Output Feedback Control Design for Strict-Feedback Stochastic Nonlinear Systems under a Risk-Sensitive Cost*

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Abstract

In this paper, we study the problem of output-feedback control design for a class of strict feedback stochastic nonlinear systems. Under a infinite-horizon risk-sensitive cost criterion, the controller designed can guarantee an arbitrary small long-term average cost for arbitrary risk-sensitivity parameter and achieve boundedness in probability for the closed-loop system, using the integrator backstepping methodology. Furthermore, the controller preserves the equilibrium at the origin of the nonlinear system.

Key words integrator backstepping, strict-feedback stochastic nonlinear systems, risk-sensitive control, bounded in probability.

1 Introduction

The design of globally stabilizing controllers for nonlinear systems has attracted a lot of research attention in recent years. A breakthrough is achieved with the introduction of the integrator backstepping design methodology [1] after the celebrated characterization of the feedback linearizable systems [2]. This methodology provides a general recursive constructive tool to design globally stabilizing controllers for nonlinear systems that are given in strict-feedback form, and for systems that are feedback equivalent to such systems. Additional results obtained for strict-feedback systems have been obtained since the early 90's, with some selective references in the area being [3, 4, 5, 6, 7, 8, 9]. See also the book [10] for an extensive list of references on the topic.

It has been known for some time that differential game problems are closely related to risk-sensitive stochastic control problems [11, 12, 13, 14, 15, 16, 17, 18, 19]. If the noise intensity vanishes,

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then a large deviation limit of the risk-sensitive stochastic control problem yields a deterministic differential game. All these connections have motivated, in recent years, accelerated research on the topic of risk-sensitive stochastic control.

Controller design for strict-feedback stochastic nonlinear systems is a research topic under intensive investigation recently [20, 21, 22]. Asymptotic stability has been the main objective of the design in [20] and [22],with the working assumption that the disturbance vector field vanishes at the origin. The paper [20] considers full state feedback, while [22] considers output feedback design. In [21], the design objective is to guarantee a particular long-term average risk-sensitive cost criterion is less than a given bound, under full state measurements. It is shown that, for arbitrary risk-sensitivity parameter, the long term average risk-sensitive cost can be guaranteed to be less than any given desired cost bound. Furthermore, the closed-loop system is bounded in probability. With the additional assumption that the disturbance vector field vanishes at the origin, the closed-loop system is asymptotically stable in the large.

In this paper, we again poise the objective of guaranteeing a particular long-term average risk-sensitive cost criterion to be less than a desired bound. Unlike [21], we consider the problem under output measurements. We show that, for arbitrary risk-sensitivity parameter, the controller achieves arbitrary small long-term average cost for the problem. Furthermore, the closed-loop system is bounded in probability. Special care is taken that the nonlinear controller preserves the equilibrium at the origin of the original system.

The balance of the paper is organized as follows. In Section 2 we formulate the problem under study and review a couple of definitions and a theorem that are needed in the subsequent development. In Section 3 we analyze in detail the output feedback risk sensitive control problem. We find the appropriate value function and derive the output-feedback risk-sensitive controller for this class of strict-feedback stochastic nonlinear systems. The paper ends with some concluding remarks in Section 4.

2 Problem Formulation

We consider the following noise-prone strict-feedback nonlinear systems given by the Itô stochastic differential equation (SDE) as in [22].

$$dx_1 = x_2\,dt + f_1(y)\,dt + \varphi_1(y)'\,dw \tag{1a}$$
$$dx_2 = x_3\,dt + f_2(y)\,dt + \varphi_2(y)'\,dw \tag{1b}$$
$$\vdots$$
$$dx_n = u\,dt + f_n(y)\,dt + \varphi_n(y)'\,dw \tag{1c}$$
$$y = x_1. \tag{1d}$$

where $x = [x_1, \ldots, x_n]' \in \mathbb{R}^n$ is the state vector with a fixed initial value $x(0)$; $u \in \mathbb{R}$ is the scalar control input; $w \in \mathbb{R}^s$ is an independent standard vector-valued Brownian Motion defined on probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the sample space, $\mathcal{F}$ is a filtration, and $\mathcal{P}$ is the probability measure; y is the scalar output. The following assumptions are basic for our ensuing study.

A1 The functions $f_i(y)$ and $\varphi_i(y)$, $1 \le i \le n$, are assumed to be smooth.

A2 $f_i(y)$, $1 \le i \le n$, are assumed to vanish at zero.

The assumption A2 insures that the origin is an equilibrium point of the undisturbed system.

The objective of the controller design is to maintain a finite long-term average value for the following risk-sensitive cost function:

$$J_\theta(u) = \lim_{T\to\infty} \frac{1}{T}\frac{2}{\theta} \ln \left\{ E\left\{ \exp\left(\frac{\theta}{2}\int_0^T q(y(t))y^4(t)\,dt \right) \right\} \right\} \tag{2}$$

that is no more than arbitrary given constant $R_l > 0$, where $q(y) \geq 0$, $\forall y \in \mathbb{R}$, is an arbitrary given function, and θ is the risk-sensitivity parameter.

