Time-Scale Separation and Robust Controller Design for Uncertain Nonlinear Singularly Perturbed Systems under Perfect State Measurements*

Zigang Pan and **Tamer Başar**

Decision and Control Laboratory
Coordinated Science Laboratory and the
Department of Electrical and Computer Engineering
University of Illinois
Urbana, IL 61801/USA

Address for Correspondence: Prof. Tamer Başar
Coordinated Science Laboratory
University of Illinois
1308 West Main Street
Urbana, IL 61801/USA
Tel: 217/333-3607; Fax: 217/244-1653

*Research supported in part by the U.S. Department of Energy under Grant DE-FG-02-88-ER-13939, and in part by the National Science Foundation under Grant ECS91-13153. An earlier version was presented at the workshop on "Robust Controller Designs and Differential Games" held at the University of California at Santa Barbara, May 28-29, 1993. Revised versions were completed on May 19, 1994 and October 27, 1994. It is under review for publication in the *International J. Robust and Nonlinear Control*.

Abstract

We study time-scale separation and robust controller design for a class of singularly perturbed nonlinear systems under perfect state measurements. The system dynamics are taken to be jointly linear in the fast state variables, control and disturbance inputs, but nonlinear in the slow state variables. Since global time-scale separation may not always be possible for nonlinear singularly perturbed systems, we restrict our attention here to some closed subset of the state space, on which a time-scale separation holds for sufficiently small values of the singular perturbation parameter. We construct a slow controller and a composite controller based on the solutions of particular slow and fast games obtained using time-scale separation. For the class of systems for which the slow controller can be selected to be robust with respect to small regular structural perturbations on the slow subsystem, we show under some growth conditions that the composite controller can achieve any desired level of performance that is larger than the maximum of the performance levels for the slow and fast subsystems. A slow controller, however, is not generally as robust as the composite controller; but, still under some conditions which are delineated in the paper, the fast dynamics can be totally ignored. The paper also presents a numerical example to illustrate the theoretical results.

Key Words—H^∞-optimal control, optimal disturbance attenuation bound, singular perturbation, robust control, nonlinear systems.

1 Introduction

After the completion of a satisfactory theory for linear H^∞-optimal control, using either frequency-domain or state-space methods, attention has been focused more recently on more general classes of problems, where the objective is either to develop a counterpart of the linear theory for nonlinear systems (see, e. g. , [1], [2], [3], [4], [5], [6], [7], [8], [9]) or to study robustness of the H^∞ theory to structural perturbations ([10], [11], [12], [13], [14].) A recent work [14] has addressed both of these issues in the context of a special class of nonlinear systems subject to regular perturbations. The objective of the present paper is to develop a counterpart of these results when the perturbations are singular, leading to a time-scale separation. More specifically, we consider a special class of nonlinear singularly perturbed systems where in the state dynamics the fast state variables, control and disturbance inputs enter linearly. Our objective is two fold: In terms of singular perturbation theory (with the small singular perturbation parameter denoted by ϵ), the objective is to obtain ϵ-independent "approximate" controllers by using a time-scale decomposition procedure, and to study the robustness of these controllers with respect to the performance of the original full-order system. In terms of "nonlinear H^∞ theory", our objective is to study the performance-robustness of H^∞ suboptimal controllers subject to unmodeled fast dynamics.

Robustness of nonlinear H^∞ suboptimal controllers with respect to small regular structural perturbations has been studied earlier in [14]. In that paper, we have characterized the set of allowable structural perturbations in terms of a set of growth conditions on the perturbation functions. The chosen H^∞-controller has then been shown to be robust for sufficiently small perturbations that lie in this set. On the other hand, H^∞-optimal control theory for singularly perturbed linear systems under perfect state measurements has also be investigated thoroughly (see [13]). In that paper, it has been shown that the H^∞-performance level for the full-order system is asymptotically, as $\epsilon \to 0^+$, equal to the maximum of the H^∞-optimal performance levels for the slow and fast subsystems under full-information and perfect state information patterns, respectively. A composite controller, which is formed by adding the H^∞-controllers for the slow and the fast subsystems based on a common asymptotically achievable performance level, achieves the desired performance level for the full-order system for sufficiently small ϵ. A slow controller, however, can achieve a desired level of performance for the full-order system provided that this performance level further exceeds the open-loop performance level for the fast subsystem.

This paper extends the results of [13] to the case when the problem is completely nonlinear in the slow state variables. One natural observation here is that the separation of fast and slow states may not hold globally when the dynamics are nonlinear. Hence, the exact counterpart of the H^∞-control theory for singularly perturbed linear quadratic problems cannot be established in this case. We thus focus our interest to some closed subset $\mathcal{S}$ of the state space. We assume that the cost incurred by any control and disturbance pair is only due to the portion of the time when the state trajectory lies in $\mathcal{S}$. Let γ_s be the optimal performance level for the slow subsystem under full-information measurement, and γ_f be the optimal performance level for fast subsystem under perfect-state measurement. Then, a composite controller and a slow controller can be derived for any desired performance level $\gamma > \max\{\gamma_s, \gamma_f\}$. Under the condition that the slow controller is robust with respect to small regular perturbations for the slow subsystem, we prove that the composite controller achieves the desired level of performance for the full-order system for sufficiently small ϵ, if some growth conditions are satisfied. Let γ_{of} be the optimal open-loop performance level for the fast subsystem. We prove that, assuming again the robustness of the slow controller for the slow subsystem and some growth conditions, the slow controller achieves a desired performance level $\gamma > \max\{\gamma_s, \gamma_{of}\}$ for the full-order system for sufficiently small ϵ. These results parallel those obtained in [13] for linear-quadratic systems.

Design of controllers for nonlinear singularly perturbed systems subject to unknown disturbances has been studied before in the literature, notably in papers [15], [16], [17], where the objective has been to obtain composite controllers that guarantee stability of the overall system. The main approach of the authors in these papers has been the construction of appropriate Lyapunov functions, in terms of which a class of stabilizing controllers has been characterized. However, no optimality properties have been associated with these controllers, which is our main concern in this paper.

The present paper is organized as follows. In the next section (Section 2) we formulate the singularly perturbed H^∞-optimal control problem and the associated zero-sum soft-constrained differential game, under perfect-state measurements; we also provide the solution to the full-order problem. In Section 3, we identify the slow and fast subsystems and the associated differential games. In Section 4, we construct a composite controller, and present two theorems that establish optimality of the composite and the slow controllers, in a particular sense. These results, as well as the various conditions under which they are valid, are then further discussed in Section 5, where we also treat as a special case problems with linearizable system dynamics. Section 6 presents some numerical results to illustrate the theory, and Section 7 provides some concluding remarks and a discussion on further extensions. The paper ends with three Appendices, the first two of which provide some auxiliary results, and the last one gives a list of most common mathematical expressions and definitions introduced in Sections 3 and 4, along with equation reference numbers, for the convenience of the reader.

2 Problem Formulation

We consider a class of systems with distinguishable slow and fast dynamics that are linear in the control and disturbance and the fast state variable, but nonlinear in the slow state variable. Using the framework of singular perturbations, we describe these dynamics as

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & a_{11}(x_1) + A_{12}(x_1)x_2 + B_1(x_1)u + D_1(x_1)w; \quad x_1(0) = 0 \\ \epsilon\dot{x}_2 & = & a_{21}(x_1) + A_{22}(x_1)x_2 + B_2(x_1)u + D_2(x_1)w; \quad x_2(0) = 0 \end{array} \right. \tag{1}$$

where $x = (x_1' \ x_2')'$ is the n-dimensional state vector; x_1 is the n_1-dimensional slow state vector and x_2 is the n_2 dimensional fast state vector, with $n = n_1 + n_2$; u is the control input, and w is the disturbance, each belonging to appropriate ($\mathcal{L}^2$) Hilbert spaces $\mathcal{H}_x$, $\mathcal{H}_u$ and $\mathcal{H}_w$, respectively. The control input u is generated by a perfect-state closed-loop control policy μ, according to

$$u(t) = \mu(x_{[0,t]}) \tag{2}$$

where $\mu : \mathcal{H}_x \to \mathcal{H}_u$ is piecewise continuous in t and locally Lipschitz continuous in x, further satisfying the given causality condition. We denote the class of all these controllers by $\mathcal{M}$.

Note that because of the presence of nonlinearity in the system equation, the separation of the slow and fast state vectors may not be globally uniform for any fixed $\epsilon > 0$. To bring out the full generality of the results, we introduce a closed set $\mathcal{S} \subset \mathcal{R}^n$ to be the set of interest in the state space; in other words, we are interested only in that portion of state trajectories that stay in $\mathcal{S}$. We assume that $\mathcal{S}$ contains the origin as an interior point. With this system, we associate the following performance index that is jointly quadratic in x_2 and u, but generally nonquadratic in x_1:

$$L(u, w) \;\; = \;\; \int_0^{T_e} (q_{11}(x_1) + 2q_{12}(x_1)x_2 + |x_2|^2_{Q_{22}(x_1)} + |u|^2)\, dt \tag{3}$$

Here, T_e denotes the first instance when the state leaves the set $\mathcal{S}$ (T_e is taken to be ∞ if the state never leaves the set $\mathcal{S}$), which is defined precisely as:

$$T_e := \inf\{t \geq 0 : \quad x(t) \in \mathcal{R}^n \setminus \mathcal{S}\}, \tag{4}$$

Let us also introduce the notation $J(\mu, w)$ to denote $L(u, w)$ when u is given by (2).

With respect to this cost function, and for each $\epsilon > 0$, we introduce the optimum disturbance attenuation level $\gamma^*(\epsilon)$ as

$$\gamma^*(\epsilon) = \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{H}_w} \{J(\mu, w)\}^{1/2} / \|w\| \tag{5}$$

One of our objectives in this paper is to study the dependence of γ^* on ϵ, and in particular its continuity at $\epsilon = 0$. Another objective is to obtain for each $\gamma > \gamma^*(0)$, an ϵ-independent controller that will assure a disturbance attenuation level of γ for the full-order system for sufficiently small $\epsilon > 0$.

The framework in which we will study the questions raised above, as well as a number of others to be posed later, is game-theoretic. Toward this end, we first introduce for the full-order system, and for each $\epsilon > 0$, an associated soft-constrained differential game, with dynamics (1) and cost function (parameterized in $\gamma > 0$):

$$L_\gamma(u, w) = L(u, w) - \gamma^2 \|w\|^2 \tag{6}$$

It is by now well-known (see [18]) that $\gamma^*(\epsilon)$ defined by (5) is the "smallest" value of $\gamma \geq 0$ for which the upper value of the differential game is *zero*.

First we present below a (partial) solution to the full-order problem (which, however, will be sufficient for our purposes), under two basic assumptions:

Assumption 1 The matrix functions $a_{i1}(x_1)$, $A_{i2}(x_1)$, $B_i(x_1)$, $D_i(x_1)$, $q_{1i}(x_1)$, $Q_{22}(x_1)$, $i = 1, 2$, are locally Lipschitz continuous in x_1 on $\mathcal{S}$. The matrix functions A_{22}, B_2, D_2 and Q_{22} are further continuously differentiable in x_1 on $\mathcal{S}$.

