta^2 E'C(\check{x} - x) \tag{50a}$$

$$\check{w}_{opt} = K\bar{\nu}_r \tag{50b}$$

where K is defined by $\begin{bmatrix} e_{s+1,1} & K' \end{bmatrix} = I_{s+1}$.

Next, we summerize in the following theorem the strong robustness property of the closed-loop system.

Theorem 1 *Consider the robust adaptive control problem formulated in Section 2 with Assumptions 1—8 holding. The robust adaptive controller μ defined by (44) with the optimal choice for the worst-case estimate $\hat{\xi}$ defined by (48) achieves the following strong robustness properties for the closed-loop system.*

1. *The controller μ achieves disturbance attenuation level γ for any uncertainty quadruple $(x(0), \theta, w_{[0,\infty)}, \check{w}_{[0,\infty)}) \in \mathcal{W}$.*

2. *Given a $c_w > 0$, there exists a constant $c_c > 0$ and a compact set $\Theta_c \subset \Theta_o$, such that for any uncertainty quadruple $(x(0), \theta, \dot{w}_{[0,\infty)}, \check{w}_{[0,\infty)}) \in \mathbb{R}^n \times \Theta_e \times \mathcal{L}_{2e} \times \mathcal{L}_{2e}$ with*

$$|x(0)| \le c_w; \quad |\dot{w}(t)| \le c_w; \quad |\check{w}(t)| \le c_w; \qquad \forall t \in [0, \infty)$$

all closed-loop state variables $x, \check{x}, \check{\theta}, \Sigma, s, \eta, \eta^{\check{w}}, \eta_d, \lambda$ are bounded as follows, $\forall t \in [0, \infty)$,

$$|x(t)| \le c_c; \quad |\check{x}(t)| \le c_c; \quad \check{\theta}(t) \in \Theta_c; \quad |\eta(t)| \le c_c; \quad |\eta_d(t)| \le c_c; \quad |\lambda(t)| \le c_c; \quad |\eta^{\check{w}_i}| \le c_c$$

$$\tfrac{1}{K_c} I \le \Sigma(t) \le \tfrac{1}{\gamma^2} Q_0^{-1}; \quad \tfrac{1}{K_c} \le s(t) \le \tfrac{1}{\gamma^2 \mathrm{Tr}(Q_0)}; \quad i = 1, 2, \cdots, s$$

3. *For any uncertainty quadruple $(x(0), \theta, \dot{w}_{[0,\infty)}, \check{w}_{[0,\infty)}) \in \mathbb{R}^n \times \Theta_e \times \mathcal{L}_{2e} \times \mathcal{L}_{2e}$ with $\dot{w}_{[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\check{w}_{[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, the output of the system, x_1, asymptotically tracks the reference trajectory, y_d, i.e.,*

$$\lim_{t \to \infty} (x_1(t) - y_d(t)) = 0$$

Proof For the frist statement, if we define

$$\begin{aligned} l_0(\check{x}_0, \check{\theta}_0) &:= \bar{V}_r(0) = \frac{1}{2}\sum_{j=1}^{r} z_j^2(0) \\ l(\tau, \theta, x, y_{[0,t]}, \check{w}_{[0,t]}) &:= |\xi - \hat{\xi}|^2_{\bar{\tilde{Q}}} - 2(\theta - \check{\theta})' P_r(\check{\theta}) + \frac{1}{4}|\bar{\varsigma}_r|^2_{\bar{\tilde{Q}}} + |\tilde{\eta}|^2_Y + \sum_{j=1}^{r} \bar{\beta}_j z_j^2 \\ &= \gamma^4 |(x - \hat{x}) - \Phi(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C \Phi} \\ &\quad -2(\theta - \check{\theta})' P_r(\check{\theta}) + \frac{1}{4}|\bar{\varsigma}_r|^2_{\bar{\tilde{Q}}} + |\tilde{\eta}|^2_Y + \sum_{j=1}^{r} \bar{\beta}_j z_j^2 \end{aligned}$$

then, we have

$$\begin{aligned} J^t &= J^t + \int_0^t \dot{\bar{U}} d\tau - \bar{U}(t) + \bar{U}(0) \\ &\leq -\bar{U}(t) \leq 0 \end{aligned}$$

It follows

$$\sup_{(x(0),\theta,w_{[0,\infty)},\check{w}_{[0,\infty)})\in\mathcal{W}} J^t \leq 0$$

This establishes the first statement.

Next, we will prove the second statement. Define $[0, t_f)$ to be the maximal interval on which the closed-loop system admits a solution[1], which lies in $\Theta_o \times \mathbb{R}^{(5+s)n+\frac{\sigma(\sigma+1)}{2}+1}$. We will show t_f is always ∞.

Fix $c_w > 0$, consider any uncertainty quadruple $(x(0), \theta, \grave{w}_{[0,\infty)}, \check{w}_{[0,\infty)})$ that satisfies:

$$|x(0)| \leq c_w; \|\grave{w}(t)| \leq c_w; \ |\check{w}(t)| \leq c_w; \quad \forall t \in [0, \infty); \quad \theta \in \Theta_e$$

By Lemma 1, we have Σ and s are uniformly upper bounded and uniformly bounded away from 0 as desired. By the differential equation for $\eta^{\check{w}}$, we have $\eta^{\check{w}}$ is also uniformly bounded over $[0, t_f)$ as desired. Then, $\Phi^{\check{w}}$ is uniformly bounded on $[0, t_f)$.

Introduce the vector of variables

$$\xi_e := [\ \check{\theta}' \quad (\tilde{x} - \Phi\tilde{\theta})' \quad \tilde{\eta}' \quad z_1 \quad \cdots \quad z_r\ ]'$$

Since $|\grave{w}(t)| \leq c_w$ and $|\check{w}(t)| \leq c_w$, we have the following inequality for the derivative of $\bar{U}$:

$$\begin{aligned} \dot{\bar{U}} &\leq -|x_1 - y_d|^2 - |\xi - \hat{\xi}|^2_{\bar{\tilde{Q}}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r} \bar{\beta}_j z_j^2 - \frac{1}{4}|\bar{\varsigma}_r|^2_{\bar{\tilde{Q}}} \\ &\quad +\gamma^2|w|^2 + \gamma^2|\check{w}|^2 \\ &\leq -|x_1 - y_d|^2 - 2|\hat{\xi} - \frac{1}{2}(\xi + \check{\xi})|^2_{\bar{\tilde{Q}}} - \frac{1}{2}|\xi - \check{\xi}|^2_{\bar{\tilde{Q}}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r} \bar{\beta}_j z_j^2 \\ &\quad +\gamma^2\|\grave{M}\|^2 c_w^2 + \gamma^2 c_w^2 \\ &\leq -\frac{\gamma^4}{2}|x - \check{x} - \Phi(\theta - \check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r} \bar{\beta}_j z_j^2 + \gamma^2\|\grave{M}\|^2 c_w^2 + \gamma^2 c_w^2 \end{aligned}$$

[1] Please see Appendix A Lemma 3

Since the $\dot{\bar{U}}$ will tend to $-\infty$, when ξ_e approach the boundary of $\Theta_o \times \mathbb{R}^{2n+r}$, there exists a compact set $\Omega_1(c_w) \subset \Theta_o \times \mathbb{R}^{2n+r}$, such that ,$\forall \xi_e \in \Theta_o \times \mathbb{R}^{2n+r} \backslash \Omega_1(c_w) \Rightarrow \dot{\bar{U}} < 0$.

Introduce the constant

$$c_\Omega := \max \left\{ \max_{\xi_e \in \Omega_1} \bar{U}_M(\xi_e),\ \bar{U}(0, \xi_e(0)) \right\} + 1$$

where

$$\bar{U}_M(\xi_e) := K_c|\theta - \check{\theta}|^2 + \gamma^2 |x - \check{x} - \Phi(\theta - \check{\theta})|^2_{\Pi^{-1}} + |\tilde{\eta}|^2_Z + \frac{1}{2}\sum_{j=1}^{r} z_j^2$$

Note that $\bar{U}_M(\xi_e) \geq \bar{U}(t, \xi_e), \forall t \in [0, t_f)$. Next we will prove $\bar{U} \leq c_\Omega, \forall t \in [0, t_f)$ by contradiction. Suppose $t'_f < t_f$ is the first time instant the vector ξ_e leaves the set $\{\bar{U} \leq c_\Omega\}$ at $\xi_{ef} := \xi_e(t'_f)$. Then, by Lemma 4 of Appendix A, we have $\bar{U}(t'_f, \xi_{ef}) = c_\Omega$. By Lemma 1, we have

$$|\theta - \check{\theta}|^2_{\Sigma^{-1}(t)} \leq \lambda_{\max}(\Sigma^{-1}(t))|\theta - \check{\theta}|^2 \leq \mathrm{Tr}(\Sigma^{-1}(t))|\theta - \check{\theta}|^2 \leq K_c|\theta - \check{\theta}|^2$$

Then,

$$c_\Omega = \bar{U}(t'_f, \xi_{ef}) \leq \bar{U}_M(\xi_{ef})$$

which further implies that $\xi_{ef} \notin \Omega_1$, in view of the definition of c_Ω. Therefore, it follows that $\dot{\bar{U}}(t'_f) < 0$, which implies that $\bar{U} \leq c_\Omega$ on $[0, t_f)$. Hence, the vector ξ_e is uniformly bounded on the interval $[0, t_f)$, namely, $\tilde{\theta}$, $\tilde{x} - \Phi\tilde{\theta}$, $\tilde{\eta}$, and $z_1, \cdots, z_r$ are uniformly bounded.

Based on the dynamics of η_d, we have η_d is uniformly bounded. Since $\tilde{\eta} = \eta - \eta_d$ is uniformly bounded, then η is also uniformly bounded. Furthermore, there is a particular linear combination of the components of η, denoted by η_L,

$$\begin{aligned} \dot{\eta} &= A_f \eta + p_n y \\ \eta_L &= T_L \eta \end{aligned}$$

which is minimum phase and has relative degree 1 with respect to y. Then the signal η_L has relative degree $r+1$ with respect to the input u, and is uniformly bounded.