Remark 1 *Note here, the weighting function for the long term average cost is $q(y)y^4$. This is chosen to allow easy controller design in the subsequent development. In [21], the weighting function y^2 (or $q(y)y^2$) is chosen, which leads to complications in dealing with the second order variation terms due to the stochastic nature of the problem. The result can be utilized to study a problem where the goal is to guarantee a long term average cost of no more than R_l for a weighting function of $q(y)y^2$, which only involve a quadratic term of y. By the inequality: $q(y)y^2 \leq \varepsilon^2/2 + q^2(y)y^4/(2\varepsilon^2)$, where ε is an arbitrary small positive constant, the first term results in a long term average cost of $\varepsilon^2/2$, and the second term results in a problem that can be treated with the methodology developed in this paper. Then, in the design, we need only to achieve a long term average cost of not more than $R_l - \varepsilon^2/2$ for a weighting function of $q^2(y)y^4/(2\varepsilon^2)$ to achieve the original design objective.*

Based on the value of θ, the risk-sensitive cost criteria can be classified into three categories. If θ is positive, the optimization will lead to a risk-averse control design. As $\theta \to 0$, the cost function converges to a standard integral cost, the problem is known as a risk-neutral problem. The case where θ is negative is called the risk-seeking problem. The larger θ is, the more conservative is the controller. When $\theta \to \infty$ and the noise intensity decreases to zero, the optimization problem converges to a differential game associated with a robust control problem. In this paper, we will study only the case where θ is positive. The case when $\theta \leq 0$ can be studied analogously.

As in [21], we introduce the following definition.

Definition 1 *For a given positive risk-sensitivity parameter θ, a controller is said to achieve a guaranteed risk-sensitive cost $R_l (R_l \geq 0)$, if*

$$J_\theta(u) = \lim_{T\to\infty} \frac{1}{T}\frac{2}{\theta} \ln \left\{ E\left\{ \exp\left(\frac{\theta}{2}\int_0^T (q(y(t))y^4(t) + l(y_{[0,t]}))\,dt \right) \right\} \right\} \leq R_l \tag{3}$$

where $l(y_{[0,t]})$ is a nonnegative function which is chosen by the designer.

We consider the following compact form of the system (1) along with the infinite-horizon risk-sensitive cost function $J_\theta(u)$ of (3): $dx = f(x)\,dt + Bu\,dt + h(y)\,dw$.

As in [21], we are interested here in designing a suboptimal control strategy rather than an optimal one. We search for control law in the form:

$$\dot{\vartheta} = a(\vartheta, y) \qquad\qquad u = \mu(\vartheta, y) \tag{4}$$

and a positive definite and radially unbounded value function $V(x,\vartheta)$ such that the HJB inequality

$$\begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial \vartheta} \end{bmatrix} \begin{bmatrix} f + B\mu \\ a \end{bmatrix} + \frac{\theta}{4}\frac{\partial V}{\partial x} hh' \left(\frac{\partial V}{\partial x}\right)' + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2} hh'\right) + Q(x,\vartheta) \leq R_l \tag{5}$$

is satisfied, where a and μ are $\mathcal{C}^1$ functions, $R_l \geq 0$ is a desired long-term average cost, and $Q(x, \vartheta) = q(y)y^4 + \check{l}(y, \vartheta)$. The variable ϑ corresponds to $\hat{x}$ in the later derivations.

The following theorem is a direct consequence of Theorems 5 and 6 of [21].

Theorem 1 *Consider the stochastic nonlinear system (1) with the risk- sensitive cost function (2). For any given $\theta > 0$ and $R_l \geq 0$, let there exists a $\mathcal{C}^2$ positive definite and radially unbounded value function $V(x, \vartheta)$, and control law (4), such that, $dV \leq \sigma\, dw - \frac{\theta}{4}\sigma\sigma'\, dt - Q\, dt + R_l\, dt$, for some $\mathcal{C}^1$ function $\sigma(x, \vartheta)$, then*

1. *there exists an a.s. $\mathcal{P}$ unique solution to the closed loop system;*

2. *the risk-sensitive cost (2) is upper bounded by R_l.*

Furthermore, if there exist positive constants c_1, c_2, such that

$$V(x, \vartheta) \leq c_1 Q(x, \vartheta) + c_2, \qquad \forall (x, \vartheta)$$

then the closed-loop system is bounded in probability.

Now, we turn to the design of the desired controller in the next section.

3 Main Results

In this section, we present the design of a dynamic output-feedback risk -sensitive controller for the class of strict-feedback nonlinear systems by using integrator backstepping methodology.

3.1 State estimator design

Since the state variables $x_2, \cdots, x_n$ of the system (1) are unmeasured and cannot be used to construct the controller, we design the following state-estimator based on the output y [22].

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + k_1(y - \hat{x}_1) + f_1(y) && \text{(6a)}\\
&\vdots \\
\dot{\hat{x}}_n &= \hat{x}_{n+1} + k_n(y - \hat{x}_1) + f_n(y) && \text{(6b)}\\
\hat{x}_{n+1} &= u && \text{(6c)}
\end{aligned}$$

where the parameters $k_1, \ldots, k_n$ are chosen such that the polynomial $s^n + k_1 s^{n-1} + \cdots + k_n$ has all roots with negative real parts.

