Backstepping Design with Local Optimality Matching *†‡

Zigang Pan § Kenan Ezal ¶ Arthur J. Krener ‖ Petar V. Kokotović **

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Abstract

In this paper, we study the nonlinear H^∞-optimal control design for strict-feedback nonlinear systems. Our objective is to construct globally stabilizing control laws, which matches locally around $x = 0$ the optimal control design up to any desired order, say m, and is inverse optimal with respect to some computable cost function. The methodology we develop here provides an explicit recursive construction procedure for a cost function and the corresponding solution to the Hamilton-Jacobi-Isaacs equation for a *fixed* strict-feedback nonlinear system such that the cost function matches a desired cost function at $x = 0$ up to any desired order in their Taylor series expansions. This construction is based on the new concept of nonlinear Cholesky factorization for a given positive definite function. When the value function for the system has a nonlinear Cholesky factorization, we show that the backstepping design procedure can be tuned to yield the optimal control design for the problem. We develop an explicit recursive computation scheme for an inversely optimal controller that matches the optimal solution up to the desired order of expansion, based on the Taylor series expansion of the value function. The recursive design involves an approximation of the Cholesky factorization of the value function and a backstepping procedure.

1 Introduction

H^∞-optimal control has been an area of intense research in the past decade, which was first formulated in the frequency domain and then extended to state-space. The problem addresses

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‡All correspondence should be addressed to Dr. Kenan Ezal, Toyon Research Corporation, 75 Aero Camino, Suite A, Goleta, CA 93117. Tel: (805) 968-6787 x180; Fax: (805) 685-8089; Email: `kezal@toyon.com`.

§Center for Control Engineering and Computation, Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106. Present address: Department of Automation, Shanghai Jiao Tong University, Shanghai 200030, P. R. China. Tel: 86-21-62932320; Fax: 86-21-62933155; Email: `zggpan@263.net`.

¶Toyon Research Corporation, 75 Aero Camino, Suite A, Goleta, CA 93117, USA. Tel: (805) 968-6787 x180; Fax: (805) 685-8089; Email: `kezal@toyon.com`.

‖Department of Mathematics, University of California, Davis, CA 95616. Tel: (916) 752-6722; Fax: (916) 752-6635; Email: `ajkrener@ucdavis.edu`.

**Center of Control Engineering and Computation, Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106. Tel: (805) 893-7011; Fax: (805) 893-3262; Email: `petar@seidel.ece.ucsb.edu`.

one of the most fundamental questions in system and control theory regarding the sensitivity to uncertainties in the system. After a successful resolution of the linear quadratic problem, recent research attention has been focused on the robust control design for nonlinear systems, with non-quadratic cost functions [2], [3], [4], [5], [6]. In these research, the dynamic game theoretic approach has been adopted, [7], as it provides the most natural setting for a study of worst-case design problems. The solution to the nonlinear H^∞-optimal control problem is then prescribed in terms of a Hamilton-Jacobi-Isaacs (HJI) equation, which is very difficult to solve in general.

Although a general solution method is not available for HJI equations at present, there exists a local solution method when the underlying nonlinear system admits a controllable Jacobi linearization and the associated cost function has a Taylor series expansion with quadratic leading terms. Similar to the solution of the Hamilton-Jacobi-Bellman equation [8], [9], the solution to the HJI equation can be found locally around the origin of the system by computing the coefficients of its Taylor series expansion [10], [11], and [12]. This Taylor series solution provides an adequate approximation to the optimal control law in the vicinity of the origin whose size depends on the radius of convergence of the Taylor series. It is observed that in general the convergence holds within a finite radius of convergence. Therefore, the solution obtained is not satisfactory for applications requiring global system performances.

Recently, a series of breakthroughs have been witnessed in nonlinear control system theory. In the 80's, a special class of nonlinear systems have been identified that can be transformed into a linear control system under a coordinate diffeomorphism and state feedback, which is now known as feedback linearizable systems [13]. In the early 90's, a systematic control design procedure – integrator backstepping, was developed in [14] for nonlinear systems described in strict-feedback form. This control design approach has been further extended and perfected in a recent book [15]. For this class of nonlinear systems, the backstepping design offers a tremendous amount of flexibility in terms of the choice of the stabilizing virtual control input and additive value function at each step of the control design. Every valid choice leads to a stabilizing control design. Yet it is not understood, heretofore, what kind of choices will lead to a "better" overall design.

In this paper, we combine the Taylor series approach and the backstepping design approach to the H^∞ optimal control problem for strict-feedback nonlinear systems. Our objective is to construct globally stabilizing control laws, which matches locally around $x = 0$ the Taylor series expansion of the optimal control design up to any desired order, say m, and is inverse optimal with respect to some computable cost function. The methodology we develop provides an explicit recursive construction procedure for a cost function and the corresponding solution to the HJI equation for a *fixed* strict-feedback nonlinear system such that the cost function matches a desired cost function at $x = 0$ up to *any* desired order of expansion. This recursive construction is based on a new concept of nonlinear Cholesky factorization for a given positive definite function, which is introduced in the paper. It is defined as an upper triangular coordinate transformation, such that the given function is equal to the sum of squares of all the state variables when expressed in the new coordinates. We obtain precise conditions under which a given positive definite nonlinear function has a nonlinear Cholesky factorization. When the value function for the system has a nonlinear Cholesky factorization, we show that the backstepping design procedure can be tuned to yield the optimal control design for the problem. This justifies that the backstepping procedure is an optimality feasible methodology. Assuming that the Taylor series expansion for the value function has been obtained, we develop explicit recursive computation schemes for an inverse optimal controller that matches the optimal solution up to any desired order of Taylor series expansion. Using this approximating function and its nonlinear Cholesky factorization, we recursively construct the desired matching controller by another backstepping procedure. A simulation example has been presented to illustrate the theoretical findings. A more detailed discussion of the first order matching design can be found in [16].

Inverse optimal controller design for nonlinear systems has been investigated in earlier studies [17, 18]. In [17], robust control problems for systems subject to bounded disturbances are studied, for which robust inverse optimal controllers that guarantee input-to-state stability of the closed-loop systems are obtained. In [18], it is shown that input-to-state stability is both necessary and sufficient for inverse optimality of nonlinear systems. The objective of this paper, on the other hand, is to design inverse optimal controllers which achieve a desired level of $\mathcal{L}_2$ disturbance attenuation with local (mth-order) optimality.

The balance of the paper is as follows. In the next section, Section 2, we state the precise mathematical formulation of the problem. In Section 3, we present important results related to nonlinear Cholesky factorization, which are utilized for the backstepping control design in Section 4. A numerical example is presented in Section 5 to illustrate the theory developed in the main body of the paper. The paper ends with some concluding remarks in Section 6.

2 Problem Formulation

We consider the nonlinear H^∞ control for the following strict-feedback nonlinear system subject to additional disturbance inputs:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1) + h_1'(x_1)w && \text{(1a)}\\
\dot{x}_2 &= x_3 + f_2(x_1, x_2) + h_2'(x_1, x_2)w && \text{(1b)}\\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= u + f_n(x_1, \cdots, x_n) + h_n'(x_1, \ldots, x_n)w && \text{(1c)}
\end{aligned}
$$

where $x = (x_1, \ldots, x_n)'$ is the n-dimensional state variable, with initial state fixed at 0; u is the scalar control input; and w is the q-dimensional disturbance input. We note that, in addition to the strict-feedback nonlinear system [15], the above system also includes a set of disturbance inputs w, which multiplies a set of vector fields that are in strict-feedback form. This nonlinear system can be compactly written as $\dot{x} = f(x) + Bu + H(x)w$, where $B = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}'$.

We assume that the control input u is generated by the feedback control law: $u = \mu(x)$, where $\mu : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz. We will denote the set of all these control laws as $\mathcal{M}_u$.

We consider that the disturbance w to be a deterministic signal which is generated by some adversary player according to $w(t) = \nu(t, x)$, where $\nu : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^p$ is piecewise continuous in t and locally Lipschitz continuous in x. We will denote the set of of all these disturbance strategies by $\mathcal{M}_w$.

Associated with this nonlinear system, we introduce a cost function $J = \int_0^\infty l(x, u)\, dt$, where $l(x, u) = q(x) + r(x)u^2$.[1]

[1] We assume that the weighting function is quadratic in u. For the more general case $l(x, u) = q(x) + p(x)'u + r(x)u^2$, we can easily remove the cross term by a simple redefinition of the control variable $\tilde{u} = u + r^{-1}p(x)$.

The objective of the nonlinear H^∞ control design is to minimize the worst case disturbance attenuation level $\inf_{\mu\in\mathcal{M}_u}\sup_{\nu\in\mathcal{M}_w} J^{1/2}/\|w\| =: \gamma^*$, where $\|.\|$ denotes the $\mathcal{L}_2$ norm of a signal; the optimal performance level is henceforth denoted by γ^*. The nonlinear H^∞ optimal control problem then consists of the evaluation of the threshold γ^* and finding a controller $\mu \in \mathcal{M}_u$ that guarantees any desired disturbance attenuation level γ larger than γ^*.

This nonlinear H^∞ optimal control problem has been shown [7] to be closely related to a class of zero-sum differential games with the cost function indexed by the desired attenuation level γ:

$$J_\gamma = J - \gamma^2\|w\|^2 = \int_0^\infty (l(x,u) - \gamma^2 w'w)\,dt \tag{2}$$

For this differential game, the control is the minimizer and the disturbance is the maximizer. We are particularly interested in the upper value of the game: $\inf_{\mu\in\mathcal{M}_u}\sup_{\nu\in\mathcal{M}_w} J_\gamma$. Assuming the initial condition of the system is zero, for any $\gamma > \gamma^*$, the upper value of this zero-sum game is *zero* [2]. On the other hand, the upper value is strictly positive for any $\gamma < \gamma^*$. Because of the above stated equivalence, we will focus on the zero-sum differential game problem (2) for a given value of γ.

The following two assumptions serve as the basis for our study.

Assumption A1: The nonlinear functions f_i, h_i, $i = 1,\dots,n$, q and r are $\mathcal{C}^\infty$ (or simply smooth) in all of their arguments. The open loop system of (1) has the origin as an equilibrium, i.e., $f_i(x_1,\dots,x_i)|_{x=0} = 0$, $i = 1,\dots,n$. $\diamond$

Assumption A2: The weighting function $l(x,u)$ is positive definite and radially unbounded for x and u, and $l(0,0) = 0$. Furthermore, the Hessian of q with respect to x evaluated at 0 is a positive definite matrix Q_l. The function $r(x)$ is positive for any x. In particular $r(0) = R_l > 0$. $\diamond$

The solution to the nonlinear zero-sum differential game problem is given by the HJI equation:

$$V_x(x)f(x) + \frac{1}{4\gamma^2}V_x(x)H(x)H'(x)V_x'(x) - \frac{1}{4}V_x(x)Br^{-1}(x)B'V_x' + q(x) = 0 \tag{3}$$

and, when the solution is available, the optimal control law is given by

$$u^*(t) = \mu^*(x) = -\frac{1}{2}r^{-1}(x)B'V_x'. \tag{4}$$

In general, $V(x)$ is very difficult to obtain in an explicit form. Yet, it is relatively simple to compute the Taylor series expansions of the function $V(x)$ and the optimal control law μ^* around $x = 0$ (see for example [8], [9], [10], [2], [11], [12]). In general, these Taylor series expansions have finite radii of convergence. Therefore, they yield a near-optimal controller when x is close to the origin. But, when $|x|$ is large, the truncated Taylor series may not guarantee stability for the nonlinear system. In this paper, we will assume that the Taylor series expansions of the value function $V(x)$, and the minimax optimal control law $\mu^*(x)$ have been obtained using the methods prescribed in the references given above. This assumption will be formally introduced after we introduce our convention for the notation.

[2] In the case when the initial condition of the system is not zero but fixed, we have the following inequality if $\gamma > \gamma^*$, $\int_0^T l(x,u)\,dt \le \gamma^2\int_0^T w'w\,dt + C$, for some constant $C > 0$, which depends only on the initial condition, and for any terminal time $T > 0$.

Our control design objective is then to match the optimal control design up to any desired order of accuracy for $|x|$ sufficiently close to zero and to guarantee global inverse robust optimality for the closed-loop system. This concept of design objective is made precise with the following two definitions.

Definition 1 *A smooth control law $\mu(x)$ is* locally optimal matching to the mth order, *if, for a desired positive integer m, the truncation of the Taylor series expansion of $\mu(x)$ up to the mth order is equal to that of the worst-case optimal control law μ^*.* $\diamond$

Definition 2 *A smooth control law $\mu(x)$ is* inversely robust optimal, *if there exists a nonnegative function $\check{l}(x,u) = \check{q}(x) + \check{r}(x)u^2$ such that the zero-sum game with cost function*

$$J_{\gamma i} = \int_0^\infty (\check{l}(x,u) - \gamma^2|w|^2)\, dt \tag{5}$$

admits $\mu(x)$ as its minimax control law. Furthermore, the value function $V_{\gamma i}$ associated with the cost function $J_{\gamma i}$ and $\check{q}$ are radially unbounded, and the function $\check{r}$ is positive for any $x \in \mathbb{R}^n$. $\diamond$

Next, we introduce three conventions that we will adopt throughout the paper. Any function symbol with an "over bar" will denote a function defined in terms of the transformed state variables, such as $\bar{a}$ denoting the equivalent form of function a (of x) in terms of transformed state variable, z. Given any polynomial function V, we will denote its mth order homogeneous terms by $V_{[m]}$, and its homogeneous terms up to mth order by $V^{[m]}$. Then, it is clear that $V^{[m]} = \sum_{i=0}^m V_{[i]}$. For a smooth function V, we abuse the notation that $V_{[m]}$ denotes the mth order homogeneous terms in the Taylor series expansion of V around the origin, and $V^{[m]}$ denotes the homogeneous terms up to mth order in the Taylor series expansion. The variable $z = (z_1, \ldots, z_n)$ denotes the transformed coordinates for the nonlinear system.

Using this notation, we make the following basic assumption for our study in this paper.

Assumption A3: The Taylor series expansion of the optimal value function is given up to $(m+1)$st order $V^{[m+1]}(x) = V_{[2]}(x) + V_{[3]}(x) + \cdots + V_{[m+1]}(x)$, and the Taylor series expansion of the minimax control law $\mu^*(x)$ is given up to mth order $\mu^{*[m]}(x) = \mu^*_{[1]}(x) + \cdots + \mu^*_{[m]}(x) = -\frac{1}{2}(r^{-1}(x)B'V_x')^{[m]}$. $\diamond$

This completes the formulation of the problem. In the next section, we present the nonlinear Cholesky factorization that is crucial in the construction of the matching control design.

3 A Nonlinear Cholesky Factorization

Based on the result for the linear quadratic matching backstepping design [16], the key step that prescribes all the backstepping coefficients is the Cholesky factorization of the quadratic value function. In this section, we study the nonlinear version of the Cholesky factorization for a given nonlinear smooth positive definite function.

