Model Simplification and Optimal Control of Stochastic Singularly Perturbed Systems under Exponentiated Quadratic Cost*

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Abstract

We study the optimal control of a general class of stochastic singularly perturbed linear systems with perfect and noisy state measurements under positively and negatively exponentiated quadratic cost. The (expected) cost function to be minimized is actually taken as the long term time average of the logarithm of the expected value of an exponentiated quadratic loss. We identify appropriate "slow" and "fast" subproblems, obtain their optimum solutions (compatible with the corresponding measurement structure), and subsequently study the performances they achieve on the full-order system as the singular perturbation parameter ϵ becomes sufficiently small, with the expressions given in all cases being exact to within $O(\sqrt{\epsilon})$. It is shown that the composite controller (obtained by appropriately combining the optimum slow and fast controllers) achieves a performance level close to the optimal one whenever the full-order problem has a solution. The slow controller, on the other hand, achieves (asymptotically, as $\epsilon \to 0$) only a finite performance level (but not necessarily optimal), provided that the fast subsystem is open-loop stable. If the intensity of the noise in the system dynamics decreases to zero, however, the slow controller also achieves a performance level close to the optimal one.

The paper also presents a more direct derivation (than heretofore available) of the solution to the LEQG problem under noisy state measurements, which allows for a general quadratic cost (with cross terms) in the exponent and correlation between system and measurement noises, and obtains both necessary and sufficient conditions for existence of an optimal solution. Such a general LEQG problem is encountered in the slow-fast decomposition of the full-order problem, even if the original problem does not feature correlated noises. In this general context, the paper also establishes a complete equivalence between the LEQG problem and the H^∞-optimal control problem with measurement feedback, though this equivalence does not extend to the slow and fast subproblems arrived at after time-scale separation.

Key Words—Linear exponential quadratic Gaussian optimal control, generalized Riccati differential equation, generalized algebraic Riccati equation, singular perturbations, H^∞-optimal control.

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1 Introduction

The problem of optimal control of stochastic linear systems under exponentiated quadratic loss (so called LEQG problem) has been studied extensively in the literature, with new interest arousing on the topic due to the recently established relationship with the H^∞-optimal control of similar systems (but with deterministic disturbances) under quadratic loss. Perhaps the first formulation of the LEQG problem was given by Jacobson ([1]), in both discrete and continuous time, and using perfect state measurements, motivated by the fact that the exponentiated quadratic cost captures risk seeking or risk averse behavior, not obtainable using the LQG formulation (which is risk neutral). Indeed it was discovered in [1] that the LEQG formulation with a positive exponent is equivalent (as far as the optimal solution goes) to a deterministic zero-sum LQ differential game, which we now know ([2]) is equivalent to an H^∞-optimal control problem, thus completing the link. The counterparts of the results of [1] in the imperfect state measurement case for discrete and continuous time were later obtained in [3], [4] and [5], with the relationship with the H^∞-optimal control problem established in a series of subsequent publications, such as [6], [7], [8]; see also the book by Whittle ([9]). Similar relationships (between exponentiated cost stochastic control and worst case designs) exist also for nonlinear problems, as established for some subclass of such problems in [10]; see also the recent paper [11] for connection with differential games in the infinite horizon. Another recent paper [12] completely establishes this equivalence in the discrete-time finite-horizon case.

Our objective in this paper is to study, under both perfect and noisy state measurements, the robustness properties of the optimal solution of the LEQG problem with respect to unmodeled fast dynamics. This study is conducted in the framework of singularly-perturbed models, with a small positive parameter ϵ quantifying the extent of coupling between the slow and fast dynamics. We seek ϵ-independent controllers that provide good (in a sense to be made precise later) approximation to the optimal controller of the full-order problem in a neighborhood of $\epsilon = 0$.

As mentioned earlier, at the full-order level there is an equivalence between the positively exponentiated subclass and a class of LQ H^∞-optimal control problems with singularly perturbed dynamics, with this latter class of problems extensively studied recently from the point of view of robustness and model reduction ([13], [14], [15], [16]). This equivalence, however, does not readily carry over to the "model-reduction" stage, and as to be seen here the end results in the two cases are considerably different. One of the reasons for this is that (as it has been studied earlier in [18]) in stochastic problems the parameter ϵ has to enter the system dynamics and the measurement equation in a certain way for the problem to be well-defined as $\epsilon \to 0$. The exact problem formulation provided in Section 2 shows that indeed in the stochastic case a time-scale separation of the full-order system becomes much more involved. Nevertheless, we still find occasion to use some of our earlier results from [13] and [14] in the present development, to simplify some of the proofs. Furthermore, in the derivation of the optimal solution to the stochastic control problem associated with the slow subsystem, we are faced with the need to obtain a clean and complete solution to the general LEQG problem with general cost structure and correlation between system and measurement noises. This motivates us into the investigation that leads to the results of Section 4, which generalize the earlier results of [5].

The paper is organized as follows. In the next section (Section 2) we formulate the LEQG problem with perfect and noisy state measurements for singularly perturbed systems. In Section 3, we study the singularly perturbed stochastic control problem under perfect state measurements, where we decompose the problem into slow and fast ϵ-free subproblems, obtain optimal controllers for these subproblems, and study the optimality of the composite controller as well as that of the slow controller in terms of the attainable performance for the full-order problem. In Section 4, we present a clean derivation (under least stringent conditions) of a complete solution to the general LEQG problem under noisy state measurements with general cost structure and correlation between system and measurement noises, and in both finite- and infinite-horizon cases. In Section 5, we study the singularly perturbed LEQG problem and its decomposition under noisy state measurements, where we identify the slow and fast subproblems to the full-order problem, obtain optimal controllers for these subproblems, construct the composite controller from these controllers, and study the optimality of these suboptimal controllers in terms of the attainable performance for the full-order problem. Three numerical examples are presented in Section 6 to illustrate the theory, and the paper ends with the concluding remarks of Section 7. Details of some of the derivations, not included here, can be found in the internal report [17].