Assumption 2 For every fixed $(x_1, x_2) \in \mathcal{S}$, the $n_2 \times n_2$ matrix $A_{22}(x_1)$ is nonsingular, the pair $(A_{22}(x_1), Q_{22}(x_1))$ is observable, and the pair $(A_{22}(x_1), B_2(x_1))$ is controllable. Furthermore, the function $q(x) := q_{11}(x_1) + 2q_{12}(x_1)x_2 + |x_2|^2_{Q_{22}(x_1)}$ is nonnegative definite.[1]

Let us further introduce the following notation:

$$
\begin{aligned}
&\mathcal{S}_1 := \{x_1 \in \mathcal{R}^{n_1} : \quad (x_1, x_2) \in \mathcal{S} \text{ for some } x_2 \in \mathcal{R}^{n_2}. \ \}; \quad B(x_1) := \begin{bmatrix} B_1(x_1) \\ B_2(x_1) \end{bmatrix}; \\
&a(x) := \begin{bmatrix} a_{11}(x_1) + A_{12}(x_1)x_2 \\ a_{21}(x_1) + A_{22}(x_1)x_2 \end{bmatrix}; \quad a_\epsilon(x) := \begin{bmatrix} a_{11}(x_1) + A_{12}(x_1)x_2 \\ \frac{1}{\epsilon}(a_{21}(x_1) + A_{22}(x_1)x_2) \end{bmatrix}; \\
&B_\epsilon(x_1) := \begin{bmatrix} B_1(x_1) \\ \frac{1}{\epsilon}B_2(x_1) \end{bmatrix}; \quad D(x_1) := \begin{bmatrix} D_1(x_1) \\ D_2(x_1) \end{bmatrix}; \quad D_\epsilon(x_1) := \begin{bmatrix} D_1(x_1) \\ \frac{1}{\epsilon}D_2(x_1) \end{bmatrix}; \\
&S(x_1; \gamma) := B(x_1)B'(x_1) - \gamma^{-2}D(x_1)D'(x_1); \\
&S_\epsilon(x_1; \gamma) := B_\epsilon(x_1)B'_\epsilon(x_1) - \gamma^{-2}D_\epsilon(x_1)D'_\epsilon(x_1)
\end{aligned}
$$

with the ij-th block of $S(x_1; \gamma)$ denoted henceforth by $S_{ij}(x_1; \gamma)$, $i, j = 1, 2$. For any functional f of a vector x, f_x denotes its gradient with respect to x, which is taken as a row vector.

For simplicity in the derivations involved, we will first assume that the condition $Q_{22}(x_1) > 0$ holds for each fixed $x_1 \in \mathcal{S}_1$. Later in Appendix B, we will show how the results can be extended also to cover the case $Q_{22} \geq 0$.

For each $\epsilon > 0$, the full-order problem is partly solved by the following proposition, which can be proved by an application of the "completion of squares" argument:

Proposition 1 *Consider the control problem (1)–(5), and let Assumptions 1 and 2 hold. For a fixed $\gamma > 0$, if there exists a nonnegative definite function $V(\epsilon; x)$ that is continuously differentiable with respect to x and satisfies the following partial differential inequality on $\mathcal{S}$:*

$$
V_x a_\epsilon - \frac{1}{4} V_x S_\epsilon V_x' + Q \leq 0, \tag{7}
$$

then, the associated soft-constrained zero-sum differential game with cost function (6) has an upper value 0, and a control policy that achieves the upper value is

$$
\mu_\gamma^*(x) = -\frac{1}{2} B_\epsilon' V_x'. \tag{8}
$$

Furthermore, $\gamma^*(\epsilon) \leq \gamma$, and the above controller attains the disturbance attenuation level γ for the full-order system.

[1] Throughout this paper, a function $f(x)$ is said to be nonnegative definite if $f(x) \geq 0$ $\forall x \in \mathcal{S}$ and $f(0) = 0$. It is said to be positive definite if it further satisfies $f(x) > 0$ $\forall x \neq 0$.

We note that the "worst" disturbance in the above "completion of squares" argument is:

$$\nu_\gamma^*(x) = \frac{1}{2} D'_\epsilon V'_x, \tag{9}$$

which, however, is not necessarily the worst possible disturbance for the system, since V is not the value function for the game but only an upper bounding function. Henceforth, we will call the disturbance (9) the *worst disturbance adapted* to V.

Even though the above theorem solves the problem formulated (at least partially),[2] the computation of V and that of a corresponding optimal or suboptimal controller for small values of $\epsilon > 0$ present serious difficulties, due to numerical stiffness. To remedy this, we pose the question of whether V above and the corresponding cost bounding controller can be determined, for small values of $\epsilon > 0$, by solving well-behaved ϵ-independent smaller-order problems, as in the case of the H^∞-optimal control of singularly perturbed linear systems ([13], [19]). This also renders better solvability of the problem since the dimension of the partial differential inequality would then be greatly reduced. Another question of interest is whether the fast subsystem dynamics can be neglected completely in the design of such controllers, and if so under what conditions. These are now addressed in the next two sections.

3 A Time-Scale Decomposition

The slow subsystem and the associated soft-constrained game

To obtain the slow dynamics associated with (1), we let $\epsilon = 0$ and solve for x_2 (to be denoted $\bar{x}_2$) in terms of $x_1 =: x_s$, $u =: u_s$, $w =: w_s$, and under the working Assumption 2:

$$\bar{x}_2 = -A_{22}^{-1}(a_{21} + B_2 u_s + D_2 w_s) \tag{10}$$

Using this in the first equation of (1), we obtain the reduced-order (slow) dynamics:

$$\dot{x}_s = a_0 + B_0 u_s + D_0 w_s \tag{11}$$

where

$$a_0 := a_{11} - A_{12}A_{22}^{-1}a_{21} \qquad B_0 := B_1 - A_{12}A_{22}^{-1}B_2 \qquad D_0 := D_1 - A_{12}A_{22}^{-1}D_2$$

Using (10) also in the cost function (3) leads to the reduced (slow) cost (with $x_1 = x_s$):

$$L_s = \int_0^{T_e} (q_{11} + 2q_{12}\bar{x}_2 + |\bar{x}_2|^2_{Q_{22}} + |u_s|^2)\, dt. \tag{12}$$

[2]Note that the Theorem provides only a sufficient condition, since we have not shown that (8) is the desired controller for all $\gamma > \gamma^*(\epsilon)$.

In view of our earlier discussion, the control problem with state equation (11) and cost function (12) is closely related to the soft-constrained zero-sum differential game with the same state equation and with cost function

$$L_{\gamma s} = L_s - \gamma^2 \|w_s\|^2 \tag{13}$$

By the results as well as the discussion of [13], the information structure that is relevant in this game is the *full information pattern*, where the control u_s is allowed to depend on the disturbance w_s directly, and hence our interest is in the lower value of the game. Proposition 1 can still be applied to the above game, after a transformation similar to that described in [13] for its linear-quadratic counterpart. Then, the subgame described by (11) and (13) will have a finite lower value (which will be *zero* as the initial state is *zero*) if

$$S_{22}(\gamma) + A_{22}Q_{22}^{-1}A'_{22} > 0 \tag{14}$$

and the following partial differential inequality admits a nonnegative definite solution on $\mathcal{S}_1$:

$$W_{x_s}\tilde{a}_0 - \frac{1}{4}W_{x_s}\tilde{S}_0 W'_{x_s} + \tilde{q} \leq 0 \tag{15}$$

where

$$\begin{aligned}
\tilde{a}_0(\gamma) &= a_{11} - A_{12}Q_{22}^{-1}q'_{12} - (S_{12} + A_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot(a_{21} - A_{22}Q_{22}^{-1}q'_{12}) && (16)\\
\tilde{S}_0(\gamma) &= S_{11} + A_{12}Q_{22}^{-1}A'_{12} - (S_{12} + A_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot(S_{21} + A_{22}Q_{22}^{-1}A'_{12}) && (17)\\
\tilde{q}(\gamma) &= q_{11} - q_{12}Q_{22}^{-1}q'_{12} + (a'_{21} - q_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot(a_{21} - A_{22}Q_{22}^{-1}q'_{12}) && (18)
\end{aligned}$$

Let us introduce the set:

$$\begin{aligned}
\tilde{\Gamma}_s := \{&\gamma' > 0 : \forall \gamma \geq \gamma', S_{22}(\gamma) + A_{22}Q_{22}^{-1}A'_{22} > 0 \text{ and (15) has a bounded} \\
&\text{nonnegative definite solution over } \mathcal{S}_1\}
\end{aligned} \tag{19}$$

and further define

$$\gamma_s := \inf\{\gamma \in \tilde{\Gamma}_s\} \tag{20}$$

Then, for $\gamma > \gamma_s$, the state feedback representation of the optimal controller is given by

$$u^*_{s\gamma} = \mu^*_{s\gamma}(x_s) = -\frac{1}{2}B'_1 W_{x_s} - B'_2 Y \tag{21}$$

and the corresponding "worst" case disturbance (adapted to W) is

$$w^*_{s\gamma} = \nu^*_{s\gamma}(x_s) = \gamma^{-2}(\frac{1}{2}D'_1 W_{x_s} + D'_2 Y) \tag{22}$$

where the vector Y is given by

$$Y = -(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1}\Big[\frac{1}{2}(S_{21} + A_{22}Q_{22}^{-1}A'_{12})W_{x_s} - (a_{21} - A_{22}Q_{22}^{-1}q'_{12})\Big] \quad (23)$$

$$\equiv \frac{1}{2}Y_1 W_{x_s} + Y_2 \quad (24)$$

Finally, substitution of the controller (21) and disturbance (22) into (10) leads to the following expression for $\bar{x}_2$:

$$\bar{x}_2 = -\frac{1}{2}Q_{22}^{-1}A'_{12}W_{x_s} - Q_{22}^{-1}q'_{12} - Q_{22}^{-1}A'_{22}Y =: -X \equiv -\frac{1}{2}X_1 W_{x_s} - X_2 \quad (25)$$

To complete the study of the decomposition of the full-order problem, we now turn to the study of the "fast game".

The fast subsystem and the associated soft-constrained game

Let (t, x_s) be frozen on $[0,\infty)\times\mathcal{S}_1$, and $x_f := x_2 - \bar{x}_2$, $u_f := u - u_s$, $w_f := w - w_s$ and $\tau = \frac{t'-t}{\epsilon}$ where we take t' to vary on the same time scale as t. Since the system is autonomous, the fast subsystem is completely independent of the frozen time t. We define the fast subsystem and the associated cost (as in the linear case; see [13]) by:

$$\frac{d}{d\tau}x_f^{(x_s)} = A_{22}(x_s)x_f^{(x_s)} + B_2(x_s)u_f^{(x_s)} + D_2(x_s)w_f^{(x_s)}; \quad x_f^{(x_s)}(0) = x_f(t) \quad (26)$$

$$L_{\gamma f}^{(x_s)} = \int_0^\infty (|x_f^{(x_s)}|^2_{Q_{22}(x_s)} + |u_f^{(x_s)}|^2 - \gamma^2|w_f^{(x_s)}|^2)\, d\tau \quad (27)$$

In the following discussion, we will drop the superscript (x_s) whenever the meaning is clear from the context.

Note that the fast game is a standard soft-constrained LQ differential game, with an associated GARE:

$$A'_{22}(x_s)Z_{f\gamma} + Z_{f\gamma}A_{22}(x_s) + Q_{22}(x_s) - Z_{f\gamma}S_{22}(x_s)Z_{f\gamma} = 0 \quad (28)$$

We now let $\gamma_f^{(x_s)}$ denote the minimax disturbance attenuation bound for the H^∞-optimal control problem defined by (26)–(27) under closed-loop information[3].

For every $\gamma > \gamma_f^{(x_s)}$, let $Z_{f\gamma}(x_s)$ be the minimal positive definite solution for (28). Then, for any $\gamma > \gamma_f^{(x_s)}$, the feedback controller that attains the finite upper value of the fast game is

$$\begin{aligned} u_{f\gamma}^{(x_s)*}(\tau) &= \mu_{f\gamma}^{(x_s)*}(x_f^{(x_s)}(\tau)) = -B'_2(x_s)Z_{f\gamma}(x_s)x_f^{(x_s)}(\tau) \\ \Rightarrow \quad \mu_{f\gamma}^*(x) &= \mu_{f\gamma}^{(x_s)*}(x_f^{(x_s)}(0)) = -B'_2(x_s)Z_{f\gamma}(x_s)x_f(t) \\ &= -B'_2(x_s)Z_{f\gamma}(x_s)(x_2(t) + X) \end{aligned} \quad (29)$$

[3]To ensure that $\gamma_f^{(x_s)} < \infty$, it will be sufficient to take the pair $(A_{22}(x_s), B_2(x_s))$ stabilizable.