Note $\Phi = \Phi^y + \Phi^u + \Phi^{\check{w}}$, Φ^y and $\Phi^{\check{w}}$ are uniformly bounded, to prove Φ is bounded, we need to prove Φ^u is uniformly bounded. Define the following equations to separate Φ^u into two part:

$$\begin{aligned} \lambda_b &= [\lambda_{b1}, \cdots, \lambda_{bn}]' \\ \dot{\lambda}_b &= A_f \lambda_b + e_r u;\ \lambda_b(0) = 0 \\ \Phi^{us} &= [\Phi_1^{us\prime}\ \cdots\ \Phi_n^{us\prime}]' \\ \dot{\Phi}^{us} &= A_f \Phi^{us} + u \begin{bmatrix} \mathbf{0}_{r\times\sigma} \\ \bar{A}_{22r} \end{bmatrix};\ \Phi^{us}(0) = \mathbf{0}_{n\times\sigma} \end{aligned}$$

Thus we have

$$\Phi^u = \Phi^{us} + [\lambda_b\ \mathbf{0}_{n\times(\sigma-1)}]$$

We observe that the relative degree for each element of Φ_1^{us} is at least $r+1$ with respect to the input u, and is the output of a stable linear system. Take η_L and u as output and input of the

reference system, we conclude Φ_1^{us} is uniformly bounded from Lemma 8 of (Pan and Başar 2000). Because the first row element of $\tilde{x} - \Phi\tilde{\theta}$ is:

$$\tilde{x}_1 - \Phi_1^{us}\tilde{\theta} - \lambda_{b1}\tilde{b}_0 - \eta' T_1\tilde{\theta} - \Phi_1^{\check{w}}\tilde{\theta}$$

we can conclude that $\tilde{x}_1 - \lambda_{b1}\tilde{b}_0$ is uniformly bounded in view of the boundedness of $\tilde{x} - \Phi\tilde{\theta}$, $\tilde{\theta}$, Φ_1^{us}, η, and $\eta^{\check{w}}$. Since $z_1 = \check{x}_1 - y_d$ and y_d are both uniformly bounded, we have that $\check{x}_1$ is also uniformly bounded.

Consider $A_f = A - \zeta^2 LC - \Pi C'C(\zeta^2 - \frac{1}{\gamma^2})$, we notice that the signal $x_1 - b_0\lambda_{b1}$ is given by:

$$\begin{aligned}
\dot{x} - b_0\dot{\lambda}_b &= A_f(x - b_0\lambda_b) + \begin{bmatrix} \mathbf{0}_{r\times 1} \\ \bar{A}_{22r}\theta \end{bmatrix} u + \bar{A}_{21}\theta y + D\grave{M}\dot{w} + \sum_{j=1}^{s} \check{w}_j \bar{A}_{23j}\theta \\
&\quad + (\zeta^2 L + \Pi C'(\zeta^2 - \frac{1}{\gamma^2}))(y - E\grave{M}\dot{w}) + \check{D}\check{w} \\
x_1 - b_0\lambda_{b1} &= C(x - b_0\lambda_b)
\end{aligned}$$

Now we will separate the above dynamics into y dependent and u dependent parts by the linearity of the system, $x_1 - b_0\lambda_{b1} := x_1^u + x_1^y$, which are respectively given by,

$$\begin{aligned}
\dot{x}^u &= A_f x^u + \begin{bmatrix} \mathbf{0}_{r\times 1} \\ \bar{A}_{22r}\theta \end{bmatrix} u \\
x_1^u &= Cx^u \\
\dot{x}^y &= A_f x^y + (\zeta^2 L + \Pi C'(\zeta^2 - \frac{1}{\gamma^2}))(y - E\grave{M}\dot{w}) \\
&\quad + \bar{A}_{21}\theta y + D\grave{M}\dot{w} + \sum_{j=1}^{s} \check{w}_j \bar{A}_{23j}\theta + \check{D}\check{w} \\
x_1^y &= Cx^y
\end{aligned}$$

We observe that the signal x_1^u has relative degree at least $r+1$ with respect to u, take η_L and u as output and input of the reference system, we conclude x_1^u is uniformly bounded by Lemma 8 in (Pan and Başar 2000). Since x_1^y has relative degree at least 1 with respect to y, take η_L and y as output and input of the reference system, we conclude x_1^y is uniformly bounded by Lemma 8 in (Pan and Başar 2000). Then it follows that $\check{x}_1 - \lambda_{b1}\check{b}_0$ is also uniformly bounded, since $\check{x}_1 - \lambda_{b1}\check{b}_0 = (x_1 - \lambda_{b1}b_0) - (\tilde{x}_1 - \lambda_{b1}\tilde{b}_0)$. Since $\check{x}_1$ is uniformly bounded, $\check{b}_0$ is uniformly bounded away from 0, then λ_{b1} is uniform bounded. That further imply Φ_1, i.e., $C\Phi$, is uniformly bounded. Furthermore, since $x_1 - b_0\lambda_{b1}$ and $\dot{w}$ are bounded, we have that the signals of x_1 and y are uniformly bounded.

Next we need to show that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, t_f)$, for some compact set $\Theta_c \subset \Theta_o$. First introduce the function:

$$\Upsilon := U + \frac{P(\check{\theta})}{\frac{1+\rho}{2} - P(\check{\theta})}$$

We notice that, when $\check{\theta}$ approached the boundary of Θ_o, $P(\check{\theta})$ approaches $\frac{1+\rho}{2}$. Then Υ approaches ∞ as ξ_e approaches the boundary of $\Theta_o \times \mathbb{R}^{2n+r}$. Consider the time derivative of Υ:

$$
\begin{aligned}
\dot{\Upsilon} &= \dot{U} + \frac{\frac{1+\rho}{2}}{\left(\frac{1+\rho}{2} - P(\check{\theta})\right)^2}\frac{\partial P}{\partial \theta}(\check{\theta})\dot{\check{\theta}} \\
&\leq -|x_1 - y_d|^2 - \gamma^4|x - \hat{x} - \Phi(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C \Phi} \\
&\quad +2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r} \bar{\beta}_j z_j^2 - \frac{1}{4}|\bar{\varsigma}_r|^2_{\bar{\tilde{Q}}} + \gamma^2 \|\grave{M}\|^2 c_w^2 + \gamma^2 c_w^2 \\
&\quad +\frac{\frac{1+\rho}{2}}{\left(\frac{1+\rho}{2} - P(\check{\theta})\right)^2}\left(-(1 - \mathbf{1}_{\Theta_e}(\check{\theta}))\left(\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\left(\frac{1+\rho}{2} - P(\check{\theta})\right)^3}\right)\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \right. \\
&\quad -\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\Phi' C'(y_d - \check{x}_1) + \frac{1}{2}\frac{\partial P}{\partial \theta}(\check{\theta})\left[\begin{array}{cc} \Sigma & \Sigma\Phi' \end{array}\right]\bar{\tilde{Q}}\bar{\varsigma}_r \\
&\quad \left. +\frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2\zeta^2\Sigma\Phi' C'(\tilde{x}_1 + E\grave{M}\grave{w})\right)
\end{aligned}
$$

Define $\bar{\varsigma}_r = [\bar{\varsigma}'_{r\sigma}\ \ \bar{\varsigma}'_{rn}]'$, where $\bar{\varsigma}_{r\sigma}$ is σ-dimensional. We observe the following facts.

$$
\begin{aligned}
\frac{\partial P}{\partial \theta}(\check{\theta})\left[\begin{array}{cc} \Sigma & \Sigma\Phi' \end{array}\right]\bar{\tilde{Q}}\bar{\varsigma}_r &= \frac{\partial P}{\partial \theta}(\check{\theta})\epsilon\Sigma\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi\bar{\varsigma}_{r\sigma} \\
-\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\Phi' C'(y_d - \check{x}_1) &\leq \left|\left(\frac{\partial P}{\partial \theta}\right)'\frac{1}{(\frac{1+\rho}{2} - P(\check{\theta}))}\right|^2 + \left|\frac{(\frac{1+\rho}{2} - P(\check{\theta}))}{2}\Sigma\Phi' C'(y_d - \check{x}_1)\right|^2 \\
\frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2\zeta^2\Sigma\Phi' C'(\tilde{x}_1 + E\grave{M}\grave{W}) &\leq \left|\left(\frac{\partial P}{\partial \theta}\right)'\frac{1}{(\frac{1+\rho}{2} - P(\check{\theta}))}\right|^2 \\
&\quad + \left|\frac{(\frac{1+\rho}{2} - P(\check{\theta}))}{2}\gamma^2\zeta^2\Sigma\Phi' C'(\tilde{x}_1 + E\grave{M}\grave{w})\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}|\bar{\varsigma}_r|^2_{\bar{\tilde{Q}}} + \frac{1+\rho}{2(\frac{1+\rho}{2} - P(\check{\theta}))^2}\frac{\partial P}{\partial \theta}(\check{\theta})\epsilon\Sigma\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi\bar{\varsigma}_{r\sigma} = \frac{1}{4}|\bar{\varsigma}_r|^2_{\Delta_1} \\
&-\frac{\epsilon}{4}(\gamma^2\zeta^2 - 1)\left|\bar{\varsigma}_{r\sigma} - \frac{1+\rho}{(\frac{1+\rho}{2} - P(\check{\theta}))^2}\Sigma\left(\frac{\partial P}{\partial \theta}\right)'\right|^2_{\Phi' C' C \Phi} + \frac{\epsilon(\gamma^2\zeta^2 - 1)(1+\rho)^2}{4(\frac{1+\rho}{2} - P(\check{\theta}))^4}\left|\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\Phi' C'\right|^2
\end{aligned}
$$

where $\Delta_1 = \bar{\Sigma}^{-1}\left[\begin{array}{cc} 0 & 0 \\ 0 & \Delta \end{array}\right]\bar{\Sigma}^{-1}$. Then there exist some constant $c > 0$ such that the following inequalities hold.