For any given positive definite and proper nonlinear function $V(x)$, we are interested in the necessary and sufficient conditions under which there exists global upper triangular diffeomorphism $z = \Phi(x)$ such that $V(x) = \Phi'(x)\Phi(x)$, where Φ is called upper triangular if it can be written as

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \phi_1(x_1, \dots, x_n) \\ \vdots \\ \phi_n(x_n) \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{6}$$

Without loss of generality, we will limit our attention to zero preserving transformations, that is, $0 = \Phi(0)$. It should be noted that it is the lower triangular transformation that is important for the backstepping control design, not the upper triangular ones. But, we pursue the upper triangular ones just to parallel the well known Cholesky factorization for positive definite matrices (quadratic functions). The result obtained below admit direct counterpart for lower triangular transformations if we reverse the ordering of the state variables $(x_1, \dots, x_n)$. It is obvious that, for any (but fixed) ordering of state variables $(x_1, \dots, x_n)$, there is an immediate counterpart for that particular ordering.

The Jacobian of the mapping Φ is given by

$$\frac{\partial \Phi}{\partial x} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & * & \cdots & * \\ 0 & \frac{\partial \phi_2}{\partial x_2} & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix}$$

where $*$ denotes terms of no particular interest. Since the mapping Φ is a diffeomorphism, the Jacobian is always nonsingular and the diagonal partial derivatives are nonzero, $\partial\phi_i/\partial x_i \neq 0 \quad \forall x \in \mathbb{R}$. Therefore, these quantities are sign definite. Without loss of generality, we consider only positive Φ, where the diagonal terms of $\dfrac{\partial \Phi}{\partial x}$ are positive. This motivates the following definition.

Definition 3 *An upper triangular diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be positive if the diagonal partial derivatives are nonzero, $\dfrac{\partial \phi_i}{\partial x_i} > 0$, $i = 1, \dots, n$ for any $x \in \mathbb{R}^n$.* $\diamond$

For a given proper function $V(x)$, where $x \in \mathbb{R}^n$. We define its nonlinear Cholesky factorization as follows.

Definition 4 *A proper $\mathcal{C}^\infty$ function V is said to admit a nonlinear Cholesky factorization if $V = \Phi'(x)\Phi(x)$ for some global positive upper triangular diffeomorphism with $\Phi(0) = 0$.* $\diamond$

It will be shown later that the nonlinear Cholesky factorization is unique within the class of positive diffeomorphisms.

Given a positive upper triangular state transformation $z = \Phi(x)$, it is still quite difficult to obtain the inverse transformation of Φ. To simplify this, we define the simple upper triangular diffeomorphism.

Definition 5 *An upper triangular diffeomorphism* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *is said to be simple if* $\dfrac{\partial \phi_i}{\partial x_i} \equiv 1$, $i = 1, \dots, n$. *Then, the individual coordinate map is given by* $\phi_i(x_i, \dots, x_n) = x_i - \alpha_i(x_{i+1}, \dots, x_n)$, $i = 1, \dots, n$, *where* α_i, $i = 1, \dots, n$ *(we have set* $\alpha_n = 0$*) are arbitrary smooth functions such that* $\alpha_i(0) = 0$. ◇

There exists an equivalence relationship between the positive upper triangular diffeomorphisms and simple ones in terms of yet another factorization. This relationship is characterized in the following proposition.

Proposition 1 *An upper triangular diffeomorphism* Φ *is positive if and only if it can be uniquely factored as* $\Phi = \Delta \Phi_s$, *where* Φ_s *is a simple upper triangular diffeomorphism; and* Δ *is a diagonal matrix*

$$\Delta = diagonal\ \{\delta_1(x_1, \dots, x_n), \delta_2(x_2, \dots, x_n), \dots, \delta_n(x_n)\} \tag{7}$$

such that, for each $i = 1, \dots, n$,

$$\delta_i(x_i, \dots, x_n) > 0 \tag{8}$$

$$\pi_{\delta i}(x_i, \dots, x_n) := \delta_i(x_i, \dots, x_n) + (x_i - \alpha_i(x_{i+1}, \dots, x_n)) \frac{\partial \delta_i}{\partial x_i}(x_i, \dots, x_n) > 0 \tag{9}$$

$$\lim_{|x_i| \to \infty} |(x_i - \alpha_i(x_{i+1}, \dots, x_n)) \delta_i(x_i, \dots, x_n)| = \infty \qquad \forall i = 1, \dots, n \tag{10}$$

for all $x \in \mathbb{R}^n$.

Proof The sufficiency part is straightforward by observing the diagonal elements of the Jacobian of $\Delta\Phi_s$ is exactly diagonal$\{\pi_{\delta 1}, \dots, \pi_{\delta n}\}$, which is nonsingular. Furthermore, the unboundedness condition (10) implies that the diffeomorphism Φ is global.

For the necessity part, we observe that $\phi_i(x_i, \dots, x_n) = 0$, for any $i = 1, \dots, n$, has a unique root, which is given by $x_i = \alpha_i(x_{i+1}, \dots, x_n)$ when $i = 1, \dots, n-1$, or $x_n = 0 =: \alpha_n$. The functions α_i, $i = 1, \dots, n$, are smooth by the positiveness of $\dfrac{\partial \phi_i}{\partial x_i}$. This uniquely defines a simple upper triangular diffeomorphism Φ_s:

$$\eta = \Phi_s(x) = \begin{bmatrix} x_1 - \alpha_1(x_2, \dots, x_n) \\ \vdots \\ x_n \end{bmatrix}$$

Then, the diagonal matrix Δ is determined, with each individual terms defined by

$$\delta_i(x_i, \dots, x_n) := \frac{\phi_i(x_i, \dots, x_n)}{x_i - \alpha_i(x_{i+1}, \dots, x_n)}$$

The matrix Δ thus defined is positive and smooth.

Evaluating the partial derivative, $\delta_i + \eta_i \frac{\partial \delta_i}{\partial x_i} = \frac{\partial \phi_i}{\partial x_i} > 0$, $i = 1, \dots, n$, where $\eta_i = x_i - \alpha_1(x_{i+1}, \dots, x_n)$. This proves that Δ satisfies (9).

The property (10) is satisfied since $\lim_{|x_i| \to \infty} |\phi_i| = \infty$.

This completes the proof. $\square$

To obtain the nonlinear Cholesky factorization for a given proper function V with n independent variables, we will go through n steps of recursive construction. The objective of each step of the recursive construction is to find one coordinate transformation to reduce the number of independent variables by 1. The following lemma is the key technical tool used in each step of the factorization.

Lemma 1 *Consider a nonlinear function $V(\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{a}$ is a scalar and $\boldsymbol{b}$ is a vector. Assume the following conditions hold:*

C1 *V is $\mathcal{C}^\infty$ in all of its arguments.*

C2 *There is a constant $C \geq 0$ such that, for each fixed $\boldsymbol{b}$, $V(\boldsymbol{a}, \boldsymbol{b}) \geq C$, $\forall \boldsymbol{a} \in \mathbb{R}$.*

C3 *For each fixed $\boldsymbol{b}$, V is radially unbounded in $\boldsymbol{a}$, i. e. , $\lim_{|\boldsymbol{a}| \to \infty} V(\boldsymbol{a}, \boldsymbol{b}) = +\infty$.*

Then, there exists a unique mapping $\phi(\boldsymbol{a}, \boldsymbol{b})$ such that

1. *ϕ is $\mathcal{C}^\infty$ in all of its the arguments.*
2. *For each fixed $\boldsymbol{b}$, $\dfrac{\partial \phi}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b}) > 0$, $\forall \boldsymbol{a}$.*
3. *For each fixed $\boldsymbol{b}$, ϕ maps $\mathbb{R}$ onto $\mathbb{R}$, i. e. , $\lim_{|\boldsymbol{a}| \to \infty} |\phi(\boldsymbol{a}, \boldsymbol{b})| = \infty$.*
4. *The function V can be decomposed as*

$$V(\boldsymbol{a}, \boldsymbol{b}) \quad = \quad \hat{V}(\boldsymbol{b}) + \phi^2(\boldsymbol{a}, \boldsymbol{b}) \tag{11}$$

for some $\mathcal{C}^\infty$ function $\hat{V}$, such that $\hat{V}(\boldsymbol{b}) \geq C$ for any $\boldsymbol{b}$.

if and only if the following condition holds:

E1 *For each fixed $\boldsymbol{b}$, $\dfrac{\partial V}{\partial \boldsymbol{a}} = 0$ has a unique root $\boldsymbol{a} = \alpha(\boldsymbol{b})$, which is simple, i. e. ,*

$$\frac{\partial V}{\partial \boldsymbol{a}}(\alpha(\boldsymbol{b}), \boldsymbol{b}) \equiv 0; \qquad \frac{\partial^2 V}{\partial \boldsymbol{a}^2}(\alpha(\boldsymbol{b}), \boldsymbol{b}) \neq 0 \qquad \forall \boldsymbol{b}$$

Proof First, we prove sufficiency. Let conditions **E1**, **C1**–**C3** hold. Conditions **C2** and **C3** imply that, for each fixed $\boldsymbol{b}$, the function V has at least a local minimum in $\boldsymbol{a}$. This, coupled with the smoothness assumption **C1** and uniqueness assumption **E1**, implies that the the minimum is achieved at $\boldsymbol{a} = \alpha(\boldsymbol{b})$. Furthermore, we have that the second partial differential $\dfrac{\partial^2 V}{\partial \boldsymbol{a}^2}(\alpha(\boldsymbol{b}), \boldsymbol{b}) > 0$, $\forall \boldsymbol{b}$, because it is a minimum. The nonsingularity of the partial derivative $\dfrac{\partial^2 V}{\partial \boldsymbol{a}^2}(\alpha(\boldsymbol{b}), \boldsymbol{b})$ guarantees that the root function $\alpha(\boldsymbol{b})$ is $\mathcal{C}^\infty$ by the implicit function theorem.

Let $\hat{V}$ and π be defined as

$$\hat{V}(\boldsymbol{b}) \quad := \quad V(\alpha(\boldsymbol{b}), \boldsymbol{b}) \tag{12}$$

$$(\boldsymbol{a} - \alpha(\boldsymbol{b}))\pi(\boldsymbol{a}, \boldsymbol{b}) \quad := \quad \frac{\partial V}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b}) \tag{13}$$

It directly follows from the above definition of $\hat{V}$ and **C2** that $\hat{V}(\boldsymbol{b}) \geq C$ for any $\boldsymbol{b}$.

We note that the function π is well defined by the fact that $\boldsymbol{a} = \alpha(\boldsymbol{b})$ is the root of the $\mathcal{C}^\infty$ function $\dfrac{\partial V}{\partial \boldsymbol{a}}$. Because of the uniqueness assumption **E1** and the fact that $\alpha(\boldsymbol{b})$ achieves the minimum, we have that $\partial V/\partial \boldsymbol{a} > 0$, $\forall \boldsymbol{a} > \alpha(\boldsymbol{b})$; $\partial V/\partial \boldsymbol{a} < 0$, $\forall \boldsymbol{a} < \alpha(\boldsymbol{b})$. Therefore, the function π is positive, i.e., $\pi(\boldsymbol{a}, \boldsymbol{b}) > 0$, $\forall (\boldsymbol{a}, \boldsymbol{b})$. This allows the following definition of a positive function $m(\boldsymbol{a}, \boldsymbol{b})$:

$$V(\boldsymbol{a}, \boldsymbol{b}) - V(\alpha(\boldsymbol{b}), \boldsymbol{b}) =: m(\boldsymbol{a}, \boldsymbol{b})(\boldsymbol{a} - \alpha(\boldsymbol{b}))^2 \tag{14}$$

since

$$V(\boldsymbol{a}, \boldsymbol{b}) - V(\alpha(\boldsymbol{b}), \boldsymbol{b}) \;=\; \int_{\alpha(\boldsymbol{b})}^{\boldsymbol{a}} (s - \alpha(\boldsymbol{b}))\pi(s, \boldsymbol{b})\, ds \tag{15}$$

By the above equality, the function $m(\boldsymbol{a}, \boldsymbol{b})$ is $\mathcal{C}^\infty$.

The mapping ϕ is then defined by

$$\phi(\boldsymbol{a}, \boldsymbol{b}) \;:=\; (\boldsymbol{a} - \alpha(\boldsymbol{b}))\delta(\boldsymbol{a}, \boldsymbol{b}); \qquad \delta(\boldsymbol{a}, \boldsymbol{b}) := \sqrt{m(\boldsymbol{a}, \boldsymbol{b})} \tag{16}$$

Statement 1 follows from the definition of ϕ and δ. Statement 3 holds by the assumption **C3** and the definition of ϕ. It is obvious that statement 4 holds by the definition of $\hat{V}$ and ϕ.

For statement 2, we evaluate the partial derivative:

$$\frac{\partial \phi}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b}) \;=\; \delta(\boldsymbol{a}, \boldsymbol{b}) + (\boldsymbol{a} - \alpha(\boldsymbol{b}))\frac{\partial \delta}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b}) =: \pi_\delta(\boldsymbol{a}, \boldsymbol{b})$$

Evaluating the partial derivative of V with respect to $\boldsymbol{a}$, yields the following relationship,

$$\begin{aligned}
\frac{\partial V}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b}) &= \frac{\partial}{\partial \boldsymbol{a}}(\delta^2(\boldsymbol{a}, \boldsymbol{b})(\boldsymbol{a} - \alpha(\boldsymbol{b}))^2) \\
&= 2(\boldsymbol{a} - \alpha(\boldsymbol{b}))\delta(\boldsymbol{a}, \boldsymbol{b})(\delta(\boldsymbol{a}, \boldsymbol{b}) + (\boldsymbol{a} - \alpha(\boldsymbol{b}))\frac{\partial \delta}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b})) \\
&= (\boldsymbol{a} - \alpha(\boldsymbol{b}))\pi(\boldsymbol{a}, \boldsymbol{b})
\end{aligned}$$

Hence, we have the identity: $\dfrac{\partial \phi}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b}) = \dfrac{\pi(\boldsymbol{a}, \boldsymbol{b})}{2\delta(\boldsymbol{a}, \boldsymbol{b})} > 0$, $\forall (\boldsymbol{a}, \boldsymbol{b})$. This then proves statement 2. Hence, the mapping $\phi(\boldsymbol{a}, \boldsymbol{b})$ constructed satisfies the statements 1–4.

