Differential Geometric Condition for Feedback Complete Linearization of Stochastic Nonlinear System *†

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Abstract

In this paper, we study the problem of feedback complete linearization for a given single input single output stochastic nonlinear system. Using the invariance under transformation rule [1], the coordinate free necessary and sufficient conditions for the solvability of the problem is derived, which is exactly the same as the necessary and sufficient conditions for the solvability of the feedback complete linearization problem for an associated deterministic uncertain nonlinear system.
Keywords: stochastic differential equation, feedback complete linearization, differential geometry

1 Introduction

Globally stabilizing control design for nonlinear systems has been an intense area of research in recent years. A class of nonlinear systems that have attracted particular interest are those that can be transformed into linear time-invariant systems under a state diffeomorphism and state feedback — to so-called feedback linearizable nonlinear systems [2]. With the introduction of the integrator backstepping design methodology in the early 90's [3] for this class of nonlinear systems, many results have been obtained in the area of nonlinear control systems [4], [5], [6], [7]. See the recent book [8] for an up-to-date coverage of this topic, with an extensive list of references.

Risk sensitive control for stochastic systems has been an area of intense research in the last 20 years, [9], [10], [11], [12], [13]. In recent papers [14], [15], investigation of the stabilization of specially structured stochastic nonlinear systems has been initiated, which is the generalization of the integrator backstepping methodology to stochastic systems. The remaining question of when a given stochastic nonlinear system is diffeomorphic to a specially structured one is solved in [1]. In that paper, an association map is introduced, which maps a stochastic nonlinear system to a deterministic uncertain nonlinear system. This association map is shown to be natural with respect to state diffeomorphisms for the stochastic nonlinear system and the deterministic uncertain nonlinear system. This fact is known as the invariance under transformation rule. In that paper,

*Paper completed on October 14, 1999. Submitted to *System and Control Letters*.
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coordinate free necessary and sufficient conditions for a given stochastic nonlinear system to be diffeomorphic to various canonical forms are obtained using the invariance under transformation rule.

In this paper, we again apply the invariance under transformation rule for the study of geometric conditions under which a given single input and single output stochastic nonlinear system is feedback equivalent to a linear and controllable one. We have shown that the solvability of the problem is equivalent to the solvability of the problem that the associated deterministic uncertain nonlinear system is feedback equivalent to a linear and controllable one. Then, the coordinate free necessary and sufficient conditions for the associated deterministic uncertain nonlinear system to be feedback equivalent to a linear and controllable system is obtained, which completes the solution to the original problem. A numerical example has been included that illustrates the theoretical findings.

The balance of the paper is organized as follows. In the next section, we introduce the problem and present the solution to the problem. In Section 3, we give a numerical example to illustrate the theory developed. The paper ends with the concluding remarks of Section 4.

2 Main Result

We consider the following stochastic differential equation:

$$dx = (a(x) + b(x)u)\,dt + g(x)d\beta_t \tag{1a}$$
$$dy = c(x) \tag{1b}$$

Here, x is the n-dimensional state vector, with initial state being $x(0)$; u is the scalar control input; $\beta_t := (\beta_t^1, \ldots, \beta_t^s)'$ is the s-dimensional vector-valued Brownian motion; and y is the scalar measurement process. The functions a, b, g, and c are assumed to be smooth functions of appropriate dimension. The underlying probability space is the triple (- , $\mathcal{F}$, $\boldsymbol{P}$), where - is the sample space of continuous functions, $\mathcal{F}$ is a filtration adapted to the Brownian motion β_t, and $\boldsymbol{P}$ is the reference probability measure on - .

For ease of ensuing discussion, we will denote the column vectors of the matrix-valued functions g as

$$g(x) = \begin{bmatrix} g_1(x) & \cdots & g_s(x) \end{bmatrix}$$

Throughout this paper, we will consider only smooth objects, and make use of the following notation. We will regard n-dimensional column vector-valued functions, such as a, b, g_i's as vector fields on $\mathbb{R}^n$. The coordinate mapping Φ is also regarded as a diffeomorphism between smooth manifolds $\mathbb{R}^n$ and $\mathbb{R}^n$. For any smooth function λ, we will denote the Lie derivative of λ along a vector field f by $L_f\lambda := \frac{\partial \lambda}{\partial x} f$. The repeated Lie derivatives will be denoted $L_f^k\lambda$, with $L_f^0\lambda = \lambda$. For any two vector fields f and g, we will denote its Lie bracket by $[f, g]$, which is also denoted by $\mathrm{ad}_f g$. The repeated Lie brackets will be denoted by $\mathrm{ad}_f^k g$, with $\mathrm{ad}_f^0 g = g$. In order to avoid any notational confusion, we will denote the deterministic differential of a function λ by $\delta\lambda$, to distinguish it from the stochastic differential operator d.