Let $x = [x_1, x_2, \ldots, x_n]'$, $\hat{x} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n]'$, and $\tilde{x} = x - \hat{x}$. Then we have

$$d\tilde{x} = \begin{bmatrix} -k_1 & & & \\ \vdots & & I & \\ \vdots & & & \\ -k_n & 0 & \cdots & 0 \end{bmatrix} \tilde{x}\, dt + \varphi(y)'\, dw =: A\tilde{x}\, dt + \varphi(y)'\, dw$$

where $\varphi(y) = [\varphi_1, \cdots, \varphi_n]$. By the choice of the parameters k_i, $1 \leq i \leq n$, the matrix A is Hurwitz. Then, there exists a positive definite matrix $P > 0$, satisfying the following Lyapunov equation:

$A'P + PA = -I$. The entire system dynamics with the state-estimator can be rewritten as follows.

$$
\begin{aligned}
d\tilde{x} &= A\tilde{x}\,dt + \varphi(y)'\,dw && \text{(7a)}\\
dy &= (\hat{x}_2 + \tilde{x}_2)\,dt + f_1(y)\,dt + \varphi_1'\,dw && \text{(7b)}\\
d\hat{x}_2 &= (\hat{x}_3 + k_2(y - \hat{x}_1) + f_2(y))\,dt && \text{(7c)}\\
&\vdots\\
d\hat{x}_n &= (u + k_n(y - \hat{x}_1) + f_n(y))\,dt && \text{(7d)}
\end{aligned}
$$

We observe that the dynamics (7) possess a lower-triangle structure, which motivates us to apply integrator backstepping methodology to design the output feedback controller.

3.2 Output feedback risk-sensitive control design

We observe that the dynamics of $\tilde{x}$ is globally asymptotically stable without the disturbance input. We will treat it as zero dynamics in the backstepping design. In the backstepping, we introduce the following transformed state vector, z, for the original variables $(y, \hat{x}_2, \ldots, \hat{x}_n)$, $z_1 = y$, $z_i = \hat{x}_i - \alpha_{i-1}(\bar{\hat{x}}_{i-1}, y)$, $2 \le i \le n$, where α_i, $1 \le i \le n-1$, are smooth virtual control laws to be determined in the design, and $\bar{\hat{x}}_i = [\hat{x}_1, \cdots, \hat{x}_i]$, $1 \le i \le n$. We will denote $\bar{z}_i = [z_1, \cdots, z_i]$, $1 \le i \le n$. Due to page limitation, we will design the controller in one compact step, rather than designing it in n steps.

In this paper, the virtual control laws $\alpha_i(\bar{\hat{x}}_i, y)$, $1 \le i \le n$, and the control law $u = \alpha_n(\bar{\hat{x}}_n, y)$ are required to satisfy the equilibrium condition: $\alpha_i(0,0) = 0$, $1 \le i \le n$, which requires special attention in the design process.

In terms of the transformed state vector z, the systems (7) can be expressed as the following:

$$
\begin{aligned}
d\tilde{x} &= A\tilde{x}\,dt + \varphi(y)'\,dw\\
dz_1 &= (z_2 + \alpha_1(z_1) + \tilde{x}_2)\,dt + F_1(y)\,dt + \Psi_1\,dw\\
&\vdots\\
dz_i &= (z_{i+1} + \alpha_i(\bar{\hat{x}}_i, y))\,dt + F_i(\bar{\hat{x}}_i, y)\,dt - \frac{\partial \alpha_{i-1}}{\partial y}\tilde{x}_2\,dt + \Psi_i(\bar{\hat{x}}_i, y)\,dw\\
&\vdots\\
dz_n &= u\,dt + F_n(\bar{\hat{x}}_n, y)\,dt - \frac{\partial \alpha_{n-1}}{\partial y}\tilde{x}_2\,dt + \Psi_n(\bar{\hat{x}}_n, y)\,dw
\end{aligned}
$$

where

$$
\begin{aligned}
F_i &= (k_i(y - \hat{x}_1) + f_i(y)) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}(z_{j+1} + \alpha_j + k_j(y - \hat{x}_1) + f_j(y))\\
&\quad - \frac{\partial \alpha_{i-1}}{\partial y}(z_2 + \alpha_1 + f_1(y)) - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\varphi_1'(y)\varphi_1(y)\right), \qquad 2 \le i \le n\\
F_1 &= f_1(y), \qquad \Psi_1 = \varphi_1', \qquad \Psi_i = -\frac{\partial \alpha_{i-1}}{\partial y}\varphi_1', \quad 2 \le i \le n
\end{aligned}
$$

The value function of the whole systems is $V_n = \phi(\tilde{x}'P\tilde{x}) + \frac{1}{4}\sum_{j=1}^{n} z_j^4$, where $\phi(\xi) = (c+\xi)^{\frac{2}{3}} - c^{\frac{2}{3}}$, where $c > 0$ is a constant yet to be determined.

Then, by using Itô's formula, we have

$$
\begin{aligned}
dV_n &= -\frac{\partial \phi}{\partial \xi}|\tilde{x}|^2\,dt + \sigma_n\,dw + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2\phi(\tilde{x}'P\tilde{x})}{\partial \tilde{x}^2}\varphi'\varphi\right)\,dt + \frac{\theta}{4}\sigma_n\sigma_n'\,dt - \frac{\theta}{4}\sigma_n\sigma_n'\,dt \\
&\quad + \sum_{i=1}^{n}\beta_i z_i^4\,dt - \sum_{i=1}^{n}\beta_i z_i^4\,dt + \sum_{j=1}^{n}\frac{3z_j^2}{2}\Psi_j\Psi_j'\,dt \\
&\quad + \sum_{j=1}^{n-1} z_j^3(z_{j+1}+\alpha_j+F_j)\,dt - z_1^3\tilde{x}_2 - \sum_{j=2}^{n} z_j^3\frac{\partial \alpha_{j-1}}{\partial y}\tilde{x}_2\,dt + z_n^3(u+F_n)\,dt
\end{aligned}
$$

where $\sigma_n = 2\frac{\partial\phi}{\partial\xi}\tilde{x}'P\varphi' + \sum_{j=1}^{n} z_j^3\Psi_j$, as in paper [21], we have add and subtracted $\frac{\theta}{4}\sigma_n\sigma_n'\,dt$ and $\sum_{i=1}^{n}\beta_i z_i^4\,dt$ to the right-hand side of the above equation, and $\beta_1, \ldots, \beta_n$ are design parameters to be determined later.