Next, we show that this mapping ϕ is unique, which further implies the function $\hat{V}$ is unique. We will prove this by contradiction. Suppose there exists another mapping ϕ_*, which corresponds to a function $\hat{V}_*$, that also satisfies the statements 1–4, and $\phi(\boldsymbol{a}_0, \boldsymbol{b}_0) \neq \phi_*(\boldsymbol{a}_0, \boldsymbol{b}_0)$, for some $(\boldsymbol{a}_0, \boldsymbol{b}_0)$. Because $V = \hat{V} + \phi^2 = \hat{V}_* + \phi_*^2$, we have *either* (a) $\hat{V}(\boldsymbol{b}_0) \neq \hat{V}_*(\boldsymbol{b}_0)$ *or* (b) $\phi(\boldsymbol{a}_0, \boldsymbol{b}_0) = -\phi_*(\boldsymbol{a}_0, \boldsymbol{b}_0) \neq 0$.

In case of (b), we observe the following equality:

$$\frac{\partial V}{\partial \boldsymbol{a}}(\boldsymbol{a}_0, \boldsymbol{b}_0) \;=\; 2\phi(\boldsymbol{a}_0, \boldsymbol{b}_0)\frac{\partial \phi}{\partial \boldsymbol{a}}(\boldsymbol{a}_0, \boldsymbol{b}_0) = 2\phi_*(\boldsymbol{a}_0, \boldsymbol{b}_0)\frac{\partial \phi_*}{\partial \boldsymbol{a}}(\boldsymbol{a}_0, \boldsymbol{b}_0)$$

This is a contradiction because both $\dfrac{\partial \phi}{\partial \boldsymbol{a}}(\boldsymbol{a}_0, \boldsymbol{b}_0)$ and $\dfrac{\partial \phi_*}{\partial \boldsymbol{a}}(\boldsymbol{a}_0, \boldsymbol{b}_0)$ are positive by statement 2.

In case of (a), we conclude that $\phi(\boldsymbol{a},\boldsymbol{b}_0) \neq \phi_*(\boldsymbol{a},\boldsymbol{b}_0)$, $\forall \boldsymbol{a}$. In particular, $0 = \phi(\alpha(\boldsymbol{b}_0),\boldsymbol{b}_0) \neq \phi_*(\alpha(\boldsymbol{b}_0),\boldsymbol{b}_0)$. This leads to the contradiction:

$$0 = 2\phi(\alpha(\boldsymbol{b}_0),\boldsymbol{b}_0)\frac{\partial \phi}{\partial \boldsymbol{a}}(\alpha(\boldsymbol{b}_0),\boldsymbol{b}_0) = \frac{\partial V}{\partial \boldsymbol{a}}(\alpha(\boldsymbol{b}_0),\boldsymbol{b}_0) \quad = \quad 2\phi_*(\alpha(\boldsymbol{b}_0),\boldsymbol{b}_0)\frac{\partial \phi_*}{\partial \boldsymbol{a}}(\alpha(\boldsymbol{b}_0),\boldsymbol{b}_0) \neq 0$$

where the last inequality follows from statement 2. Hence, in both cases, it leads to contradictions. Consequently, the hypothesis is not valid, and the mapping ϕ is uniquely defined. This completes the sufficiency proof.

To prove necessity, we assume that there exists a mapping ϕ satisfying the statements 1–4. Then, $\frac{\partial V}{\partial \boldsymbol{a}}(\boldsymbol{a},\boldsymbol{b}) = 2\phi(\boldsymbol{a},\boldsymbol{b})\frac{\partial \phi}{\partial \boldsymbol{a}}(\boldsymbol{a},\boldsymbol{b})$. It is concluded that $\frac{\partial V}{\partial \boldsymbol{a}}(\boldsymbol{a},\boldsymbol{b})$ vanishes if and only if $\phi(\boldsymbol{a},\boldsymbol{b}) = 0$, because of statement 2. Again by statement 2, ϕ is strictly increasing in $\boldsymbol{a}$. Coupled with radially unboundedness statement 3, we conclude that there is a unique root $\boldsymbol{a} = \alpha(\boldsymbol{b})$ for the equation $\phi(\boldsymbol{a},\boldsymbol{b}) = 0$. This coupled with conditions **C2** and **C3** implies that, for each fixed $\boldsymbol{b}$, the function V has a unique minimum in $\boldsymbol{a}$. Partial differentiate it with respect to $\boldsymbol{a}$ once more, we obtain: $\frac{\partial^2 V}{\partial \boldsymbol{a}^2}(\alpha(\boldsymbol{b}),\boldsymbol{b}) = 2\left(\frac{\partial \phi}{\partial \boldsymbol{a}}(\alpha(\boldsymbol{b}),\boldsymbol{b})\right)^2 > 0$. Therefore, the result E1 follows, and consequently, the necessity part of the lemma.

This completes the proof for this lemma. □

We observe that the construction of the mapping ϕ in the above lemma involves the following steps. First, we solve for the unique smooth root function $\alpha(\boldsymbol{b})$, which is guaranteed to exist by the condition **E1**. Next, we can define the function $\hat{V}$ by (12). Then, mapping ϕ is given by Equation (16).

Now, we are in a position to present the n steps of recursive construction for the nonlinear Cholesky factorization of a given smooth proper function V.

Step 1: For notational consistency in the recursive construction, we let $V(x) =: V_1(x)$. Since $V(x)$ is $\mathcal{C}^\infty$ and proper, we can easily check that the conditions **C1**–**C3** of Lemma 1 are satisfied for V, with $\boldsymbol{a} = x_1$ and $\boldsymbol{b} = (x_2,\dots,x_n)'$.

As delineated in the lemma, in order for a smooth transformation $\phi_1(x)$ to exist we make the following assumption:

B1 For any fixed $n-1$ dimensional vector $(x_2,\dots,x_n)'$, the algebraic equation $\frac{\partial V_1}{\partial x_1} = 0$ has a unique root $x_1 = \alpha_1(x_2,\dots,x_n)$, which is simple, i. e., $\frac{\partial^2 V_1}{\partial x_1^2}(\alpha_1(x_2,\dots,x_n),x_2,\dots,x_n) \neq 0$, $\forall (x_2,\dots,x_n)'$.

Under this assumption, there exists a smooth mapping $z_1 = \phi_1(x_1,\dots,x_n)$ and a smooth nonlinear function $V_2(x_2,\dots,x_n)$ such that

R1.1 For any fixed $(x_2,\dots,x_n)'$, $\frac{\partial \phi_1}{\partial x_1}(x_1,\dots,x_n) > 0$, $\forall x_1 \in \mathbb{R}$.

R1.2 For each fixed $(x_2,\dots,x_n)'$, ϕ_1 maps $\mathbb{R}$ onto $\mathbb{R}$, i. e., $\lim_{|x_1|\to\infty} |\phi_1(x_1,\dots,x_n)| = \infty$.

R1.3 The function V_1 can be decomposed as $V_1(x_1,\dots,x_n) = V_2(x_2,\dots,x_n) + \phi_1^2(x_1,\dots,x_n)$.

We note here that the assumption **B1** is also necessary for the existence of the transformation ϕ_1 as proved in Lemma 1. Because the zero vector is the global minimum of the function V, we have:

R1.4 The mapping ϕ_1 is zero preserving, i. e., $\phi_1(0, \dots, 0) = 0$.

By the fact that V_1 is proper and smooth, we can conclude that the nonlinear function V_2 is also proper (when restricted to the $n-1$ dimensional subspace $(x_2, \dots, x_n)'$) and smooth, based on the equality: $V_1(\alpha_1(x_2, \dots, x_n), x_2, \dots, x_n) \equiv V_2(x_2, \dots, x_n)$. This nonlinear function V_2 is uniquely defined by Lemma 1, and hence, the construction of the later steps is *independent* of the construction at this first step.

This completes the first step of the recursive construction.

Step i**,** $2 \le i \le n$: By induction argument, we directly consider the ith step, for some $i = 2, \dots, n$. We assume inductively that the mappings $\phi_j(x_j, \dots, x_n)$ are constructed from the previous steps $1, \dots, i-1$, and we are given a proper smooth nonlinear function $V_i(x_i, \dots, x_n)$, that satisfies $V(x) = \phi_1^2 + \cdots + \phi_{i-1}^2 + V_i(x_i, \dots, x_n)$.

The properness and smoothness of V_i implies conditions **C1**–**C3** of Lemma 1. We make the following assumption, in order for a smooth mapping $\phi_i(x)$ to exist,

Bi For any fixed $n-i$ dimensional vector $(x_{i+1}, \dots, x_n)'$, the partial derivative $\dfrac{\partial V_i}{\partial x_i} = 0$ has a unique root $x_i = \alpha_i(x_{i+1}, \dots, x_n)$, which is simple, i. e., $\dfrac{\partial^2 V_i}{\partial x_i^2}(\alpha_i(x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) \neq 0$, $\forall (x_{i+1}, \dots, x_n)'$.

Under this assumption, there exists a smooth transformation $z_i = \phi_1(x_i, \dots, x_n)$ and a smooth nonlinear function $V_{i+1}(x_{i+1}, \dots, x_n)$ such that

Ri**.1** For any fixed $(x_{i+1}, \dots, x_n)'$, $\dfrac{\partial \phi_i}{\partial x_i}(x_i, \dots, x_n) > 0$, $\forall x_i \in \mathbb{R}$.

Ri**.2** For each fixed $(x_i, \dots, x_n)'$, ϕ_i maps $\mathbb{R}$ onto $\mathbb{R}$, i. e., $\lim_{|x_i| \to \infty} |\phi_i(x_i, \dots, x_n)| = \infty$.

Ri**.3** The function V_i can be decomposed as $V_i(x_i, \dots, x_n) = V_{i+1}(x_{i+1}, \dots, x_n) + \phi_i^2(x_i, \dots, x_n)$.

Again, the assumption **B**i is necessary for the existence of the transformation ϕ_i as shown in Lemma 1. The mapping ϕ_i is also zero preserving, since V_i admits a global minimum at $(x_i, \dots, x_n) = (0, \dots, 0)$.

Ri**.4** The mapping ϕ_i is zero preserving, i. e., $\phi_i(x_i, \dots, x_n)|_{(x_i, \dots, x_n) = (0, \dots, 0)} = 0$.

The following equality holds, as prescribed by the construction of Lemma 1:

$$V_i(\alpha_i(x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) \equiv V_{i+1}(x_{i+1}, \dots, x_n)$$

Therefore, by the fact that V_i is proper and smooth, we can conclude that the nonlinear function V_{i+1} is also proper and smooth. Furthermore, this nonlinear function V_{i+1} is uniquely defined by Lemma 1, and hence, the construction of the later steps is *independent* of the construction at this ith step.

Furthermore, on the nth step, we have $V_{n+1} \equiv 0$, which is equivalent to

$$V(x) \equiv \phi_1^2 + \cdots + \phi_n^2 \tag{17}$$

This completes the recursive construction of the nonlinear Cholesky factorization.

Summarizing the above derivation, we can state the following nonlinear Cholesky factorization theorem:

Theorem 1 *Consider a positive definite, proper and smooth nonlinear function $V(x)$, where $x \in \mathbb{R}^n$. The function V has a unique nonlinear Cholesky factorization $z = \Phi(x)$, as defined in Definition 4, if and only if the assumptions* **B1**–**B**n *are satisfied. Furthermore, if V can be uniquely factored, then, it can be factored in terms of a simple upper triangular diffeomorphism $\Phi_s(x)$, as*

$$V(x) = \Phi_s'(x)\Delta(x)\Delta(x)\Phi_s(x) \tag{18}$$

where Δ is a diagonal matrix that satisfies (7)–(10).

Proof We first prove sufficiency. Let assumptions **B1**–**B**n hold. By the recursive construction described preceding the theorem, there exists triangular mappings $\Phi = (\phi_1, \dots, \phi_n)'$ such that the conditions **R1.1**–**R1.4**, ... , **R**n**.1**–**R**n**.4** are satisfied. Furthermore, the equality (17) holds. These conditions directly imply that Φ is a positive upper triangular diffeomorphism. This completes the sufficiency proof.

Next, we prove the necessity part of the theorem by using the necessity result of Lemma 1. Let the function V admit a nonlinear Cholesky factorization. We will first consider the mapping ϕ_1. Define $V_2(x_2, \dots, x_n) = z_2^2 + \dots + z_n^2$. Since V is proper and smooth, it satisfies conditions **C1**–**C3** of Lemma 1. In addition to this, the definition of the nonlinear Cholesky factorization implies that ϕ_1 and V_2 satisfies the statements 1–4 of the Lemma. Then, Lemma 1 concludes that the condition **E1** holds, which is equivalent to **B1**. Therefore, the assumption **B1** is necessary for the existence of a nonlinear Cholesky factorization. When condition **B1** is satisfied, the mapping ϕ_1 is uniquely defined, and henceforth the function V_2.

Arguing inductively for each $i = 2, \dots, n$, we have that, for each i, the function V_i is determined, and is independent of the construction of the previous mapping. The function V_i is proper and smooth. Then, by Lemma 1, the unique mapping ϕ_i exists only if assumption **B**i holds. Therefore, assumptions **B1**–**B**n are necessary for the existence of a nonlinear Cholesky factorization.

By the uniqueness of the mappings ϕ_i, $i = 1, \dots, n$, at each step of the recursive construction, it follows that the diffeomorphism Φ is uniquely defined whenever it exists.

The second part of the theorem follows directly from the Proposition 1.

This completes the proof of the theorem. □

Although the Definition 4 is conceptually simple, it does not lend itself to simple computation of the inverse mapping Φ^{-1}. On the other hand, the factorization using simple diffeomorphism (18) is in general more computationally feasible. The computation of the inverse mapping Φ_s^{-1} involves only substitutions. Yet, the form of the factorization is more complicated. This is the motivation for the introduction of the simple diffeomorphism.

Next, we consider two simple examples to illustrate the factorization result. The first example concerns quadratic functions, and the second one involves a 4th order polynomial.

Example 1 In the case of $V = x'Px$, for some positive definite matrix P, the nonlinear Cholesky factorization mapping Φ is given by $z = Ux$, where the upper triangular matrix U is the Cholesky factorization of the matrix P, $P = U'U$.

In the Definition 4, the ordering of the coordinate variables $(x_1, \dots, x_n)$ is critical for the existence of the factorization. To illustrate this aspect, we consider the following example.

Example 2 Consider the nonlinear function $V(x_1, x_2) = (x_1 + x_2^2)^2 + x_2^2$. It is smooth and proper. The expression indicates that V admits a nonlinear Cholesky factorization in the ordering (x_1, x_2), where the mapping Φ is given by $\begin{bmatrix} z_1 & z_2 \end{bmatrix}' := \begin{bmatrix} x_1 + x_2^2 & x_2 \end{bmatrix}'$. By the theorem, we know that this factorization is unique (in the ordering of (x_1, x_2)). If we switch the coordinate variables (x_1, x_2), we obtain the function $W(x_1, x_2) := V(x_2, x_1) = (x_2 + x_1^2)^2 + x_1^2$. Applying the above result to this function, we first evaluate the partial derivative $\dfrac{\partial W}{\partial x_1} = 4x_1^3 + 4x_1x_2 + 2x_1$. It is easy to check that the equation $\frac{\partial W}{\partial x_1} = 0$ has three distinct roots when $x_2 = -1$. Therefore, the theorem concludes that it does not admit a nonlinear Cholesky factorization.