2 Problem Formulation

The system under consideration, with slow and fast dynamics, is described in the "singularly perturbed" form by

$$\left\{\begin{array}{rcl} dx_1 &=& (A_{11}x_1 + A_{12}x_2 + B_1 u_t)\,dt + G_1\,dw_t; \quad x_1(0) = x_{10} \\ \epsilon dx_2 &=& (\epsilon^{\alpha}A_{21}x_1 + A_{22}x_2 + B_2 u_t)\,dt + \epsilon^{1/2}G_2\,dw_t; \quad x_2(0) = x_{20} \end{array}\right. \tag{1}$$

$$\left\{\begin{array}{rcl} dy_1 &=& (C_{11}x_1 + C_{12}x_2)\,dt + E_1\,dw_t; \quad y_1(0) = 0 \\ dy_2 &=& (\epsilon^{1/2}C_{21}x_1 + C_{22}x_2)\,dt + \epsilon^{1/2}E_2\,dw_t; \quad y_2(0) = 0 \end{array}\right. \tag{2}$$

where $x := (x_1', x_2')'$ is the n-dimensional state vector, with x_1 of dimension n_1, and x_2 of dimension $n_2 := n - n_1$; $y := (y_1', y_2')'$ is the m-dimensional measurement process, with y_1 of dimension m_1, and y_2 of dimension $m_2 := m - m_1$; $\{u_t\}$ is the p-dimensional control input, and $\{w_t\}$ is an r-dimensional vector-valued standard Wiener process with $w_0 = 0$ with probability 1, which is independent of the initial condition; ϵ is a small positive scalar, denoting the singular perturbation parameter; the underlying probability space is the triplet $(\Omega, \mathcal{F}, \boldsymbol{P})$; and the parameter α is taken to be equal to $1/2$, except in the perfect state measurements case, when it is taken to be equal to 0. The specific way the parameter ϵ enters equations (1) and (2) is crucial for the system and measurement dynamics to be well defined in the limit as $\epsilon \to 0$, as otherwise (that is, for other powers of ϵ) either the noise dominates in the limit (and hence the optimization problem loses its significance) or the stochastic terms disappear (again making the problem uninteresting) – as extensively discussed in [18] in the context of the singularly perturbed LQG problem, whose arguments equally apply here as the nature of the cost function was irrelevant in that analysis.

Associated with the system (1), we now introduce the infinite-horizon exponentiated quadratic (*risk-sensitive*) cost function:

$$J_\theta(\mu) = \lim_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (x'Qx + u'u)\,dt \right) \right] \right\} \right\} \tag{3}$$

Here ln denotes the natural logarithm, and the scalar $\theta \neq 0$ is the *risk-sensitivity* parameter, in terms of which we will parametrize the solution.

We will consider two different information structures for the controller: perfect state measurements (where the current and past values of the state are available), and the noisy (imperfect) state measurements, where the measurement equations are as given in (2) above. Thus, in the former case, the control input $u = \{u_t\}$ is generated by a closed-loop control policy μ, according to

$$u_t = \mu(t, x_{[0,t]})\,, \quad t \geq 0\,, \tag{4}$$

where $\mu \in \mathcal{M}$ is an admissible controller, satisfying the standard conditions of Lipschitz-continuity in x and piecewise-continuity in t. Furthermore, as indicated earlier, we take in this case $\alpha = 0$, which makes the system dynamics well-defined as $\epsilon \to 0$, as shown in [18]. Our objective is to find an optimal (minimizing) controller with respect to the cost (3), i.e., a $u_t^* = \mu^*(t, x_{[0,t]})$ such that

$$J_\theta(\mu^*) = \min_{\mu \in \mathcal{M}} J_\theta(\mu) := J_\theta^* \tag{5}$$

Here, the initial state x_0 is taken as a fixed vector in $\mathbf{R}^n$

In the noisy state measurements case, the initial state is taken to be a Gaussian random vector with mean $\bar{x}_0$ and covariance Σ_0, where Σ_0 is assumed to be positive definite, and it depends on ϵ in a way to be specified shortly. In this case, the control input $u \in \mathcal{H}_u$ is generated by a control policy μ_I, according to[1]

$$u_t = \mu_I(t, y_{[0,t]})\,, \quad t \geq 0\,, \tag{6}$$

where $\mu_I : [0, t_f] \times \mathcal{H}_y \to \mathcal{H}_u$ is piecewise continuous in t and Lipschitz continuous in $y := (y_1' \quad y_2')' \in \mathcal{H}_y$, further satisfying the given causality condition. Let us denote the class of all admissible controllers in this case by $\mathcal{M}_I$, which is defined in precise mathematical terms in Section 4 later. For the limiting system and measurement equations to be well defined (as $\epsilon \to 0$), we take $\alpha = 1/2$ in this case. Denoting the cost function (3) for the noisy state measurements case by $J_{I\theta}(\mu_I)$, we again seek an optimal controller with respect to $J_{I\theta}(\mu_I)$, that is a $u_t^* = \mu_I^*(t, y_{[0,t]})\,,\ t \geq 0\,,$ such that

$$J_{I\theta}(\mu_I^*) = \min_{\mu_I \in \mathcal{M}_I} J_{I\theta}(\mu_I) := J_{I\theta}^* \tag{7}$$

To complete the formulation of this risk sensitive stochastic control problem, we introduce two basic assumptions, which will be required to hold throughout.