In further analysis, we need the following Young's inequality [22].

Lemma 1 *For any vector $x, y \in R^n$, we have $x'y \le \varepsilon^p|x|^p/p + |y|^q/(q\varepsilon^q)$, where $\varepsilon > 0$, and $p > 1, q > 1$ satisfying: $\frac{1}{q} + \frac{1}{p} = 1$.*

Let $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum eigenvalue and the minimum eigenvalue of the matrix P, respectively. Define function $\Xi_{c,P}(\tilde{x}) = |\tilde{x}|^{\frac{4}{3}}\lambda_{\max}^{-\frac{1}{3}}(P) - |\tilde{x}|^2/\sqrt[3]{c+\tilde{x}'P\tilde{x}}$. It is easy to prove that the maximum of $\Xi_{c,P}(\tilde{x})$ exists and denote it $\mu(c,P) := \max_{\tilde{x}}\Xi_{c,P}(\tilde{x}) < \infty$ which is positive.

Note that, for positive constants ε_{n1}, ε_{n2}, and ε_{n3}, which are to be determined later,

$$
\begin{aligned}
\sigma_n\sigma_n' &= 4\left(\frac{\partial\phi}{\partial\xi}\right)^2\tilde{x}'P\varphi^T\varphi P\tilde{x} + 4\frac{\partial\phi}{\partial\xi}\tilde{x}'P\varphi'\sum_{j=1}^{n} z_j^3\Psi_j' + \left(\sum_{j=1}^{n} z_j^3\Psi_j\right)\left(\sum_{j=1}^{n} z_j^3\Psi_j\right)' \\
&\le \frac{16\tilde{x}'P\varphi'(0)\varphi(0)P\tilde{x}}{9(c+\tilde{x}'P\tilde{x})^{\frac{2}{3}}} + \frac{32|\tilde{x}|^2|P|^2|\varphi(0)||\Phi(y)|y}{9(c+\tilde{x}'P\tilde{x})^{\frac{2}{3}}} + \frac{16|\tilde{x}|^2|P|^2|\Phi'(y)\Phi(y)|y^2}{9(c+\tilde{x}'P\tilde{x})^{\frac{2}{3}}} \\
&\quad + \frac{8\tilde{x}'}{3\sqrt[3]{c+\tilde{x}'P\tilde{x}}}P\varphi'(y)\sum_{j=1}^{n}\Psi_j' z_j^3 + \left(\sum_{j=1}^{n} z_j^3\Psi_j\right)\left(\sum_{j=1}^{n} z_j^3\Psi_j\right)' \\
&\le \frac{16|P\varphi'(0)\varphi(0)P||\tilde{x}|^2}{9\sqrt[3]{c}\sqrt[3]{c+\tilde{x}'P\tilde{x}}} + \frac{32|P|^2}{9}\left(\frac{|\varphi(0)|^4|\Phi(y)|^4y^4}{4\varepsilon_{n1}^4} + \frac{3\varepsilon_{n1}^{\frac{4}{3}}|\tilde{x}|^{\frac{8}{3}}}{4(c+\tilde{x}'P\tilde{x})^{\frac{8}{9}}}\right) \\
&\quad + \frac{16|P|^2}{9}\left(\frac{|\Phi'(y)\Phi(y)|^2y^4}{2\varepsilon_{n2}^2} + \frac{\varepsilon_{n2}^2|\tilde{x}|^4}{2(c+\tilde{x}'P\tilde{x})^{\frac{4}{3}}}\right) + 4\left(\frac{2\varepsilon_{n3}^2|\tilde{x}|^2}{9(c+\tilde{x}'P\tilde{x})^{\frac{2}{3}}} + \frac{1}{2\varepsilon_{n3}^2}\left(\sum_{j=1}^{n}\Psi_j z_j^3\right)\right. \\
&\quad \left.\cdot\,\varphi(y)PP\varphi'(y)\left(\sum_{j=1}^{n}\Psi_j' z_j^3\right)\right) + \left(\sum_{j=1}^{n} z_j^3\Psi_j\right)\left(\sum_{j=1}^{n} z_j^3\Psi_j\right)' \\
&\le \frac{16|P\varphi'(0)\varphi(0)P||\tilde{x}|^2}{9\sqrt[3]{c}\sqrt[3]{c+\tilde{x}'P\tilde{x}}} + \frac{8|P|^2|\varphi(0)|^4|\Phi(y)|^4y^4}{9\varepsilon_{n1}^4} + \frac{8\varepsilon_{n1}^{\frac{4}{3}}|P|^2|\tilde{x}|^2}{3\sqrt[9]{c^2}\lambda_{\min}^{\frac{1}{3}}(P)\sqrt[3]{c+\tilde{x}'P\tilde{x}}}
\end{aligned}
$$