We are interested in finding a state feedback of the form

$$u = \alpha(x) + \kappa(x)v \tag{2}$$

and a local state diffeomorphism $z = \Phi(x)$ around a point $x_0 \in \mathbb{R}^n$, with $\kappa(x_0) \neq 0$, such that, in the z coordinates, the system is completely linear and controllable from the control input v, that is,

$$dz = (Az + Bv)\,dt + Gd\beta_t \tag{3a}$$
$$y = Cz \tag{3b}$$

where A, B, G, and C are constant matrices of appropriate dimensions and the pair (A, B) is controllable. This problem will be called the *feedback complete linearization problem* for the stochastic nonlinear system (1).

As defined in [1], the associated uncertain nonlinear system of (1) is given by

$$\dot{\eta} = f(\eta) + b(\eta)u + g(\eta)w \tag{4a}$$
$$\xi = c(\eta) \tag{4b}$$

where

$$f(\eta) = a(\eta) - \frac{1}{2}\sum_{i=1}^{s} \frac{\partial g_i}{\partial \eta}(\eta) g_i(\eta)$$

The *feedback complete linearization problem* for this uncertain nonlinear system is finding a feedback of the form

$$u = \grave{\alpha}(\eta) + \grave{\kappa}(\eta)v \tag{5}$$

and a local coordinate diffeomorphism $\lambda = \grave{\Phi}(\eta)$ around the point $x_0 \in \mathbb{R}^n$, with $\grave{\kappa}(x_0) \neq 0$, such that, in the λ coordinates, the system is described by

$$\dot{\lambda} = \grave{A}\lambda + \grave{B}v + \grave{G}w \tag{6a}$$
$$y = \grave{C}\lambda \tag{6b}$$

where $\grave{A}$, $\grave{B}$, $\grave{G}$, and $\grave{C}$ are constant matrices of appropriate dimension and the pair $(\grave{A}, \grave{B})$ is controllable.

We have the following theorem stating the equivalence between the solvability of the feedback complete linearization problems for the stochastic nonlinear system and the associated deterministic uncertain nonlinear system.

Theorem 1 *The feedback complete linearization problem for the stochastic nonlinear system (1) is solvable around x_0 with feedback law $u = \alpha(x) + \kappa(x)v$ and local state diffeomorphism $z = \Phi(x)$ if and only if the same feedback law $u = \alpha(\eta) + \kappa(\eta)v$ and the same local state diffeomorphism $\lambda = \Phi(\eta)$ solves the feedback complete linearization problem for the associated deterministic uncertain nonlinear system (4) around x_0.*

Proof We first prove the "sufficiency" part of the theorem. Assume that the feedback complete linearization problem for the associated deterministic uncertain nonlinear system (4) is solved around x_0 with $u = \alpha(\eta) + \kappa(\eta)v$ and $\lambda = \Phi(\eta)$. Under the feedback law $u = \alpha(x) + \kappa(x)v$, the stochastic nonlinear system is given by

$$dx = (a(x) + b(x)\alpha(x) + b(x)\kappa(x)v)\,dt + g(x)\,d\beta_t \tag{7a}$$
$$y = c(x) \tag{7b}$$

The associated deterministic uncertain nonlinear system for (7) is

$$\dot{\eta} = f(\eta) + b(\eta)\alpha(\eta) + b(\eta)\kappa(\eta)v + g(\eta)w \tag{8a}$$
$$\xi = c(\eta) \tag{8b}$$

which is the system (4) under the feedback $u = \alpha(\eta) + \kappa(\eta)v$. It is known that, under the local diffeomorphism $\lambda = \Phi(\eta)$, the system dynamics (8) is described by (6). Then, under the local diffeomorphism $z = \Phi(x)$, the stochastic nonlinear system (7) is again associated with the deterministic uncertain nonlinear system (6), and is given by

$$dz = (\grave{A}z + \grave{B}v)\,dt + \grave{G}d\beta_t \tag{9a}$$
$$y = \grave{C}z \tag{9b}$$

around x_0.