A1 Σ_0 and Q in (3) are partitioned as

$$\Sigma_0 = \begin{bmatrix} \Sigma_{011} & \sqrt{\epsilon}\,\Sigma_{012} \\ \sqrt{\epsilon}\,\Sigma_{021} & \Sigma_{022} \end{bmatrix}; \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

where in each case the 11-block is of dimensions $n_1 \times n_1$, and the 22-block of $n_2 \times n_2$.

A2 The matrices A_{22}, Q_{22}, $G_2 G_2'$ and $N := EE'$ are invertible, where $E' = [E_1'\, E_2']$. [2] The system noise and the measurement noise are uncorrelated, i.e., $G_1 E' = 0$ and $G_2 E' = 0$.

[1] $\mathcal{H}_u$ denotes the Hilbert space of p-dimensional square-integrable functions (the controls). Likewise, $\mathcal{H}_y$ denotes the Hilbert space of m-dimensional square-integrable functions (the measurements).

[2] The conditions of invertibility of Q_{22} and $G_2 G_2'$ can be further relaxed to the conditions of observability of the pairs (A_{22}, Q_{22}) and (A_{22}, G_2). Results similar to those (to be) presented in this paper can be derived under these relaxed conditions, by perturbing the matrices Q_{22} and $G_2 G_2'$ by λI, for some scalar $\lambda > 0$, applying the results of this paper, and then letting $\lambda \to 0$; the limiting quantities are all well defined, as shown in [19].

3 Model Simplification under Perfect State Measurements

The full-order solution to the (infinite-horizon) LEQG problem with state feedback can be obtained by taking an appropriate limit of the finite-horizon solution first obtained in [1]. Toward presenting this solution, let us first introduce the following useful notation:

$$A_\epsilon := \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{bmatrix}; \qquad B_\epsilon := \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}; \qquad G_\epsilon := \begin{bmatrix} G_1 \\ \frac{1}{\sqrt{\epsilon}}G_2 \end{bmatrix} \tag{8}$$

Note that in terms of this notation, the n-dimensional system dynamics can be written in the compact form:

$$dx = (A_\epsilon x + B_\epsilon u_t)\,dt + G_\epsilon\,dw_t\,; \quad x(0) = x_0\,.$$

Introduce the quantity

$$S_\epsilon(\theta) := B_\epsilon B_\epsilon' - \theta G_\epsilon G_\epsilon'\,, \tag{9}$$

and consider the generalized algebraic Riccati equation (GARE)

$$A_\epsilon'\tilde{Z} + \tilde{Z}A_\epsilon - \tilde{Z}S_\epsilon\tilde{Z} + Q = 0\,. \tag{10}$$

Further introduce the quantity

$$\theta^*(\epsilon) \;:=\; \sup\{\theta \in \boldsymbol{R} : \text{the GARE (10) admits a positive definite solution } \tilde{Z}(\epsilon)\}. \tag{11}$$

Assume that (A_ϵ, B_ϵ) is controllable and (A_ϵ, Q) is observable for every $\epsilon > 0$. Then it follows by taking an appropriate limit of the finite-horizon result of [1] (see Theorem 7 of [17]) that, for each $\epsilon > 0$, $\theta^*(\epsilon)$ is positive, and for $\theta < \theta^*(\epsilon)$ the LEQG problem (with perfect state measurements) admits an optimal state-feedback solution:

$$u^* = \mu^*(x) = -B_\epsilon'\tilde{Z}(\epsilon)\,x, \quad t \geq 0. \tag{12}$$

with the optimal (minimum) cost being

$$J_\theta^*(\epsilon) = \mathrm{Tr}(G_\epsilon G_\epsilon'\tilde{Z}(\epsilon))\,. \tag{13}$$

Furthermore, the feedback matrix $A_\epsilon - B_\epsilon B_\epsilon'\tilde{Z}(\epsilon)$ is Hurwitz.

Thus completing presentation of the full-order solution (for all $\epsilon > 0$), we now return to the original goal of this paper, which is the derivation of the optimal solution as $\epsilon \to 0$, via model simplification. Toward the end of obtaining ϵ-free solutions, we first decompose the system into slow and fast modes as in [13].

3.1 Time-Scale Decomposition

Slow subsystem

The slow subsystem is obtained by letting $\epsilon = 0$ in the system dynamics, and solving for x_2 (to be denoted $\bar{x}_2$) in terms of $x_1 =: x_s$, $u =: u_s$, and under the working assumption A2:

$$\bar{x}_2 = -A_{22}^{-1}(A_{21}x_s + B_2u_s). \tag{14}$$

Using this in the first equation of (1), we obtain the reduced-order (slow) dynamics:

$$dx_{st} = (A_0 x_{st} + B_0 u_{st})\, dt + G_1\, dw_t \tag{15}$$

where $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$. Use of (14) also in the cost function (3) leads to the reduced (slow) cost (with $x_1 = x_s$):

$$\begin{aligned} J_{s\theta}(\mu_s) &= \lim_{t_f \to \infty} \frac{2}{\theta t_f} \ln\{\boldsymbol{E}\{\exp[\frac{\theta}{2}(\int_0^{t_f} (|x_s|^2_{Q_{11}} + x_s' Q_{12}\bar{x}_2 + \bar{x}_2' Q_{21} x_s + |\bar{x}_2|^2_{Q_{22}} \\ &\qquad + |u_s|^2)\, dt)]\}\} \end{aligned} \tag{16}$$