$$+\frac{8|P|^2|\Phi(y)|^4y^4}{9\varepsilon_{n2}^2}+\frac{8\varepsilon_{n2}^2|P|^2|\tilde{x}|^2}{9\lambda_{\min}(P)\sqrt[3]{c+\tilde{x}'P\tilde{x}}}+\frac{8\varepsilon_{n3}^2|\tilde{x}|^2}{9\sqrt[3]{c}\sqrt[3]{c+\tilde{x}'P\tilde{x}}}+\frac{2}{\varepsilon_{n3}^2}\sum_{i=1}^{n}z_i^3\Psi_i\varphi(y)PP\varphi'(y)$$
$$\cdot\left(2\sum_{j=1}^{i-1}\Psi_j'z_j^3+\Psi_i'z_i^3\right)+\sum_{i=1}^{n}z_i^3\Psi_i\left(2\sum_{j=1}^{i-1}\Psi_j'z_j^3+\Psi_i'z_i^3\right)$$

where $\Phi(y)$ is defined by the equation $\varphi(y)=\varphi(0)+y\Phi(y)$.

Note the following inequalities.

$$\begin{aligned}
\sum_{j=2}^{n}z_j^3\frac{\partial\alpha_{j-1}}{\partial y}\tilde{x}_2 &\le \sum_{j=2}^{n}\left(\frac{z_i^{12}}{4\varepsilon_{n4j}^4}\left|\frac{\partial\alpha_{j-1}}{\partial y}\right|^4+\frac{3\varepsilon_{n4j}^{\frac{4}{3}}}{4}|\tilde{x}|^{\frac{4}{3}}\right);\quad z_1^3\tilde{x}_2\le\frac{z_1^{12}}{4\varepsilon_{n41}^4}+\frac{3\varepsilon_{n41}^{\frac{4}{3}}|\tilde{x}|^{\frac{4}{3}}}{4}\\
\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2\phi(\tilde{x}'P\tilde{x})}{\partial\tilde{x}^2}\varphi'\varphi\right) &\le \frac{2\mathrm{Tr}(P\varphi'\varphi)}{3\sqrt[3]{c+\tilde{x}'P\tilde{x}}}-\frac{2\mathrm{Tr}(P\tilde{x}\tilde{x}'P\varphi'\varphi)}{9(c+\tilde{x}'P\tilde{x})^{\frac{4}{3}}}\le\frac{2}{3\sqrt[3]{c}}\left(\mathrm{Tr}(\varphi(0)P\varphi'(0))+\frac{3\varepsilon_{n5}^{\frac{4}{3}}}{2}\right.\\
&\quad\left.+\frac{\mathrm{Tr}^4(\Phi P\varphi'(0))y^4}{2\varepsilon_{n5}^4}+\frac{\mathrm{Tr}^2(\Phi P\Phi')y^4}{2\varepsilon_{n5}^2}+\frac{\varepsilon_{n5}^2}{2}\right)\\
\sum_{j=1}^{n}\frac{3z_j^2}{2}\Psi_j\Psi_j' &\le \frac{3}{4}\sum_{j=1}^{n}\left(\frac{1}{\varepsilon_{n7j}^2}z_j^4|\Psi_j|^4+\varepsilon_{n7j}^2\right);\quad\sum_{j=1}^{n-1}z_j^3z_{j+1}\le\sum_{j=1}^{n-1}\left(\frac{z_{j+1}^4}{4}+\frac{3z_j^4}{4}\right)
\end{aligned}$$

where the positive constants ε_{n41}, …, ε_{n4n}, ε_{n5}, ε_{n71}, …, ε_{n7n} are yet to be determined.