$$
\begin{aligned}
\dot{\Upsilon} &\leq -|x_1 - y_d|^2 - \frac{\gamma^4}{2}|x - \check{x} - \Phi(\theta - \check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r}\bar{\beta}_j z_j^2 \\
&\quad -\frac{1}{K_c}(1 - \mathbf{1}_{\Theta_e}(\check{\theta}))\left(\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{(\frac{1+\rho}{2} - P(\check{\theta}))^5}\right)\left|\left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)'\right|^2 + \frac{1}{(\frac{1+\rho}{2} - P(\check{\theta}))^4}\left|\left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)'\right|^2 c + c
\end{aligned}
$$

$$\leq -\frac{\gamma^4}{2}|x-\check{x}-\Phi(\theta-\check{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}}+2(\theta-\check{\theta})'P_r(\check{\theta})-|\tilde{\eta}|^2_Y-\sum_{j=1}^{r}\bar{\beta}_j z_j^2$$
$$-\frac{\left|\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)'\right|^2}{(\frac{1+\rho}{2}-P(\check{\theta}))^4}\left(\frac{1}{K_c}(1-\mathbf{1}_{\Theta_e}(\check{\theta}))\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\frac{1+\rho}{2}-P(\check{\theta})}-c\right)+c$$

Notice $\frac{1}{K_c}(1-\mathbf{1}_{\Theta_e}(\check{\theta}))\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\frac{1+\rho}{2}-P(\check{\theta})}$ approach ∞ when $\check{\theta}$ approach the boundary of Θ_o. Then, there exists a compact set $\Omega_c \subset \Theta_o \times \mathbb{R}^{2n+r}$ such that $\dot{\Upsilon} < 0$ for $\forall \xi_e \in \Theta_o \times \mathbb{R}^{2n+r}\backslash\Omega_c$. It can be concluded that $\exists c_1 > 0$, such that, $\forall \xi_e \in \Theta_o \times \mathbb{R}^{2n+r}$ with $\Upsilon(\xi_e) > c_1$, implies $\dot{\Upsilon} < 0$. Then, Υ is uniformly bounded on $[0, t_f)$, $\check{\theta}$ is uniformly bounded away from the boundary of Θ_o. It follows that $\check{\theta}$ is bounded inside a compact set $\Theta_c \subset \Theta_o$. Then, $P_r(\check{\theta})$ is uniformly bounded.

Take x_1 and u as the output and input of reference system, x_1 is minimum phase and has relative degree r with respect to input u. Use the similar bounding analysis for x_1 as discussed above, we can deduce that Φ_2^{us}, $\tilde{x}_2 - \lambda_{b2}\tilde{b}_0$, $\check{x}_2$, $x_2 - b_0\lambda_{b2}$, λ_{b2}, and x_2 are uniformly bounded. We can find a linear combination of x_1 and x_2 that is minimum phase and has relative degree $r-1$ with respect to u. Use it as the output of the reference system, we can derive the boundedness of $\check{x}_3$, x_3, λ_{b3}. Inductively, we may deduce the boundedness of $\check{x}_4$, x_4, λ_{b4}, $\cdots$, $\check{x}_r$, x_r, λ_{br}. The state x can be viewed as stably filtered output signals of u and y, it is then uniformly bounded.

Also, η, λ are some stably filtered signals of u and y, they are uniformly bounded. Since $\eta^{\check{w}}$ is uniformly bounded, Φ is uniformly bounded. Then we can conclude $\check{x}$ is uniformly bounded from the boundedness of $\tilde{x} - \Phi\tilde{\theta}$. This further implies that the control input u is uniformly bounded.

Then we can get the conclusion that the complete system states and u are uniformly bounded on $[0, t_f)$, Σ, s are uniformly bounded and bounded away from 0, and $\check{\theta}$ is uniformly bounded away from the boundary of the set Θ_o. Therefore, it follows $t_f = \infty$ and the complete system states are uniformly bounded on $[0, \infty)$.

Last, we will establish the third statement. By the following inequality

$$\int_0^\infty \dot{U}d\tau \leq \int_0^\infty(-|x_1-y_d|^2+\gamma^2|\grave{M}\grave{w}|^2+\gamma^2|\check{w}|^2)d\tau$$

it follows that

$$\int_0^\infty |x_1-y_d|^2 d\tau \leq \int_0^\infty \left(\gamma^2|\grave{M}\grave{w}|^2+\gamma^2|\check{w}|^2\right)d\tau + U(0) < +\infty$$

By the second statement, we notice that

$$\sup_{0\leq t<\infty}|\dot{x}_1-\dot{y}_d| < \infty.$$

By Lemma 5 in Appendix A,

$$\lim_{t\to\infty}|x_1(t)-y_d(t)| = 0$$

This complete the proof of the theorem. □

From the Equations (41e) and (48), we notice that $\hat{\xi}^*$ is generally very complicated. In pratical situations, we can simply choose $\xi_c = 0$, i.e., $\hat{\xi} = \check{\xi}$, which will result in a simplified identifier that resembles the standard identifiers, such as the least squares or least mean squares identifiers. Then, we state the following corollary to Theorem 1.

Corollary 1 *Consider the robust adaptive control problem formulated in Section 2, under the same assumptions as those of Theorem 1. Then, the same result of Theorem 1 hold for the robust adaptive controller $\bar{\mu}$ defined by (44), with the worst-case estimate $\hat{\xi}$ generated by the suboptimal policy $\hat{\xi} = \check{\xi}$.*

Proof Consider the value function defined by (46), its time derivative is given by

$$\begin{aligned}\dot{U} &= -|x_1 - y_d|^2 - \gamma^4 \left|x - \check{x} - \Phi(\theta - \check{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \check{\theta}\right|^2_{\Phi' C' C \Phi} \\ &\quad +2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r} \bar{\beta}_j z_j^2 + \gamma^2|w|^2 + \gamma^2|\check{w}|^2 \\ &\quad -\gamma^2 \left|w - w^*(t, u_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, \xi_{[0,t]}, \hat{\xi}_{[0,t]}, \check{\xi}_{[0,t]})\right|^2 - \gamma^2|v - \bar{\nu}_r|^2 \end{aligned} \tag{51}$$

where the only difference between (47) and (51) is that the terms involving $\bar{\varsigma}_r$ in (47) vanish in (51). Then, the closed-loop system is dissipative with storage function U and supply rate

$$-|x_1 - y_d|^2 + \gamma^2|w|^2 + \gamma^2|\check{w}|^2 \tag{52}$$

Following the same line of reasoning as that of Theorem 1, we can deduce that η_L, Φ_1^{us}, $\tilde{x}_1 - \lambda_{b1}\tilde{b}_0$, z_1, y_d, $x_1 - b_0\lambda_{b1}$, $\check{x}_1 - \check{b}_0\lambda_{b1}$, $\check{x}_1$, x_1, λ_{b1}, y, $C\Phi$ are uniformly bounded on the maximum-length interval that the closed-loop system admits a continuous solution such that $\check{\theta}$ always belongs to Θ_o.

To show the desired boundedness result for $\check{\theta}$, we consider the total time derivative of the function of $P(\check{\theta})$,

$$\begin{aligned}\frac{d}{dt}P(\check{\theta}) &= \frac{\partial P}{\partial \theta}(\check{\theta})\left(-\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'(y_d - C\check{x}) + \gamma^2\Sigma\Phi' C'\zeta e'_{s+1,1} v\right) \\ &\leq -\frac{1}{2K_c}(1 - \mathbf{1}_{\Theta_e}(\check{\theta}))\frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\left(\frac{1+\rho}{2} - P(\check{\theta})\right)^3}\left|\left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)'\right|^2 + c\end{aligned}$$

for some constant $c > 0$. Since the derivative of $P(\check{\theta})$ approaches $-\infty$ as $\check{\theta}$ approaches the boundary of the set Θ_o, $\check{\theta}$ is bounded inside Θ_o and uniformly bounded away from the boundary of Θ_o. Hence, $\check{\theta}$ is always bounded inside a compact set $\Theta_c \subset \Theta_o$. Then the projection signal $P_r(\check{\theta})$ is uniformly bounded.

By a line of reasoning that is similar to that of Theorem 1, we can prove that all states of the closed-loop system are uniformly bounded on $[0, \infty)$ as desired. Furthemore, the asymptotic tracking result can be established similarly as that of Theorem 1.

This complete the proof of this corollary. □

Based on Theorem 1 and Corollary 1, we note that the above controller μ defined by (44) can achieve disturbance attenuation level γ with respect to w and $\check{w}$. Then, the time derivative of value function U satisfies:

$$\dot{U} \leq -|x_1 - y_d|^2 + \gamma^2|w|^2 + \gamma^2|\check{w}|^2$$

Consider $\gamma^2|\check{w}|^2 = \check{\gamma}^2\left|\frac{\gamma}{\check{\gamma}}\check{w}\right|^2$, we can achieve arbitrary disturbance attenuation level $\check{\gamma}$ with respect to $\check{w}$ by multipling a scalar to $\check{w}$. However, $\check{\gamma}$ have to be positive. To achieve disturbance attenuation level 0 with respect to $\check{w}$, we need the following assumption.