¿From the above example, we observe that the existence of a nonlinear Cholesky factorization in general depends critically on the ordering of the coordinate variables.

Here, a legitimate question one can ask is how can assumptions **B1**–**B**n be checked without actually carrying out the recursive construction. It turns out that these conditions hold under the additional assumption that V is strictly convex. To prove this proposition, we need the following corollary to Lemma 1.

Corollary 1 *Consider a nonlinear function $V(\boldsymbol{a}, \boldsymbol{b})$ that satisfies the conditions* **C1**–**C3** *of Lemma 1. Assume that*

C4 *The function V is strictly convex in $(\boldsymbol{a}, \boldsymbol{b})$, i. e.,* $\dfrac{\partial^2 V}{\partial(\boldsymbol{a}, \boldsymbol{b})^2}(\boldsymbol{a}, \boldsymbol{b}) > 0$, $\forall(\boldsymbol{a}, \boldsymbol{b})$.

Then, the condition **E1** *is satisfied and statements 1–4 of the Lemma 1 hold. Furthermore, the nonlinear function $\hat{V}(\boldsymbol{b})$ of the decomposition (11) is also strictly convex in $\boldsymbol{b}$, i. e.,* $\dfrac{\partial^2 \hat{V}}{\partial \boldsymbol{b}^2}(\boldsymbol{b}) > 0$, $\forall \boldsymbol{b}$.

Proof We will show first that condition **E1** holds under the convexity assumption. Assumption **C4** implies that V is strictly convex in $\boldsymbol{a}$. Hence, there is at most one root for the equation $\dfrac{\partial V}{\partial \boldsymbol{a}} = 0$. By **C2** and **C3**, there exists at least one root because V admits a global minimum for any fixed $\boldsymbol{b}$. Therefore, there exists a unique root for the equation $\dfrac{\partial V}{\partial \boldsymbol{a}} = 0$, and it is simple because $\dfrac{\partial^2 V}{\partial \boldsymbol{a}^2} > 0$. Hence, **E1** holds.

To prove that the matrix $\dfrac{\partial^2 \hat{V}}{\partial \boldsymbol{b}^2}(\boldsymbol{b})$ is positive definite $\forall \boldsymbol{b}$, we will show that $p' \frac{\partial^2 \hat{V}}{\partial \boldsymbol{b}^2}(\boldsymbol{b}) p > 0$, for any nonzero vector p and any fixed vector $\boldsymbol{b}$.

For each $\boldsymbol{b}$ and nonzero vector p, there exists a constant δ such that the vector $\bar{p} := \begin{bmatrix} \delta & p' \end{bmatrix}'$ satisfies $\dfrac{\partial \phi}{\partial(\boldsymbol{a}, \boldsymbol{b})}(\alpha(\boldsymbol{b}), \boldsymbol{b})\bar{p} = 0$, where $\alpha(\boldsymbol{b})$ is the unique root of $\phi(\boldsymbol{a}, \boldsymbol{b}) = 0$.

Using this nonzero vector $\bar{p}$, we have the following

$$\begin{aligned} 0 &< \bar{p}' \frac{\partial^2 V}{\partial(\boldsymbol{a}, \boldsymbol{b})^2}(\alpha(\boldsymbol{b}), \boldsymbol{b})\bar{p} = p' \frac{\partial^2 \hat{V}}{\partial \boldsymbol{b}^2}(\boldsymbol{b}) p + 2\bar{p}' \left(\frac{\partial \phi}{\partial(\boldsymbol{a}, \boldsymbol{b})}(\alpha(\boldsymbol{b}), \boldsymbol{b}) \right)' \left(\frac{\partial \phi}{\partial(\boldsymbol{a}, \boldsymbol{b})}(\alpha(\boldsymbol{b}), \boldsymbol{b}) \right) \bar{p} \\ &= p' \frac{\partial^2 \hat{V}}{\partial \boldsymbol{b}^2}(\boldsymbol{b}) p \end{aligned}$$

This implies the strict convexity of $\hat{V}(\boldsymbol{b})$, and completes the proof of this corollary. □

Using Corollary 1 repeatedly in the recursive construction of the nonlinear Cholesky factorization, we have the following corollary to Theorem 1.

Corollary 2 *Consider a positive definite, proper and smooth nonlinear function $V(x)$, where $x \in \mathbb{R}^n$. The function V admits a nonlinear Cholesky factorization $z = \Phi(x)$, as defined in Definition 4, if it is strictly convex, i. e., $\partial^2 V/\partial x^2(x) > 0 \qquad \forall x \in \mathbb{R}^n$.*

An interesting consequence of the above Corollary is that, for a strictly convex function V, there is a nonlinear Cholesky factorization for any ordering of the coordinate variables $(x_1, \dots, x_n)$.

So far, we have obtained conditions and a procedure to obtain a nonlinear Cholesky factorization for a given function. For the nonlinear robust control design problem formulated in Section 2, we can only obtain the local Taylor series expansion of the value function V, rather than the entire function. In the remainder of this section, we present results concerning the approximate nonlinear Cholesky factorization for a function V, which is prescribed locally around $x = 0$ in terms of its Taylor series expansion. The objective of the factorization is to derive a factorizable function W that matches the function V up to any given order of accuracy. We assume that the quadratic approximation of V is positive definite, i.e., $V^{[2]}(x) = x'Px$, where the matrix P is positive definite. Our goal is to find, for any positive integer m, a proper smooth function $W(x)$ such that $W^{[m]}(x) = V^{[m]}(x)$ and W has a nonlinear Cholesky factorization. Theoretically, this approximate nonlinear Cholesky factorization can be easily shown to exist by adding high order terms to make $W = V + \Delta V$ strictly convex and then applying the result of Corollary 1. However, this will result in serious difficulty in the computation of the factorization for the resulting polynomial function. To avoid this, we present an explicit construction for the approximate function W and the factorizing diffeomorphism $\Phi(x)$, which involves milder nonlinear growth as compared with the approximate function resulting from the simplistic convexification approach.

We will now follow n steps of a recursive construction for an approximate nonlinear Cholesky factorization for a locally prescribed function V with a positive definite quadratic approximation. Using an induction argument similar to that used in the proof of Theorem 1, we can directly deal with a single step of the recursive construction.

The following illustrates one of the typical steps in the recursive construction. Consider a polynomial function $V(\boldsymbol{a}, \boldsymbol{b})$ of order m, where $\boldsymbol{a}$ is a scalar and $\boldsymbol{b}$ is a vector of appropriate dimension. Assume that $V^{[2]}(\boldsymbol{a}, \boldsymbol{b}) = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b}' \end{bmatrix} P \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b}' \end{bmatrix}'$ with the matrix P being positive definite. Partition the matrix P in a way that is compatible with the partitioning of $(\boldsymbol{a}, \boldsymbol{b})$,

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix}$$

where P_{11} is a positive scalar.

Our goal is to find a mapping $\phi(\boldsymbol{a}, \boldsymbol{b})$ explicitly such that it satisfies the statements 1–4 of Lemma 1. Furthermore, the resulting function $\hat{V}(\boldsymbol{b})$ as given by (11) is such that $\hat{V}^{[2]}(\boldsymbol{b}) = \boldsymbol{b}'\hat{P}\boldsymbol{b}$, where $\hat{P}$ is a positive definite matrix.

First, we will evaluate the root of $\dfrac{\partial V}{\partial \boldsymbol{a}} = 0$ approximately. Let $V_p(\boldsymbol{a}, \boldsymbol{b}) := \dfrac{\partial V^{[m]}}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b})$. Then, V_p is an $(m-1)$th order polynomial. We note that $V_p^{[1]} = 2 \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b}' \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \end{bmatrix}'$. This implies that the equation $V_p^{[1]}(\boldsymbol{a}, \boldsymbol{b}) = 0$ admits a unique root $\boldsymbol{a} = -\dfrac{1}{P_{11}} P_{12} \boldsymbol{b}$.

Although, there may be many roots to the equation

$$\frac{\partial V^{[m]}}{\partial \boldsymbol{a}} = 0 \tag{19}$$

we are only interested in the root around $\boldsymbol{a} = -\frac{1}{P_{11}} P_{12}\boldsymbol{b}$. The Taylor series expansion of the root function up to $(m-1)$st order can be computed by equating homogeneous terms of the same order. Let the root function be $\alpha(\boldsymbol{b})$. It is obvious that $\alpha^{[1]}(\boldsymbol{b}) = -\frac{1}{P_{11}} P_{12}\boldsymbol{b}$. Then, we obtain higher-order expansion terms

$$\alpha_{[i]}(\boldsymbol{b}) \quad = \quad -\frac{1}{2P_{11}} \left(\sum_{j=2}^{i} V_{p[j]}(\alpha^{[i-1]}(\boldsymbol{b}), \boldsymbol{b}) \right)_{[i]} \quad ; \qquad i = 2, \dots, m-1 \tag{20}$$

Therefore, we have

$$\left(V_p(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}) \right)^{[m-1]} \quad = \quad 0 \tag{21}$$

Next, we will add higher order correction terms which enforce $\alpha^{[m-1]}(\boldsymbol{b})$ to be the unique root of (19), and in order to satisfy conditions **C1**–**C3** and **E1** of Lemma 1. Toward this end, let $\eta := \boldsymbol{a} - \alpha^{[m-1]}(\boldsymbol{b})$. We define a function $\bar{\Pi}(\eta, \boldsymbol{b})$ through

$$\eta \bar{\Pi}(\eta, \boldsymbol{b}) := V_p(\eta + \alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}) - V_p(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}) \tag{22}$$

By the approximation (21), it is obvious that $\left((\boldsymbol{a} - \alpha^{[m-1]}(\boldsymbol{b})) \bar{\Pi}(\boldsymbol{a} - \alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}) \right)^{[m-1]} = \frac{\partial V^{[m]}}{\partial \boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{b})$ and

$$\left(\int_0^{\eta} (s\bar{\Pi}(s, \boldsymbol{b}))^{[m-1]} \, ds \bigg|_{\eta = \boldsymbol{a} - \alpha^{[m-1]}(\boldsymbol{b})} \right)^{[m]} = V^{[m]}(\boldsymbol{a}, \boldsymbol{b}) - (V(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}))^{[m]} \tag{23}$$

Next, we proceed to modify the high order polynomial terms of the function $\bar{\Pi}$ to guarantee that it is larger than a positive constant for all $(\boldsymbol{a}, \boldsymbol{b})$. This then leads to the satisfaction of conditions **C2**, **C3**, and **E1** of Lemma 1 for the modified function.

We now modify the high-order terms of $\bar{\Pi}$ in the $(\eta, \boldsymbol{b})$ coordinates, which is equivalent to modifying them in $(\boldsymbol{a}, \boldsymbol{b})$ coordinates. Consider the function $\bar{\Pi}(\eta, \boldsymbol{b})$, we want to find a polynomial $\bar{\Pi}_s(\eta, \boldsymbol{b})$ of order $(m-2)$nd such that

$$\left(\bar{\Pi}_s^2(\eta, \boldsymbol{b}) \right)^{[m-2]} + C \quad = \quad \bar{\Pi}^{[m-2]}(\eta, \boldsymbol{b}) \tag{24}$$

for some constant $C > 0$.

We note the fact $\bar{\Pi}(0, 0) = 2P_{11}$, which follows directly from equating the second order partial derivatives at the origin. Therefore, for the root function $\bar{\Pi}_s$ to exist around the origin, we must choose the desired constant C to be a fraction of P_{11}, say $\frac{1}{m} P_{11}$. The function $\bar{\Pi}_s$ is expanded as

$$\bar{\Pi}_s(\eta, \boldsymbol{b}) = \sum_{i=0}^{m-2} \bar{\Pi}_{s[i]}(\eta, \boldsymbol{b}) \tag{25}$$

Substitution of this structure of $\bar{\Pi}_s$ into (24), and equating polynomials of the same order, we obtain the following expressions:

$$\begin{aligned}\bar{\Pi}_{s[0]} &= \sqrt{2P_{11}-C} \\ \bar{\Pi}_{s[i]} &= \frac{1}{2\sqrt{2P_{11}-C}}(\bar{\Pi}_{[i]} - \sum_{j=1}^{i-1} \bar{\Pi}_{s[j]}\bar{\Pi}_{s[i-j]}) \qquad i = 1, \dots, m-2\end{aligned}$$

Therefore, by (23) and (24), we have

$$V^{[m]}(\boldsymbol{a}, \boldsymbol{b}) = \left(\int_0^{\eta} s(\bar{\Pi}_s^2(s, \boldsymbol{b}) + C)\, ds \Big|_{\eta = \boldsymbol{a} - \alpha^{[m-1]}(\boldsymbol{b})} \right)^{[m]} + \left(V(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}) \right)^{[m]} \tag{26}$$

For large values of m, the polynomials $\bar{\Pi}_s^2(\boldsymbol{a}, \boldsymbol{b})$ and $\alpha^{[m-1]}(\boldsymbol{b})$ involve high-order polynomials that grow rapidly in magnitude as the state variable x deviates from the origin. To alleviate this difficulty, we propose the following two smooth scaling functions $S(\boldsymbol{b})$ and $T(\boldsymbol{a}, \boldsymbol{b})$, which satisfy

$$S^{[m-1]}(\boldsymbol{b}) \equiv 1 \tag{27}$$

$$T^{[m-2]}(\boldsymbol{a}, \boldsymbol{b}) \equiv 1 \qquad T(\boldsymbol{a}, \boldsymbol{b}) > 0 \quad \forall (\boldsymbol{a}, \boldsymbol{b}) \tag{28}$$

Using these scaling functions in the equation (26), we have the following identity

$$\begin{aligned} V^{[m]}(\boldsymbol{a}, \boldsymbol{b}) &= \left(\int_0^{\eta} s(\bar{\Pi}_s^2(s, \boldsymbol{b}) + C)\bar{T}^2(s, \boldsymbol{b})\, ds \Big|_{\eta = \boldsymbol{a} - \alpha^{[m-1]}(\boldsymbol{b})S(\boldsymbol{b})} \right)^{[m]} \\ &\quad + \left(V(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}) \right)^{[m]} \end{aligned} \tag{29}$$

where S multiplies $\alpha^{[m-1]}$ and $\bar{T}^2$ multiplies $\Pi_s^2(\boldsymbol{a}, \boldsymbol{b}) + C$. We note that multiplier S has been removed in the term $\left(V(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b})\right)^{[m]}$ because $\left(V(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b})\right)^{[m]} = \left(V(\alpha^{[m-1]}(\boldsymbol{b})S(\boldsymbol{b}), \boldsymbol{b})\right)^{[m]}$.