Note that this is another LEQG problem, which this time has a cross term in the cost between the control and the state; but a simple linear state-feedback transformation on the control brings it into the same form as the full-order LEQG problem, and hence the theory described at the beginning of this section (for the full-order problem) applies, with some obvious modifications. Hence, the *slow* LEQG problem admits an optimal solution if the following GARE,

$$\tilde{A}_0' \tilde{Z}_s + \tilde{Z}_s \tilde{A}_0 - \tilde{Z}_s S_0 \tilde{Z}_s + \tilde{Q} = 0\,, \tag{17}$$

admits a minimal positive definite solution $Z_{s\theta}$, such that the matrix $\tilde{A}_0 - S_0 Z_{s\theta}$ is Hurwitz, where the coefficient matrices above are explicit functions of the parameter θ, and are written as (see [13] for details):

$$\begin{aligned} \tilde{A}_0(\theta) &= A_{11} - A_{12}Q_{22}^{-1}Q_{21} - (S_{12} + A_{12}Q_{22}^{-1}A_{22}')(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1} \\ &\quad \cdot(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) \end{aligned} \tag{18}$$

$$\begin{aligned} \tilde{Q} &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (A_{21}' - Q_{12}Q_{22}^{-1}A_{22}')(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1} \\ &\quad \cdot(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) \end{aligned} \tag{19}$$

$$\begin{aligned} S_0(\theta) &= S_{11} + A_{12}Q_{22}^{-1}A_{12}' - (S_{12} + A_{12}Q_{22}^{-1}A_{22}')(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1} \\ &\quad \cdot(S_{21} + A_{22}Q_{22}^{-1}A_{12}') \end{aligned} \tag{20}$$

where S_{ij} denotes the ij'th block of $S(\theta)$, of dimensions $n_i \times n_j$, $i, j = 1, 2$, where the latter is defined by

$$S(\theta) := BB' - \theta GG'\,; \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \quad G := \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \tag{21}$$

In view of this , let us define

$$\theta_s \ := \ \sup\{\theta \in \boldsymbol{R} : \text{the GARE (17) admits a positive definite solution.}\} \tag{22}$$

Then, the transformed LEQG problem admits an optimal solution if $\theta < \theta_s$. For $\theta < \theta_s$, let $Z_{s\theta}$ be the unique positive definite solution of (17). Then, the optimal controller (after the inverse transformation) is given by (see [17])

$$\begin{aligned} u_{s\theta}^* &= \mu_{s\theta}^*(t, x_s) = (-B_1' Z_{s\theta} + B_2'(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\theta} \\ &\quad - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_s \end{aligned} \tag{23}$$

Fast subsystem

To arrive at the fast subproblem, let $x_f := x_2 - \bar{x}_2$, $u_f := u - u_s$ and $\tau = \frac{t'-t}{\epsilon}$, where we take t to be frozen, and t' to vary on the same scale as t. We define the fast subsystem and the associated cost (as in the standard regulator problem; see [23]) by:

$$\frac{d}{d\tau}x_f^t = A_{22}x_f^t + B_2 u_f^t; \quad x_f^t(0) = x_f \tag{24}$$

$$J_{f\theta}^t(\mu_f^t) = \frac{2}{\theta}\ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2}\left(\int_0^\infty (|x_f^t|_{Q_{22}}^2 + |u_f^t|^2)\, d\tau\right)\right]\right\}\right\} \tag{25}$$

This is a deterministic strictly convex optimal control problem,[3] which admits a unique optimal controller that does not depend on the parameter θ:

$$\begin{aligned} u_f^{t*}(\tau) &= \mu_f^{t*}(x_f^t(\tau)) = -B_2' Z_f x_f^t(\tau) \\ \Rightarrow \quad \mu_f^* &= \mu_f^{t*}(x_f^t(0)) = -B_2' Z_f x_f = -B_2' Z_f (x_2 - \bar{x}_2) \end{aligned} \tag{26}$$

where Z_f is the positive definite solution to the ARE [4]

$$A_{22}' Z_f + Z_f A_{22} + Q_{22} - Z_f S_{22} Z_f = 0\,. \tag{27}$$

Substitute (14) and (23) into (26), to obtain

$$\begin{aligned} \mu_{f\theta}^*(t,x) \;=\; & -B_2' Z_f x_2 - B_2' Z_f Q_{22}^{-1}(A_{12}' Z_{s\theta} + Q_{21} - A_{22}'(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1} \\ & \cdot((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\theta} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_1 \end{aligned} \tag{28}$$

Also, introduce the following Lyapunov equation, when the matrix A_{22} is Hurwitz:

$$A_{22}' Z_{of} + Z_{of} A_{22} + Q_{22} = 0 \tag{29}$$

whose relevance to our problem will be seen shortly.

3.2 Composite Controller

We now introduce the composite controller:

$$\mu_{c\theta}^*(t,x) = \mu_{s\theta}^*(t,x) + \mu_{f\theta}^*(t,x) \tag{30}$$

where $\mu_{s\theta}^*$ and μ_f^* were defined by (23) and (28) respectively, for $\theta < \theta_s$. After some manipulations, this composite controller can be written as

$$\mu_{c\theta}^*(t,x) = -B' \begin{bmatrix} Z_{s\theta} & 0 \\ Z_c & Z_f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{31}$$

where

$$\begin{aligned} Z_c \;:=\; & Z_f Q_{22}^{-1}(A_{12}' Z_{s\theta} + Q_{21}) - (I + Z_f Q_{22}^{-1} A_{22}')(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1} \\ & \cdot((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\theta} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})) \end{aligned} \tag{32}$$

and

$$B := (B_1' \; B_2')\,.$$

[3] Recall that by our standard assumption **A1**, $Q_{22} > 0$.