This completes the proof of the sufficiency.

The necessity of the theorem can be proved along a line that is similar to the above argument, and is therefore omitted. This completes the proof of the theorem. □

Next, we derive coordinate-free conditions for the solvability of the feedback complete linearization problem for the stochastic nonlinear system (1). The Theorem 1 suggests that we need only to derive conditions for the associated deterministic system (4).

Corollary 1 *Consider the stochastic nonlinear system (1). For a fixed $x_0 \in \mathbb{R}^n$, assume $f(x_0) = 0$, $h(x_0) = 0$ and $h \not\equiv 0$. The feedback complete linearization problem for the stochastic nonlinear system around x_0 is solvable, if and only if, all of the following four conditions hold,*

1. *there exists an integer r, $0 < r \leq n$, such that $L_b c = L_b L_f c = \cdots = L_b L_f^{r-2} c = 0 \quad L_b L_f^{r-1} c \neq 0$ on an open neighborhood of x_0;*
2. *The matrix $\left[\begin{array}{cccc} b(x_0) & \mathrm{ad}_f b(x_0) & \cdots & \mathrm{ad}_f^{n-1} b(x_0) \end{array}\right]$ has rank n;*
3. *the vector fields $\check{f}(x) := f(x) + b(x)\check{\alpha}(x)$ and $\check{b}(x) := b(x)\check{\kappa}(x)$, with*

   $$\check{\alpha}(x) = -\frac{L_f^r c(x)}{L_b L_f^{r-1} c(x)} \qquad \check{\kappa}(x) = \frac{1}{L_b L_f^{r-1} c(x)}$$

   are such that $\left[\mathrm{ad}_{\check{f}}^i \check{b}, \mathrm{ad}_{\check{f}}^j \check{b}\right](x) = 0$ for all $0 \leq i, j \leq n$, on an open neighborhood of x_0.
4. *$\left[g_i, \mathrm{ad}_{\check{f}}^j \check{b}\right](x) = 0$ for all $1 \leq i \leq s$ and $0 \leq j \leq n-1$, on an open neighborhood of x_0.*

Remark 1 *The first condition is equivalent to the assumption that the associated deterministic uncertain nonlinear system (4) has relative degree r with respect to the control input u.* ◇

Proof We first prove the "necessity." Let $u = \alpha(x) + \kappa(x)v$ be the desired feedback and $z = \Phi(x)$ be the desired state diffeomorphism that solves the feedback complete linearization problem for the stochastic nonlinear system (1). By Theorem 1, the same feedback and coordinate transformation solves the feedback complete linearization problem for the deterministic uncertain nonlinear system (4). In the $\lambda = \Phi(\eta)$ coordinate, the vector field f, b, g_1, …, g_s, and the function c are given by

$$\begin{aligned}
\grave{f}(\lambda) &= \grave{A}\lambda - \grave{B}\left.\frac{\alpha}{\kappa}\right|_{\eta=\Phi^{-1}(\lambda)} \\
\grave{b}(\lambda) &= \frac{\grave{B}}{\kappa(\Phi^{-1}(\lambda))} \\
\grave{g}_i(\lambda) &= \grave{G}_i \qquad i = 1, \ldots, s \\
\grave{c}(\lambda) &= \grave{C}\lambda
\end{aligned}$$

where

$$\grave{G} = \left[\ \grave{G}_1 \ \cdots \ \grave{G}_s\ \right]$$

Then, there exists an integer r, $0 < r \leq n$, such that

$$\grave{C}\grave{A}^i\grave{B} = 0 \qquad i = 0, 1, \ldots, r-2 \qquad \grave{C}\grave{A}^{r-1}\grave{B} \neq 0$$

It is easy to verify that

$$\begin{aligned}
&L^i_{\grave{f}}\grave{c} = \grave{C}\grave{A}^i\lambda \qquad i = 0, 1, \ldots, r-1 \\
&L_{\grave{b}}L^i_{\grave{f}}\grave{c} = 0 \qquad i = 0, 1, \ldots, r-2 \qquad L_{\grave{b}}L^{r-1}_{\grave{f}}\grave{c} = \frac{\grave{C}\grave{A}^{r-1}\grave{B}}{\kappa(\Phi^{-1}(\lambda))}
\end{aligned}$$

This proves the first statement.

The second and third statement are then necessary by Theorem 4.8.3 of [2].