We now define

$$\gamma_f := \sup_{x_s \in \mathcal{S}_1} \gamma_f^{(x_s)} \tag{30}$$

Then, for every $\gamma > \gamma_f$, the GARE (28) admits a positive definite solution for all $x_1 \in \mathcal{S}_1$. Let

$$\bar{\gamma} := \max\{\gamma_f, \gamma_s\} \tag{31}$$

This value will play an important role in the characterization of the composite controller in Section 4, as in the linear case discussed earlier in [13].

To obtain the conditions under which the fast phenomenon can be completely neglected in the controller design, we have to study the fast subgame under open-loop information. The GARE related to this information structure is

$$A'_{22}(x_s)Z_{of\gamma} + Z_{of\gamma}A_{22}(x_s) + Q_{22}(x_s) + \gamma^{-2}Z_{of\gamma}D_2(x_s)D'_2(x_s)Z_{of\gamma} = 0. \tag{32}$$

Let $\gamma_{of}^{(x_s)}$ denote the infimum of all γ for which the GARE above admits a positive definite solution $Z_{of\gamma}$. Clearly, $\gamma_{of}^{(x_s)} \geq \gamma_f^{(x_s)}$ (see [18]). Define

$$\gamma_{of} := \sup_{x_s \in \mathcal{S}_1} \gamma_{of}^{(x_s)}, \tag{33}$$

which we will have occasion to use in the next section.

4 Main Results

We now study the performance of a composite controller (introduced below), as well as that of the slow controller, when applied to the original full-order problem.

The composite controller

Introduce a composite controller as:

$$\mu^*_{c\gamma}(x) = \mu^*_{s\gamma}(x) + \mu^*_{f\gamma}(x), \tag{34}$$

where $\mu^*_{s\gamma}$ and $\mu^*_{f\gamma}$ were defined by (21) and (29), respectively, and $\gamma > \bar{\gamma}$ where the latter was defined by (31). This the composite controller can be written in the following compact form, which will be convenient to work with:

$$\mu^*_{c\gamma}(x) = -B'\begin{bmatrix} \frac{1}{2}W_{x_1} \\ V_c + Z_{f\gamma}x_2 \end{bmatrix} \tag{35}$$

where

$$V_c \;:=\; Z_{f\gamma}X + Y \tag{36}$$

We now show that under certain conditions (to be introduced below), the controller (35) guarantees a disturbance attenuation level of γ for sufficiently small $\epsilon > 0$. Toward this end, we first introduce the the following condition on the slow subsystem, which is motivated by Assumption 7 of [14]:

Assumption 3 There exists a positive definite function $\Delta_\gamma(x_1)$ on $\mathcal{S}_1$ such that the following partial differential inequality admits a positive definite solution $\bar{W}(x_1)$ on $\mathcal{S}_1$:

$$\bar{W}_{x_1}(\tilde{a}_0 - \frac{1}{2}\tilde{S}_0 W'_{x_1}) + \Delta_\gamma \leq 0, \tag{37}$$

It can be shown by direct manipulation that the system $\dot{x}_1 = \tilde{a}_0 - \frac{1}{2}\tilde{S}_0 W'_{x_1}$ is precisely the slow subsystem driven by the controller $\mu^*_{s\gamma}$ given by (21) and the corresponding disturbance $\nu^*_{s\gamma}$ given by (22). Assumption 3 directly implies that this resulting closed-loop slow subsystem admits a Lyapunov function $\bar{W}$ with its time derivative along the nonlinear system

$$\dot{x}_1 = \tilde{a}_0 - \frac{1}{2}\tilde{S}_0 W'_{x_1}$$

being less than or equal to $-\Delta_\gamma$.

For each $x_1 \in \mathcal{S}_1$, we have $\gamma > \gamma_f^{x_1}$, which implies that the matrix $A_{22}(x_1) - S_{22}(x_1)Z_{f\gamma}(x_1)$ is Hurwitz. This in turn implies that there exists an $n_2 \times n_2$ matrix $\bar{\delta}(x_1) > 0$[4] such that the following GARE

$$\begin{aligned} Z_{f\gamma\delta}(A_{22} - B_2B_2'Z_{f\gamma}) + (A_{22} - B_2B_2'Z_{f\gamma})'Z_{f\gamma\delta} + \gamma^{-2}Z_{f\gamma\delta}D_2D_2'Z_{f\gamma\delta} + Q_{22} \\ + Z_{f\gamma}B_2B_2'Z_{f\gamma} + \bar{\delta} = 0 \end{aligned} \tag{38}$$

admits a positive-definite solution $Z_{f\gamma\delta}(x_1)$. Thus, we can take a continuously differentiable matrix function $\bar{\delta}(x_1)$ on $\mathcal{S}_1$, which is positive definite for each fixed x_1, such that the GARE (38) admits a continuously differentiable positive-definite matrix function $Z_{f\gamma\delta}(x_1)$ on $\mathcal{S}_1$, since A_{22}, B_2, D_2 and Q_{22} are continuously differentiable. This further implies that $Z_{f\gamma}(x_1)$ is also continuously differentiable in x_1 on $\mathcal{S}_1$.

Now, applying the composite controller $\mu^*_{c\gamma}(x)$ to the full-order system, we arrive at the following maximization problem:

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} - \frac{1}{2}B_1B_1'W_{x_1} - B_1B_2'V_c + (A_{12} - B_1B_2'Z_{f\gamma})x_2 \\ a_{21} - \frac{1}{2}B_2B_1'W_{x_1} - B_2B_2'V_c + (A_{22} - B_2B_2'Z_{f\gamma})x_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w \tag{39}$$

$$L_\gamma = \int_0^{T_e} (q(x) + |\frac{1}{2}B_1'W_{x_1} + B_2'V_c + B_2'Z_{f\gamma}x_2|^2 - \gamma^2|w|^2)\,dt \quad \to \quad \max \tag{40}$$

[4]Such a matrix $\bar{\delta}(x_1)$ always exists, by an immediate application of the Implicit Function Theorem. A candidate for such a matrix is one of the form $\bar{\delta}I_{n_2}$, where $\bar{\delta} > 0$ is a scalar. Here, we allow $\bar{\delta}(x_1)$ to take very general forms so that there will be flexibility in the choice of the matrix $Z_{f\gamma\delta}(x_1)$. It is preferred to have a $Z_{f\gamma\delta}$ that is independent of x_1, as to been seen shortly. Note that for a valid choice of $\bar{\delta}(x_1)$, any matrix that is less than or equal to it is also a valid choice.

Introduce the notation

$$\bar{a}_\epsilon = \begin{bmatrix} a_{11} - \frac{1}{2}B_1B_1'W_{x_1} - B_1B_2'V_c + (A_{12} - B_1B_2'Z_{f\gamma})x_2 \\ \frac{1}{\epsilon}(a_{21} - \frac{1}{2}B_2B_1'W_{x_1} - B_2B_2'V_c + (A_{22} - B_2B_2'Z_{f\gamma})x_2) \end{bmatrix}$$

$$=: \begin{bmatrix} \bar{a}_{11} + \bar{A}_{12}x_2 \\ \frac{1}{\epsilon}(\bar{a}_{21} + \bar{A}_{22}x_2) \end{bmatrix} \tag{41}$$

$$\bar{q} = q + |\frac{1}{2}B_1'W_{x_1} + B_2'V_c + B_2'Z_{f\gamma}x_2|^2 =: \bar{q}_{11}(x_1) + 2\bar{q}_{12}(x_1)x_2 + x_2'\bar{q}_{22}(x_1)x_2 \tag{42}$$

To show that the composite controller $\mu_{c\gamma}^*$ achieves the disturbance attenuation level γ for the full-order system, it will be sufficient to show that the supremum of (40) under (39) is 0. Thus, in view of Proposition 1, we seek a nonnegative definite function $V(\epsilon, x)$ such that

$$V_x\bar{a}_\epsilon + \frac{1}{4\gamma^2}V_xD_\epsilon D_\epsilon'V_x' + \bar{q} \leq 0 \tag{43}$$

We establish the existence of such a function in a number of steps. First, we introduce a candidate function:

$$V(\epsilon, x) = \tilde{W}(x_1) + 2\epsilon\bar{V}_c(x_1)'x_2 + \epsilon x_2'Z_{f\gamma\delta}(x_1)x_2 \tag{44}$$

where

$$\tilde{W} = W + \eta\bar{W}; \quad \bar{V}_c = Z_{f\gamma\delta}X + Y;$$

and $\eta > 0$ is a scalar to be chosen later.

We next invoke a growth condition on $\bar{W}$, so that V defined above is nonnegative definite on $\mathcal{S}$:

Assumption 4 There is a constant $M > 0$ such that

$$M\bar{W} \geq \bar{V}_c'Z_{f\gamma\delta}^{-1}\bar{V}_c; \quad \forall x_1 \in \mathcal{S}_1$$

This assumption immediately leads to the inequality

$$V \geq W + \epsilon|x_2 + Z_{f\gamma\delta}^{-1}\bar{V}_c|^2_{Z_{f\gamma\delta}},$$

for sufficiently small $\epsilon > 0$.

Now the last step is to show that this nonnegative-definite function satisfies (43). Substituting (44) into the left-hand side (LHS) of (43), we first obtain:

$$\begin{aligned} \text{LHS} = {}& (\tilde{W}_{x_1}\bar{a}_{11} + 2\bar{V}_c'\bar{a}_{21} + \frac{1}{4\gamma^2}\tilde{W}_{x_1}D_1D_1'\tilde{W}_{x_1}' + \frac{1}{\gamma^2}\tilde{W}_{x_1}D_1D_2'\bar{V}_c + \frac{1}{\gamma^2}\bar{V}_c'D_2D_2'\bar{V}_c \\ & +\bar{q}_{11}) + 2(\epsilon\bar{\alpha} + \frac{1}{2}\tilde{W}_{x_1}\bar{A}_{12} + \bar{V}_c'\bar{A}_{22} + \bar{a}_{21}'Z_{f\gamma\delta} + \frac{1}{2\gamma^2}\tilde{W}_{x_1}D_1D_2'Z_{f\gamma\delta} + \frac{1}{\gamma^2}\bar{V}_c'D_2D_2' \\ & \cdot Z_{f\gamma\delta} + \bar{q}_{12})x_2 + x_2'(Z_{f\gamma\delta}\bar{A}_{22} + \bar{A}_{22}'Z_{f\gamma\delta} + Z_{f\gamma\delta}\frac{1}{\gamma^2}D_2D_2'Z_{f\gamma\delta} + \bar{q}_{22})x_2 + \epsilon\bar{\beta} \end{aligned} \tag{45}$$

where

$$\bar{\alpha}(x_1) := \bar{a}_{11}\bar{V}_{cx_1} + \frac{1}{2\gamma^2}\tilde{W}_{x_1}D_1D_1'\bar{V}_{cx_1} + \frac{1}{\gamma^2}\bar{V}_c'D_2D_1'\bar{V}_{cx_1} \tag{46}$$

$$\begin{aligned}\bar{\beta}(\epsilon, x_1, x_2) := {} & x_2'(\bar{V}_{cx_1}'\bar{A}_{12} + \bar{A}_{12}'\bar{V}_{cx_1} + \frac{\epsilon}{\gamma^2}\bar{V}_{cx_1}'D_1D_1'\bar{V}_{cx_1} + \frac{1}{\gamma^2}\bar{V}_{cx_1}'D_1D_2'Z_{f\gamma\delta} \\ & + \frac{1}{\gamma^2}Z_{f\gamma\delta}D_2D_1'\bar{V}_{cx_1})x_2 + (x_2'Z_{f\gamma\delta}x_2)_{x_1}(\bar{a}_{11} + \bar{A}_{12}x_2 + \frac{1}{4\gamma^2}D_1D_1' \\ & \cdot(\tilde{W}_{x_1}' + 2\epsilon\bar{V}_{cx_1}'x_2 + \epsilon(x_2'Z_{f\gamma\delta}x_2)_{x_1}) + \frac{1}{\gamma^2}D_1D_2'(\bar{V}_c +' Z_{f\gamma\delta}x_2))\end{aligned} \tag{47}$$