[4] This ARE admits a unique positive-definite solution if the pair (A_{22}, B_2) is controllable.

3.3 Performances of the Suboptimal Controllers

The next important set of issues is the derivation of the conditions under which the performances attained by the composite and slow controllers are finite, and study of the limiting behavior of these expected cost values as $\epsilon \to 0$. A complete answer to these questions is given in the following theorem.

Theorem 1 *For the singularly perturbed system (1) with state-feedback information and with the cost function (3), let the relevant parts of assumptions A1–A2 be satisfied, the pairs (A_0, B_0) and (A_{22}, B_2) be controllable and the pair $(A_0, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ be observable. Then,*

1. $\lim_{\epsilon\to 0^+} \theta^*(\epsilon) = \theta_s$.
2. $\forall\theta < \theta_s$, $\exists\epsilon_\theta > 0$ *such that* $\forall\epsilon \in [0, \epsilon_\theta)$, *the GARE (10) admits a positive definite solution, and consequently, the problem has an optimal solution, and the optimal cost for the problem can be approximated by*
$$J^*_{\theta\infty}(\epsilon) = Tr(G_1G_1'Z_{s\theta} + G_2G_2'Z_f) + O(\sqrt{\epsilon}) \tag{33}$$
3. $\forall\theta < \theta_s$, *if the composite controller $\mu^*_{c\theta}$ is applied to the system, then* $\exists\epsilon'_\theta > 0$ *such that* $\forall\epsilon \in [0, \epsilon'_\theta)$,
$$J^c_\theta := J_\theta(\mu^*_{c\theta}) = J^*_\theta(\epsilon) + O(\sqrt{\epsilon}) \tag{34}$$
4. $\forall\theta < \theta_s$, *if, in addition, the matrix A_{22} is Hurwitz, and the slow controller $\mu^*_{s\theta}$ is applied to the system, then* $\exists\tilde\epsilon_\theta > 0$ *such that* $\forall\epsilon \in [0, \tilde\epsilon_\theta)$,
$$J^s_\theta := J_\theta(\mu^*_{s\theta}) = J^*_\theta(\epsilon) + Tr(G_2G_2'(Z_{of} - Z_f)) + O(\sqrt{\epsilon}) \tag{35}$$

Proof By Theorem 1 in [13], result 1) follows, and there exits an $\epsilon_\theta > 0$, such that the full-order GARE (10) admits a positive definite solution, for $\epsilon \in [0, \epsilon_\theta)$, which can be approximated by
$$\tilde Z = \begin{bmatrix} Z_{s\theta} + O(\sqrt{\epsilon}) & \epsilon Z_c' + O(\epsilon^{3/2}) \\ \epsilon Z_c + O(\epsilon^{3/2}) & \epsilon Z_f + O(\epsilon^{3/2}) \end{bmatrix}.$$
By Corollary 2 in [16], we have that the matrix $\tilde A_0 - S_0 Z_{s\theta}$ is Hurwitz for $\theta < \theta_{s\infty}$, which leads to 2).

For 3) and 4), we substitute the controllers $\mu^*_{c\theta}$ and $\mu^*_{s\theta}$ into the full-order system to obtain the resulting control-free LEQG problems. We then use the detailed derivations that led to Theorem 1 of [13] to establish 3) and 4). □

3.4 A Large Deviation Form

Consider now a large deviation form of the problem considered in this section, which is the case when the system noise intensity asymptotically approaches zero. To formally illustrate this situation, we consider the following set-up for the problem:
state dynamics :
$$\left\{ \begin{array}{rcl} dx_1 &=& (A_{11}x_1 + A_{12}x_2)\,dt + B_1 u + \xi G_1\,dw_t; \quad x_1(0) = x_{10} \\ \epsilon dx_2 &=& (A_{21}x_1 + A_{22}x_2 + B_2 u)\,dt + \sqrt{\epsilon}\xi G_2\,dw; \quad x_2(0) = x_{20} \end{array} \right. \tag{36}$$

cost function :

$$J_{\theta\infty}(\mu,\xi) = \lim_{t_f\to\infty} \frac{2\xi^2}{\theta t_f} \ln\left\{ \boldsymbol{E}\left\{ \exp\left[\frac{\theta}{2\xi^2}\left(\int_0^{t_f}(x'Qx+u'u)\,dt\right)\right]\right\}\right\} \tag{37}$$

where ξ is a small scalar parameter to be varied. We will study the solution as $\xi \to 0$. Note that this problem is equivalent to the one considered earlier in this section, if we introduce the substitutions:

$$\theta \leftarrow \frac{\theta}{\xi^2}; \quad G_\epsilon \leftarrow \xi G_\epsilon \tag{38}$$

Assume that (A_ϵ, B_ϵ) is controllable, and (A_ϵ, Q) is observable for every $\epsilon > 0$. Under assumptions A1–A2, we know that the optimal solution exists, for each fixed $\epsilon > 0$, if the GARE (10) admits a positive-definite solution, in which case the optimal controller is given by (12) independent of the parameter ξ. We define the quantity $\theta^*(\epsilon)$ in the same way as in (11). Thus, $\forall\theta < \theta^*(\epsilon)$, the optimal cost is given by

$$J^*_{\theta\infty}(\epsilon,\xi) = \xi^2 \mathrm{Tr}(G_\epsilon G_\epsilon' \tilde{Z}(t;\epsilon)) \tag{39}$$

where $\tilde{Z}(\epsilon)$ is the minimal positive-definite solution to the GARE (10). Hence, $J^*_{\theta\infty}(\epsilon,\xi) \to 0$ as $\xi \to 0$.