It is easy to verify that

$$L^r_{\grave{f}}\grave{c} = \grave{C}\grave{A}^r\lambda - \grave{C}\grave{A}^{r-1}\grave{B}\left.\frac{\alpha}{\kappa}\right|_{\eta=\Phi^{-1}(\lambda)}$$

The vector fields $\check{f}$ and $\check{b}$ can be expressed in the λ coordinate as

$$\begin{aligned}
\grave{\check{f}} &= \left(\grave{A} - \frac{\grave{B}\grave{C}\grave{A}^r}{\grave{C}\grave{A}^{r-1}\grave{B}}\right)\lambda \\
\grave{\check{b}} &= \frac{1}{\grave{C}\grave{A}^{r-1}\grave{B}}\grave{B}
\end{aligned}$$

This implies that

$$\mathrm{ad}^j_{\grave{\check{f}}}\grave{\check{b}} = \left(\grave{A} - \frac{\grave{B}\grave{C}\grave{A}^r}{\grave{C}\grave{A}^{r-1}\grave{B}}\right)^j \frac{\grave{B}}{\grave{C}\grave{A}^{r-1}\grave{B}} \qquad j = 0, \ldots, n-1$$

Therefore, in the λ coordinates, we have

$$\left[\grave{G}_i, \mathrm{ad}^j_{\grave{\check{f}}}\grave{\check{b}}\right] = 0 \qquad 1 \leq i \leq s \quad 0 \leq j \leq n-1$$

This proves the fourth statement and completes the proof for necessity.

Next, we show "sufficiency" part of the theorem. Assume statements 1 – 4 holds. By Theorem 1, we only need to show that the feedback complete linearization problem for the deterministic uncertain nonlinear system (4) is solvable. By Theorem 4.8.3 of [2], there exists a feedback $u = \alpha(\eta) + \kappa(\eta)v$ and a state diffeomorphism $\lambda = \Phi(\eta)$, defined locally around x_0, transforming the system (4) without the disturbance w into a linear and controllable system and $0 = \Phi(x_0)$

$$\begin{aligned} \dot{\lambda} &= \grave{A}\lambda + \grave{B}v \\ \xi &= \grave{C}\lambda \end{aligned}$$

That is, in the λ coordinates, the vector fields f, b, and the function c are given by

$$\begin{aligned} \grave{f}(\lambda) &= \grave{A}\lambda - \grave{B}\left.\frac{\alpha}{\kappa}\right|_{\eta=\Phi^{-1}(\lambda)} \\ \grave{b}(\lambda) &= \frac{\grave{B}}{\kappa(\Phi^{-1}(\lambda))} \\ \grave{c}(\lambda) &= \grave{C}\lambda \end{aligned}$$

Assume that the vector fields $g_1(\eta)$, …, $g_s(\eta)$ are expressed by, in the λ coordinate, $\grave{g}_1(\lambda)$, …, $\grave{g}_s(\lambda)$. Then, we obtain

$$\begin{aligned} & L_{\grave{f}}^i \grave{c} = \grave{C}\grave{A}^i\lambda \qquad i = 0, 1, \ldots, r-1 \\ & L_{\grave{b}} L_{\grave{f}}^i \grave{c} = 0 \qquad i = 0, 1, \ldots, r-2 \qquad L_{\grave{b}} L_{\grave{f}}^{r-1} \grave{c} = \frac{\grave{C}\grave{A}^{r-1}\grave{B}}{\kappa(\Phi^{-1}(\lambda))} \neq 0 \\ & L_{\grave{f}}^r \grave{c} = \grave{C}\grave{A}^r\lambda - \grave{C}\grave{A}^{r-1}\grave{B}\left.\frac{\alpha}{\kappa}\right|_{\eta=\Phi^{-1}(\lambda)} \end{aligned}$$

The vector fields $\check{f}$ and $\check{b}$ can be expressed in the λ coordinate as

$$\begin{aligned} \grave{\check{f}} &= \left(\grave{A} - \frac{\grave{B}\grave{C}\grave{A}^r}{\grave{C}\grave{A}^{r-1}\grave{B}}\right)\lambda \\ \grave{\check{b}} &= \frac{1}{\grave{C}\grave{A}^{r-1}\grave{B}}\grave{B} \end{aligned}$$

This implies that

$$\mathrm{ad}_{\grave{\check{f}}}^j \grave{\check{b}} = \left(\grave{A} - \frac{\grave{B}\grave{C}\grave{A}^r}{\grave{C}\grave{A}^{r-1}\grave{B}}\right)^j \frac{\grave{B}}{\grave{C}\grave{A}^{r-1}\grave{B}} \qquad j = 0, \ldots, n-1$$