By GARE (38), repeated applications of Lemma 1, and several steps of the "completion of squares" argument, we deduce the bounds:

$$\begin{aligned}\text{LHS} \le {} & -X'\bar{\delta}X - \eta\Delta_\gamma + \eta\bar{W}_{x_1}(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)X + \frac{\eta^2}{4\gamma^2}\bar{W}_{x_1}D_1D_1'\bar{W}_{x_1}' \\ & -2X'\bar{\delta}x_2 + \eta\bar{W}_{x_1}(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)x_2 + 2\epsilon\bar{\alpha}x_2 - x_2'\bar{\delta}x_2 + \epsilon\bar{\beta} \\ = {} & -\eta\Delta_\gamma + \frac{\lambda}{1-\lambda}X'\bar{\delta}X + \frac{\eta^2}{4\gamma^2}\bar{W}_{x_1}D_1D_1'\bar{W}_{x_1}' - |x_2 - \frac{3\epsilon}{\lambda}\bar{\delta}^{-1}\bar{\alpha}'|^2_{\frac{\lambda}{3}\bar{\delta}} - \frac{\lambda}{3}x_2'\bar{\delta}x_2 \\ & + \frac{3\epsilon^2}{\lambda}\bar{\alpha}\bar{\delta}^{-1}\bar{\alpha}' + \epsilon\bar{\beta} - |x_2 - \frac{3\eta}{2}\bar{\delta}^{-1}(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)'\bar{W}_{x_1}'|^2_{\frac{\lambda}{3}\bar{\delta}} \\ & + \frac{3\lambda\eta^2}{4}|(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)'\bar{W}_{x_1}'|^2_{\bar{\delta}^{-1}} \\ & - |x_2 + \frac{1}{1-\lambda}X - \frac{\eta}{2}\bar{\delta}^{-1}(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)'\bar{W}_{x_1}'|^2_{(1-\lambda)\bar{\delta}} \\ & + \frac{(1-\lambda)\eta^2}{4}|(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)'\bar{W}_{x_1}'|^2_{\bar{\delta}^{-1}} \\ \le {} & -\eta\Delta_\gamma + \frac{\lambda}{1-\lambda}X'\bar{\delta}X + \frac{\eta^2}{4\gamma^2}\bar{W}_{x_1}D_1D_1'\bar{W}_{x_1}' + \frac{3\epsilon^2}{\lambda}\bar{\alpha}\bar{\delta}^{-1}\bar{\alpha}' - \frac{\lambda}{3}x_2'\bar{\delta}x_2 + \epsilon\bar{\beta} \\ & + \frac{(1+2\lambda)\eta^2}{4}|(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)'\bar{W}_{x_1}'|^2_{\bar{\delta}^{-1}}\end{aligned}$$

where λ is some constant in $(0,1)$, which is to be determined shortly, and $\bar{Z}_\delta := Z_{f\gamma\delta} - Z_{f\gamma}$. The last bound motivates the following condition, which guarantees the nonpositive definiteness of the LHS.

Assumption 5 There exists a constant $M > 0$ such that the following growth conditions are satisfied on the set $\mathcal{S}$:[5]

$$
\begin{aligned}
&M\Delta_\gamma(x_1) \geq \bar{W}_{x_1}(x_1)D_1(x_1)D_1'(x_1)\bar{W}_{x_1}'(x_1); \qquad M\Delta_\gamma(x_1) \geq X'(x_1)\bar{\delta}(x_1)X(x_1);\\
&M\Delta_\gamma(x_1) \geq \bar{W}_{x_1}(x_1)(A_{12}(x_1) - S_{12}(x_1)Z_{f\gamma}(x_1) + \tfrac{1}{\gamma^2}D_1(x_1)D_2'(x_1)\bar{Z}_\delta(x_1))\bar{\delta}^{-1}(x_1)\\
&\qquad\qquad (A_{12}(x_1) - S_{12}(x_1)Z_{f\gamma}(x_1) + \gamma^{-2}D_1(x_1)D_2'(x_1)\bar{Z}_\delta(x_1))'\bar{W}_{x_1}'(x_1);\\
&M\Delta_\gamma(x_1) \geq \bar{\alpha}(x_1)\bar{\delta}^{-1}(x_1)\bar{\alpha}'(x_1); \qquad Mx_2'\bar{\delta}(x_1)x_2 \geq \bar{\beta}(x_1,x_2);
\end{aligned}
$$

Under Assumption 5, we can first choose a sufficiently small $\bar{\eta}_\gamma > 0$ such that

$$\frac{\bar{\eta}_\gamma}{4}\Delta_\gamma \geq \frac{\bar{\eta}_\gamma^2}{4\gamma^2}\bar{W}_{x_1}D_1D_1'\bar{W}_{x_1}'$$

and

$$\frac{\bar{\eta}_\gamma}{4}\Delta_\gamma \geq \frac{(1+2\lambda)\bar{\eta}_\gamma^2}{4}|(A_{12} - S_{12}Z_{f\gamma} + \frac{1}{\gamma^2}D_1D_2'\bar{Z}_\delta)'\bar{W}_{x_1}'|_{\bar{\delta}^{-1}}^2.$$

Next we can choose a sufficiently small $\bar{\lambda}_\gamma > 0$ such that

$$\frac{\bar{\eta}_\gamma}{4}\Delta_\gamma \geq \frac{\bar{\lambda}_\gamma}{1-\bar{\lambda}_\gamma}X'\bar{\delta}X,$$

and finally we can choose a sufficiently small $\bar{\epsilon}_\gamma > 0$ such that

$$\frac{\bar{\eta}_\gamma}{4}\Delta_\gamma \geq \frac{3\epsilon^2}{\bar{\lambda}}\bar{\alpha}\bar{\delta}^{-1}\bar{\alpha}'; \quad \frac{\bar{\lambda}}{3}x_2'\bar{\delta}x_2 \geq \epsilon\bar{\beta}$$

for all $\epsilon \in (0, \bar{\epsilon}_\gamma]$. Then, the function proposed indeed solves (43), and hence the composite controller achieves the disturbance attenuation level γ for the full-order system for sufficiently small $\epsilon > 0$.

We thus arrive at the following theorem.

Theorem 1 *Consider the singularly perturbed system (1)–(6) under Assumptions 1 and 2. Given a $\gamma > \bar{\gamma}$, if Assumptions 3–5 hold, then there exists an $\bar{\epsilon}_\gamma > 0$ such that the composite controller $\mu_{c\gamma}^*$ defined in (35) achieves the disturbance attenuation level γ for the full-order system for $0 < \epsilon \leq \bar{\epsilon}_\gamma$; hence, $\gamma \geq \gamma^*(\epsilon)$ for $0 < \epsilon \leq \bar{\epsilon}_\gamma$, and the soft-constrained differential game has an upper value 0.* □

[5] The function $\bar{\beta}$ to be introduced below contains cubic terms in x_2. Hence, for $Mx_2'\bar{\delta}x_2 \geq \bar{\beta}$ to hold, $\mathcal{S}$ cannot be unbounded for a fixed $x_1 \in \mathcal{S}_1$ unless $(x_2'Z_{f\gamma\delta}x_2)_{x_1} \equiv 0$ for that fixed x_1.

The slow controller

To the study optimality of the slow controller $\mu^*_{s\gamma}$, defined in (21), we first choose a fixed a $\gamma > \max\{\gamma_s, \gamma_{of}\}$. Then, following the reasoning used as in the composite controller case, we can show the existence of a continuously differentiable $n_2 \times n_2$ positive definite matrix function $\hat{\delta}(x_1)$ on $\mathcal{S}_1$, such that the following GARE:

$$Z_{of\gamma\delta}A_{22} + A'_{22}Z_{of\gamma\delta} + \gamma^{-2}Z_{of\gamma\delta}D_2D'_2Z_{of\gamma\delta} + Q_{22} + \hat{\delta} = 0 \tag{48}$$

admits a positive-definite solution $Z_{of\gamma\delta}(x_1)$ for each fixed $x_1 \in \mathcal{S}_1$, which is further continuously differentiable on $\mathcal{S}_1$.

Invoking Assumption 3 again, and substituting the slow controller into the full-order system, we arrive at the following maximization problem:

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} - \frac{1}{2}B_1B'_1W'_{x_1} - B_1B'_2Y + A_{12}x_2 \\ a_{21} - \frac{1}{2}B_2B'_1W'_{x_1} - B_2B'_2Y + A_{22}x_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w \tag{49}$$

$$J = \int_0^{T_e} (q(x) + |\frac{1}{2}B'_1Z_{s\gamma} + B'_2Y|^2 - \gamma^2|w|^2)\, dt \quad \rightarrow \quad \max \tag{50}$$

Now introduce the notation:

$$\hat{a}_\epsilon = \begin{bmatrix} a_{11} - \frac{1}{2}B_1B'_1W'_{x_1} - B_1B'_2Y + A_{12}x_2 \\ \frac{1}{\epsilon}(a_{21} - \frac{1}{2}B_2B'_1W'_{x_1} - B_2B'_2Y + A_{22}x_2) \end{bmatrix} =: \begin{bmatrix} \hat{a}_{11} + A_{12}x_2 \\ \frac{1}{\epsilon}(\hat{a}_{21} + A_{22}x_2) \end{bmatrix} \tag{51}$$

$$\hat{q} = q + |\frac{1}{2}B'_1Z_{s\gamma} + B'_2Y|^2 =: \hat{q}_{11}(x_1) + 2\hat{q}_{12}(x_1)x_2 + x'_2\hat{q}_{22}(x_1)x_2 \tag{52}$$

Arguments similar to those in the case for the composite controller lead to the conclusion that, for the slow controller to achieve the disturbance attenuation level γ for the full-order system, it will be sufficient to find a nonnegative definite function V such that the following partial differential inequality is satisfied on $\mathcal{S}$:

$$V_x\hat{a}_\epsilon + \frac{1}{4\gamma^2}V_xD_\epsilon D'_\epsilon V_x + \hat{q} \leq 0 \tag{53}$$

We choose a function:

$$V(\epsilon, x) = \tilde{W}(x_1) + 2\epsilon\hat{V}_c(x_1)'x_2 + \epsilon x'_2Z_{of\gamma\delta}(x_1)x_2 \tag{54}$$

where

$$\tilde{W} = W + \eta\bar{W}; \quad \hat{V}_c = Z_{of\gamma\delta}X + Y; \tag{55}$$

and $\eta > 0$ is a scalar to be chosen shortly.

For V defined above to be nonnegative definite on $\mathcal{S}$, we introduce the following counterpart of Assumption 4:

Assumption 6 There exists a constant $M > 0$ such that

$$M\bar{W} \geq \hat{V}'_cZ^{-1}_{of\gamma\delta}\hat{V}_c; \quad \forall x_1 \in \mathcal{S}_1$$

Under this assumption,

$$V \geq W + \epsilon |x_2 + Z_{of\gamma\delta}^{-1}\hat{V}_c|^2_{Z_{of\gamma\delta}}$$

for sufficiently small ϵ.