To obtain ϵ-free solutions to the problem, we decompose the system into slow and fast subsystems. The slow subsystem is obtained by setting $\epsilon = 0$ in the state dynamics, as well as in the cost function. This slow problem admits an optimal solution if the GARE (17) admits a positive definite solution $Z_{s\theta}$, such that the matrix $\tilde{A}_0 - S_0 Z_{s\theta}$ is Hurwitz. We define θ_s the same way as in (22). Then, for $\theta < \theta_s$, the optimal control for the slow subsystem is again given by (23).

To obtain the fast subsystem, we use the same notation as in subsection 3.1. The fast dynamics are the same as (24) and the cost function is the same as (25) under the substitutions (38). Then, the optimal control is given by (26), where Z_f is the unique positive definite solution to ARE (27). Substitution of (14) and (23) into (26) yields the fast controller, which is precisely (28). We also introduce the matrix Z_{of} to be the solution of Lyapunov equation (29).

Then, we form the composite controller as in (30), which leads to (31). We now summarize this result below, as a corollary to Theorem 1:

Corollary 1 *For the singularly perturbed system (36), with state-feedback information and with the cost function (37), let assumptions A1–A2 be satisfied, the pairs (A_0, B_0) and (A_{22}, B_2) be controllable, and the pair $(A_0, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ be observable. Then,*

1. $\lim_{\epsilon\to 0^+} \theta^*(\epsilon) = \theta_s$.

2. *$\forall\theta < \theta_s$, $\exists\,\epsilon_\theta > 0$ such that $\forall\epsilon \in [0,\epsilon_\theta)$, the GARE (10) admits a positive definite solution, and consequently, the problem has an optimal solution, and the optimal cost for the problem can be approximated by*

$$J^*_\theta(\epsilon,\xi) = \xi^2(Tr(G_1G_1'Z_{s\theta} + G_2G_2'Z_f) + O(\sqrt{\epsilon})) \tag{40}$$

3. *$\forall\theta < \theta_s$, if the composite controller $\mu^*_{c\theta}$ is applied to the system, then $\exists\,\epsilon'_\theta > 0$ such that $\forall\epsilon \in [0,\epsilon'_\theta)$,*

$$J^c_\theta(\xi) := J_\theta(\mu^*_{c\theta},\xi) = J^*_\theta(\epsilon,\xi) + O(\xi^2\sqrt{\epsilon}) \tag{41}$$

4. $\forall\theta < \theta_s$, *if, in addition, the matrix* A_{22} *is Hurwitz, and the slow controller* $\mu^*_{s\theta}$ *is applied to the system, then* $\exists\tilde{\epsilon}_\theta > 0$ *such that* $\forall\epsilon \in [0, \tilde{\epsilon}_\theta)$,

$$J^s_\theta(\xi) := J_\theta(\mu^*_{s\theta}, \xi) = J^*_\theta(\epsilon, \xi) + O(\xi^2) \tag{42}$$

□

4 Imperfect State Measurements : Full-Order Solution

In order to obtain the counterparts of the results of Section 3 in the noisy state measurements case, we will first need the solution to the full-order case, which is derived in this section. The derivation proceeds in two steps: First the solution to the finite-horizon LEQG problem is obtained, and then an appropriate limit of that solution is taken.

The model we adopt in this section involves, by necessity, a more general version of the problem formulated in Section 2 (for fixed $\epsilon > 0$), where the system and measurement noises are correlated, and the exponent of the loss function contains an additional cross term between the state and the control. Such a generalized model will be encountered in the next section, when we study the slow subproblem. Hence, the solution derived in this section will serve two purposes: It provides a solution to the full-order problem, and also a solution to the slow subproblem which is the counterpart of the one of subsection 3.1 in the noisy measurement case.

Accordingly, we now consider the following system and measurement dynamics:

$$\left\{ \begin{array}{lcl} dx & = & (Ax + Bu_t)\, dt + G\, dw_t; \quad x(0) = x_0 \\ dy & = & Cx\, dt + E\, dw_t; \quad y(0) = 0 \end{array} \right. \tag{43}$$

where the correlation between the system and measurement noises is given by $L := GE' \neq 0$, and $x_0 \sim N(\bar{x}_0, \Sigma_0)$, $\Sigma_0 > 0$, $N := EE' > 0$.

4.1 The Finite-Horizon Problem

Along with the system described by (43), consider the cost function

$$J_{I\theta}(\mu_I) = \frac{2}{\theta} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \left(x'_{t_f} Q_f x_{t_f} + \int_0^{t_f} (x'Qx + 2x'Pu + u'Ru)\, dt \right) \right] \right\} \right\} \tag{44}$$

where $R > 0$, $Q_f \geq 0$, and $Q - PR^{-1}P \geq 0$.[5]

Introduce the following notation,

$$\begin{array}{lll} \bar{A} := A - BR^{-1}P'; & \bar{Q} := Q - PR^{-1}P'; & \bar{S} := BR^{-1}B' - \theta GG'; \\ \tilde{A} := A - LN^{-1}C; & \tilde{M} := GG' - LN^{-1}L'; & \tilde{R} := C'N^{-1}C - \theta Q \end{array}$$

and in terms of this the backward generalized (game) Riccati differential equation (GRDE):

$$\dot{\tilde{Z}} + \bar{A}'\tilde{Z} + \tilde{Z}\bar{A} - \tilde{Z}\bar{S}\tilde{Z} + \bar{Q} = 0; \quad \tilde{Z}(t_f) = Q_f\,, \tag{45}$$

[5]The analysis and results of this subsection are valid even if the system and cost matrices are time varying, satisfying some natural smoothness conditions, such as continuous differentiability in t; see [17].