Assumption 9 *The transfer function from $\check{w}_i$ to y has relative degree greater than or equal to r, $i = 1, \cdots, s$.*

Definition 2 *A controller μ is said to achieve* disturbance attenuation level γ with respect to w and disturbance attenuation level 0 with respect to $\check{w}$, *if there exist nonnegative functions $l(t, \theta, x, y_{[0,t]}, \check{w}_{[0,t]})$ and $l_0(\check{x}_0, \check{\theta}_0)$ such that*

$$\sup_{(x(0),\theta,w_{[0,\infty)},\check{w}_{[0,\infty)})\in\mathcal{W}} J_0^t \leq 0; \qquad \forall t \geq 0. \tag{53}$$

where

$$J_0^t := \int_0^t ((x_1 - y_d)^2 + l(\tau, \theta, x(\tau), y_{[0,\tau]}, \check{w}_{[0,\tau]}) - \gamma^2|w|^2)d\tau - \gamma^2|\theta - \check{\theta}_0|^2_{Q_0} \\ -\gamma^2|x(0) - \check{x}_0|^2_{\Pi_0^{-1}} - l_0(\check{x}_0, \check{\theta}_0) \tag{54}$$

We have the following results for this objective.

Corollary 2 *Consider the robust adaptive control problem formulated in Section 2, under the assumptions of Theorem 1 and Assumption 9, the result of Theorem 1 holds for the control law $\bar{\mu}$ defined by (44) and the worst-case estimate $\hat{\xi}$ generated by (48). In addition,*

1. *the controller achieves disturbance attenuation level γ with respect to w and disturbance attenuation level* 0 *with respect to $\check{w}$.*

2. *for any uncertainty quadruple $(x(0), \theta, \grave{w}_{[0,\infty)}, \check{w}_{[0,\infty)}) \in \mathbb{R}^n \times \Theta_e \times \mathcal{L}_{2e} \times \mathcal{L}_{2e}$ with $\grave{w}_{[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\check{w}_{[0,\infty)} \in \mathcal{L}_\infty$, the output of the system, x_1, asymptotically tracks the reference trajectory, y_d, i.e.,*

$$\lim_{t\to\infty}(x_1(t) - y_d(t)) = 0$$

Proof Clearly, the result of Theorem 1 holds. We note that the coefficent matrix of $\check{w}$ in the design model (2) has the following structure under Assumption 9,

$$\check{D} = \begin{bmatrix} \mathbf{0}_{(r-1)\times\sigma} \\ \check{D}_r \end{bmatrix}; \quad \bar{A}_{231} = \begin{bmatrix} \mathbf{0}_{(r-1)\times\sigma} \\ \bar{A}_{231,r} \end{bmatrix}; \quad \cdots; \quad \bar{A}_{23s} = \begin{bmatrix} \mathbf{0}_{(r-1)\times\sigma} \\ \bar{A}_{23s,r} \end{bmatrix}$$

Then, by Equation (31), we note κ, h_1, $\cdots$, h_{r-1} have the following structure

$$\kappa = \begin{bmatrix} \kappa_1 \\ \mathbf{0}_{s\times\sigma} \end{bmatrix}'; \quad h_1 = \begin{bmatrix} h_{1,1} \\ \mathbf{0}_{s\times 1} \end{bmatrix}'; \quad \cdots; \quad h_{r-1} = \begin{bmatrix} h_{r-1,1} \\ \mathbf{0}_{s\times 1} \end{bmatrix}'$$

where $h_{1,1}$, $\cdots$, $h_{r-1,1}$ are scalars, and κ_1 is a σ-dimensional row vector.

Based on the Equations (35), (37b), (39c), (41c) and (45b), we note that ν_0, $\cdots$, ν_r have the following structure

$$\bar{\nu}_0 = \begin{bmatrix} \bar{\nu}_{0,1} \\ \mathbf{0}_{s\times 1} \end{bmatrix}; \quad \cdots; \quad \bar{\nu}_r = \begin{bmatrix} \bar{\nu}_{r,1} \\ \mathbf{0}_{s\times 1} \end{bmatrix}$$

where $\bar{\nu}_{0,1}, \cdots, \bar{\nu}_{r,1}$ are scalars.

Consider the value function U defined in (46), its derivative is given by

$$\begin{aligned}
\dot{U} &= -|x_1 - y_d|^2 - \gamma^4 \left| x - \hat{x} - \Phi(\theta - \hat{\theta}) \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1) \left| \theta - \hat{\theta} \right|^2_{\Phi' C' C \Phi} \\
&\quad + 2(\theta - \check{\theta})' P_r(\check{\theta}) + |\xi_c + \frac{1}{2}\bar{\varsigma}_r|^2_{\bar{\tilde{Q}}} - \frac{1}{4}|\bar{\varsigma}_r|^2_{\bar{\tilde{Q}}} - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r} \bar{\beta}_j z_j^2 + \gamma^2 |w|^2 \\
&\quad - \gamma^2 \left| w - w^*(t, u_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, \xi_{[0,t]}, \hat{\xi}_{[0,t]}, \check{\xi}_{[0,t]}) \right|^2 - \gamma^2 |\zeta(y - \check{x}_1) - \bar{\nu}_{r,1}|^2 \\
&\leq -|x_1 - y_d|^2 + \gamma^2 |w|^2 \qquad (55)
\end{aligned}$$

Therefore, the adaptive controller (44), with the worst-case estimate $\hat{\xi}$ defined by (48) achieves disturbance attenuation level γ with respect to w and disturbance attenuation level 0 with respect to $\check{w}$.

The second statement is immediate from Equation (55) by a line of reasoning that is similar to that for the third statement of Theorem 1.

This complete the proof this Corollary. □

Due to the same reason of introducing the Corollary 1, we state the following corollary for a simplified choice of the worst-case estimate $\hat{\xi} = \check{\xi}$.

Corollary 3 *Consider the robust adaptive control problem formulated in Section 2, under the assumptions of Theorem 1 and Assumption 9, then the result of Corollary 2 holds for the adaptive controller μ defined by (44) and the worst-case estimate $\hat{\xi}$ generated by the suboptimal policy*

Proof This corollary can be proved by a line of reasoning that is similar to that of Corollaries 1 and 2. □

Remark 1 *The adaptive control design can be extended to treat linear systems with partially known control vector fields. Consider the following design model*

$$\dot{x} = Ax + (y\bar{A}_{21} + u\bar{A}_{22})\theta + \sum_{j=1}^{s} \check{w}_j \bar{A}_{23j}\theta + Bu + Dw + \check{D}\check{w} \qquad (56a)$$

$$y = Cx + Ew \qquad (56b)$$

Compared to the model (2), there is a term Bu in (56a), where B is known and has the following structure

$$B = [\ \mathbf{0}_{1\times(r-1)} \quad b_{p0} \quad b_{p1} \quad \cdots \quad b_{pm}\ ] \qquad (57)$$

If the high frequency gain b_0 is unknown, b_{p0} will be absorbed into θ. In this case, the identification step will involve equations (21a)-(21g), and the following state estimation dynamics

$$\begin{aligned}
\dot{\check{x}} &= -\Phi\Sigma P_r(\check{\theta}) + A\check{x} + Bu - (\frac{1}{\gamma^2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) - \left[\ \Phi\Sigma \quad \tfrac{1}{\gamma^2}\Pi + \Phi\Sigma\Phi'\ \right] \bar{\tilde{Q}}\xi_c \\
&\quad + (y\bar{A}_{21} + u\bar{A}_{22} + \sum_{j=1}^{s} \check{w}_j \bar{A}_{23j})\check{\theta} + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi'C' + L)(y - C\check{x}) \\
&\quad + \check{D}\check{w}; \qquad \check{x}(0) = \check{x}_0 \qquad (58)
\end{aligned}$$

In the control design, we can follow the same steps 0 *to* $r-1$*. At step* r*, if high frequency gain* b_0 *is unknown, the control law* μ *is defined as (44). If the high frequency gain* b_0 *is known, the term* $\check{b}_0$ *will be changed into* b_{p0} *in (44). The same results as Theorem 1 and Corollaries 1 − 3 still hold.*

5 Example

In this section, we present two examples to illustrate the main results of this paper. The designs were carried out using MATLAB symbolic computation tools, and the closed-loop systems were simulated using SIMULINK.

5.1 Example 1

Consider the following first-order system:

$$\begin{aligned} \dot{\grave{x}}_1 &= \grave{x}_1 + u + w_e + [\ 0.1 \quad 0\ ]\grave{w} + \check{w} \\ y &= \grave{x}_1 + [\ 0 \quad 0.1\ ]\grave{w} \end{aligned}$$

Here we assume that the system is completely known, and the exogenous disturbance input consists of three components: a dominant sinusoidal waveform w_e, a measured waveform $\check{w}$, and an arbitrary waveform of smaller magnitude $\grave{w}$. The magnitude, frequency, and phase of the sinusoidal waveform w_e are all unknown. The coefficent terms, 0.1, reflect the *a priori* knowledge that the arbitrary disturbance is weak in power relative to that of the sinusoidal disturbance and the measured disturbance. The sinusodial component w_e can be modeled by a second order linear system:

$$\begin{aligned} \dot{\grave{x}}_2 &= \grave{x}_3 \\ \dot{\grave{x}}_3 &= \theta \grave{x}_2 \\ w_e &= \grave{x}_2 \end{aligned}$$

where the initial conditions of $\grave{x}_2$, $\grave{x}_3$, and θ determine the phase and magnitude of the sinusoid, and the parameter θ determines the frequency of the sinusoid. We note that the Assumptions 1, 2, and 9 are satisfied by the true system. The adaptive controller will achieve zero disturbance attenuation level with respect to $\check{w}$. For simulation purposes, we will set $\grave{x}(0) = [1; 1; 2]'$ and $\theta^* = -4$, which coresponds to $w_e(t) = \sqrt{2}sin(2t + \frac{\pi}{4})$

Introduce the following state and disturbance transformation,

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{bmatrix} \grave{x}; \qquad w = \begin{bmatrix} 1 & 0 \\ 0 & -\theta \\ -\theta & \theta \\ 0 & 1 \end{bmatrix} \grave{w}$$

we obtain the design model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \theta \\ -\theta \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ -\theta \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ -\theta \end{bmatrix} \check{w} + \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} 0 & 0 & 0 & 0.1 \end{bmatrix} w \end{aligned}$$

The ultimate performance lower bound for this system is 0.1 with respect to w. For the adaptive control design, we set the desired disturbance attenuation level $\gamma = 0.2$. The parameter θ is assumed to belong to the set $[-20, 0]$, with the projection function $P(\theta)$ chosen as

$$P(\theta) = \frac{1}{100}(\theta + 10)^2$$

The reference trajectory, y_d, is generated by the following linear system

$$\begin{aligned} \dot{x}_d &= -x_d + d; \qquad x_d(0) = 0 \\ y_d &= x_d \end{aligned}$$

where d is the command input signal.