Define $\bar{\delta}(\eta, \boldsymbol{b}) = \sqrt{\frac{1}{\eta^2} \int_0^{\eta} s(\bar{\Pi}_s^2(s, \boldsymbol{b}) + C)\bar{T}^2(s, \boldsymbol{b})\, ds}$ and $\hat{V}(\boldsymbol{b}) = (V(\alpha^{[m-1]}(\boldsymbol{b}), \boldsymbol{b}))^{[m]}$. We note that $\bar{\delta}$ is a smooth positive function of $(\boldsymbol{a}, \boldsymbol{b})$, and $\hat{V}$ is an mth order polynomial of $\boldsymbol{b}$. Then, $V^{[m]}(\boldsymbol{a}, \boldsymbol{b}) = W^{[m]}(\boldsymbol{a}, \boldsymbol{b})$, where

$$W(\boldsymbol{a}, \boldsymbol{b}) := \left(\eta^2 \bar{\delta}^2(\eta, \boldsymbol{b}) \big|_{\eta = \boldsymbol{a} - \alpha^{[m-1]}(\boldsymbol{b})S(\boldsymbol{b})} \right)^{[m]} + \hat{V}(\boldsymbol{b}) \tag{30}$$

Since $\eta = 0$ is the unique zero of $\eta(\bar{\Pi}_s^2(\eta, \boldsymbol{b}) + C)\bar{T}^2(\eta, \boldsymbol{b}) = 0$ and $\frac{\partial}{\partial \eta}(\eta(\bar{\Pi}_s^2(\eta, \boldsymbol{b}) + C)\bar{T}^2(\eta, \boldsymbol{b}))\Big|_{\eta=0} = (\bar{\Pi}_s^2(0, \boldsymbol{b}) + C)\bar{T}^2(0, \boldsymbol{b}) > 0$, condition **E1** is satisfied for the function W. By simple inspection, condition **C2** is satisfied. For condition **C3** to hold, a sufficient condition is that the function $\bar{T}$ satisfies $\bar{T}^2 \geq \hat{C}\frac{1}{(1+\eta^2)(C + \bar{\Pi}_s^2(\eta, \boldsymbol{b}))}$, $\forall (\eta, \boldsymbol{b})$, for some positive constant $\hat{C}$.

With the introduction of the scaling function S, we can completely shape the growth rate of the function $\hat{\alpha}(\boldsymbol{b}) := \alpha^{[m-1]}(\boldsymbol{b})S(\boldsymbol{b})$, which corresponds to the virtual control law in the backstepping design. With the scaling function $\bar{T}$, we can indirectly shape the growth rate of the function $\delta(\boldsymbol{a}, \boldsymbol{b})$, which further determines the growth rate of the approximate function and the virtual control law in the backstepping design.

For induction purpose, we verify that the quadratic part of the function $\hat{V}$ is $\hat{V}^{[2]}(\boldsymbol{b}) = \boldsymbol{b}'\hat{P}\boldsymbol{b}$, where $\hat{P} = P_{22} - P_{12}'P_{11}^{-1}P_{12}$ is positive definite due to the positive definiteness of P.

By repeated application of the above process, we can prove the following theorem.

Theorem 2 *Consider a smooth nonlinear function $V(x)$, where $x \in \mathbb{R}^n$. Assume V is locally quadratic $V^{[2]}(x) = x'Px$, where the matrix P is positive definite. Then, for any desired matching order $m \geq 2$, a positive integer, there exists a proper smooth function $W(x)$ which admits a nonlinear Cholesky factorization such that $W^{[m]}(x) = V^{[m]}(x)$ and $W = \Phi'\Phi = \Phi_s'\Delta\Delta\Phi_s$, where Φ_s, a simple upper triangular diffeomorphism, and Δ, a diagonal matrix, are recursively constructed explicitly by the procedure outlined preceding the theorem.*

In the case when V is an analytic function of x and admits a Cholesky factorization $V = \Phi_{Vs}'\Delta_V\Delta_V\Phi_{Vs}$, then, there exists an $\epsilon_0 > 0$, such that, $\forall |x| \leq \epsilon_0$, $\lim_{m\to\infty} W(x) = V(x)$, where the function W is constructed following the steps depicted preceding the theorem, under the additional assumption that the scaling functions S and T, introduced at each step of the construction, are such that they converge to 1 on the set $\{x : \ |x| \leq \epsilon_0\}$ as $m \to \infty$.

Proof The first part of the theorem has been proved by the construction prior to the statement of the theorem. Now, since V is positive definite and analytic, we conclude that functions Δ_V and Φ_{Vs} are analytic. This conclusion is made based on the fact that the elements of Φ_{Vs} are roots of n analytic equations. By the analytic version of the implicit function theorem, they are also analytic under the necessary and sufficient conditions of Theorem 1.

For the second part of the theorem, we first consider the case when the scaling functions S and T are set to 1 at each step of recursive construction. Then, the simple diffeomorphism Φ_s is the truncated Taylor series approximation to Φ_{Vs} up to the order of $m-1$. By the analyticity of Φ_{Vs}, it converges to Φ_s as $m \to \infty$ within a radius of convergence.

The nonlinear function $\bar{\Pi}_s$ as defined in (25) is again the truncated Taylor series approximation to the corresponding nonlinear function for the function V. It then converges again due to the analyticity of Δ_V as $m \to \infty$ within a radius of convergence. This further implies that Δ_{Vs} converges to Δ_V as $m \to \infty$ within the same radius of convergence.

Hence, the function W converges to V as $m \to \infty$ within the common radius of convergence, since each factor converges.

For the general case when the scaling factors S and T converges to 1 as $m \to \infty$, we can conclude the same convergence result because of the fact that product converges when all factors converge.

This completes the proof of the theorem. □

An important comment to make regarding the recursive construction procedure is that the function generated is in general not convex. Therefore the approximate function W includes lower order polynomial terms as compared with the approximation resulting from the simplistic convexification approach.

Equipped with this high-order matching result for any given Taylor series expansion of a locally positive definite quadratic function V, we can now turn to present, in the next section, the design procedure for an inverse optimal control law that matches the optimal design up to any given order in their Taylor series expansion.

4 Higher Order Matching Control Design

In this section, we present the high-order matching control design method for the nonlinear system (1) and the cost function (2). First, we present a theorem that establishes the optimality of the integrator backstepping design methodology.

Theorem 3 *Consider the nonlinear robust control problem with system dynamics (1) and cost function (2), under Assumptions* **A1** *and* **A2***. Assume that there exists a smooth value function $V(x)$ satisfying the HJI equation (3), which has a nonlinear Cholesky factorization $V = \Phi_s' \Delta\Delta\Phi_s$ in the reverse order of state variables, $(x_n, \ldots, x_1)$. Then, there exists an integrator backstepping procedure that yields the value function, minimax control law (4) and the corresponding worst-case disturbance for the problem.*

The proof is quite straightforward and is omitted here. For details of the proof, please see [1].

Theorem 3 predicts that the integrator backstepping procedure can produce the optimal solution to a given problem as long as the value function solution is factorizable in the lower triangular fashion. Yet, it requires the knowledge of the value function $V(x)$ for the design of the backstepping algorithm, which is very difficult to obtain, if not impossible. Therefore, we pursue the inverse optimal design with locally optimal matching as prescribed in Definitions 1 and 2.

Let m be the desired matching order for the control design, $m \geq 1$. By the result of [8], we will need the Taylor series expansion of V up to $(m+1)$st order. Then, by Theorem 2, we can find a function $W_l(x)$ that matches V up to $(m+1)$st order in their Taylor series expansions and can be factorized as $W_l(x) = \Phi_s(x)'\Delta(x)\Delta(x)\Phi_s(x)$. We have used the subscript l to denote functions to be matched in the backstepping design described later. Then, it is obvious that W_l satisfies the HJI equation (3) up to $(m+1)$th order. Introduce the transformed state variable $z_l = \Phi_s(x)$. We further assume that the Φ_s and Δ are given by

$$\Phi_s(x) = \begin{bmatrix} x_1 \\ x_2 - \alpha_{l1}(x_1) \\ \vdots \\ x_n - \alpha_{l\,n-1}(x_1, \ldots, x_{n-1}) \end{bmatrix}; \qquad \Delta(x) = \text{diagonal}\{\delta_{l1}(x_1), \ldots, \delta_{ln}(x)\} \tag{31}$$

where the subscript l again indicates that this nonlinear function is to be matched in the backstepping design.

We can express the system dynamics in the z_l coordinate as

$$\begin{aligned} \dot{z}_{l1} &= z_{l2} + \bar{\hat{f}}_{l1}(z_{l1}) + \bar{\hat{h}}_{l1}'(z_{l1})w \\ \vdots &= \quad \vdots \\ \dot{z}_{ln} &= u + \bar{\hat{f}}_{ln}(z_{l1}, \ldots, z_{ln}) + \bar{\hat{h}}_{ln}'(z_{l1}, \ldots, z_{ln})w \\ J_\gamma &= \int_0^\infty (\bar{q}_l(z_l) + \bar{r}_l(z_l)u^2 - \gamma^2 w'w)\, dt \end{aligned}$$

In compact form, we can write

$$\dot{z}_l = \bar{\hat{f}}_l(z_l) + Bu + \bar{\hat{H}}_l(z_l)w$$

Then, the function W_l can be expressed as $\bar{W}_l(z_l) = \sum_{i=1}^n \bar{\delta}_{li}^2 z_{li}^2$ and satisfies the HJI equation (3) up to $(m+1)$th order:

$$\left(\frac{\partial \bar{W}_l}{\partial z_l}\bar{\hat{f}}_l - \frac{1}{4}\frac{\partial \bar{W}_l}{\partial z_l}(B\bar{r}_l^{-1}B' - \frac{1}{\gamma^2}\bar{\hat{H}}_l\bar{\hat{H}}_l')\left(\frac{\partial \bar{W}_l}{\partial z_l}\right)' + \bar{q}_l\right)^{[m+1]} = 0 \tag{32}$$

As a consequence of the construction of the approximating function W_l, we have $\bar{\pi}_i(z_{l\{1\}}^{\{i\}}) := \bar{\delta}_{li}^2 + z_{li}\bar{\delta}_{li}\dfrac{\partial \bar{\delta}_{li}}{\partial z_{li}} > 0$, for any state z_l and all $i = 1, \dots, n$.

Again, setting $z_{l\,i+1} = \cdots = z_{ln} = 0$ yields the following approximate HJI equation for $\bar{W}_{li}$, for any $i = 1, \dots, n-1$,

$$\left(\frac{\partial \bar{W}_{li}}{\partial z_{l\{1\}}^{\{i\}}}\bar{\hat{f}}_{l\{1\}}^{\{i\}} + \frac{1}{4\gamma^2}\frac{\partial \bar{W}_{li}}{\partial z_{l\{1\}}^{\{i\}}}\bar{\hat{H}}_{l\{1\}}^{\{i\}}\,\bar{\hat{H}}_{l\{1\}}^{\{i\}\,\prime}\left(\frac{\partial \bar{W}_{li}}{\partial z_{l\{1\}}^{\{i\}}}\right)' + \bar{q}_{l\{1\}}^{\{i\}}\right)^{[m+1]} = 0 \tag{33}$$

where

$$\bar{W}_{li} := \sum_{j=1}^{i}\bar{\delta}_{lj}^2 z_{lj}^2 \qquad \bar{q}_{l\{1\}}^{\{i\}} := \bar{q}_l(z_{l1}, \dots, z_{li}, 0, \dots, 0)$$

$$\bar{\hat{f}}_{l\{1\}}^{\{i\}} := \begin{bmatrix} z_{l2} + \bar{\hat{f}}_{l1}(z_{l1}) \\ \vdots \\ z_{l\,i} + \bar{\hat{f}}_{li-1}(z_{l\{1\}}^{\{i-1\}}) \\ \bar{\hat{f}}_{li}(z_{l\{1\}}^{\{i\}}) \end{bmatrix}; \qquad \bar{\hat{H}}_{l\{1\}}^{\{i\}} := \begin{bmatrix} \bar{\hat{h}}_{l1}'(z_{l1}) \\ \vdots \\ \bar{\hat{h}}_{li}'(z_{l\{1\}}^{\{i\}}) \end{bmatrix}; \qquad z_{l\{1\}}^{\{i\}} = \begin{bmatrix} z_{l1} \\ \vdots \\ z_{li} \end{bmatrix}$$

We now present the backstepping design procedure that leads to an inverse optimal matching control design.

Step 1: Define $z_1 := z_{l1} = x_1$. Let the value function for this step be defined as: $\bar{W}_1(z_{l1}) := \bar{\delta}_{l1}^2(z_1)z_1^2$. The virtual control law that is prescribed by the nonlinear Cholesky factorization of W_l is $x_2 = \bar{\alpha}_{l1}(z_1)$. Under this control law, the value function $\bar{W}_1$ satisfies the HJI equation (33), with $i = 1$. Therefore, the derivative of $\bar{W}_1$ is given by

$$\dot{\bar{W}}_1 = -\bar{q}_{l\{1\}}^{\{1\}} + \gamma^2 w'w - \gamma^2\left|w - \frac{1}{2\gamma^2}\bar{\hat{H}}_{l\{1\}}^{\{1\}\,\prime}\left(\frac{\partial \bar{W}_1}{\partial z_1}\right)'\right|^2 + \bar{\xi}_1(z_1) + 2z_1\bar{\pi}_1(z_1)z_{l2}$$

where $\bar{\xi}_1$ is the remainder of the approximate HJI equality (33), i. e.,

$$\bar{\xi}_1(z_1) = \frac{\partial \bar{W}_1}{\partial z_1}\bar{\hat{f}}_{l\{1\}}^{\{1\}} + \frac{1}{4\gamma^2}\frac{\partial \bar{W}_1}{\partial z_1}\bar{\hat{H}}_{l\{1\}}^{\{1\}}\,\bar{\hat{H}}_{l\{1\}}^{\{1\}\,\prime}\left(\frac{\partial \bar{W}_1}{\partial z_1}\right)' + \bar{q}_{l\{1\}}^{\{1\}}$$

This suggests the following smooth virtual control law for z_{l2}: $z_{l2} = -\dfrac{1}{2\bar{\pi}_1(z_1)z_1}\bar{\xi}_1(z_1)$, which is equivalent to a virtual control law for x_2:

$$x_2 = \bar{\alpha}_{l1}(z_1) - \frac{1}{2\bar{\pi}_1(z_1)z_1}\bar{\xi}_1(z_1) =: \bar{\alpha}_{l1}(z_1) + \tilde{\bar{\alpha}}_1(z_1) := \bar{\alpha}_1(z_1) \tag{34}$$

Because of the fact that $\bar{\xi}_1^{[m+1]} = 0$, and $\bar{\pi}_1$ is positive, we conclude that $\tilde{\bar{\alpha}}_1^{[m]} = 0$, and hence is higher-order than the desired matching order.