Substituting the function V of (54) into the LHS of the inequality (53), we obtain:

$$\begin{aligned}\text{LHS} = &(\tilde{W}_{x_1}\hat{a}_{11} + 2\hat{V}_c'\hat{a}_{21} + \frac{1}{4\gamma^2}\tilde{W}_{x_1}D_1D_1'\tilde{W}_{x_1}' + \frac{1}{\gamma^2}\tilde{W}_{x_1}D_1D_2'\hat{V}_c + \frac{1}{\gamma^2}\hat{V}_c'D_2D_2'\hat{V}_c + \hat{q}_{11})\\ &+2(\epsilon\hat{\alpha} + \frac{1}{2}\tilde{W}_{x_1}A_{12} + \hat{V}_c'A_{22} + \hat{a}_{21}'Z_{of\gamma\delta} + \frac{1}{2\gamma^2}\tilde{W}_{x_1}D_1D_2'Z_{of\gamma\delta} + \frac{1}{\gamma^2}\hat{V}_c'D_2D_2'Z_{of\gamma\delta}\\ &+q_{12})x_2 + x_2'(Z_{of\gamma\delta}A_{22} + A_{22}'Z_{of\gamma\delta} + \frac{1}{\gamma^2}Z_{of\gamma\delta}D_2D_2'Z_{of\gamma\delta} + Q_{22})x_2 + \epsilon\hat{\beta}\end{aligned} \tag{56}$$

where

$$\hat{\alpha}(x_1) := \hat{a}_{11}\hat{V}_{cx_1} + \frac{1}{2\gamma^2}\tilde{W}_{x_1}D_1D_1'\hat{V}_{cx_1} + \frac{1}{\gamma^2}\hat{V}_c'D_2D_1'\hat{V}_{cx_1} \tag{57}$$

$$\begin{aligned}\hat{\beta}(\epsilon, x_1, x_2) := &\, x_2'(\hat{V}_{cx_1}'A_{12} + A_{12}'\hat{V}_{cx_1} + \frac{\epsilon}{\gamma^2}\hat{V}_{cx_1}'D_1D_1'\hat{V}_{cx_1} + \frac{1}{\gamma^2}\hat{V}_{cx_1}'D_1D_2'Z_{of\gamma\delta}\\ &+\frac{1}{\gamma^2}Z_{of\gamma\delta}D_2D_1'\hat{V}_{cx_1})x_2 + (x_2'Z_{of\gamma\delta}x_2)_{x_1}(\hat{a}_{11} + A_{12}x_2 + \frac{1}{4\gamma^2}D_1D_1'\\ &\cdot(\tilde{W}_{x_1}' + 2\epsilon\hat{V}_{cx_1}'x_2 + \epsilon(x_2'Z_{of\gamma\delta}x_2)_{x_1}) + \frac{1}{\gamma^2}D_1D_2'(\hat{V}_c +' Z_{of\gamma\delta}x_2))\end{aligned} \tag{58}$$

Using arguments similar to those in the composite controller case, the following growth condition is sufficient for nonpositive definiteness of the LHS for sufficiently small $\epsilon > 0$:

Assumption 7 There exists a constant $M > 0$ such that the following growth conditions are satisfied on the set $\mathcal{S}$:

$$\begin{aligned}&M\Delta_\gamma(x_1) \geq \bar{W}_{x_1}(x_1)D_1(x_1)D_1'(x_1)\bar{W}_{x_1}'(x_1); \qquad M\Delta_\gamma(x_1) \geq X'(x_1)\hat{\delta}(x_1)X(x_1);\\ &M\Delta_\gamma(x_1) \geq \bar{W}_{x_1}(x_1)(A_{12}(x_1) + \tfrac{1}{\gamma^2}D_1(x_1)D_2'(x_1)Z_{of\gamma\delta}(x_1))\bar{\delta}^{-1}(x_1)(A_{12}(x_1)\\ &\qquad +\gamma^{-2}D_1(x_1)D_2'(x_1)Z_{of\gamma\delta}(x_1))'\bar{W}_{x_1}';\\ &M\Delta_\gamma(x_1) \geq \hat{\alpha}(x_1)\hat{\delta}^{-1}(x_1)\hat{\alpha}'(x_1); \qquad Mx_2'\hat{\delta}(x_1)x_2 \geq \hat{\beta}(x_1, x_2).\end{aligned}$$

This takes us to the following counterpart of Theorem 1 for the slow controller.

Theorem 2 *Consider the singularly perturbed system (1)–(6) under Assumptions 1 and 2. Given a $\gamma > \max\{\gamma_s, \gamma_{of}\}$, if Assumptions 3, 6 and 7 hold, then there exists an $\hat{\epsilon}_\gamma > 0$ such that the slow controller $\mu_{s\gamma}^*$ defined in (21) achieves the disturbance attenuation level γ for the full-order system for $0 < \epsilon \leq \hat{\epsilon}_\gamma$; hence $\gamma \geq \gamma^*(\epsilon)$ for $0 < \epsilon \leq \hat{\epsilon}_\gamma$, and the soft-constrained differential game has an upper value 0.* □

5 Discussion of the Main Results

In this section, we discuss a number of important special cases where the Theorems presented in the previous section find immediate applications and various conditions introduced there can readily be tested.

The linear quadratic case. Let the original problem be of the linear quadratic type, i.e., a_{i1} and q_{12} are linear functions of x_1, A_{i2}, B_i, D_i and Q_{22} are constant matrices, and q_{11} is a quadratic function of x_1, for $i = 1, 2$. In this case, we can choose $\mathcal{S}$ to be the entire $\mathcal{R}^n$. The partial differential inequality (15) can be solved with equality by a quadratic function, $W = x_1' Z_{s\gamma} x_1$, where $Z_{s\gamma}$ is a nonnegative definite matrix. The linear system $\dot{x}_s = \tilde{a}_0 - \tilde{S}_0 W_{x_s}$ is asymptotically stable for $\gamma > \gamma_s$ under some stabilizability and observability conditions as in [13]. Thus, Assumption 3 is satisfied by some positive-definite quadratic function $\bar{W} = x_1' \bar{Z}_{s\gamma} x_1$ with $\Delta_\gamma = x_1' x_1$. For the fast subsystem, $\bar{\delta}$ and $\hat{\delta}$ can be chosen as some constant positive-definite matrices, when $\gamma > \gamma_f$ and $\gamma > \gamma_{of}$, respectively. Then, the matrix functions X and Y are linear in x_1, and hence Assumptions 4–7 easily hold for some sufficiently large M. Hence, we have the following corollary:

Corollary 1 *Theorems 1 and 2 subsume the sufficiency results of [13] in the infinite-horizon case.* □

The linearizable case. Given a nonlinear-nonquadratic problem of the type formulated in Section 2, let its dynamics be linearizable around $x_1 = 0$, and its cost function admit in its expansion at $x_1 = 0$ a leading quadratic term. The resulting linear-quadratic problem will then be called the "linear-quadratic approximant" of the original one. Consider the class of problems for which the linear-quadratic approximants satisfy the assumptions of Theorem 1 of [13], which we will refer to as *linearizable* problems.

In this case, we will take the set $\mathcal{S}$ to be any closed and bounded subset of $\mathcal{R}^n$. For $\gamma > \gamma_s$, the partial differential inequality (15) admits a nonnegative definite solution W_γ. By Theorem 23 of [3], it is reasonable to assume that the function W satisfies the condition $\frac{\partial^2}{\partial x_1^2} W(x_1 = 0) = Z_{s\gamma}$, where $Z_{s\gamma}$ is a constant positive-definite matrix. Hence, linearization of the system $\dot{x}_s = \tilde{a}_0 - \tilde{S}_0 W_{x_s}$ at $x_1 = 0$ is asymptotically stable. Suppose that, in addition to satisfaction of Assumption 3, the functions $\bar{W}_\gamma$ and Δ_γ are such that $\frac{\partial^2}{\partial x_1^2} \bar{W}(x_1 = 0) = \bar{Z}_{s\gamma}$ and $\frac{\partial^2}{\partial x_1^2} \Delta_\gamma(x_1 = 0) = P_\gamma$, where $\bar{Z}_{s\gamma}$ and P_γ are some constant positive-definite matrices. Under these assumptions, we can find a sufficiently small $\varrho > 0$ such that Assumptions 4–7 are satisfied on the set $\mathcal{S} \bigcap \{|x_1| < \varrho\}$. This is true, because the set is closed and bounded, and the functions W, $\bar{W}$ and Δ_γ are always positive on the set. Hence, we have the following corollary:

Corollary 2 *Consider the singularly perturbed control system (1)–(6) under Assumptions 1 and 2. Let the problem admit a linear-quadratic approximant at $x_1 = 0$, which further satisfies the assumptions of Theorem 1 of [13]. For a $\gamma > \bar{\gamma}$, if Assumptions 3 hold and*

$\frac{\partial^2}{\partial x_1^2}W(x_1=0)=Z_{s\gamma}\geq 0$, $\frac{\partial^2}{\partial x_1^2}\bar{W}(x_1=0)=\bar{Z}_{s\gamma}>0$ and $\frac{\partial^2}{\partial x_1^2}\Delta_\gamma(x_1=0)=P_\gamma>0$, then Assumptions 4 and 5 are satisfied, and there exists an $\bar{\epsilon}_\gamma>0$ such that the composite controller $\mu^*_{c\gamma}$ defined in (35) achieves the disturbance attenuation level γ for the full-order system for $0<\epsilon\leq\bar{\epsilon}_\gamma$; hence $\gamma\geq\gamma^*(\epsilon)$ for $0<\epsilon\leq\bar{\epsilon}_\gamma$, and the soft-constrained differential game has an upper value 0.

For a $\gamma>\max\{\gamma_s,\gamma_{of}\}$, if Assumption 3 holds, and $\frac{\partial^2}{\partial x_1^2}W(x_1=0)=Z_{s\gamma}\geq 0$, $\frac{\partial^2}{\partial x_1^2}\bar{W}(x_1=0)=\bar{Z}_{s\gamma}>0$ and $\frac{\partial^2}{\partial x_1^2}\Delta_\gamma(x_1=0)=P_\gamma>0$, then Assumptions 6 and 7 are satisfied, and there exists an $\hat{\epsilon}_\gamma>0$ such that the slow controller $\mu^*_{s\gamma}$ defined in (21) achieves the disturbance attenuation level γ for the full-order system for $0<\epsilon\leq\hat{\epsilon}_\gamma$; hence $\gamma\geq\gamma^*(\epsilon)$ for $0<\epsilon\leq\hat{\epsilon}_\gamma$, and the soft-constrained differential game has an upper value 0. □

The choice of $\mathcal{S}$. We have thus seen that the main results of the previous section are directly applicable to linear-quadratic problems, where the set of interest, $\mathcal{S}$, can be chosen to be the entire $\mathcal{R}^n$, and also to nonlinear but *linearizable* systems, where the set $\mathcal{S}$ is taken to be closed and bounded. For problems that are not *linearizable*, however, no general conclusions can be made, due to lack of useful bounds for the solution to the partial differential inequality (15). In spite of this, in all the examples we have attempted, the results have turned out to hold with a proper choice of the set $\mathcal{S}$. In the next section, we present one such example where the system is not *linearizable* at $x_1=0$.

In general, the choice of the set $\mathcal{S}$ is very much problem dependent. A general procedure is to first choose $\mathcal{S}_1$ such that the partial differential inequality (15) is solved on this set; next for each $x_1\in\mathcal{S}_1$ to determine the set $\mathcal{S}_{2,x_1}$ for x_2 such that the corresponding theorems (of the previous section) can be applied. The set $\mathcal{S}_1$ may have to be chosen to be closed and bounded for the assumptions to hold. It should be noted that the set $\mathcal{S}_{2,x_1}$ could be unbounded for Assumption 5 to be satisfied only if $\frac{\partial}{\partial x_1}Z_{f\gamma\delta}(x_1)$ is zero at x_1 (correspondingly, for Assumption 7 to be satisfied only if $\frac{\partial}{\partial x_1}Z_{of\gamma\delta}(x_1)$ is zero at x_1). But, as the value of ϵ decreases, the set $\mathcal{S}_{2,x_1}$ can be expanded to be arbitrarily large. Furthermore, the set $\mathcal{S}$ can be chosen to be of the strip form $\mathcal{S}=\{(x_1,x_2)\in\mathcal{R}^n:\quad x_1\in\mathcal{S}_1\}$ if we can solve the GAREs (38) and (48) for x_1-independent solutions. To obtain such solutions, the existing theory on quadratic stability for linear uncertain systems [11] can be applied.