Then, the fourth condition implies that

$$\left[\grave{g}_i, \mathrm{ad}_{\grave{\check{f}}}^j \grave{\check{b}}\right] = -\frac{\partial \grave{g}_i}{\partial \lambda}\left(\grave{A} - \frac{\grave{B}\grave{C}\grave{A}^r}{\grave{C}\grave{A}^{r-1}\grave{B}}\right)^j \frac{\grave{B}}{\grave{C}\grave{A}^{r-1}\grave{B}} = 0 \qquad i = 1, \ldots, s, \quad j = 0, \ldots, n-1$$

For each $i = 1, \ldots, s$, we have

$$\begin{aligned} 0 &= \frac{\partial \grave{g}_i}{\partial \lambda}\left[\begin{array}{ccc} \frac{\grave{B}}{\grave{C}\grave{A}^{r-1}\grave{B}} & \cdots & \left(\grave{A} - \frac{\grave{B}\grave{C}\grave{A}^r}{\grave{C}\grave{A}^{r-1}\grave{B}}\right)^{n-1} \frac{\grave{B}}{\grave{C}\grave{A}^{r-1}\grave{B}} \end{array}\right] \\ &= \frac{\partial \grave{g}_i}{\partial \lambda}\frac{1}{\grave{C}\grave{A}^{r-1}\grave{B}}\left[\begin{array}{ccc} \grave{B} & \cdots & \grave{A}^{n-1}\grave{B} \end{array}\right]\left[\begin{array}{cccc} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array}\right] \end{aligned}$$

where $*$ denotes constants of no interest. Since the pair $(\grave{A}, \grave{B})$ is controllable, then the matrix

$$\begin{bmatrix} \grave{B} & \cdots & \grave{A}^{n-1}\grave{B} \end{bmatrix}$$

is nonsingular. Then, we have $\frac{\partial \grave{g}_i}{\partial \lambda} = 0$, or equivalently, $\grave{g}_i$ is constant, denoted by $\grave{G}_i$. This proves that the feedback law $u = \alpha(\eta) + \kappa(\eta)v$ and the diffeomorphism $\lambda = \Phi(\eta)$ such that, in the λ coordinates, the system (4) is described by (6). This completes the proof of the sufficiency. □

3 Example

In this section, we will illustrate the theoretical findings of this paper by a numerical example. Consider the following stochastic nonlinear system,

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1^2 \\ 1 - 2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 + x_1^2 \end{bmatrix} u\,dt + \begin{bmatrix} 1 \\ 1 + 2x_1 \end{bmatrix} d\beta_t \quad (10)$$

$$y = x_1 - x_1^2 + x_2 \quad (11)$$

Thus, the vector fields a, b, g, and the function c are given by

$$a(x) = \begin{bmatrix} x_2 - x_1^2 \\ 1 - 2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} \qquad b(x) = \begin{bmatrix} 0 \\ 1 + x_1^2 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 1 \\ 1 + 2x_1 \end{bmatrix}$$

$$c(x) = x_1 - x_1^2 + x_2$$

The vector field f for the associated deterministic uncertain nonlinear system is given by

$$\begin{aligned} f(x) &= \begin{bmatrix} x_2 - x_1^2 \\ 1 - 2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 + 2x_1 \end{bmatrix} \\ &= \begin{bmatrix} x_2 - x_1^2 \\ -2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} \end{aligned}$$

Consider the point $x_0 = (0, 0)'$, with $f(x_0) = 0$ and $c(x_0) = 0$. Then,

$$L_b c = \begin{bmatrix} 1 - 2x_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 + x_1^2 \end{bmatrix} = 1 + x_1^2 \neq 0$$

This implies that the first condition of Corollary 1 holds with $r = 1$.

Next, we have

$$\begin{aligned} \mathrm{ad}_f b &= \begin{bmatrix} 0 & 0 \\ 2x_1 & 0 \end{bmatrix} \begin{bmatrix} x_2 - x_1^2 \\ 1 - 2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} -2x_1 & 1 \\ -6x_1^2 + 2x_2 - 2x_1x_2 & -1 + 2x_1 - x_1^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 + x_1^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 - x_1^2 \\ 1 - 2x_1 + 2x_1^2 - 4x_1^3 + x_1^4 + 2x_1x_2 \end{bmatrix} \end{aligned}$$

Then, the matrix

$$\begin{bmatrix} b & \mathrm{ad}_f b \end{bmatrix}(x_0) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

has rank 2. This shows that the second condition of Corollary 1 holds.