Then, under this control law, we have that

$$\dot{\bar{W}}_1 = -\bar{q}_{l\{1\}}^{\{i\}} + \gamma^2 w'w - \gamma^2 \left| w - \frac{1}{2\gamma^2} \hat{\bar{H}}_{l\{1\}}^{\{i\}\,\prime} \left(\frac{\partial \bar{W}_1}{\partial z_1} \right)' \right|^2 + 2z_1\bar{\pi}_1(z_1)(x_2 - \bar{\alpha}_1(z_1))$$

This completes the first step of the control design.

Step i, $1 < i < n$**:** At this step, we assume that from the previous steps, we have defined a value function $\bar{W}_{i-1} = \sum_{j=1}^{i-1} \bar{\delta}_{lj}^2 z_j^2$ and a virtual control law $\bar{\alpha}_{i-1}$ for x_i, which can be decomposed into a matching part and a high-order part as follows: $\bar{\alpha}_{i-1}(z_{\{1\}}^{\{i-1\}}) = \bar{\alpha}_{l\,i-1}(z_{\{1\}}^{\{i-1\}}) + \tilde{\bar{\alpha}}_{i-1}(z_{\{1\}}^{\{i-1\}})$. The derivative of $\bar{W}_{i-1}$ is equal to

$$\begin{aligned} \dot{\bar{W}}_{i-1} &= -\bar{q}_{l\{1\}}^{\{i-1\}}(z_{\{1\}}^{\{i-1\}}) + \gamma^2 w'w - \gamma^2 \left| w - \bar{\nu}_{i-1}(z_{\{1\}}^{\{i\}}) \right|^2 \\ &\quad + 2z_{i-1}\bar{\pi}_{i-1}(x_i - \bar{\alpha}_{i-1}(z_{\{1\}}^{\{i-1\}})) \end{aligned} \tag{35}$$

where $\bar{\nu}_{i-1}$ is the corresponding worst-case disturbance, given by

$$\bar{\nu}_{i-1}(z_{\{1\}}^{\{i-1\}}) = \frac{1}{2\gamma^2} \left(\hat{\bar{H}}_{l\{1\}}^{\{i-1\}} + \hat{\bar{N}}_{\{1\}}^{\{i-1\}} \right)' \left(\frac{\partial \bar{W}_{i-1}}{\partial z_{\{1\}}^{\{i-1\}}} \right)' \tag{36}$$

$$\hat{\bar{N}}_{\{1\}}^{\{i-1\}}(z_{\{1\}}^{\{i-1\}}) = \begin{bmatrix} \hat{\bar{N}}_1(z_1) \\ \vdots \\ \hat{\bar{N}}_{i-1}(z_{\{1\}}^{\{i-1\}}) \end{bmatrix}$$

and the dynamics of the transformed state variable $z_{\{1\}}^{\{i-1\}}$ are given by

$$\dot{z}_1 = z_2 + \hat{\bar{f}}_{l1}(z_1) + \hat{\bar{h}}_{l1}'(z_1)w + \hat{\bar{M}}_1(z_1) + \hat{\bar{N}}_1(z_1)w \tag{37a}$$

$$\vdots = \vdots$$

$$\begin{aligned} \dot{z}_{i-1} &= x_i - \bar{\alpha}_{i-1}(z_{\{1\}}^{\{i-1\}}) + \hat{\bar{f}}_{l\,i-1}(z_{\{1\}}^{\{i-1\}}) + \hat{\bar{h}}_{l\,i-1}'(z_{\{1\}}^{\{i-1\}})w + \hat{\bar{M}}_{i-1}(z_{\{1\}}^{\{i-1\}}) \\ &\quad + \hat{\bar{N}}_{i-1}(z_{\{1\}}^{\{i-1\}})w \end{aligned} \tag{37b}$$

where the functions (possibly vector-valued) $\hat{\bar{M}}_j(z_{\{1\}}^{\{j\}})$ and $\hat{\bar{N}}_j(z_{\{1\}}^{\{j\}})$, $j = 1, \ldots, i-1$ are high-order nonlinear functions.

It is hypothesized that the transformation z_j matches exactly to z_{lj}, $j = 1, \ldots, i-1$, up to mth order.

On the ith step, we define the new coordinate $z_i := x_i - \bar{\alpha}_{i-1}(z_{\{1\}}^{\{i-1\}})$, which is the deviation of x_i from its desired virtual control trajectory.

Because of the matching between $\bar{\alpha}_{i-1}$ and $\bar{\alpha}_{l\,i-1}$, and the matching between $z_{\{1\}}^{\{i-1\}}$ and $z_{l\{1\}}^{\{i-1\}}$, the dynamics of z_i is then given by

$$\dot{z}_i = \hat{\bar{f}}_{li}(z_{\{1\}}^{\{i\}}) + z_{l\,i+1} + \hat{\bar{h}}_{li}'(z_{\{1\}}^{\{i\}})w + \hat{\bar{M}}_{bi}(z_{\{1\}}^{\{i\}}) + \hat{\bar{N}}_i(z_{\{1\}}^{\{i\}})w$$

where $\bar{\hat{M}}_{bi}$ and $\bar{\hat{N}}_i$ are again high-order nonlinear functions.

Again, based on the approximate factorization, we recursively define the value function for this step of control design as: $\bar{W}_i(z_{\{1\}}^{\{i\}}) := \bar{W}_{i-1}(z_{\{1\}}^{\{i-1\}}) + \bar{\delta}_{li}^2(z_{\{1\}}^{\{i\}})z_i^2$. The derivative of $\bar{W}_i$ can be expressed as follows, after some algebraic manipulations using the relationship (33):

$$\dot{\bar{W}}_i = -\bar{q}_{l\{1\}}^{\{i\}}(z_{\{1\}}^{\{i\}}) + \gamma^2 w'w - \gamma^2 \left| w - \bar{\nu}_i(z_{\{1\}}^{\{i\}}) \right|^2 + 2z_i\bar{\pi}_i z_{l\,i+1} + \bar{\xi}_i(z_{\{1\}}^{\{i\}})$$

where

$$\begin{aligned} \bar{\xi}_i(z_{\{1\}}^{\{i\}}) &= \bar{q}_{l\{1\}}^{\{i\}} - \bar{q}_{l\{1\}}^{\{i-1\}} + \gamma^2\bar{\nu}_i'\bar{\nu}_i - \gamma^2\bar{\nu}_{i-1}'\bar{\nu}_{i-1} + \sum_{j=1}^{i-1} 2z_i^2\bar{\delta}_{li}\frac{\partial\bar{\delta}_{li}}{\partial z_j}(z_{j+1} + \bar{\hat{f}}_{lj} + \bar{\hat{M}}_j) \\ &\quad +2z_i\bar{\pi}_i(\bar{\hat{f}}_{li} + \bar{\hat{M}}_{bi}) + 2z_{i-1}\bar{\pi}_{i-1}z_i \end{aligned} \tag{38}$$

$$\bar{\nu}_i(z_{\{1\}}^{\{i\}}) = \frac{1}{2\gamma^2}\left(\bar{\hat{H}}_{l\{1\}}^{\{i\}} + \bar{\hat{N}}_{\{1\}}^{\{i\}}\right)'\left(\frac{\partial\bar{W}_i}{\partial z_{\{1\}}^{\{i\}}}\right)' \qquad \bar{\hat{N}}_{\{1\}}^{\{i\}}(z_{\{1\}}^{\{i\}}) = \begin{bmatrix} \bar{\hat{N}}_1(z_1) \\ \vdots \\ \bar{\hat{N}}_i(z_{\{1\}}^{\{i\}}) \end{bmatrix} \tag{39}$$

By the approximate HJI equality (33), we conclude that $\left(\bar{\xi}_i(z_{\{1\}}^{\{i\}})\right)^{[m+1]} = 0$. We also conclude that, from the expressions (38) (39) and (36), $\bar{\xi}_i(z_{\{1\}}^{\{i\}})$ contains z_i as a factor. Hence, the smooth virtual control law for $z_{l\,i+1}$ is $z_{l\,i+1} = -\frac{1}{2z_i\bar{\pi}_i}\bar{\xi}_i(z_{\{1\}}^{\{i\}})$.

This leads to the satisfaction of a dissipation inequality for the $x_{\{1\}}^{\{i\}}$ dynamics with supply rate, $\bar{q}_{l\{1\}}^{\{i\}}(z_{\{1\}}^{\{i\}}) - \gamma^2 w'w$. The equivalent virtual control law for x_{i+1} is

$$x_{i+1} = \bar{\alpha}_i(z_{\{1\}}^{\{i\}}) := \bar{\alpha}_{li}(z_{\{1\}}^{\{i\}}) - \frac{1}{2z_i\bar{\pi}_i}\bar{\xi}_i(z_{\{1\}}^{\{i\}}) =: \bar{\alpha}_{li}(z_{\{1\}}^{\{i\}}) + \tilde{\bar{\alpha}}_i(z_{\{1\}}^{\{i\}})$$

Therefore, the matching property of the virtual control law is verified.

Under this control law, the derivative of $\bar{W}_i$ is given by

$$\dot{\bar{W}}_i = -\bar{q}_{l\{1\}}^{\{i\}}(z_{\{1\}}^{\{i\}}) + \gamma^2 w'w - \gamma^2 \left| w - \bar{\nu}_i(z_{\{1\}}^{\{i\}}) \right|^2 + 2z_i\bar{\pi}_i(x_{i+1} - \bar{\alpha}_i(z_{\{1\}}^{\{i\}}))$$

This completes the design for this step.

Step n: Recursively, we can complete the control design up to $n-1$ steps. For the final step, the nth step, we define $z_n := x_n - \bar{\alpha}_{n-1}(z_{\{1\}}^{\{n-1\}})$, which is again the deviation of x_n from its desired virtual control trajectory.

We can compute the derivative of z_n, and the actual control variable u appears.

$$\dot{z}_n = \bar{\hat{f}}_{ln}(z) + u + \bar{\hat{h}}_{ln}'(z)w + \bar{\hat{M}}_n(z) + \bar{\hat{N}}_n(z)w$$

where $\bar{\hat{M}}_n$ and $\bar{\hat{N}}_n$ are again higher-order nonlinear functions that are present due to the higher-order mismatches between z and z_l.

We define recursively the value function for this final step, which becomes the value function for the complete system, $\bar{W}_n(z) := \bar{W}_{n-1}(z_{\{1\}}^{\{n-1\}}) + \bar{\delta}_{ln}^2(z)z_n^2 =: \bar{W}(z)$. Because of the approximate

HJI equality (32), the derivative of $\bar{W}$ is equal to the following expression, after two "completion of squares" arguments.

$$\begin{aligned}\dot{\bar{W}} &= -\bar{q}_l(z) - \breve{\bar{r}}u^2 + \gamma^2 w'w + \breve{\bar{r}}\left|u + \breve{\bar{r}}^{-1}\bar{\pi}_n(z)z_n\right|^2 - \gamma^2\left|w - \bar{\nu}_n(z)\right|^2 \\ &\quad -(\breve{\bar{r}}^{-1}(z) - \bar{r}_l^{-1}(z))\bar{\pi}_n^2(z)z_n^2 + \bar{\xi}_n(z)\end{aligned}$$

where

$$\begin{aligned}\bar{\xi}_n(z) &= \bar{q}_l - \bar{q}_{l\{1\}}^{\{n-1\}} + \gamma^2\bar{\nu}_n'\bar{\nu}_n - \gamma^2\bar{\nu}_{n-1}'\bar{\nu}_{n-1} + \sum_{j=1}^{n-1} 2z_n^2\bar{\delta}_{ln}\frac{\partial\bar{\delta}_{ln}}{\partial z_j}(z_{j+1} + \hat{\bar{f}}_{lj} + \hat{\bar{M}}_j) \\ &\quad +2z_n\bar{\pi}_n(\hat{\bar{f}}_{ln} + \hat{\bar{M}}_n) + 2z_{n-1}\bar{\pi}_{n-1}z_n - \bar{r}_l^{-1}(z)\bar{\pi}_n^2(z)z_n^2 \\ \bar{\nu}_n(z) &= \frac{1}{2\gamma^2}\left(\hat{\bar{H}}_l' + \hat{\bar{N}}\right)'\left(\frac{\partial\bar{W}}{\partial z}\right)' \qquad \hat{\bar{N}}(z) = \begin{bmatrix}\hat{\bar{N}}_1(z_1) \\ \vdots \\ \hat{\bar{N}}_n(z)\end{bmatrix}\end{aligned}$$

Because the approximating value function W satisfies the approximate HJI equality (32), the function $\bar{\xi}_n$ is higher-order such that $\bar{\xi}_n^{[m+1]} = 0$. Furthermore, by the definition of $\bar{\xi}_n$, we conclude that $\bar{\xi}_n$ contains z_n as a factor. Let $\bar{\xi}_f(z)$ be defined as $\bar{\xi}_f(z) := \bar{\xi}_n(z)/z_n$. Then, $\bar{\xi}_f$ is a smooth function such that $\bar{\xi}_f^{[m]} = 0$.

Choose $\breve{\bar{r}}$ as following

$$\breve{\bar{r}}(z) = \begin{cases} \dfrac{2\bar{q}_l(z)\bar{\pi}_n^2(z)z_n^2\bar{r}_l(z)}{(\bar{q}_l(z)/4 - \bar{\xi}_f(z)z_n)^2\bar{r}_l(z) + 2\bar{q}_l(z)\bar{\pi}_n^2(z)z_n^2} & \text{if } \bar{q}_l(z)/4 < \bar{\xi}_f(z)z_n \\ \bar{r}_l(z) & \text{if } \begin{array}{l}\bar{q}_l(z)/2 - \bar{r}_l^{-1}(z)\bar{\pi}_n^2(z)z_n^2 - \epsilon \\ \leq \bar{\xi}_f(z)z_n \leq \bar{q}_l(z)/4\end{array} \\ (\bar{r}_l \cdot (1 + (\bar{q}_l/2 - \bar{r}_l^{-1}\bar{\pi}_n^2 z_n^2 - \epsilon - \bar{\xi}_f z_n)^2\bar{\pi}_n^2 z_n^2))(z) & \text{if } \begin{array}{l}\bar{q}_l(z)/2 - \bar{r}_l^{-1}(z)\bar{\pi}_n^2(z)z_n^2 - \epsilon \\ > \bar{\xi}_f(z)z_n\end{array}\end{cases} \tag{40}$$

where ϵ is any positive constant. It is clear that this choice of $\breve{\bar{r}}$ is locally Lipschitz continuous.