In some cases, it may be easier to obtain an x_1-independent quadratically stabilizing controller for the fast subsystem, which is tantamount to obtaining an x_1-independent solution to the following Riccati type algebraic inequality:

$$\tilde{Z}_{f\gamma}A_{22}+A'_{22}\tilde{Z}_{f\gamma}+\tilde{Z}_{f\gamma}S_{22}\tilde{Z}_{f\gamma}+Q_{22}<0 \tag{59}$$

Then, we simply let

$$\bar{\delta}:=-(\tilde{Z}_{f\gamma}A_{22}+A'_{22}\tilde{Z}_{f\gamma}+\tilde{Z}_{f\gamma}S_{22}\tilde{Z}_{f\gamma}+Q_{22})$$

and replace both $Z_{f\gamma}$ and $Z_{f\gamma\delta}$ by $\tilde{Z}_{f\gamma}$ in all the derivations of the previous section. With this modification, all the earlier derivations for the composite controller remain valid, as well as the proof of its optimality. Hence, we can again choose the set $\mathcal{S}$ to be of strip form if Theorem 1 is applicable.

6 Example

To illustrate the applicability of the results developed in Section 4, we present here a numerical example which involves a system that is not linearizable (using the terminology of Section 5). Consider the following system and the performance index:

$$\dot{x}_1 = x_1^3 - x_1 x_2 + 2x_1 u - 3x_1 w \tag{60}$$

$$\epsilon \dot{x}_2 = x_1^2 - x_2 + u - 2w \tag{61}$$

$$L_\gamma = \int_0^{T_e} (x_1^4 - x_1^2 x_2 + x_2^2 + |u|^2 - \gamma^2 |w|^2)\, dt \tag{62}$$

Notice that for a fixed $\epsilon > 0$, the "slow" state variable x_1 will not be slow compared with x_2 when x_1 is sufficiently large. This observation cautions us that a two time-scale analysis will not generally yield a global result. Hence, x_1 must be forced to stay in a bounded set. The stability of the fast subsystem, however, is independent of x_1; hence, no *a priori* bound is need on x_2. Accordingly, we take the set of interest in the state space to be $\mathcal{S} := \{(x_1, x_2) \in \mathcal{R}^2 : \quad |x_1| \leq 1\}$.

Note that the fast subsystem is open-loop stable. By solving the slow and fast subsystems analytically, we obtain the three basic performance levels in this case to be:

$$\gamma_s = \sqrt{2}; \quad \gamma_f = \sqrt{2}; \quad \gamma_{of} = 2$$

Note that $\gamma_f = \gamma_s < \gamma_{of}$.

First we choose $\gamma = 1.5 > \max\{\gamma_s, \gamma_f\}$, and design the controllers for the slow and fast subsystems based on this value of γ:

$$\begin{aligned} W(x_1) &= 1.5x_1^2; \quad \bar{W}(x_1) = 1.5x_1^2; \quad \Delta_\gamma(x_1) = 3x_1^4; \\ \mu_{s\gamma}^*(x_1) &= -3x_1^2; \quad Z_{f\gamma} = 0.67962; \quad \bar{\delta} = 0.125; \quad Z_{f\gamma\delta} = 0.94479; \\ \mu_{c\gamma}^*(x_1, x_2) &= -1.6408x_1^2 - 0.67962x_2 \end{aligned}$$

By Theorem 1, we can choose an upper bounding function as

$$V(x_1, x_2) = 1.7166x_1^2 - 3.7792\epsilon x_1^2 x_2 + 0.94479\epsilon x_2^2;$$

then, for $0 < \epsilon \leq 0.016531$, the composite controller achieves the performance level 1.5 for the full-order system on $\mathcal{S}$. Theorem 2 is not applicable here since $1.5 < \gamma_{of}$. From the results of [13], we would in fact not expect the slow control to achieve that performance bound for the full-order system on $\mathcal{S}$.

We have conducted several simulations of the behavior of the full-order system with slow and composite controllers, under constant and sinusoidal disturbance inputs. We took $x_1(0) = 0.5$ and $x_2(0) = 0$, and $\epsilon = 0.015$. The disturbance input was first chosen to be $w(t) \equiv -6$. Plots of the control signals, state trajectories and performances for the full-order system under composite and slow controllers are given in Figure 1. We see that both controllers stabilize the system, but the composite controller leads to a better system response and keeps the state in the set $\mathcal{S}$. More evidently, the composite controller leads to a growing negative game cost, which shows that it achieves the performance bound 1.5 for the full-order system under this particular disturbance input. On the other hand, the slow controller leads to a growing positive game cost, which shows that it does not achieve the desired performance level for the full-order system under this particular disturbance input.

Next, we considered a sinusoidal disturbance input : $w(t) = 4\sin(2\pi t)$. Plots of the control signals, state trajectories and performance levels for the full-order system under composite and slow controllers are given in Figure 2. Observations similar to those in the first case can be made here; the composite controller achieves the desired performance level for the full-order system under this particular disturbance input, but the slow controller does not.

If we want to ignore the fast dynamics in the design of the controller, then we should choose γ to be greater than γ_{of}. For instance, if we choose $\gamma = 2.1$, we obtain the values:

$$\begin{aligned} W(x_1) &= 0.98576x_1^2; \quad \bar{W}(x_1) = 0.98576x_1^2; \quad \Delta_\gamma(x_1) = 1.4646x_1^4; \\ \mu^*_{s\gamma}(x_1) &= -0.95035x_1^2; \quad Z_{f\gamma} = 0.48889; \quad Z_{of\gamma} = 0.76634; \\ \bar{\delta} &= 1.205; \quad Z_{f\gamma\delta} = 1.6415; \quad \hat{\delta} = 0.1025; \quad Z_{of\gamma\delta} = 1.1025; \\ \mu^*_{c\gamma}(x_1, x_2) &= -0.72321x_1^2 - 0.48889x_2 \end{aligned}$$

By Theorem 1, we can choose an upper bounding function as

$$V(x_1, x_2) = 1.3229x_1^2 - 3.5676\epsilon x_1^2 x_2 + 1.6415\epsilon x_2^2;$$

then, for $0 < \epsilon \le 0.11421$, the composite controller achieves the performance level 2.1 for the full-order system on $\mathcal{S}$. By Theorem 2, we can choose for the slow game an upper bounding function

$$V(x_1, x_2) = 1.0651x_1^2 - 3.0668\epsilon x_1^2 x_2 + 1.1025\epsilon x_2^2,$$

under which, for $0 < \epsilon \le 0.0096206$, the slow controller achieves the performance level 2.1 for the full-order system on $\mathcal{S}$.

To present some simulation results for this case also, we took the same initial conditions for the state, but changed the value of ϵ to 0.0095. Plots of the control signals, state trajectories and performance levels for the full-order system under composite and slow controllers are given in Figure 3 under the constant disturbance input $w(t) \equiv -6$. We see that both controllers stabilize the system, but neither of them keeps the state in the set $\mathcal{S}$; the composite controller though leads to better system response. Both controllers lead to a growing negative game cost, which shows that both achieve the performance bound 2.1 for the full-order system under this particular disturbance input, even though the state leaves the set $\mathcal{S}$. Notice that the game cost incurred by the composite controller goes to negative infinity at a much higher rate than that of the slow controller. This means that the composite controller is more robust than the slow controller. We also see that under the slow controller the game cost grows at a slower rate when the state x_1 exceeds a certain threshold. This phenomenon indicates that the slow controller is less robust for large values of $|x_1|$, which further implies that the two-time-scale analysis is less accurate for $|x_1|$ large. This corroborates the observation we had made earlier.

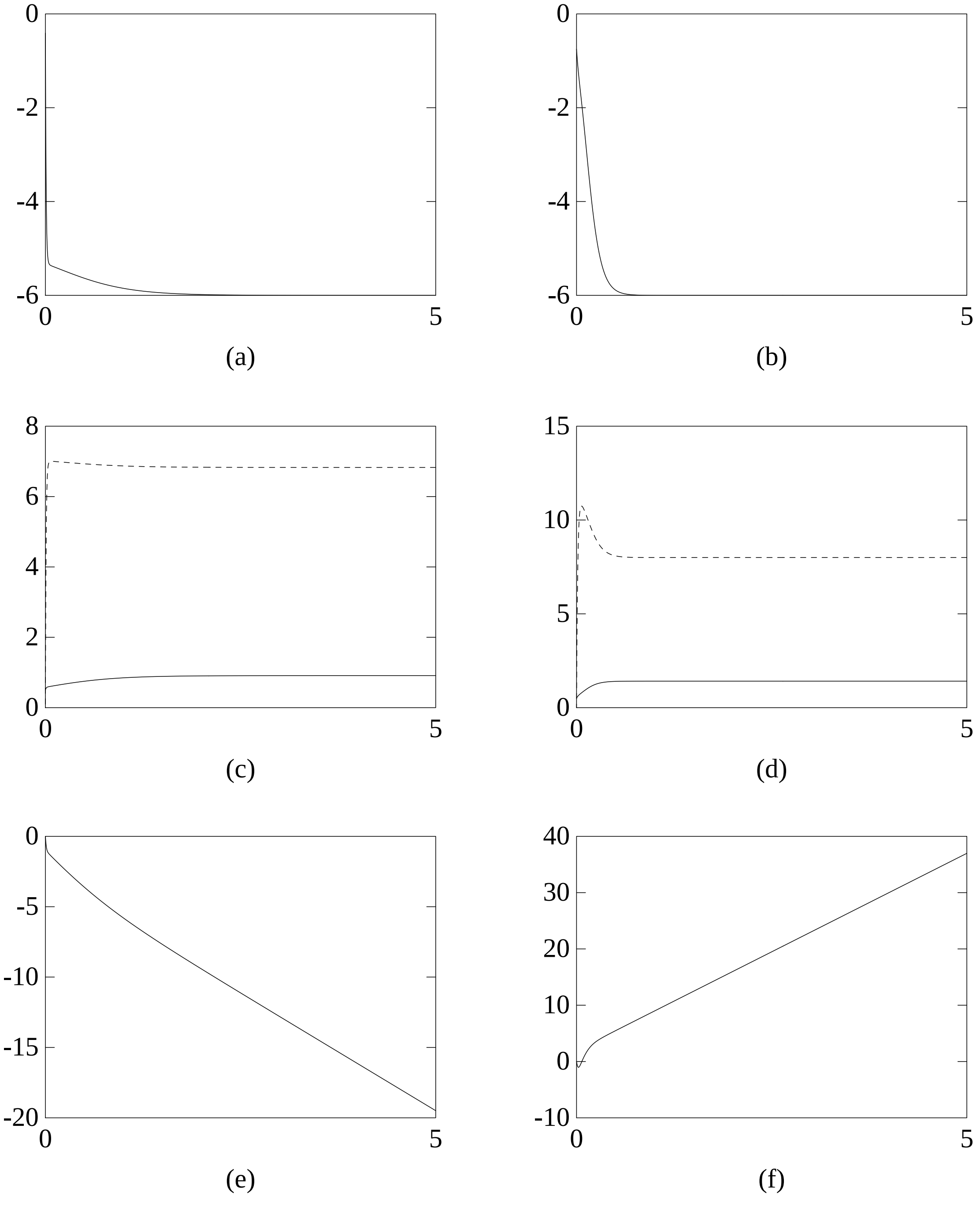

Figure 1: System responses for a constant disturbance; $\gamma = 1.5$
(a) composite control input; (b) slow control input; (c) state trajectories under composite controller; (d) state trajectories under slow controller; (e) game cost incurred under composite controller; (f) game cost incurred under slow controller.