Thus, we have

$$\begin{aligned}
dV_n &\le \sigma_n\,dw-\frac{\theta}{4}\sigma_n\sigma_n'\,dt-\sum_{i=1}^{n}\beta_iz_i^4\,dt-q(y)y^4\,dt\\
&\quad-\frac{2|\tilde{x}|^2}{3\sqrt[3]{c+\tilde{x}'P\tilde{x}}}\,dt+q(y)y^4\,dt+\sum_{i=1}^{n}\beta_iz_i^4\,dt+\frac{2}{3\sqrt[3]{c}}\left(\mathrm{Tr}(\varphi(0)P\varphi'(0))+\frac{\mathrm{Tr}^4(\Phi P\varphi'(0))y^4}{2\varepsilon_{n5}^4}\right.\\
&\quad\left.+\frac{3\varepsilon_{n5}^{\frac{4}{3}}}{2}+\frac{\mathrm{Tr}^2(\Phi P\Phi')y^4}{2\varepsilon_{n5}^2}+\frac{\varepsilon_{n5}^2}{2}\right)\,dt+\sum_{j=1}^{n-1}z_j^3(\alpha_j+F_j)\,dt+z_n^3(u+F_n)\,dt+\sum_{j=1}^{n-1}\left(\frac{z_{j+1}^4}{4}\right.\\
&\quad\left.+\frac{3}{4}z_j^4\right)\,dt+\frac{3}{4}\sum_{j=1}^{n}\left(\frac{1}{\varepsilon_{n7j}^2}|\Psi_j|^4z_j^4+\varepsilon_{n7j}^2\right)\,dt+\frac{z_1^{12}\,dt}{4\varepsilon_{n41}^4}+\frac{3\varepsilon_{n41}^{\frac{4}{3}}|\tilde{x}|^{\frac{4}{3}}\,dt}{4}+\sum_{j=2}^{n}\left(\frac{z_j^{12}}{4\varepsilon_{n4j}^4}\right.\\
&\quad\left.\cdot\left|\frac{\partial\alpha_{j-1}}{\partial y}\right|^4+\frac{3\varepsilon_{n4j}^{\frac{4}{3}}}{4}|\tilde{x}|^{\frac{4}{3}}\right)\,dt+\frac{4\theta|P\varphi'(0)\varphi(0)P||\tilde{x}|^2\,dt}{9\sqrt[3]{c}\sqrt[3]{c+\tilde{x}'P\tilde{x}}}+\frac{2\theta|P|^2|\varphi(0)|^4|\Phi|^4y^4\,dt}{9\varepsilon_{n1}^4}\\
&\quad+\frac{2\theta\varepsilon_{n1}^{\frac{4}{3}}|P|^2|\tilde{x}|^2\,dt}{3\sqrt[9]{c^2}\lambda_{\min}^{\frac{1}{3}}(P)\sqrt[3]{c+\tilde{x}'P\tilde{x}}}+\frac{2\theta|P|^2|\Phi(y)|^4y^4\,dt}{9\varepsilon_{n2}^2}+\frac{2\theta\varepsilon_{n2}^2|P|^2|\tilde{x}|^2\,dt}{9\lambda_{\min}(P)\sqrt[3]{c+\tilde{x}'P\tilde{x}}}\\
&\quad+\frac{2\theta\varepsilon_{n3}^2|\tilde{x}|^2\,dt}{9\sqrt[3]{c}\sqrt[3]{c+\tilde{x}'P\tilde{x}}}+\frac{\theta}{2\varepsilon_{n3}^2}\sum_{i=1}^{n}z_i^3\Psi_i\varphi(y)PP\varphi'(y)\left(2\sum_{j=1}^{i-1}\Psi_j'z_j^3+\Psi_i'z_i^3\right)\,dt\\
&\quad+\frac{\theta}{4}\sum_{i=1}^{n}z_i^3\Psi_i\left(2\sum_{j=1}^{i-1}\Psi_j'z_j^3+\Psi_i'z_i^3\right)\,dt
\end{aligned}\tag{8}$$

Now, we design our virtual control laws and the actual control law as follows.

The objective of the choice of the first virtual control law, α_1, is to cancel all terms involving y^3 (or z_1^3) as a factor in the right hand side of (8) except the first line that can be determined without the knowledge of the virtual control laws $\alpha_2, \ldots, \alpha_{n-1}$. This leads to

$$\begin{aligned}\alpha_1(y) &= -q(y)y - \beta_1 z_1 - \frac{2}{3\sqrt[3]{c}}\left(\frac{\mathrm{Tr}^4(\Phi P\varphi'(0))y}{2\varepsilon_{n5}^4} + \frac{\mathrm{Tr}^2(\Phi P\Phi)y}{2\varepsilon_{n5}^2}\right) - F_1 - \frac{3}{4\varepsilon_{n71}^2}z_1|\Psi_1|^4 - \frac{3}{4}z_1 \\ &\quad -\frac{z_1^9}{4\varepsilon_{n41}^4} - \frac{2\theta|P|^2|\varphi(0)|^4|\Phi|^4 z_1}{9\varepsilon_{n1}^4} - \frac{2\theta|P|^2|\Phi|^4 z_1}{9\varepsilon_{n2}^2} - \frac{\theta}{2\varepsilon_{n3}^2}\Psi_1\varphi P P\varphi'\Psi_1' z_1^3 - \frac{\theta}{4}\Psi_1\Psi_1' z_1^3\end{aligned}$$

At step i, $2 \le i \le n-1$, we design the following function for the virtual control law α_i, which cancels all terms involving z_i^3 as a factor in the right hand side of (8) except the first line that does not depend on the choice of virtual control laws to be determined at later steps.

$$\begin{aligned}\bar{\alpha}_i &= -\beta_i z_i - F_i - \frac{3}{4\varepsilon_{n7i}^2}z_i|\Psi_i|^4 - z_i - \frac{z_i^9}{4\varepsilon_{n4i}^4}\left|\frac{\partial\alpha_{i-1}}{\partial y}\right|^4 \\ &\quad -\frac{\theta}{2\varepsilon_{n3}^2}\Psi_i\varphi(y)PP\varphi'(y)\left(2\sum_{j=1}^{i-1}\Psi_j' z_j^3 + \Psi_i' z_i^3\right) - \frac{\theta}{4}\Psi_i\left(2\sum_{j=1}^{i-1}\Psi_j' z_j^3 + \Psi_i' z_i^3\right)\end{aligned}$$

Unlike the first step, this choice of virtual control law does not guarantee the equilibrium condition at the origin of the closed-loop system. To establish the equilibrium condition, as in [21], we design the virtual control α_i as, $\alpha_i(\bar{\hat{x}}_i, y) = \bar{\alpha}_i(\bar{\hat{x}}_i, y) - \bar{\alpha}_i(0,0)$. This choice of the virtual control law cancels the desired terms except a remainder term $z_i^3\bar{\alpha}_i(0,0)$, which can be dominated by $z_i^3\bar{\alpha}_i(0,0) \le z_i^4/(4\varepsilon_{n6i}^4) + 3\varepsilon_{n6i}^{\frac{4}{3}}|\bar{\alpha}_i(0,0)|^{\frac{4}{3}}/4$, where ε_{n6i} is a positive constant to be determined.