For other design and simulation parameters, we select

$$\check{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \check{\theta}_0 = -1; \quad Q_0 = 1; \quad K_c = \frac{1}{10}; \quad \Delta = I_3; \quad \bar{\beta}_1 = 0.1; \quad Y = 0.2773 I_3$$

Then, we obtain

$$Z = \begin{bmatrix} 0.3546 & -0.1464 & -0.2697 \\ -0.1464 & 0.3284 & -0.2493 \\ -0.2697 & -0.2493 & 0.94094 \end{bmatrix}; \quad \Pi = \begin{bmatrix} 0.06176 & 0.05628 & 0.02582 \\ 0.05628 & 0.17859 & 0.09378 \\ 0.02582 & 0.09378 & 0.10899 \end{bmatrix}$$

We present two sets of simulation results in this example. Since the Assumption 9 is satisfied, the robust adaptive controller achieves zero $\mathcal{L}_2$ attenuation with repect to the measured disturbance $\check{w}$. In the first set of simulation, we set $d(t) = 0$, $\check{w} = \sin(4t)$, and $\grave{w}(t) = 0$. This simulation is to demonstrate the regulatory behaviour of the adaptive controller. The results are shown in Figure 1. We observe from Figure 1, even though $\check{w}(t) \neq 0$, the output tracking error asymptotically converges to zero and the parameter estimate asymptotically converges to its true value -4. The transient of the system is well-behaved, and the control magnitude is upper bounded by 6.

The second set of simulation results is to demonstrate the robustness of the adaptive controller to unmodeled exogenous disturbance inputs. We set $d(t) = \sin(t)$, $\check{w}(t) = \sin(4t)$, $\grave{w}_1(t) = \sin(3t + \frac{\pi}{2})$, and $\grave{w}_2(t) = 0.5\sin(5t + \frac{\pi}{4})$. The simulation results are presented in Figure 2. We observe that the parameter estimate no longer converges to the true value -4, but it asymtotically oscillates around the true value. The output tracking error satisfies the targeted attenuation level as desired. The transient of the system is well-behaved, and the control magnitude is again upper bounded by 6.

5.2 Example 2

Consider the following circuit problem in Figure 3, where v_i is the input voltage source; v_o is the measured output; v_e is an unknown sinusoidal voltage source; v_{w1} is an unmeasured exogenous voltage source; v_{w2} is an unmeasured exogenous voltage disturbance in the output channel; i_s is a measured exogenous current source. The objective is to achieve asymptotic tracking of $v_o - v_{w2}$ to the reference trajectory y_d.

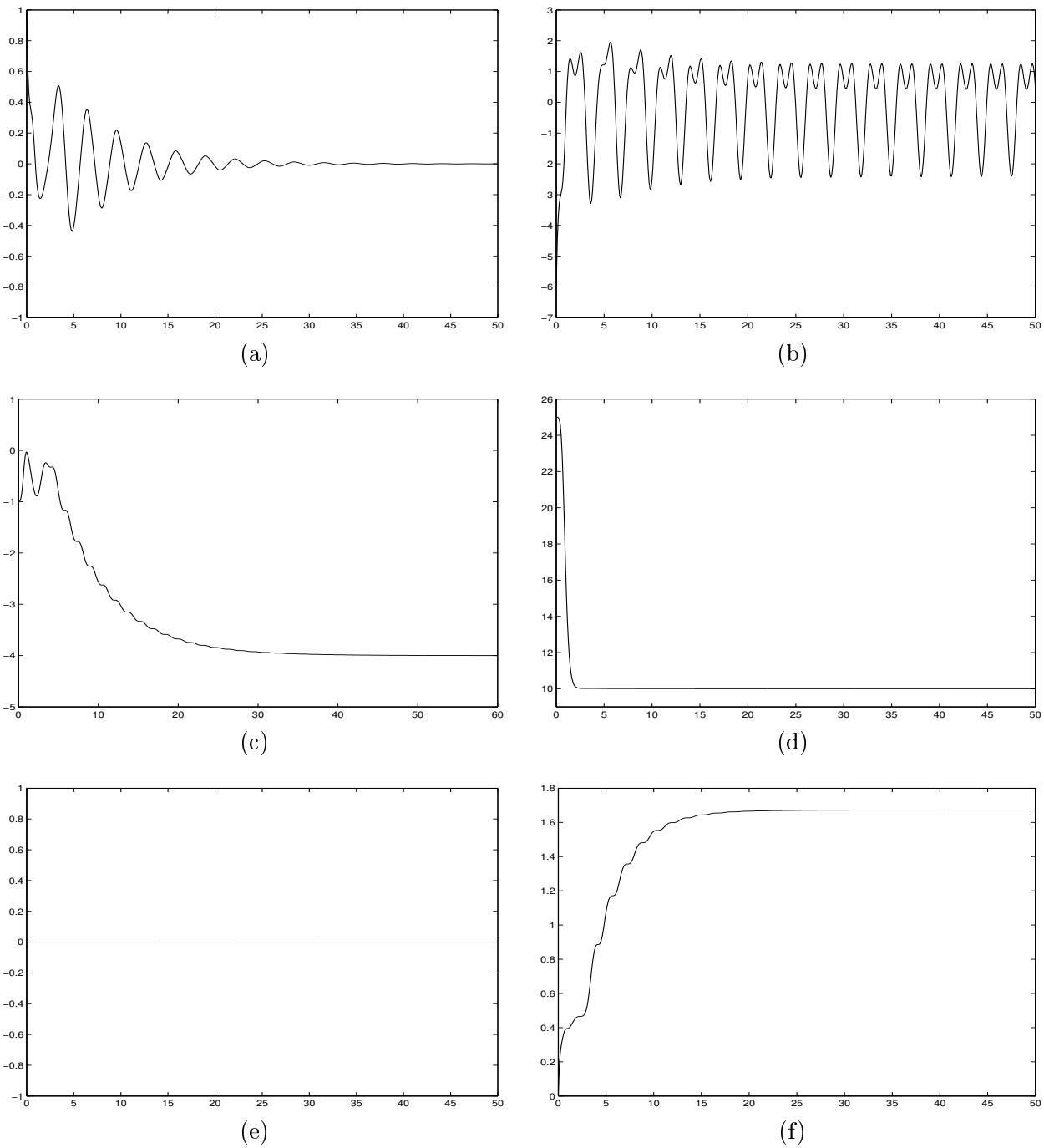


Figure 1: System response for Example 1 under command input $d(t) = 0$, $\check{w}(t) = \sin(4t)$, and $\dot{\check{w}} = 0$.
(a) Tracking error; (b) Control input; (c) Parameter estimate; (d) Worst-case error covariance for parameter estimate; (e) Reference trajectory; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