It is easy to compute that

$$\begin{aligned} L_f c &= \begin{bmatrix} 1-2x_1 & 1 \end{bmatrix} \begin{bmatrix} x_2 - x_1^2 \\ -2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} = -x_1^2 - x_1^2 x_2 \\ \check{f} &= f - b\frac{L_f c}{L_b c} = \begin{bmatrix} x_2 - x_1^2 \\ x_1^2 - 2x_1^3 - x_2 + 2x_1x_2 \end{bmatrix} \\ \check{b} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{ad}_{\check{f}}\check{b} &= -\begin{bmatrix} -2x_1 & 1 \\ 2x_1 - 6x_1^2 + 2x_2 & -1+2x_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1-2x_1 \end{bmatrix} \\ \mathrm{ad}^2_{\check{f}}\check{b} &= \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_2 - x_1^2 \\ -2x_1^3 - x_2 + 2x_1x_2 - x_1^2x_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} -2x_1 & 1 \\ 2x_1 - 6x_1^2 + 2x_2 & -1+2x_1 \end{bmatrix} \begin{bmatrix} -1 \\ 1-2x_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1-2x_1 \end{bmatrix} \end{aligned}$$

It is easy to check that $\left[\mathrm{ad}^i_{\check{f}}\check{b}, \mathrm{ad}^j_{\check{f}}\check{b}\right] = 0,\ 0 \le i, j \le 2$. This shows that the third condition of Corollary 1 holds.

$$\begin{aligned} [g, \check{b}] &= -\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left[g, \mathrm{ad}_{\check{f}}\check{b}\right] &= \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1+2x_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1-2x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

This shows that the fourth condition of Corollary 1 holds.

To find the desired feedback law and diffeomorphism, we solve the partial differential equation

$$\delta h, \check{b}\rangle = 0$$

An obvious solution is $h(x) = x_1^3$. Then,

$$\left\langle \delta h, \mathrm{ad}_{\check{f}}\check{b} \right\rangle = -3x_1^2 = -3h(x)^{2/3}$$

The ψ function can be solved by

$$\frac{\partial \psi}{\partial \zeta} = -\frac{1}{3\zeta^{2/3}} \qquad \Rightarrow \qquad \psi(\zeta) = -\zeta^{1/3}$$

This yields $\check{h}(x) = -\psi(h) = -x_1$. Hence, the desired diffeomorphism is

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \check{h} \\ L_{\check{f}}\check{h} \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_1^2 - x_2 \end{bmatrix}$$

It can be evaluated

$$L_{\check{b}} L_{\check{f}} \check{h} = -1 \qquad L^2_{\check{f}}\check{h} = -x_1^2 + x_2$$

Then, the desired feedback law is

$$u = -\frac{L_f c}{L_b c} + \frac{1}{L_b c}\left(-\frac{L_{\check{f}}^2 \check{h}}{L_{\check{b}} L_{\check{f}} \check{h}} + \frac{1}{L_{\check{b}} L_{\check{f}} \check{h}} v\right) = x_2 - \frac{1}{1+x_1^2} v$$

In the $z = (z_1, z_2)'$ coordinates, we have

$$\begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v\, dt + \begin{bmatrix} -1 \\ -1 \end{bmatrix} d\beta_t$$

$$y = \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

This corroborates the results of Theorem 1 and Corollary 1.

4 Conclusion

In this paper, we have presented the coordinate free necessary and sufficient conditions for the solvability of the feedback complete linearization problem for single input and single output stochastic nonlinear systems. By utilizing the invariance under transformation rule introduced in [1], it is shown that the solvability of the problem is equivalent to the solvability of the feedback complete linearization problem for the deterministic uncertain nonlinear system that is associated with the stochastic nonlinear system. Then, the coordinate free necessary and sufficient conditions are exactly those for the solvability of the feedback complete linearization problem for the associated deterministic uncertain nonlinear system. A numerical example has been provided to illustrate the theoretical result. Additional canonical forms for stochastic nonlinear systems are currently under study.

Acknowledgement

The author want to express his gratitude for the warm accommodation by the Department of Automatic Control of Shanghai Jiao Tong University during the preparation of this paper.

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