At step n, we design the following function for the actual control law u, which cancels all terms involving z_n^3 as a factor in the right hand side of (8) except the first line.

$$\begin{aligned}\bar{u} &= -\beta_n z_n - F_n - \frac{3}{4\varepsilon_{n7n}^2}z_n|\Psi_n|^4 - \frac{1}{4}z_n - \frac{z_n^9}{4\varepsilon_{n4n}^4}\left|\frac{\partial\alpha_{n-1}}{\partial y}\right|^4 \\ &\quad -\frac{\theta}{2\varepsilon_{n3}^2}\Psi_n\varphi(y)PP\varphi'(y)\left(2\sum_{j=1}^{n-1}\Psi_j' z_j^3 + \Psi_n' z_n^3\right) - \frac{\theta}{4}\Psi_n\left(2\sum_{j=1}^{n-1}\Psi_j' z_j^3 + \Psi_n' z_n^3\right)\end{aligned}$$

Again, this choice of the control law does not guarantee the equilibrium condition at the origin of the closed-loop system. To establish the equilibrium condition, as in [21], we design the controller as follows

$$u(\bar{\hat{x}}_n, y) = \bar{u}(\bar{\hat{x}}_n, 0) - \bar{u}(0,0) \tag{9}$$

This choice of the control law cancels the desired terms except a remainder term $z_n^3\bar{u}(0,0)$, which can be dominated by $z_n^3\bar{u}(0,0) \le z_n^4/(4\varepsilon_{n6n}^4) + 3\varepsilon_{n6n}^{\frac{4}{3}}|\bar{u}(0,0)|^{\frac{4}{3}}/4$, where ε_{n6n} is a positive constant to be determined.

With the above defined virtual control laws and control law, we have $dV \le \sigma_n\,dw - \frac{\theta}{4}\sigma_n\sigma_n'\,dt - Q(x, \bar{\hat{x}}_n)\,dt + R\,dt$, where $Q(x,\bar{\hat{x}}_n) = q(y)y^4 + l(x,\bar{\hat{x}}_n)$, $l(x,\bar{\hat{x}}_n) = \delta|\tilde{x}|^2/\sqrt[3]{c + \tilde{x}'P\tilde{x}} + \sum_{i=1}^{n}\delta_i z_i^4$,

$\delta_1 = \beta_1$, $\delta_i = \beta_i - \frac{1}{4\varepsilon_{n6i}^4}$, $2 \le i \le n$,

$$
\begin{aligned}
R &= \frac{2}{3\sqrt[3]{c}}\left(\mathrm{Tr}(\varphi(0)P\varphi'(0)) + \frac{3\varepsilon_{n5}^{\frac{4}{3}}}{2} + \frac{\varepsilon_{n5}^2}{2}\right) + \frac{3}{4}\sum_{j=1}^{n}\varepsilon_{n7j}^2 + \lambda_{\max}^{\frac{1}{3}}(P)\sum_{j=1}^{n}\left(\frac{3\varepsilon_{n4j}^{\frac{4}{3}}}{4}\right)M(c,P) \\
&\quad + \sum_{j=2}^{n-1}\frac{3\varepsilon_{n6j}^{\frac{4}{3}}|\bar{\alpha}_j(0,0)|^{\frac{4}{3}}}{4} + \frac{3\varepsilon_{n6n}^{\frac{4}{3}}|\bar{u}(0,0)|^{\frac{4}{3}}}{4} \\
\delta &= \frac{2}{3} - \lambda_{\max}^{\frac{1}{3}}(P)\sum_{j=1}^{n}\left(\frac{3\varepsilon_{n4j}^{\frac{4}{3}}}{4}\right) - \frac{4\theta|P\varphi'(0)\varphi(0)P|}{9\sqrt[3]{c}} - \frac{2\theta\varepsilon_{n1}^{\frac{4}{3}}|P|^2}{3\sqrt[9]{c^2}\lambda_{\min}^{\frac{1}{3}}(P)} - \frac{2\theta\varepsilon_{n2}^2|P|^2}{9\lambda_{\min}(P)} - \frac{2\theta\varepsilon_{n3}^2}{9\sqrt[3]{c}}
\end{aligned}
$$

To achieve the design goal, we need only to choose c sufficiently large, and ε_{n1}, ε_{n2}, ε_{n3}, ε_{n41}, $\ldots$, ε_{n4n}, ε_{n5}, ε_{n62}, $\ldots$, ε_{n6n}, ε_{n71}, $\ldots$, ε_{n7n} sufficiently small, such that

$$\delta > 0 \qquad \text{and} \qquad R \le R_l \tag{10a}$$

and choose $\beta_1, \ldots, \beta_n$ sufficiently large, such that

$$\delta_1 > 0, \quad \cdots, \quad \delta_n > 0 \tag{10b}$$

By Theorem 1, we obtain the following theorem.