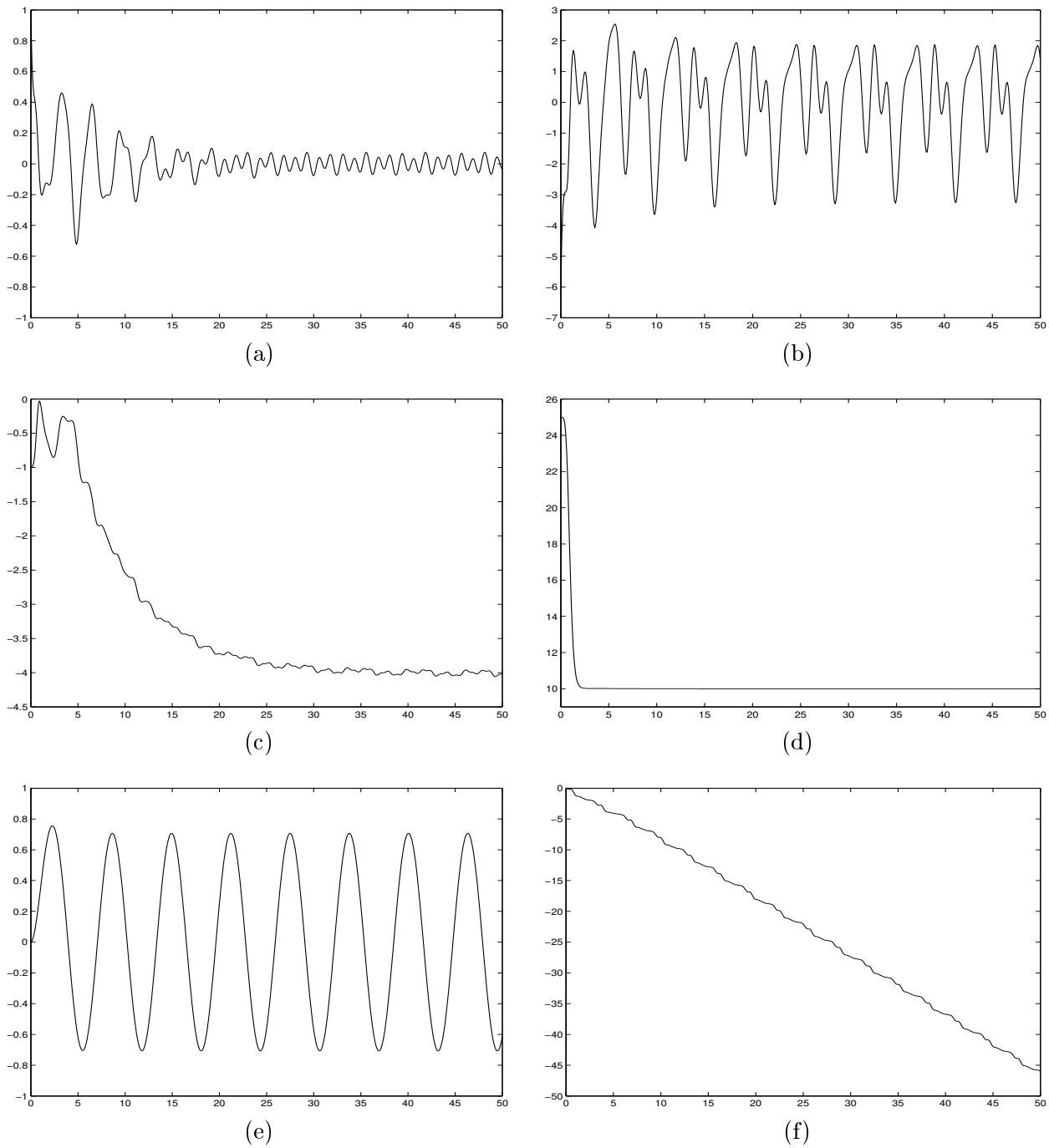


Figure 2: System response for Example 1 under command input $d(t) = \sin(t)$, $\check{w}(t) = \sin(4t)$, $\grave{w}_1(t) = \sin(3t + \frac{\pi}{2})$, and $\grave{w}_2(t) = 0.5\sin(5t + \frac{\pi}{4})$.
(a) Tracking error; (b) Control input; (c) Parameter estimate; (d) Worst-case error covariance for parameter estimate; (e) Reference trajectory; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

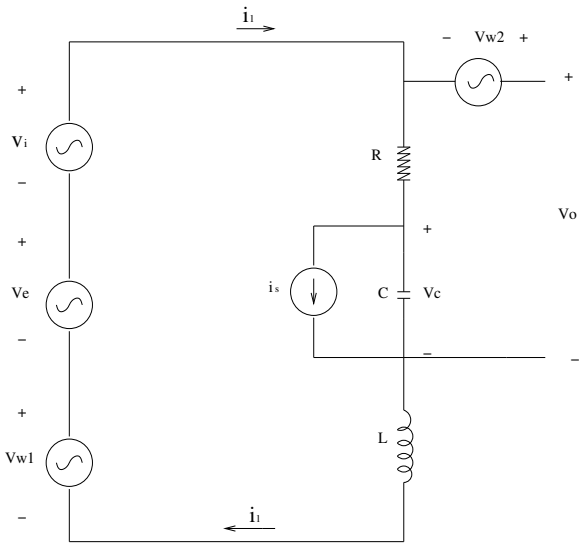


Figure 3: Diagram of a circuit.

The equations that describe the circuit are obtained as

$$\begin{aligned}
\dot{\grave{x}}_1 &= \frac{1}{L}\left(u + v_e + \grave{w}_1 - R\grave{x}_1 - \grave{x}_2\right); && \grave{x}_1(0) = 1;\\
\dot{\grave{x}}_2 &= \frac{1}{C}\left(\grave{x}_1 - \breve{w}\right); && \grave{x}_2(0) = 1;\\
y &= R\grave{x}_1 + \grave{x}_2 + \grave{w}_2
\end{aligned}$$

where $\grave{x}_1 = i_1$, $\grave{x}_2 = v_c$, $u = v_i$, $y = v_o$, $\breve{w} = i_s$, $\grave{w}_1 = v_{w1}$, and $\grave{w}_2 = v_{w2}$. v_e can be modeled as the output of a second-order linear system as in Example 1,

$$\begin{aligned}
\dot{\grave{x}}_3 &= \grave{x}_4; && \grave{x}_3(0) = 1;\\
\dot{\grave{x}}_4 &= \theta_2\grave{x}_3; && \grave{x}_4(0) = 2;
\end{aligned}$$

The initial conditions are set for illustration purposes. Note that the true system satisfies the Assumptions 1, 2, and 9. The adaptive controller will achieve zero disturbance attenuation level with respect to $\breve{w}$. We assume that $R = 1\Omega$ and $L = 1\text{H}$. We define $\theta = [\theta_1; \theta_2; \theta_3]$, where $\theta_1 = \frac{1}{C}$ and $\theta_3 = \frac{\theta_2}{C}$. Introduce the state and disturbance transformations,

$$x = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \theta_1 & 0 & 1 & 0 \\ -\theta_2 & \theta_2 & \theta_1 & 1 \\ -\theta_3 & 0 & 0 & \theta_1 \end{bmatrix} \grave{x}; \qquad w = \frac{1}{2.5}\begin{bmatrix} 1 & 0 \\ \theta_1/12 & (-\theta_1+\theta_2)/12 \\ -\theta_1/6 & \theta_2/6 \\ -\theta_3/12 & \theta_3/12 \\ 0 & 1 \end{bmatrix} \grave{w}$$

we obtain the design model for the adaptive controller

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} y + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} u + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \breve{w} \right) \theta$$

$$+\begin{bmatrix}1\\0\\0\\0\end{bmatrix}u+\begin{bmatrix}2.5&0&0&0&0\\0&30&0&0&0\\0&0&15&0&0\\0&0&0&30&0\end{bmatrix}w$$

$$y = \begin{bmatrix}1&0&0&0\end{bmatrix}x+\begin{bmatrix}0&0&0&0&2.5\end{bmatrix}w$$

The true value for the parameter θ is $[1,-4,-4]$. For the adaptive control design, the ultimate lower bound for the achievable disturbance attenuation level is 2.5 with respect to w. We set the desired disturbance attenuation level $\gamma = 12.5$. It is known that θ_1 belongs to the interval $[0.5, 6.5]$, θ_2 belongs to $[-6, 0]$, and θ_3 belongs to $[-12, 0]$, with the projection function $P(\theta)$ chosen as

$$P(\theta) = \frac{1}{3}\left(\frac{(\theta-3.5)^2}{9}+\frac{(\theta_2+3)^2}{9}+\frac{(\theta_3+6)^2}{36}\right)$$

The reference trajectory, y_d, is generated by the following linear system

$$\begin{aligned}\dot{x}_d &= -x_d + d; \qquad x_d(0) = 1\\ y_d &= x_d\end{aligned}$$

where d is the command input signal.

For other design and simulation parameters, we select

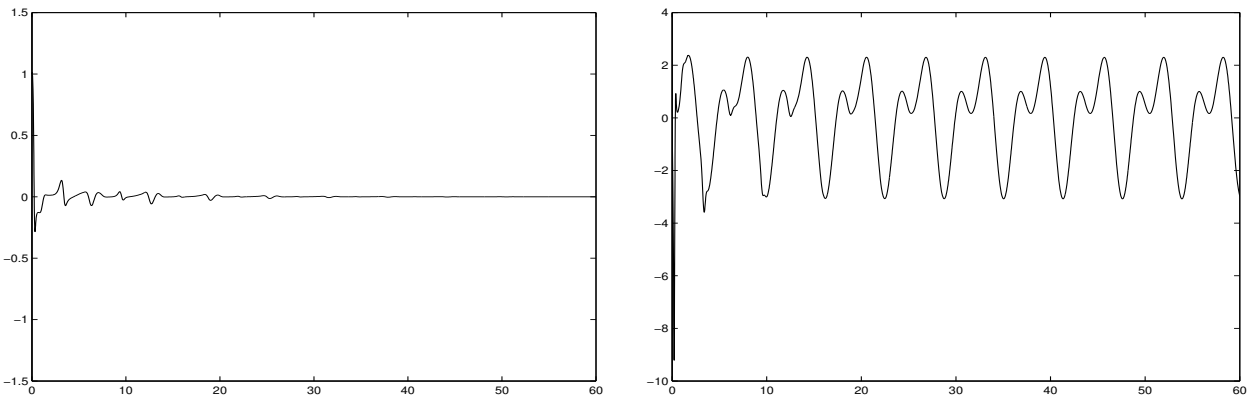
$$\check{x}_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}; \quad \check{\theta}_0 = \begin{bmatrix}2\\-1\\-2\end{bmatrix}; \quad Q_0 = 0.00008 I_3; \quad K_c = 0.038; \quad \bar{\beta}_1 = 0.1; \quad Y = 1.7988 I_3$$

Then, we obtain

$$Z = \begin{bmatrix}10.2283&-1.0070&-3.2067&1.6019\\-1.0070&3.0364&-0.9669&-2.3191\\-3.2067&-0.9669&2.8220&-1.8365\\1.6019&-2.3191&-1.8365&6.8450\end{bmatrix}; \Pi = \begin{bmatrix}0.0496&0.1574&0.1691&0.0829\\0.1574&1.1878&1.3746&0.7149\\0.1691&1.3746&3.3729&2.0049\\0.0829&0.7149&2.0049&2.1536\end{bmatrix}\times 10^3$$

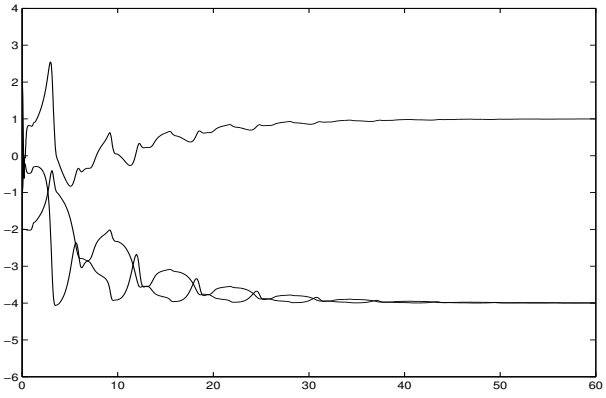
Again, we present two sets of simulation results for this example. The first set is to illustrate the regulatory behaviour of the adaptive controller. We set $d(t) = 0$, $\check{w}(t) = 2.5\sin(t)$, and $\dot{w}(t) = 0$. The result are shown in Figure 4. We observe that the parameter estimates converge to their true values, and the tracking error converges to 0. The transient of the system is well-behaved, and the control magnitude is upper bounded by 10.