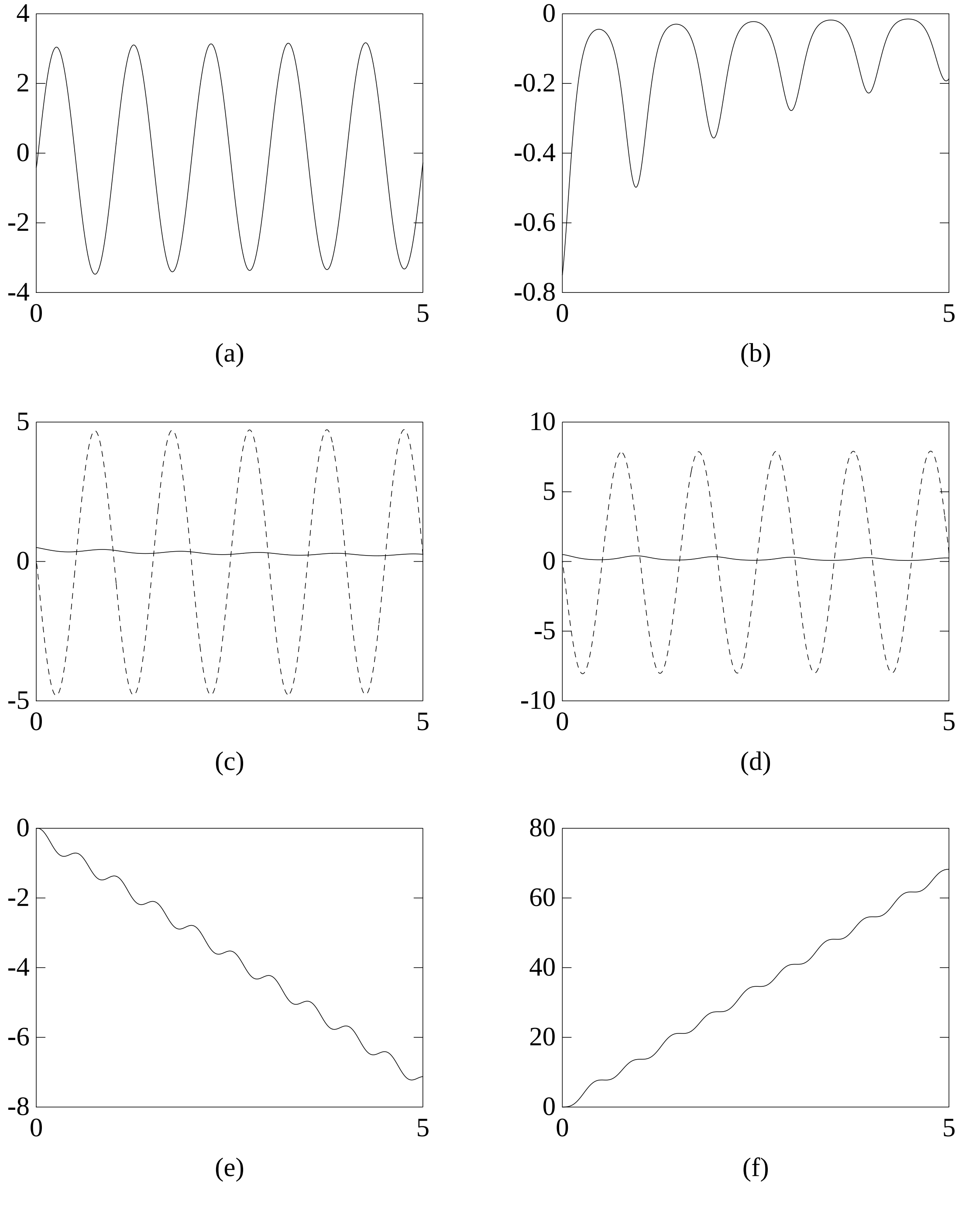

Figure 2: System responses for a sinusoidal disturbance; $\gamma = 1.5$
(a) composite control input; (b) slow control input; (c) state trajectories under composite controller; (d) state trajectories under slow controller; (e) game cost incurred under composite controller; (f) game cost incurred under slow controller.

Finally, we show in Figure 4 the plots of the control signals, state trajectories and performance levels for the full-order system under composite and slow controllers, under the sinusoidal disturbance $w(t) = 4\sin(2\pi t)$. We see that both controllers achieve the desired performance level for the full-order system under this particular disturbance input, and they keep the state in the set $\mathcal{S}$. Again, the composite controller leads to a better system response, and is more robust than the slow controller, which can be deduced from the rate at which the game cost goes to $-\infty$. $\diamond$

7 Conclusions

In this paper, we have provided a two-time-scale analysis for a class of singularly perturbed nonlinear control problems, where the dynamics are jointly linear in the fast state variables, control and disturbance inputs, but nonlinear in the slow state variables. We have taken the information structure to be perfect state, and have considered the infinite horizon formulation. One of the main results of the paper is the existence and construction of a composite controller, independent of the singular perturbation parameter, that achieves a reasonable performance for the overall system. More specifically, under the assumption that the slow controller is robust with respect to small structural perturbations for the slow subsystem, the composite controller leads to a specific (pre-computable) disturbance attenuation bound for the full-order control problem for sufficiently small values of the singular perturbation parameter under some growth conditions on some of the nonlinear terms. In the paper, we have also obtained a set of conditions under which a controller designed based on the slow subsystem only can achieve a desired performance level. These conditions ensure that a slow controller design is robust with respect to fast unmodeled dynamics.

As in [14], *robustness* is one reason why we have formulated and studied the class of problems addressed in this paper. In this interpretation, there is a nominal, known system, and one wishes to design a controller that yields good performance for a family of systems which are perturbed versions of the original one. Yet another motivation for our formulation has been the desire to obtain simpler controllers for complex nonlinear systems. In this interpretation the perturbed system could be considered as the complex system, and the nominal one is some lower-order approximation of it. What has been shown here is that, for the class of systems considered, and under some technical conditions, nonlinear systems (which are not necessarily linearizable) can be "approximated" well by simpler nonlinear systems.

Counterparts of these results have been obtained for the finite horizon case as well, and can be found in the fuller version of this paper (available from the authors). Several other extensions of these results are also possible. One future topic would be to study the relationship of these results with those obtainable from the singularly perturbed stochastic risk-sensitive optimal control problem ([20]) so as to obtain the counterparts of the results in [21] for a class of nonlinear systems. Another extension would be to study the problem under imperfect state measurements, so as to obtain the counterparts of the results in [19]; at the present this seems to be a challenging problem in its full generality unless some simplifying

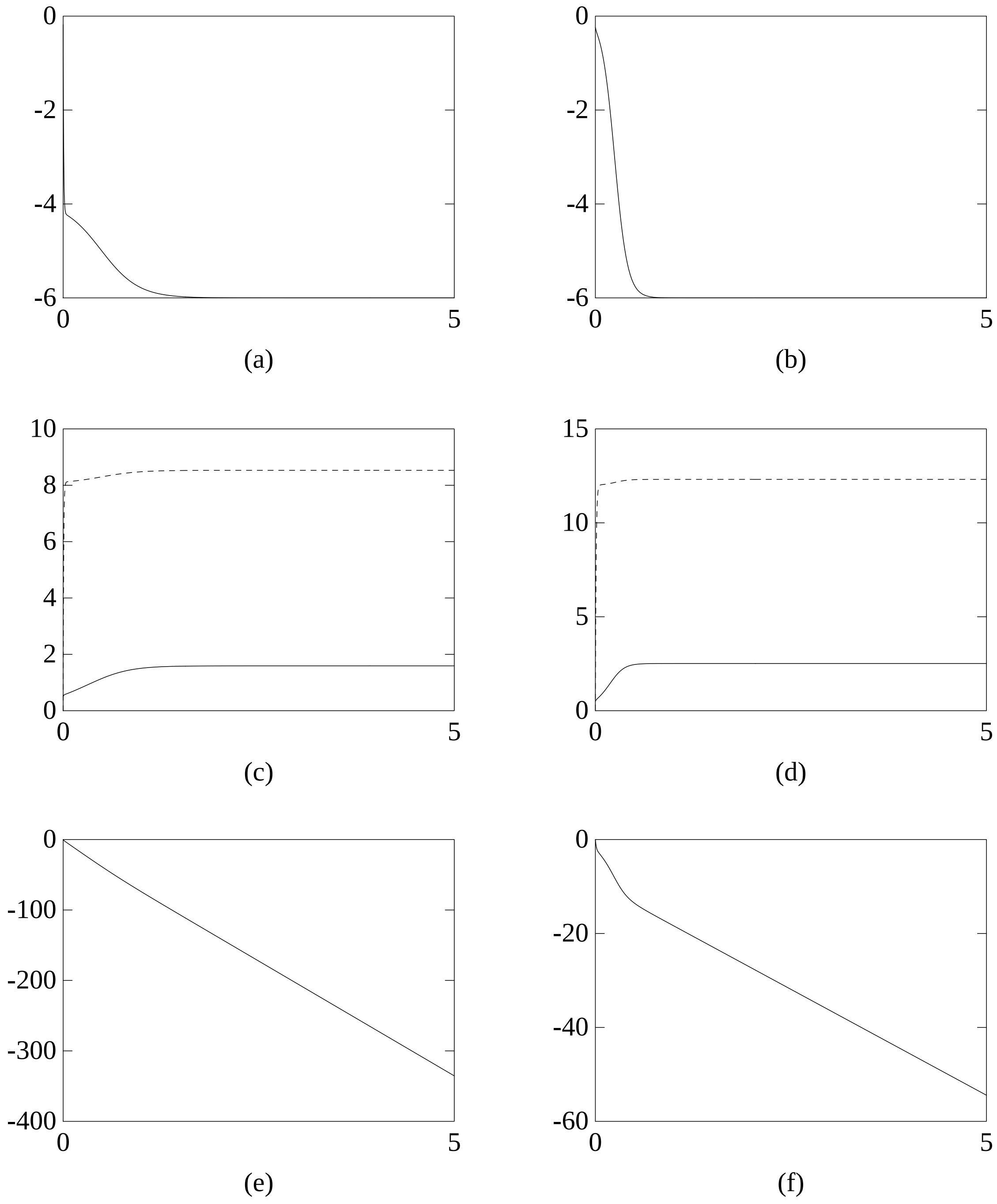

Figure 3: System responses for a constant disturbance; $\gamma = 2.1$
(a) composite control input; (b) slow control input; (c) state trajectories under composite controller; (d) state trajectories under slow controller; (e) game cost incurred under composite controller; (f) game cost incurred under slow controller.