Theorem 2 *Consider the stochastic nonlinear systems (1) with the risk-sensitive cost function (2). Let $\theta > 0$ and $R_l > 0$ be given, and design the output-feedback controller as in (6) and (9) with constants c, ε_{n1}, ε_{n2}, ε_{n3}, ε_{n41}, $\ldots$, ε_{n4n}, ε_{n5}, ε_{n62}, $\ldots$, ε_{n6n}, ε_{n71}, $\ldots$, ε_{n7n}, β_1, $\ldots$, β_n chosen such that (10) is satisfied. Then,*

1. *there exists an a.s. $\mathcal{P}$ unique solution to the closed-loop system;*
2. *the controller achieves a guaranteed risk-sensitive cost R_l;*
3. *the closed-loop system is bounded in probability.*

Proof: From the design procedure discussed preceding the theorem, the statements 1 and 2 follows.

Obviously, there exist positive constants c_1 and c_2, such that, $V_n \le -c_1 Q + c_2$, $\forall (x, \bar{\hat{x}}_n) \in \mathbb{R}^n \times \mathbb{R}^n$. Then, the closed-loop system is bounded in probability.

Remark 2 *Because of the form of the chosen value function, we can not guarantee a zero risk-sensitive cost even if the disturbance vector field vanishes at the origin. Therefore, asymptotic stability in the large can not be achieved for this class of systems with the additional assumption that the disturbance vector field vanishes at $y = 0$ under this design process.*

4 Conclusion

In this paper, we considered the output-feedback control design for strict feedback stochastic nonlinear systems under a risk-sensitive cost criterion. Using integrator backstepping approach, a controller has been designed that can guarantee arbitrary small long-term average cost for arbitrary risk-sensitivity parameter. The controller also preserves the equilibrium at origin for the original system. Furthermore, the controller is shown to achieve bounded in probability for the closed-loop system. Due to our particular choice of the form of the value function, the controller can not guarantee asymptotic stability in the large for the closed-loop system even if the disturbance vector field vanishes at the origin. For future research direction, we are currently investigating the problem of output feedback control design of stochastic nonlinear systems with stable zero dynamics.

References

[1] I. Kanellakopoulos, P. V. Kokotović, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1241–1253, 1991.

[2] A. Isidori, *Nonlinear Control Systems.* London: Springer-Verlag, 3rd ed., 1995.

[3] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, "Nonlinear design of adaptive controllers for linear systems," *IEEE Transactions on Automatic Control*, vol. 39, pp. 738–752, April 1994.

[4] D. Seto, A. M. Annaswamy, and J. Baillieul, "Adaptive control of nonlinear systems with triangular structure," *IEEE Transactions on Automatic Control*, vol. 39, pp. 1411–1428, July 1994.

[5] S. Jain and F. Khorrami, "Decentralized adaptive control of a class of large-scale interconnected nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 42, no. 2, pp. 136–154, 1997.

[6] Z. P. Jiang and J.-B. Pomet, "Backstepping-based adaptive controllers for uncertain nonholonomic systems," in *Proceedings of the 34th IEEE Conference on Decision and Control*, (New Orleans, LA), pp. 1573–1578, 1995.

[7] Z. Pan and T. Başar, "Adaptive controller design for tracking and disturbance attenuation in parametric-strict-feedback nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 43, pp. 1066–1083, August 1998.

[8] Z.-P. Jiang and H. Nijmeijer, "A recursive technique for tracking control of nonholonomic systems in chained form," *IEEE Transactions on Automatic Control*, vol. 44, pp. 265–279, February 1999.

[9] K. Ezal, Z. Pan, and P. V. Kokotović, "Locally optimal and robust backstepping design," *IEEE Transactions on Automatic Control*, vol. 45, pp. 260–271, February 2000.

[10] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design.* New York, NY: Wiley, 1995.

[11] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Boston, MA: Birkhäuser, 2nd ed., 1995.

[12] D. H. Jacobson, "Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games," *IEEE Transactions on Automatic Control*, vol. 18, pp. 167–172, 1973.

[13] M. R. James, J. Baras, and R. J. Elliott, "Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 39, pp. 780–792, 1994.

[14] T. Runolfsson, "The equivalence between infinite horizon control of stochastic systems with exponential-of-integral performance index and stochastic differential games," *IEEE Transactions on Automatic Control*, vol. 39, pp. 1551–1563, 1994.

[15] P. Whittle, *Risk-Sensitive Optimal Control*. Chichester, NY: John Wiley and Sons, 1990.

[16] W. H. Fleming and W. M. McEneaney, "Risk-sensitive control on an infinite time horizon," *SIAM J. Control and Optimization*, vol. 33, pp. 1881–1915, 1995.

[17] W. H. Fleming and W. M. McEneaney, "Risk sensitive control with ergodic cost criteria," in *Proceedings of the 31st Conference on Decision and Control*, (Tucson, AZ), pp. 2048–2052, 1992.

[18] H. Nagai, "Bellman equations of risk-sensitive control," *SIAM J. Control and Optimization*, vol. 34, no. 1, pp. 74–101, 1996.

[19] Z. Pan and T. Başar, "Model simplification and optimal control of stochastic singularly perturbed systems under exponentiated quadratic cost," *SIAM J. Control and Optimization*, vol. 34, pp. 1734–1766, September 1996.

[20] P. Florchinger, "Lyapunov-like techniques for stochastic stability," *SIAM Journal on Control and Optimization*, vol. 33, pp. 1151–1169, July 1995.

[21] Z. Pan and T. Başar, "Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost," *SIAM J. Control and Optimization*, vol. 37, no. 3, pp. 957–995, 1999.

[22] H. Deng and M. Krstić, "Output-feedback stochastic nonlinear stabilization," *IEEE Transactions on Automatic Control*, vol. 44, pp. 328–333, February 1999.