The second set of simulation results is to illustrate the robustness property of the adaptive controller. We set $d(t) = 0.2\sin(t)$, $\check{w}(t) = \sin(4t)$, $\dot{w}_1(t) = 0.1\sin(t+\frac{\pi}{2})$, and $\dot{w}_2(t) = 0.5\sin(0.7t+\frac{\pi}{4})$. The simulation results are shown in the Figure 5. We observe that the parameter estimates no longer converge to the true values, but the output tracking error satisfies the targeted attenuation level as desired. The transient of the system is well-behaved, and the control magnitude is again upper bounded by 6.



(a)

(b)



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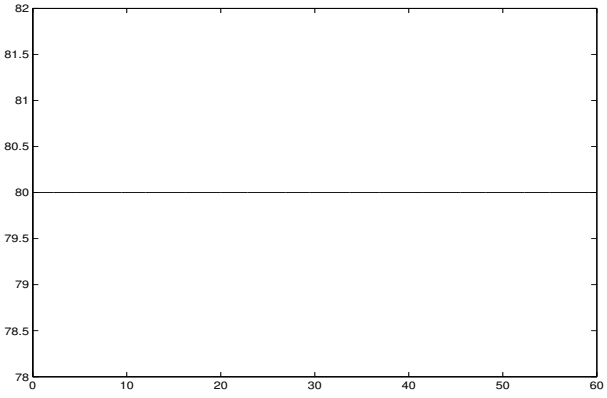
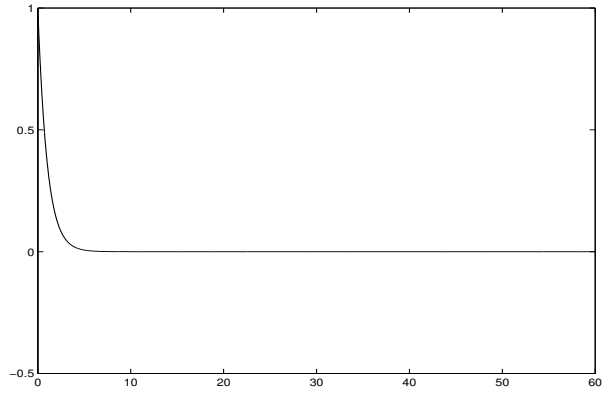


(d)



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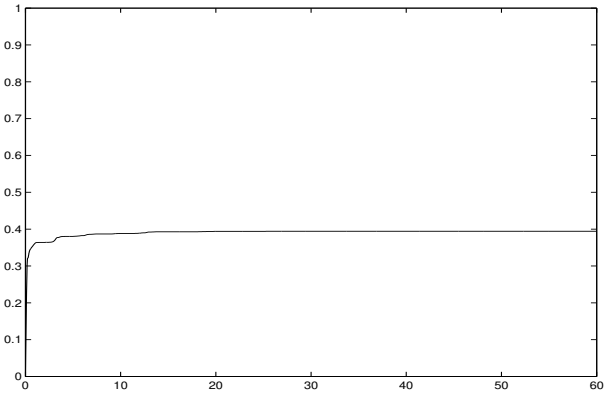

(f)

Figure 4: System response for Example 2 under command input $d(t) = 0$, $\check{w}(t) = 2.5\sin(t)$, $\dot{\check{w}}_1(t) = 0$, and $\dot{\check{w}}_2(t) = 0$.
(a) Tracking error; (b) Control input; (c) Parameter estimate; (d) Maximum eigenvalue of the worst-case error covariance matrix for parameter estimate; (e) Reference trajectory; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2 |w|^2) d\tau$.

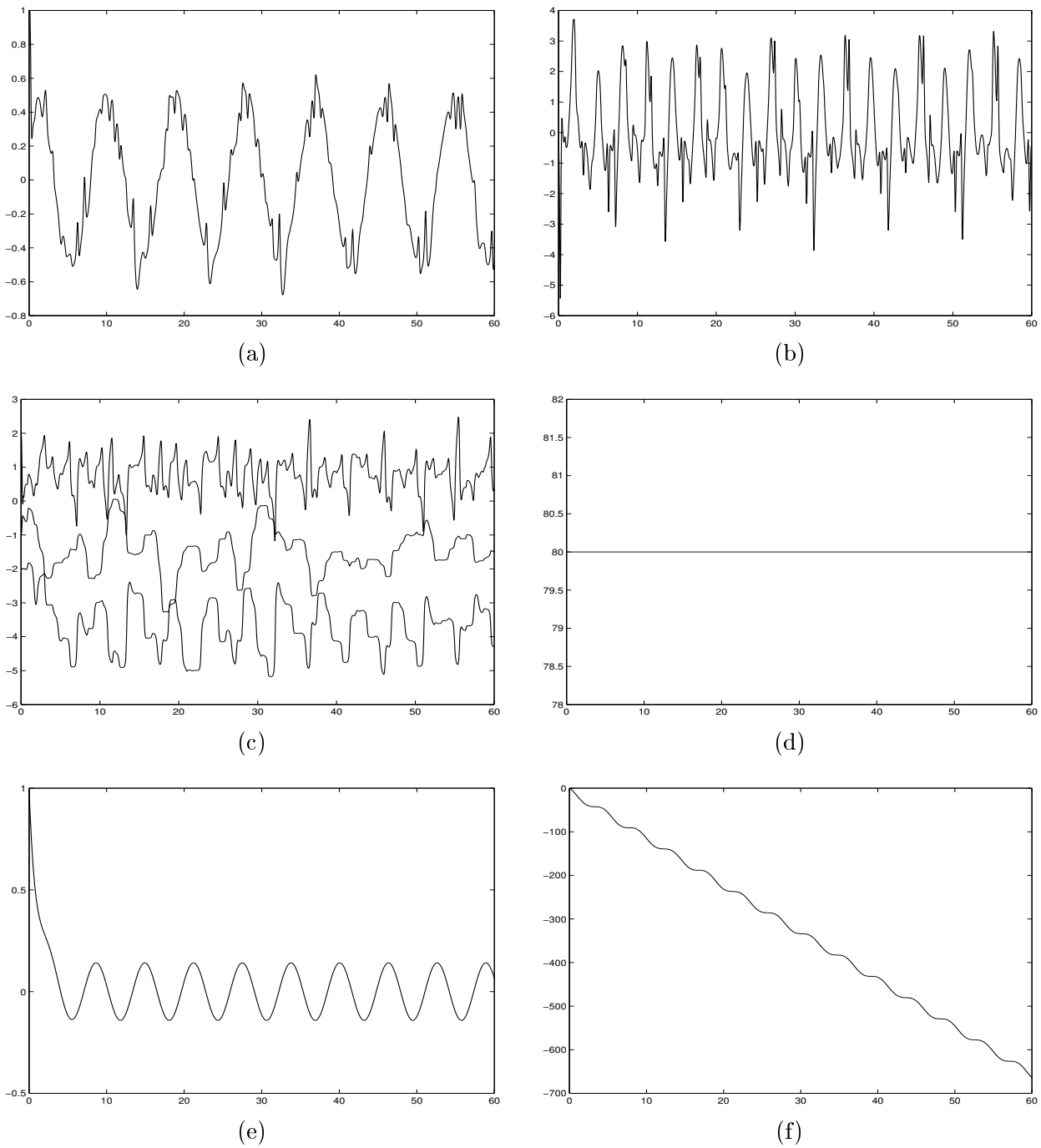

Figure 5: System response for Example 2 under command input $d(t) = 0.2\sin(t)$, $\check{w}(t) = \sin(4t)$, $\grave{w}_1(t) = 0.1\sin(0.1t + \frac{\pi}{2})$, and $\grave{w}_2(t) = 0.5\sin(0.7t + \frac{\pi}{4})$.
(a) Tracking error; (b) Control input; (c) Parameter estimate; (d) Maximum eigenvalue of the worst-case error covariance matrix for parameter estimate; (e) Reference trajectory; (f) $\int_0^t ((x_1 - y_d)^2 - \gamma^2|w|^2)d\tau$.

6 Conclusions

In this paper, we studied the adaptive control design for tracking and disturbance attenuation for SISO linear systems with noisy output measurements and partly measured disturbances. We consider the same linear system as (Pan and Başar 2000) except that part of the disturbance inputs are measured. We make the same assumptions as (Pan and Başar 2000), which includes a known upper bound of the dynamic order, a known relative degree, a strictly minimum phase transfer function from the control input to the output, known sign of the high-freqency gain with nonvanishing bound away from zero. The same design paradigm is used as (Pan and Başar 2000) to obtain the adaptive controller, where cost-to-come function analysis is first applied to obtain the finite dimensional estimator and the integrator backstepping methodology is then applied to obtain the controller. The controller then achieves the desired disturbance attenuation level, with the ultimate lower bound of the attenuation level being the noise intensity in the measurement channel. It gurantees the total stability of the closed-loop system and achieves asymptotic tracking of the refence trajectory when the disturbance is of finite energy and uniformly bounded. These results are the same as those of (Pan and Başar 2000). By proper scaling, one may achieve arbitrary positive disturbance attenuation level with respect to the measured disturbances. The advantage of the measurements of some disturbance inputs is that, when the relative degrees from the measured disturbances to the output are greater than or equal to that from the control input, the controllers guarantee *zero* disturbance attenuation level with respect to measured disturbance inputs. This means that the measured disturbances have no effect on the squared $\mathcal{L}_2$ norm of the tracking error. Furthemore, asymptotic tracking is achieved even though the measured disturbances are uniformly bounded without being of finite energy. These theoretical results are illustrated by numerical examples in the paper.