Therefore, the control law,

$$\bar{\mu}(z) = -\breve{\bar{r}}^{-1}\bar{\pi}_n(z)z_n \tag{41}$$

is the desired inverse optimal matching control design, and it is inverse robust optimal with respect to the cost function $\breve{\bar{q}} + \breve{\bar{r}}u^2$, where

$$\breve{\bar{q}} := \bar{q}_l(z) + (\breve{\bar{r}}^{-1}(z) - \bar{r}_l^{-1}(z))\bar{\pi}_n^2(z)z_n^2 - \bar{\xi}_n(z) \tag{42}$$

and $\breve{\bar{r}}$ is as defined in (40).

With this choice of $\breve{\bar{r}}$, we note that, in a neighborhood of the origin, $\bar{q}_l(z)/2 \geq c_1|z|^2 \geq c_2|z|^3 \geq \bar{\xi}_f(z)z_n$ for some positive constants c_1 and c_2, and $\epsilon \geq \bar{q}_l(z)/2 - \breve{\bar{r}}_l^{-1}(z)\bar{\pi}_n^2(z)z_n^2 - \bar{\xi}_f(z)z_n$. Therefore, in a neighborhood of the origin, we have $\breve{\bar{r}}(z) = \bar{r}_l(z)$. This implies that the control law (41) satisfies the matching requirement. In the case when $\bar{q}_l(z)/2 - \bar{r}_l^{-1}(z)\bar{\pi}_n^2(z)z_n^2 - \epsilon > \bar{\xi}_f(z)z_n$, we have $\check{q}(z) > \bar{q}_l(z)/2 + \epsilon$. In the case when $\bar{q}_l(z)/2 < \bar{\xi}_f(z)z_n$, we have $|z| > 0$, which implies that $\bar{q}_l(z)/2 > 0$, and therefore $z_n \neq 0$. Then, $\check{r}(z) > 0$ and

$$(\check{r}^{-1}(z) - \bar{r}_l^{-1}(z))\bar{\pi}_n^2(z)z_n^2 - \bar{\xi}_f(z)z_n + \frac{3}{4}\bar{q}_l(z) = \frac{1}{2}\bar{q}_l(z)\left|1 - \frac{z_n\bar{\xi}_f - \bar{q}_l(z)/4}{\bar{q}_l(z)}\right|^2 \geq 0$$

This completes the control design. This result is summarized in the following theorem.

Theorem 4 *Consider the nonlinear system (1) and cost function (2) with Assumptions* **A1**, **A2**, *and* **A3** *holding. Let the function W_l be any proper smooth function such that it admits exactly the same Taylor series expansion as the value function V up to $(m+1)$th order and it has a nonlinear Cholesky factorization in the reverse order of state variables $(x_n, \ldots, x_1)$. Then, the control law (41) is locally optimal matching up to mth order and is inverse robust optimal with respect to the cost function (5), where $\check{q}$ is a positive definite and radially unbounded weighting function defined by (42), and $\check{r}$ is a positive function defined by (40).*

Proof We note that the constructed value function $\bar{W}$ is positive definite and proper because W_l is assumed to be positive definite and proper, and $\bar{W}(z) = \bar{W}_l(z)$. The weighting function $\check{q}$ is positive definite and radially unbounded because it is larger than or equal to $q(x)/4$. Furthermore, the function $\bar{W}$ satisfies the HJI equation and the matching requirements by the recursive construction. This completes the proof of the theorem. □

In the above recursive construction, we can start from any given factorizable function W_l that approximates the value function V. Clearly, for different W_l's, the construction leads to, in general, different controller designs, that all satisfy the inverse robust optimality requirement and the local matching property. A question that remains to be answered is whether there is any criterion to distinguish which W_l is more appropriate for a particular problem.

5 Example

In this section, we will illustrate the theory developed in the previous section on a numerical example.

We consider a second order nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 + w \\ \dot{x}_2 &= u \end{aligned}$$

Associated with this nonlinear system is a cost function for the zero-sum differential game:

$$J_\gamma = \int_0^\infty (x_1^2 + x_2^2 + u^2 - \gamma^2 w^2)\,dt$$

In particular, we fix the desired disturbance attenuation level to be $\gamma = \sqrt{2}$.

The HJI equation associated with this zero-sum game is

$$V_{x_1}(x_1^2 + x_2) + \frac{1}{4\gamma^2}V_{x_1}^2 - \frac{1}{4}V_{x_2}^2 + x_1^2 + x_2^2 = 0 \tag{43}$$

where $V(x_1, x_2)$ is the value function for the game, if it exists.

Using the standard method of [2], the Taylor series expansion of the function $V(x_1, x_2)$ around the origin up to 3rd order is given by

$$V^{[3]}(x_1, x_2) = 6.680x_1^2 + 9.657x_1x_2 + 4.724x_2^2 + 89.92x_1^3 + 189.4x_1^2x_2 + 134.7x_1x_2^2 + 32.50x_2^3$$

1st order matching

If we are interested to obtain an inverse robust optimal controller that is optimal matching to the 1st order, we have $m = 1$, and is only concerned the 2nd order Taylor series expansion of the value function

$$V^{[2]}(x_1, x_2) = 6.680x_1^2 + 9.657x_1x_2 + 4.724x_2^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 6.680 & 4.828 \\ 4.828 & 4.724 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The controller design starts by first obtaining a factorizable function W that matches the value function V up to 2nd order. It turns out that the function W is exactly $V^{[2]}$. The nonlinear Cholesky factorization is equivalent to the Cholesky factorization for positive definite matrices in this case. We have

$$V^{[2]}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 + 1.022x_1 \end{bmatrix}' \begin{bmatrix} 1.323^2 & 0 \\ 0 & 2.174^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 + 1.022x_1 \end{bmatrix}$$

Hence, the simple diffeomorphism is given by $\begin{bmatrix} z_{l1} & z_{l2} \end{bmatrix}' = \begin{bmatrix} x_1 & x_2 + 1.022x_1 \end{bmatrix}'$. In the $z_l = (z_{l1}, z_{l2})'$ coordinates, the original system is given by

$$\begin{aligned} \dot{z}_{l1} &= -1.022z_{l1} + z_{l1}^2 + z_{l2} + w \\ \dot{z}_{l2} &= -1.045z_{l1} + 1.022z_{l1}^2 + 1.022z_{l2} + u + 1.022w \end{aligned}$$

The cost function can be equivalently expressed, in this coordinate, as

$$J_\gamma = \int_0^\infty (2.045z_{l1}^2 - 2.044z_{l1}z_{l2} + z_{l2}^2 + u^2 - 2w^2)\, dt$$

Next, we proceed to two steps of integrator backstepping for the controller design. In the first step, we set $z_1 = z_{l1}$, and choose a value function $V_1(z_1) = 1.323^2 z_1^2 = 1.751z_1^2$.

Then, we have

$$\begin{aligned} \dot{V}_1 &= 3.502z_1(z_{l2} - 1.022z_1 + z_1^2 + w) \\ &= -2.045z_1^2 + 3.502z_1(z_{l2} + z_1^2) + 2w^2 - 2(w - 0.8755z_1)^2 \end{aligned}$$

Hence, the desired virtual control law for z_{l2} is $-z_1^2$. The worst case disturbance input for this step is $\bar{\nu}_1^* = 0.8755z_1$.

This completes the first step of backstepping.

In the second step, which is also the last step, we define the state variable $z_2 = z_{l2} + z_1^2$. The dynamics for z_2 is then given by

$$\dot{z}_2 = u - 0.1502z_1 + 1.022z_2 + 2z_1z_2 - 0.2940z_1^2 + (1.022 + 2z_1)(w - 0.8755z_1)$$

The value function for this step of design is $V_2(z_1, z_2) = V_1 + 2.174^2 z_2^2 = V_1 + 4.724 z_2^2$.

After some straightforward algebraic manipulation, we arrive at the following expression for the derivative of V_2,

$$\begin{aligned}\dot{V}_2 &= -2.045z_1^2 + 2.044z_1z_2 - z_2^2 + 2w^2 - 2(w - 0.8755z_1 - 2.415z_2 - 4.728z_1z_2)^2 \\ &\quad -\bar{\check{r}}u^2 + \bar{\check{r}}(u + 4.726\bar{\check{r}}^{-1}z_2)^2 - 22.34z_2^2(\bar{\check{r}}^{-1} - 1) + 64.58z_1z_2^2 + 44.71z_1^4 - 2.779z_1^2z_2\end{aligned}$$

Therefore, we have $\bar{\xi}_2 = 64.58z_1z_2^2 + 44.71z_1^4 - 2.779z_1^2z_2$. Following (40), we define the weighting function

$$\bar{\check{r}}(z_1, z_2) = \begin{cases} \dfrac{44.68(2.045z_1^2 - 2.044z_1z_2 + z_2^2)z_2^2}{(0.5115z_1^2 - 0.5110z_1z_2 + 0.25z_2^2 - \bar{\xi}_2(z))^2 + 44.68(2.045z_1^2 - 2.044z_1z_2 + z_2^2)z_2^2} \\ \qquad \text{if} \;\; 0.5115z_1^2 - 0.5110z_1z_2 + 0.25z_2^2 < \bar{\xi}_2(z) \\ 1 \qquad \text{if} \;\; \begin{array}{l} 1.023z_1^2 - 1.022z_1z_2 + 0.5z_2^2 - 22.34z_2^2 - \epsilon \leq \bar{\xi}_2(z) \\ \leq 0.5115z_1^2 - 0.5110z_1z_2 + 0.25z_2^2 \end{array} \\ 1 + 499.1(1.023z_1^2 - 1.022z_1z_2 + 0.5z_2^2 - 22.34z_2^2 - \epsilon - \bar{\xi}_2(z))^2 z_2^2 \\ \qquad \text{if} \;\; 1.023z_1^2 - 1.022z_1z_2 + 0.5z_2^2 - 22.34z_2^2 - \epsilon > \bar{\xi}_2(z) \end{cases}$$

where $\epsilon = 0.1$. Then, the inverse optimal control law with optimality matching up to 1st order is given by

$$u^* \;\; = \;\; -4.726\check{r}^{-1}(x)(x_2 + 1.022x_1 + x_1^2)$$

The worst case disturbance is given by $\bar{\nu}_2^* = 0.8755z_1 + 2.415z_2 + 4.728z_1z_2$.

2nd order matching

In this case, we are interested to obtain an inverse robust optimal controller that is optimal matching to the 2nd order, we have $m = 2$, and will be concerned with the 3rd order Taylor series expansion of the value function V.

Again, we start by obtaining a factorizable function W that matches the value function V up to 3rd order. We follow the recursive procedure of Section 3 to construct the function W that is factorizable in the order (x_2, x_1). Let

$$V_p(x_2, x_1) = \frac{\partial V^{[3]}}{\partial x_2} = 9.657x_1 + 9.448x_2 + 189.4x_1^2 + 269.4x_1x_2 + 97.50x_2^2$$

The root function $\alpha(x_1)$ up to second order for x_2 is given, using Equation (20), by $\alpha(x_1) = -1.022x_1 - 1.684x_1^2$. In terms of $\eta := x_2 + 1.022x_1 + 1.684x_1^2$ and x_1, the function $\bar{\Pi}$ is obtained as $\bar{\Pi}(\eta, x_1) = 9.448 + 70.11x_1 + 97.50\eta - 328.4x_1^2$. Choosing the constant $C = 0.448$, we obtain the function $\bar{\Pi}_s$ as $\bar{\Pi}_s(\eta, x_1) = 3 + 11.69x_1 + 16.25\eta$.

Then, we have the following identity

$$\begin{aligned}V^{[3]}(x_1, x_2) =& \left(\int_0^\eta s((3 + 11.69x_1 + 16.25s)^2 + 0.448)\bar{T}_1^2(s, x_1)\, ds \Big|_{\eta = x_2 + 1.022x_1 + 1.684x_1^2} \right)^{[3]} \\ &+1.745x_1^2 + 2.351x_1^3\end{aligned}$$

where $\bar{T}_1^2 := \frac{1}{1+p_1 s^2}$ and $p_1 \geq 0$ is a design parameter.

Hence, we have $\hat{V}(x_1) = 1.745x_1^2 + 2.351x_1^3$, which is the function to be matched at the second step of the nonlinear Cholesky factorization.

Let $V_p(x_1) = \frac{\partial \hat{V}}{\partial x_1} = 3.490x_1 + 7.053x_1^2$. For this last step, the root function is fixed to be $x_1 = 0$. Then, we have $\bar{\Pi}(x_1) = 3.490 + 7.053x_1$. Choose the constant $C = 0.250$, the function $\bar{\Pi}_s$ is given by $\bar{\Pi}_s(x_1) = 1.8 + 1.959x_1$. Thus, the function $\hat{V}$ matches the following function up to 3rd order.

$$
\begin{aligned}
\hat{V}(x_1) &= \left(\int_0^\eta s((1.8 + 1.959s)^2 + 0.250)\bar{T}_2^2(s)\, ds \Big|_{\eta = x_1} \right)^{[3]} \\
&= \left(5\left[(0.349 - 0.384p_2^{-1})\ln(1+p_2x_1^2) - 1.41\tan^{-1}(p_2^{0.5}x_1)p_2^{-0.5} + 1.41x_1 + 0.384x_1^2\right]p_2^{-1}\right)^{[3]}
\end{aligned}
$$

where $\bar{T}_2^2 := \frac{1}{1+p_2 s^2}$ and $p_2 \geq 0$ is a design parameter.

The functions $\bar{T}_1$ and $\bar{T}_2$ are chosen to reduce the control effort for the inverse optimal matching design. A possibly better choice for $\bar{T}_1^2$ could be $\frac{1}{1+p_{11}x_1^2+p_{12}x_1 s+p_{13}s^2}$, which may further reduce the control effort for the inverse optimal matching design. Yet, the design expressions for this choice of $\bar{T}_1$ is too complicated for the symbolic mathematics computation software that we are using to handle. Therefore, we choose the functions $\bar{T}_1$ and $\bar{T}_2$ as above, which results in much reduced control effort for the inverse optimal matching design as compared with that for a design with the simple choice $\bar{T}_1 \equiv 1$ and $\bar{T}_2 \equiv 1$

This completes the recursive design step for the matching function $W(x_1, x_2)$, which is given by

$$
W(x_1, x_2) = \hat{W}_1(\eta)\Big|_{\eta = x_1} + \hat{W}_2(x_1, \eta)\Big|_{\eta = x_2 + 1.022x_1 + 1.684x_1^2}
$$

where

$$
\begin{aligned}
\hat{W}_1(\eta) &:= \int_0^\eta s((1.8 + 1.959s)^2 + 0.250)T_2^2(s)\, ds \\
\hat{W}_2(x_1, \eta) &:= \int_0^\eta s((3 + 11.69x_1 + 16.25s)^2 + 0.448)T_1^2(s, x_1)\, ds
\end{aligned}
$$

and which have the following properties $\frac{\partial \hat{W}_1}{\partial x_1}(x_1) = 2x_1\bar{\pi}_1(x_1)$, $\frac{\partial \hat{W}_2}{\partial \eta}(x_1, \eta) = 2\eta\bar{\pi}_2(x_1, \eta)$, $\frac{\partial \hat{W}_2}{\partial x_1}(x_1, \eta) = I(x_1, \eta)$, where

$$
\begin{aligned}
\bar{\pi}_1(x_1) &:= 0.5\left((1.8 + 1.959x_1)^2 + 0.25\right)/(1 + p_2x_1^2) > 0 \\
\bar{\pi}_2(x_1, \eta) &:= 0.5\left((3 + 11.69x_1 + 16.25\eta)^2 + 0.448\right)/(1 + p_1\eta^2) > 0 \\
I(x_1, \eta) &:= -5.0\left[-75.985\eta + 75.985\tan^{-1}(p_1^{0.5}\eta)p_1^{-0.5}\right. \\
&\quad \left. -7.014\ln(1 + p_1\eta^2) - 27.33\ln(1 + p_1\eta^2)z_1\right]p_1^{-1}
\end{aligned}
$$

This function W then matches the function V up to the 3rd order.