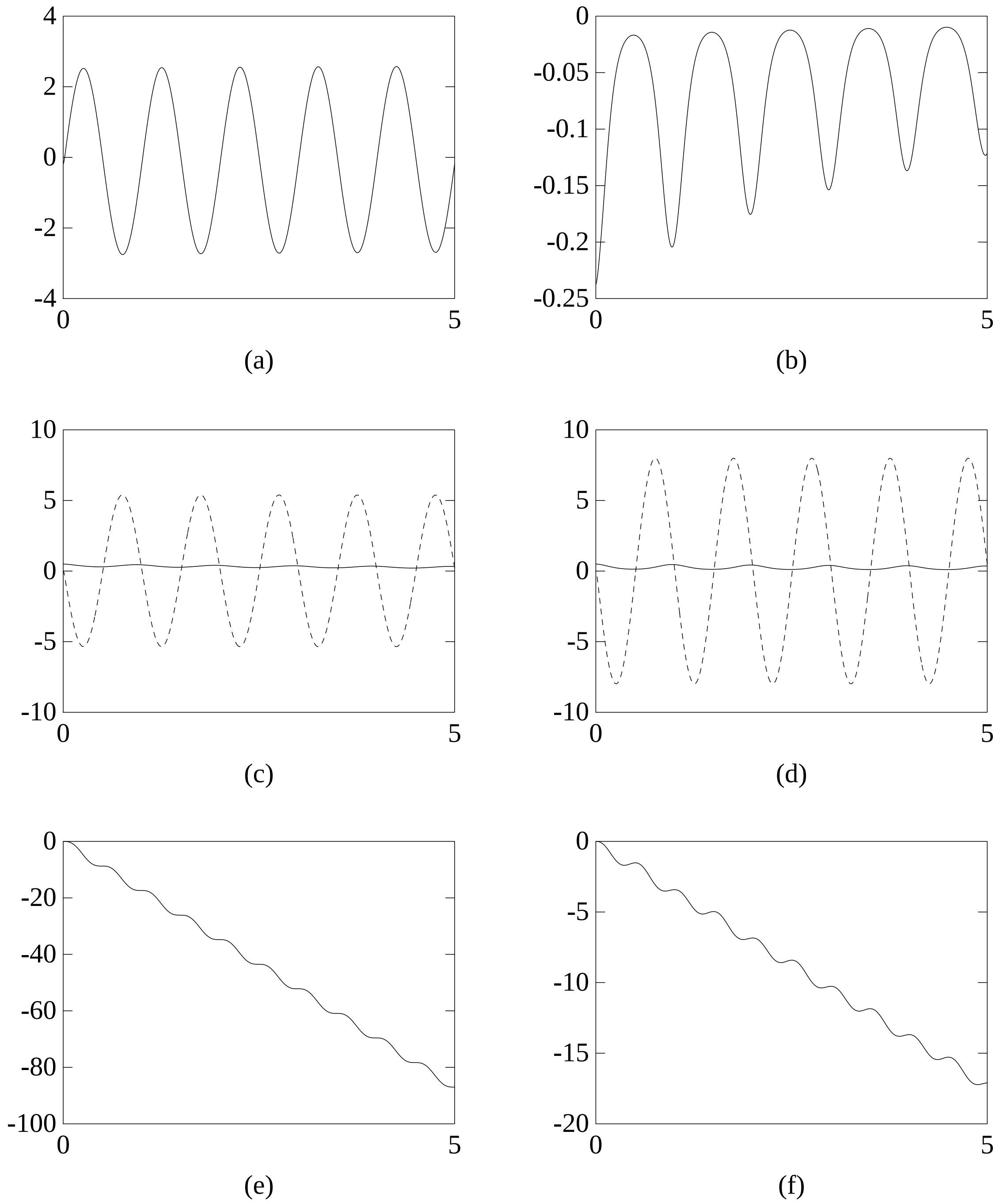

Figure 4: System responses for a sinusoidal disturbance; $\gamma = 2.1$
(a) composite control input; (b) slow control input; (c) state trajectories under composite controller; (d) state trajectories under slow controller; (e) game cost incurred under composite controller; (f) game cost incurred under slow controller.

structural assumptions are made.

Appendices

A A Lemma

Here we present (without proof) an important lemma which is repeatedly used in the derivation of some of the results in the main body of the paper, as well as in the next Appendix.

Lemma 1 *Let X_1, X_2, Y_1 and Y_2 be defined as in (24) and (25). Then the following relationships hold:*

1. $A_{22}X_1 + S_{22}Y_1 = -S_{21}$
2. $A_{22}X_2 + S_{22}Y_2 = a_{21}$
3. $Q_{22}X_2 - A'_{22}Y_2 = q'_{12}$
4. $Q_{22}X_1 - A'_{22}Y_1 = A'_{12}$
5. $A_{22}X + S_{22}Y = a_{21} - \frac{1}{2}S_{21}W_{x_1}$
6. $Q_{22}X - A'_{22}Y = q'_{12} + \frac{1}{2}A'_{12}W_{x_1}$

B Relaxation of the condition on Q_2

In this Appendix, we show that the statements of Theorems 1 and 2 hold also in the case when the matrix $Q_{22}(x_1)$ is not necessarily positive definite (but is nonnegative definite) on $\mathcal{S}_1$. For this, it will be sufficient to show that the functions $\tilde{a}_0$, $\tilde{S}_0$, $\tilde{q}$, X_1, X_2, Y_1 and Y_2 (some of which are matrix valued) are well-defined in the limit as Q_{22} becomes singular.

For a singular Q_{22}, perturb it by κI, $\kappa > 0$, so that $Q_{22} + \kappa I > 0$. By a possible abuse of notation, let us denote this new matrix (which is positive definite for all $\kappa > 0$) again by Q_{22}. The functions $\tilde{a}_0$, $\tilde{S}_0$, $\tilde{q}$, X_1, X_2, Y_1 and Y_2 are then well-defined for the new Q_{22}. We need only show that they converge to well-defined matrices as $\kappa \downarrow 0$.

Note that the function $q_{12}Q_{22}^{-1}$ converges as $\kappa \downarrow 0$, since q is nonnegative definite on $\mathcal{S}$. Then, the convergence of the functions $\tilde{q}$, $\tilde{S}_0$ and Y_2 follows. By Lemma 1, we have the relationship

$$X_2 = -A_{22}^{-1}(a_{21} - S_{22}Y_2).$$

Thus, the function X_2 also converges.

The function $\tilde{a}_0$ admits the following alternate expression, which can be proved by simple matrix manipulations:

$$\tilde{a}_0 \;=\; a_0 - \left[\begin{array}{cc} B_0 & D_0 \end{array}\right] \left[\begin{array}{c} B_2' \\ -\frac{1}{\gamma^2} D_2' \end{array}\right] (S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} (a_{21} - A_{22} Q_{22}^{-1} q_{12}')$$

Then, it is easy to see that the function $\tilde{a}_0$ is also well defined in the limit.

The convergence of Y_1 follows from the following three identities:

$$\begin{aligned}
&(S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} = (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} + (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} D_2 (\gamma^2 I \\
&\quad - D_2' (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} D_2)^{-1} D_2' (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} \\
&Y_1 = -(S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} S_{21} - (I + (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} D_2 (\gamma^2 I - D_2' (A_{22} Q_{22}^{-1} A_{22}' \\
&\quad + B_2 B_2')^{-1} D_2)^{-1} D_2')(A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} A_{22} Q_{22}^{-1} A_{12}' \\
&(A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} A_{22} Q_{22}^{-1} A_{12}' = ({A_{22}'}^{-1} Q_{22} A_{22}^{-1} - {A_{22}'}^{-1} Q_{22} A_{22}^{-1} B_2 (I + B_2' {A_{22}'}^{-1} Q_{22} \\
&\quad \cdot A_{22}^{-1} B_2)^{-1} B_2' {A_{22}'}^{-1} Q_{22} A_{22}^{-1}) A_{22} Q_{22}^{-1} A_{12}' \\
&\;=\; (I - {A_{22}'}^{-1} Q_{22} A_{22}^{-1} B_2 (I + B_2' {A_{22}'}^{-1} Q_{22} A_{22}^{-1} B_2)^{-1} B_2') {A_{22}'}^{-1} A_{12}'
\end{aligned}$$

The convergence of X_1 follows again from Lemma 1.

Thus, we see that all the derivations in the paper are valid as $\kappa \downarrow 0$. This then extends the proofs of Theorems 1 and 2 to the case when Q_{22} is not necessarily invertible.

Remark 1 *In view of the above, the assumption $Q_{22} > 0$, made in the previous work [13] can be similarly relaxed to the condition of observability of the pair (A_{22}, Q_{22}), for both infinite and finite horizon cases.* □

C A listing of some notation used

We list here some of the commonly used nonlinear functions introduced in Sections 3 and 4 of the paper, along with equation references. This should make it easier for the reader to follow the derivations given in these two sections.

Notation	*Reference*
$\tilde{a}_0$, $\tilde{S}_0$, $\tilde{q}$	(16)–(18)
Y, Y_1, Y_2	(24)
X, X_1, X_2	(25)
V_c	(36)
$Z_{f\gamma\delta}$	(38)
$\bar{a}_\epsilon$, $\bar{a}_{11}$, $\bar{A}_{12}$, $\bar{a}_{21}$, $\bar{A}_{22}$	(41)
$\bar{q}$, $\bar{q}_{11}$, $\bar{q}_{12}$, $\bar{q}_{22}$	(42)
$\bar{V}_c$, $\tilde{W}$	(44)
$\bar{\alpha}$, $\bar{\beta}$	(46)–(47)
$\hat{a}_\epsilon$, $\hat{a}_{11}$, $\hat{A}_{12}$, $\hat{a}_{21}$, $\hat{A}_{22}$	(51)
$\hat{q}$, $\hat{q}_{11}$, $\hat{q}_{12}$, $\hat{q}_{22}$	(52)
$Z_{of\gamma\delta}$	(48)
$\hat{V}_c$	(55)
$\hat{\alpha}$, $\hat{\beta}$	(57)–(58)

References

[1] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems," *IEEE Transactions on Automatic Control*, vol. AC-37, pp. 1283–1293, September 1992.

[2] A. J. Van der Schaft, "On a state-space approach to nonlinear H_∞ control," *Systems & Control Letters*, vol. 16, pp. 1–8, 1991.

[3] A. J. Van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear H_∞ control," *IEEE Transactions on Automatic Control*, vol. AC-37, pp. 770–784, 1992.

[4] A. Isidori, "New results on nonlinear H_∞-control via measurement feedback," in *Advances in Dynamic Games and Applications* (T. Başar and A. Haurie, eds.), pp. 56–69, Boston, MA: Birkhäuser, 1993.

[5] A. Isidori, "H_∞-control via measurement feedback for affine nonlinear systems," tech. rep., Washington University, St. Louis, Missouri, 1992.

[6] A. Isidori, "Feedback control of nonlinear systems," in *Proceedings of the 1st European Control Conference*, (Grenoble, France), pp. 1001–1012, July 1991.

[7] G. Didinsky, T. Başar, and P. Bernhard, "Structural properties of minimax controllers for a class of differential games arising in nonlinear H^∞-control," *Systems & Control Letters*, vol. 21, pp. 433–441, December 1993.

[8] J. A. Ball and J. W. Helton, "H^∞ control for nonlinear plants: Connections with differential games," in *Proceedings of the IEEE 28th Conference on Decision and Control*, (Tampa, FL), pp. 956–962, December 13-15, 1989.

[9] J. A. Ball, J. W. Helton, and M. Walker, "H^∞ control for nonlinear systems with output feedback," *IEEE Transactions on Automatic Control*, vol. AC-38, pp. 546–559, April 1993.

[10] L. Xie, C. E. de Souza, and M. D. Fragoso, "H_∞ filtering for linear periodic systems with parameter uncertainty," *Systems & Control Letters*, vol. 17, pp. 343–350, 1991.

[11] P. P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ control theory," *IEEE Transactions on Automatic Control*, vol. AC-35, pp. 356–361, 1990.

[12] L. Xie, C. E. de Souza, and M. Fu, "H_∞ estimation for linear discrete-time uncertain systems," *Int. J. Robust & Nonlinear Control*, vol. 2, pp. 111–123, 1991.

[13] Z. Pan and T. Başar, "H^∞-optimal control for singularly perturbed systems. Part I: Perfect state measurements," *IEEE Transactions on Automatic Control*, vol. AC-29, pp. 401–423, March 1993.

[14] Z. Pan and T. Başar, "Robustness of H^∞ controllers to nonlinear perturbations," in *Proceedings of the IEEE 32th Conference on Decision and Control*, vol. 2, (San Antonio, TX), pp. 1638–1643, December 1993.

[15] F. Garofalo and G. Leitmann, "A composite controller ensuring ultimate boundedness for a class of singularly perturbed uncertain systems," *Dynamics and Stability of Systems*, vol. 3, no. 3 & 4, pp. 135–145, 1988.

[16] M. Corless, G. Leitmann, and E. P. Ryan, "Control of uncertain systems with neglected dynamics," in *Deterministic Control of Uncertain Systems* (A. S. I. Zinober, ed.), ch. 12, pp. 252–268, London: IEE Press, 1990.

[17] F. Garofalo and G. Leitmann, "Nonlinear composite control of a class of nominally linear singularly perturbed uncertain systems," in *Deterministic Control of Uncertain Systems* (A. S. I. Zinober, ed.), ch. 13, pp. 269–288, London: IEE Press, 1990.

[18] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Boston, MA: Birkhäuser, 1991.

[19] Z. Pan and T. Başar, "H^∞-optimal control for singularly perturbed systems. Part II: Imperfect state measurements," *IEEE Transactions on Automatic Control*, vol. AC-39, pp. 280–299, February 1994.

[20] P. Whittle, "Likelihood and cost as path integrals," *J. R. Statist. Soc. B*, vol. 53, no. 3, pp. 505–538, 1991.

[21] Z. Pan and T. Başar, "Optimal control of stochastic singularly perturbed systems with exponentiated quadratic cost," in *Proceedings of the 12nd IFAC World Congress*, vol. IX, (Sydney, Australia), pp. 293–298, July 18-23, 1993.