Future research directions that are of interest are described as follows. One fruitful directioin lies in the extension of the results to more general multiple-input multiple-output systems. Another direction lies in the generalization of the results to nonlinear system.

A Three Lemmas

Lemma 3 *Consider the following state equation:*

$$\dot{x} = f(t, x); \quad x(t_0) = x_0 \tag{59}$$

Let $D \subset \mathbb{R}^n$ be open, $f : [t_0, \infty) \times D \to \mathbb{R}^n$ be locally Lipscitz in x on D uniformly over $[t_0, t_c]$, $\forall t_c \geq t_0$, and continuous in t, then there exists a maximal interval $[t_0, t_f)$ such that (1) has a solution and the solution is unique.

Proof Since $f(t, x)$ is locally Lipschitz in x on D, then there exists some $t_\delta > 0$, such that the differential equation has a unique solution over $[t_0, t_\delta]$. Define $T := \{t_\delta \geq t_0 : \text{there exists a solution over } [t_0, t_\delta]\}$, and $t_f := \sup T$.

First, we will show any time interval $[t_0, t'] \subset [t_0, t_f)$ has a solution. From the definition of t_f, there exists $t_1 \in T$, such that $t' \leq t_1 < t_f$. Then there exists a solution $x(t)$ over $[t_0, t_1]$. Define $X(t) = x(t)$ over $[t_0, t']$, since $X(t)$ satisfies differential equation (1), then it is a solution over $[t_0, t']$.

Next, we will show that if $x_1(t)$ is a solution over $[t_0, t_1]$, $x_2(t)$ is a solution over $[t_0, t_2]$, and $t_0 \leq t_1 \leq t_2 < t_f$, then $x_1(t) = x_2(t)\ \forall t \in [t_0, t_1]$.

Define $T_1 := \{t' \in [t_0, t_1] : x_1(t) = x_2(t), \forall t \in [t_0, t']\}$ and $t_3 := \sup T_1$. Then we have the following 2 cases.

Case 1: $t_0 \leq t_3 < t_1$. Case $1a : x_1(t_3) \neq x_2(t_3)$. From the definition of t_3, we know $x_1(t') = x_2(t')$ for $\forall t' \in [t_0, t_3)$. Since $x_1(t)$ and $x_2(t)$ are continuous, then:

$$x_1(t_3) = \lim_{t \to t_3^-} x_1(t) = \lim_{t \to t_3^-} x_2(t) = x_2(t_3)$$

This is a contradiction. Hence this subcase is impossible. Case $1b : x_1(t_3) = x_2(t_3)$. Consider the following differential equation:

$$\dot{x} = f(t, x); \quad x(t_3) = x_1(t_3) \tag{60}$$

Since $f(t, x)$ is locally Lipscitz in x on D uniformly over $[t_0, t_c]$, $\forall t_c \geq t_0$, and continuous in t, there exist some $\delta > 0$, satisfying $\delta + t_3 < t_1$, such that there exists a unique solution $x_3(t)$ to (60) over $[t_3, t_3 + \delta]$. It follows $x_1(t) = x_2(t) = x_3(t)$ over $t \in [t_3, t_3 + \delta]$, and $x_1(t) = x_2(t)$ over $t \in [t_0, t_3 + \delta]$, which contradict the definition of t_3. Then this subcase is impossible, thus case 1 is impossible.

Case 2: $t_3 = t_1$. By the similar method of Case 1a: we can prove $x_1(t_3) = x_2(t_3)$.

Thus we have proved that $x_1(t)$ and $x_2(t)$ agree on $[t_0, t_1]$.

Next we will prove $t_f \notin T$ by contradiction. Suppose $t_f \in T$, from the definition of t_f, there exists a solution $x_1(t)$ over $[t_0, t_f]$. From the above proof, the solution is unique. Then consider the following differential equation:

$$\dot{x} = f(t, x); \quad x(t_f) = x_1(t_f) \tag{61}$$

There exists some δ , such that there exists a unique solution $x_2(t)$ over $[t_f, t_f + \delta]$. Define

$$X(t) = \begin{cases} x_1(t) & t \in [t_0, t_f] \\ x_2(t) & t \in [t_f, t_f + \delta] \end{cases}$$

then $X(t)$ is a solution to (59) over $[t_0, t_f + \delta]$, which contradicts the definition of t_f.

Last, define $X : [t_0, t_f) \to D$, $X(t) = x(t)$ where $x : [t_0, t] \to D$ is a solution to (1). Then $\forall t \in [t_0, t_f)$ and $\forall t' \in (t, t_f)$, we can get a $t_1 \in T$ satisfied $t_1 \geq t' > t$ and there exists $x : [t_0, t_1] \to D$ is a solution to (1) then $X(t') = x(t'')$ for $\forall t'' \in [t_0, t_1]$. So $X(t)$ is a solution over $[t_0, t_f)$.

This complete the proof of this lemma. □

Lemma 4 *Let $D \subset \mathbb{R}^n$ be open, $U : [t_0, t_f) \times D \to \mathbb{R}^+$ be continuous, and $\xi_e : [t_0, t_f) \to \mathbb{R}^n$ be continuous in t. Define $T = \{t' \in [t_0, t_f) : U(t, \xi_e(t)) \leq c_\Omega, \forall t \in [t_0, t']\}$ and $t'_f = \sup T$. If $t'_f < t_f$, then $U(t'_f, \xi_e(t'_f)) = c_\Omega$.*

Proof First, we will show $t'_f \in T$ by contradiction. Suppose $t'_f \notin T$, we have $U(t'_f, \xi_e(t'_f)) > c_\Omega$. Then $\forall t' < t'_f$ there exists a $t_1 \in T$, such that $t' \leq t_1 < t'_f$ and $U(t, \xi_e(t)) \leq c_\Omega$ for $t \in [t_0, t_1]$, then $U(t', \xi_e(t')) \leq c_\Omega$. It follows $\lim_{t \to t'^-_f} U(t, \xi_e(t)) \leq c_\Omega$. Since $U(t, \xi_e(t))$ is continuous in t, we have $U(t'_f, \xi_e(t'_f)) = \lim_{t \to t'^-_f} U(t, \xi_e(t)) \leq c_\Omega$, which is contradiction. Then we have $U(t'_f, \xi_e(t'_f)) \leq c_\Omega$.

Suppose $U(t'_f, \xi_e(t'_f)) = c_1 < c_\Omega$. Since $U(t, \xi_e(t))$ is continuous in t, let $\varepsilon = c_\Omega - c_1 > 0$, there exists $\delta \in (0, t_f - t'_f)$ such that for t satisfies $|t - t'_f| < \delta$, $|U(t, \xi_e(t)) - U(t'_f, \xi_e(t'_f))| < \varepsilon$. It follows $U(t, \xi_e(t)) \leq c_\Omega$ for $t \in [t_0, t'_f + \delta)$. Then $\sup T \geq t'_f + \delta$, which is a contradiction. Then we have $U(t'_f, \xi_e(t'_f)) = c_\Omega$.

This complete the proof of this lemma. □.

Lemma 5 *If $f : [0, \infty) \to \mathbb{R}$ is continuous in t, and satisfies:*

$$\int_0^\infty |f(\tau)|^2 d\tau < \infty$$
$$|\dot{f}(t)| \leq M < \infty$$

then $\lim_{t\to\infty} f(t) = 0$.

Proof We prove it by contradiction. Suppose $\lim_{t\to\infty} f(\tau) \neq 0$, then $\exists \epsilon > 0$, $\forall T > 0$ exists $t_\epsilon > T$ such that $|f(t_\epsilon)| > \epsilon$.

Given a such t_ϵ, from the mean value theorem, we have

$$\frac{|f(t) - f(t_\epsilon)|}{|t - t_\epsilon|} = |f'(\xi)| \leq M \quad \text{for some} \quad \xi \in (t, t_\epsilon) \quad \text{or} \quad (t_\epsilon, t);$$

It follows $|f(t)| \geq |f(t_\epsilon)| - M|t - t_\epsilon|$. Then we have $|f(t)| \geq \frac{\epsilon}{2}$ when $t \in [t_\epsilon - \frac{\epsilon}{2M}, t_\epsilon + \frac{\epsilon}{2M}]$.

Let $\delta = \frac{\epsilon}{2M}$, first we take $T_1 = \delta$, then we may obtain $t_1 > T_1$ and a time interval $[t_1 - \delta, t_1 + \delta]$, such that $f(t_1) > \epsilon$ and $f(t) \geq \frac{\epsilon}{2} \quad \forall t \in [t_1 - \delta, t_1 + \delta]$. Next take $T_2 = t_1 + 2\delta$, then we can get a $t_2 > T_2$ and a time interval $[t_2 - \delta, t_2 + \delta]$. Continue such procedure, we can get a sequence $\{t_n\}_{n=1}^\infty$ and corresponding time interval $[t_n - \delta, t_n + \delta]$. Then

$$\begin{aligned}
\int_0^\infty |f(\tau)|^2 d\tau &\geq \int_{t_1-\delta}^{t_1+\delta} |f(\tau)|^2 d\tau + \int_{t_2-\delta}^{t_2+\delta} |f(\tau)|^2 d\tau + \cdots + \int_{t_n-\delta}^{t_n+\delta} |f(\tau)|^2 d\tau + \cdots \\
&\geq \int_{t_1-\delta}^{t_1+\delta} |\frac{1}{2}\epsilon|^2 d\tau + \int_{t_2-\delta}^{t_2+\delta} |\frac{1}{2}\epsilon|^2 d\tau + \cdots + \int_{t_n-\delta}^{t_n+\delta} |\frac{1}{2}\epsilon|^2 d\tau + \cdots \\
&\geq (\frac{1}{2}\epsilon)^2 2\delta N \ (N \to \infty)
\end{aligned}$$

which contradict $\int_0^\infty |f(\tau)|^2 d\tau < \infty$.

Therefore $\lim_{t\to\infty} f(t) = 0$. □

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