and the forward GRDE:

$$\dot{\tilde{\Sigma}} = \tilde{\Sigma}\tilde{A}' + \tilde{A}\tilde{\Sigma} - \tilde{\Sigma}\tilde{R}\tilde{\Sigma} + \tilde{M}; \quad \tilde{\Sigma}(0) = \Sigma_0. \tag{46}$$

Define the quantity:

$$\begin{aligned} \theta_I^* \quad := \quad & \sup\{\theta \in \boldsymbol{R} : \text{the GRDEs (45) and (46) admit nonnegative definite solutions} \\ & \tilde{Z} \text{ and } \tilde{\Sigma}, \text{ respectively, on } [0,t_f], \text{ and that the matrix } I - \theta\tilde{\Sigma}\tilde{Z} \\ & \text{has only positive eigenvalues, for each } t \in [0,t_f]\ \}, \end{aligned} \tag{47}$$

and for $\theta < \theta_I^*$, introduce the filter:

$$d\check{x} \quad = \quad (A + \theta\tilde{\Sigma}Q)\check{x}\,dt + (B + \theta\tilde{\Sigma}P)u\,dt + (\tilde{\Sigma}C' + L)N^{-1}(dy - C\check{x}\,dt) \tag{48}$$

with initial state $\check{x}(0) = \bar{x}_0$. Letting $\hat{x} := (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\check{x}$, it can be shown (see [17]) that $\hat{x}$ is generated by the following dynamics:

$$\begin{aligned} d\hat{x} \quad = \quad & (\bar{A} - \bar{S}\tilde{Z})\hat{x}\,dt + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(B + \theta\tilde{\Sigma}P)\tilde{u}\,dt + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(\tilde{\Sigma}C' + L)N^{-1} \\ & (dy - (C + \theta L'\tilde{Z})\hat{x}\,dt) \end{aligned} \tag{49}$$

where $\tilde{u} := u + R^{-1}(B'\tilde{Z} + P')\hat{x}$.

Let $\varepsilon := x - \check{x}$ and $e := x - \hat{x}$. We will now restrict our attention to the class of controllers such that the following process ζ defines a martingale on $[0,t_f]$:

$$\begin{aligned} \zeta \quad = \quad & \exp\{\int_0^t (\varepsilon'\tilde{\Sigma}^{-1}G - e'\tilde{\Sigma}^{-1}(\tilde{\Sigma}C' + L)N^{-1}E dw_t - \frac{1}{2}\int_0^t |G'\tilde{\Sigma}^{-1}\varepsilon \\ & \qquad - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}e|^2\,dt\} \end{aligned} \tag{50}$$

Hence, we define the set of admissible controllers $\mathcal{M}_I$ to be all mappings $\mu_I : [0,t_f] \times \mathcal{H}_y \to \mathcal{H}_u$ that are piecewise continuous in t and Lipschitz continuous in y, and further satisfying the given causality condition such that ζ is a martingale on $[0,t_f]$.

The above condition will be needed in the application of the Girsanov Theorem [21] for a change of probability measures. A sufficient condition for this condition to be satisfied is the existence of positive constants δ and κ such that

$$\boldsymbol{E}\{\exp\{\delta|G'\tilde{\Sigma}^{-1}\varepsilon - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}e|^2\}\} \leq \kappa, \quad \forall t \in [0,t_f].$$

It should be obvious that any linear control law renders this condition valid, and hence the class of admissible controllers is bigger than the class of all linear controllers. We refer the reader to the recent book [21] for a thorough coverage of this topic.

We note at this point that this definition of the admissible controllers involves only one martingale condition, whereas the one of [5] involved two such (separate) conditions. We can do away with the second condition because of fact that in the proof to follow, we pursue a line of reasoning motivated by the "completions of squares" proof for the H^∞ optimal control problem, which avoids recasting of the problem in a new probability space, where the processes $\{w_t\}$ and $\{y_t\}$ are independent Wiener processes. As a result of this alternative approach, the existence of an optimal controller is not restricted by the additional condition of existence of a solution to GRDE (3.5) of [5] which clearly is unnecessary. Hence, the ensuing derivation is much simpler than that of [5], and it is more complete in the sense that it leads to both necessary and sufficient conditions for the existence of an optimal controller.

Now, we prove the following result:

Theorem 2 *Consider the general finite-horizon LEQG problem described by (43) and (44). Assume that $\Sigma_0 > 0$, $R > 0$, $Q - PR^{-1}P' \geq 0$ and $N > 0$. Then, for each $\theta < \theta_I^*$, the optimal controller is given by*

$$u_{It}^* = \mu_I^*(t, y_{[0,t]}) = -R^{-1}(B'\tilde{Z} + P')(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\check{x} \equiv -R^{-1}(B'\tilde{Z} + P')\hat{x} \tag{51}$$

where $\check{x}$ is generated by the filter (48), or equivalently, $\hat{x}$ is generated by the filter (49). The optimal cost can be written as:

$$\begin{aligned} J_{I\theta}^* &= \inf_{\mu_I \in \mathcal{M}_I} J_{I\theta}(\mu_I) = \bar{x}_0'\tilde{Z}(0)(I - \theta\Sigma_0\tilde{Z}(0))^{-1}\bar{x}_0 - \frac{1}{\theta}\ln\{\det(I - \theta\tilde{\Sigma}(t_f)Q_f)\} \\ &\quad + \int_0^{t_f} Tr(\tilde{\Sigma}Q + (\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L')\tilde{Z}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1})\, dt \end{aligned} \tag{52}$$

Furthermore, the above controller is also conditionally optimal. [6]