The simple diffeomorphism involved in the nonlinear Cholesky factorization for the function W is given by

$$
\begin{bmatrix} z_{l1} \\ z_{l2} \end{bmatrix} = \Phi_s(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 + 1.022x_1 + 1.684x_1^2 \end{bmatrix}
$$

In terms of the z_l coordinates, the system dynamics is given by

$$\begin{aligned}
\dot{z}_{l1} &= -1.022z_{l1} - 0.684z_{l1}^2 + z_{l2} + w \\
\dot{z}_{l2} &= -1.044z_{l1} - 4.141z_{l1}^2 - 2.304z_{l1}^3 + 3.368z_{l1}z_{l2} + 1.022z_{l2} + u + (1.022 + 3.368z_{l1})w
\end{aligned}$$

The approximate value function W can be expressed as

$$\bar{W} = \hat{W}_1(z_{l1}) + \hat{W}_2(z_{l1}, z_{l2})$$

and the cost function is equivalent to, in this coordinates,

$$J_\gamma = \int_0^\infty (2.044z_{l1}^2 + 3.442z_{l1}^3 + 2.836z_{l1}^4 + (-2.044z_{l1} - 3.368z_{l1}^2)z_{l2} + z_{l2}^2 + u^2 - 2w^2)\, dt$$

With the function W available, we now proceed to the two steps of backstepping for the construction of an inversely robust optimal controller that matches the optimal control law up to 2nd order.

For the first step, we define $z_1 = z_{l1}$, and introduce the value function $V_1(z_1) = \hat{W}_1(z_1)$. The derivative of V_1 is given by

$$\dot{V}_1 = 2w^2 - 2(w - 0.5\bar{\pi}_1 z_1)^2 - 2.044z_1^2 - 3.442z_1^3 - 2.836z_1^4 + 2z_1\bar{\pi}_1(z_{l2} + \bar{\alpha}(z_1)),$$

where $(\bar{\alpha}(z_1))^{[2]} = 0$. Hence, the desired virtual control input for z_{l2} is $z_{l2} = -\bar{\alpha}(z_1)$. The worst case disturbance input for this step is given by $\bar{\nu}_1^* = 0.5\bar{\pi}_1(z_1)z_1$.

This completes the first step of the backstepping.

For the second step, we define a new state variable $z_2 = z_{l2} + \bar{\alpha}(z_1)$. Then, it satisfies the following dynamics

$$\begin{aligned}
\dot{z}_2 = & -1.044z_1 - 4.141z_1^2 + 3.368z_1z_2 + 1.022z_2 + u + H.O.T^{[3]} \\
& +(1.022 + 3.368z_1 + H.O.T^{[2]})w
\end{aligned}$$

where $H.O.T^{[i]}$ denotes high order terms whose Taylor series expansion is ith order and above.

The value function for this step is given by

$$V_2(z_1, z_2) = V_1(z_1) + \hat{W}_2(z_1, z_2).$$

The derivative of V_2 is given by

$$\begin{aligned}
\dot{V}_2 = & 2w^2 - 2(w - 0.5\bar{\pi}_1 z_1 - \frac{1}{2}(1.022 + 3.368z_1 + H.O.T^{[2]})\bar{\pi}_2 z_2 - \frac{1}{4}I(z_1, z_2))^2 \\
& -2.044z_1^2 - 3.442z_1^3 - 2.836z_1^4 + (2.051z_1 + 3.368z_1^2)z_2 - z_2^2 \\
& -\bar{\check{r}}u^2 + \bar{\check{r}}(u + \bar{\check{r}}^{-1}\bar{\pi}_2 z_2)^2 - (\bar{\check{r}}^{-1} - 1)\bar{\pi}_2^2 z_2^2 + z_2\xi_f(z_1, z_2)
\end{aligned}$$

where $\xi_f^{[2]}(z_1, z_2) = 0$.

The weighting function $\check{r}$ can be chosen according to (40) of the previous section. The inverse robust optimal control law is then given by

$$u^* = -\bar{\check{r}}^{-1}\bar{\pi}_2(z_1, z_2)z_2$$

and the corresponding worst-case disturbance input is

$$\bar{\nu}_2^* \;=\; 0.5\bar{\pi}_1 z_1 + \frac{1}{2}(1.022 + 3.368 z_1 + H.O.T^{[2]})\bar{\pi}_2(z_1, z_2) z_2 + \frac{1}{4} I(z_1, z_2)$$

The performance of the 2nd-order matching controller has been simulated to illustrate the theoretical findings. The design parameters p_1 and p_2 are optimized to reduce the control magnitude, which are fixed at $p_1 = 2000$ and $p_2 = 4$. For comparison, the performance of the 2nd-order local controller has also been simulated, where the control law is given by $u_{local} = -V_p/2$. The phase-portraits for the disturbance-free closed-loop systems under the two controllers are shown in Figure 1. We observe that manifold M_{s2} is the stability boundary for the local controller, while the matching controller guarantees global asymptotic stability. For the trajectories starting at point A_2, we have depicted the state trajectories in Figure 2 and the control trajectories in Figure 3. We observe that the control magnitude for the matching controller is some what larger than that for the local controller over the initial time horizon. This is the expense to be paid for global robustness with respect to the exogenous disturbance input.

6 Conclusion

In this paper, we have developed a combined approach of the Taylor series expansion and integrator backstepping to the problem of nonlinear H^∞ optimal control for strict-feedback nonlinear systems. We have developed a methodology to recursively construct a cost function and the corresponding solution to the HJI equation for a *fixed* strict-feedback nonlinear system such that the cost function matches a desired cost function at $x = 0$ up to *any* desired order of the Taylor series expansion. What lies at the heart of the recursive construction is the new concept of nonlinear Cholesky factorization. The solution obtained here is applicable directly to the nonlinear regulator problem. The nonlinear Cholesky factorization for a given positive definite function is defined as an upper triangular coordinate transformation such that the given function is equal to the sum of squares of all the state variables in the new coordinates. We have obtained precise conditions under which a given nonlinear function has a nonlinear Cholesky factorization. When the value function for the problem has a nonlinear Cholesky factorization, we have shown that the backstepping design procedure can be tuned to lead to the optimal control design for the problem, thus justifing backstepping as an optimality feasible design method. Making use of the Taylor series expansion for the value function, we have developed explicit recursive computation schemes for an inversely optimal controller that matches the optimal solution up to any desired order. We have developed a recursive procedure to compute an approximating function, which has a nonlinear Cholesky factorization, for the value function of the problem. Using this approximating function and its nonlinear Cholesky factorization, we have prescribed a recursive procedure to construct the desired matching controller by another backstepping procedure. An simulation example has been included to illustrate the theoretical findings.

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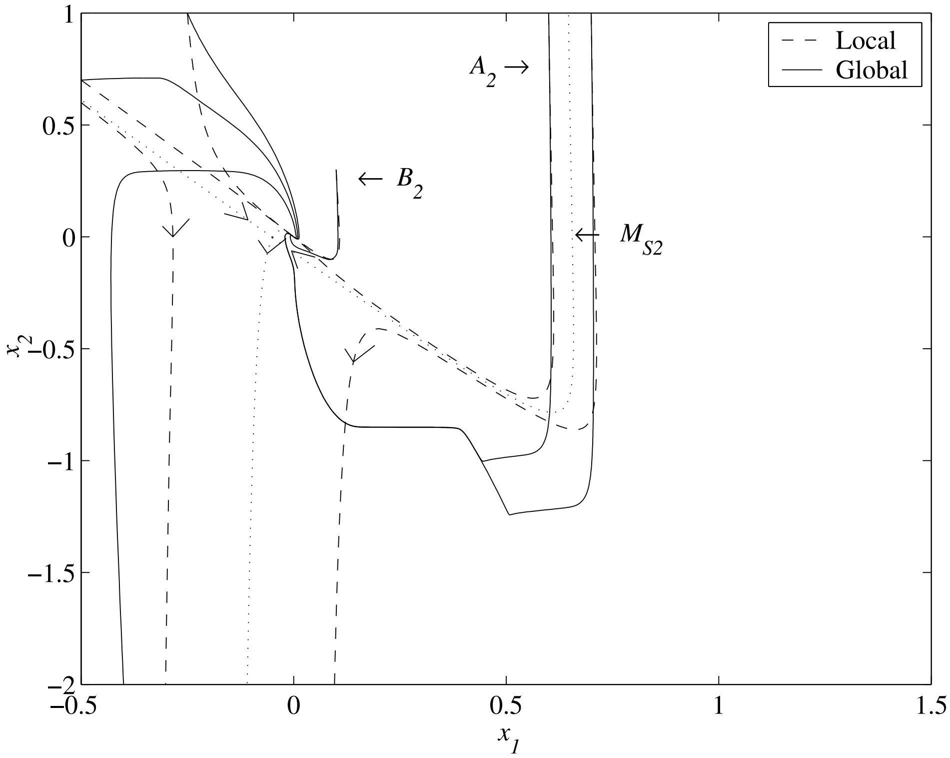

Figure 1: Phase-portraits.

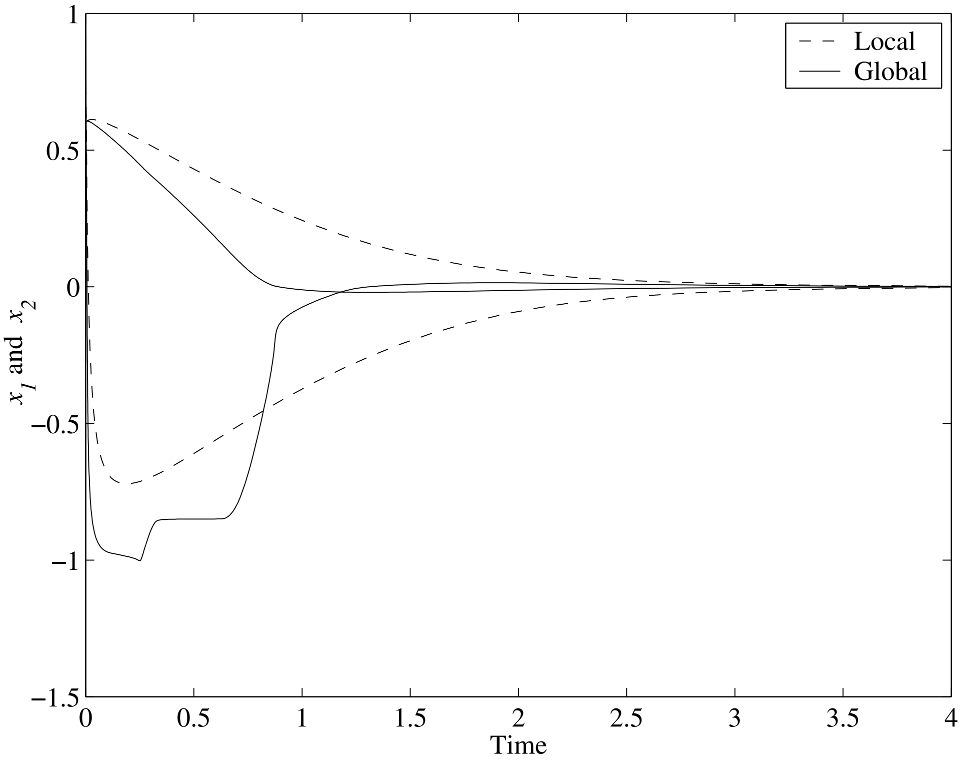

Figure 2: State trajectory.

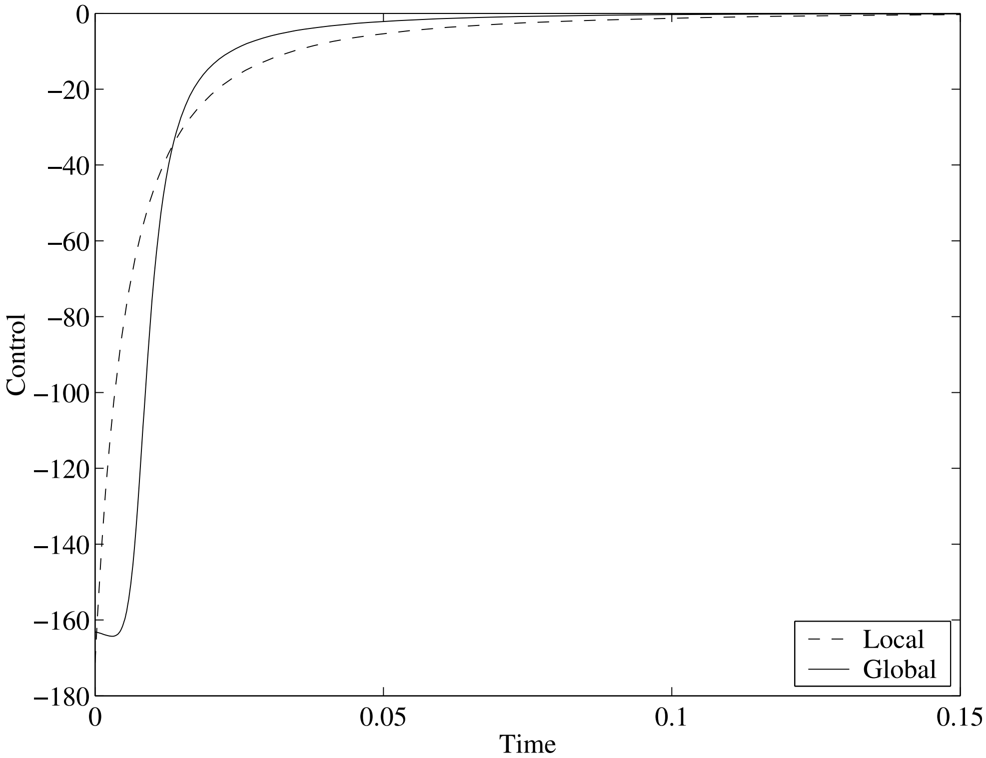

Figure 3: Control efforts.

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