Proof The differential equation for ε is easily obtained to be:

$$d\varepsilon = (\tilde{A} - \tilde{\Sigma}C'N^{-1}C)\varepsilon\, dt - \theta\tilde{\Sigma}Q\check{x}\, dt - \theta\tilde{\Sigma}Pu\, dt + (G - (\tilde{\Sigma}C' + L)N^{-1}E)\, dw_t$$

Let $\tilde{\Psi} := \tilde{Z}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}$, and define

$$\Upsilon(t, \varepsilon, \check{x}) := |\varepsilon|^2_{\frac{1}{\theta}\tilde{\Sigma}^{-1}} + |\check{x}|^2_{\tilde{\Psi}} =: \Upsilon_1(t, \varepsilon) + \Upsilon_2(t, \check{x})$$

Note that the matrix $\tilde{\Psi}$ satisfies the following GRDE:

$$\begin{aligned} &\dot{\tilde{\Psi}} + \tilde{\Psi}(A + \theta\tilde{\Sigma}Q) + (A + \theta\tilde{\Sigma}Q)'\tilde{\Psi} + Q + \theta\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L')\tilde{\Psi} \\ &\quad -((I - \theta\tilde{Z}\tilde{\Sigma})^{-1}P + \tilde{\Psi}B)R^{-1}(P'(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} + B'\tilde{\Psi}) = 0 \end{aligned} \tag{53}$$

This, with some additional lengthy but straightforward algebraic manipulations, leads to following expression of the differential for Υ:

$$\begin{aligned} d\Upsilon &= \mathrm{Tr}(E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}E + (G' - E'N^{-1}(C\tilde{\Sigma} + L'))\frac{1}{\theta}\tilde{\Sigma}^{-1}(G \\ &\quad -(\tilde{\Sigma}C' + L)N^{-1}E))\, dt - (x'Qx + 2x'Pu + u'Ru)\, dt + |\tilde{u}|^2_R\, dt + \frac{2}{\theta}(\varepsilon'\tilde{\Sigma}^{-1}G \\ &\quad -e'\tilde{\Sigma}^{-1}(\tilde{\Sigma}C' + L)N^{-1}E dw_t - \frac{1}{\theta}|G'\tilde{\Sigma}^{-1}\varepsilon - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}e|^2\, dt \end{aligned} \tag{54}$$

Adding the identically zero quantity $(2/\theta)\int_0^{t_f} d\Upsilon - (2/\theta)(\Upsilon(t_f, \varepsilon(t_f), \check{x}(t_f)) + \Upsilon(0, \varepsilon(0), \bar{x}_0))$ to the exponent of $J_{I\theta}$, yields after some re-arrangement:

$$\begin{aligned} J_{I\theta} &= |\bar{x}_0|^2_{\tilde{\Psi}} + \int_0^{t_f} \mathrm{Tr}(\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L') + \frac{1}{\theta}\tilde{\Sigma}^{-1}(\tilde{M} + \tilde{\Sigma}C'N^{-1}C\tilde{\Sigma})\, dt \\ &\quad + \frac{2}{\theta}\ln\{\boldsymbol{E}\{\exp\{\frac{1}{2}|\varepsilon(0)|^2_{\Sigma_0^{-1}} - \frac{1}{2}|e(t_f)|^2_{\tilde{\Sigma}(t_f)^{-1} - \theta Q_f} + \frac{\theta}{2}\int_0^{t_f} |\tilde{u}|^2_R\, dt\}\zeta(t_f)\}\} \end{aligned} \tag{55}$$

Introduce the change of probability [21]:

$$\frac{d\tilde{\boldsymbol{P}}}{d\boldsymbol{P}} = \zeta(t_f) \tag{56}$$

[6] For a precise definition, see [5]. This property is also referred to as *strong time consistency* [22].

The measure $\tilde{\boldsymbol{P}}$ is a valid probability measure for all $\mu_I \in \mathcal{M}_I$, since ζ is a martingale on $[0,t_f]$ by our construction of $\mathcal{M}_I$.

Under the new probability measure $\tilde{\boldsymbol{P}}$, the process v_t, defined by:

$$v_t := w_t - \int_0^{t_f} (G'\tilde{\Sigma}^{-1}\varepsilon - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}e)\, dt$$

is a standard Wiener process starting at 0, and it is independent of x_0.

It is straightforward to derive the following expression for the stochastic differential equation satisfied by y, under the new measure $\tilde{\boldsymbol{P}}$:

$$dy \;=\; (C + \theta L'\tilde{Z})\hat{x}\, dt + E\, dv_t$$

Hence, we conclude the following equivalence of sigma-fields:

$$\boldsymbol{Y}_0^t := \sigma\{y_s : \quad 0 \le s \le t\} = \sigma\{Ev_s : \quad 0 \le s \le t\}\,,$$

and that $\boldsymbol{Y}_0^t$ is independent of x_0, for each $t \in [0,t_f]$.

Let the expectation with respect to the probability measure $\tilde{\boldsymbol{P}}$ be denoted by $\tilde{\boldsymbol{E}}$. Then,

$$\bar{J}_{I\theta} \;=\; \frac{2}{\theta}\ln\{\tilde{\boldsymbol{E}}\{\exp\{\frac{\theta}{2}\int_0^{t_f} |\tilde{u}|_R^2\, dt\}\tilde{\boldsymbol{E}}\{\exp\{\frac{1}{2}|\varepsilon(0)|^2_{\Sigma_0^{-1}} - \frac{1}{2}|e(t_f)|^2_{\tilde{\Sigma}(t_f)^{-1}-\theta Q_f}\}|\boldsymbol{Y}_0^{t_f}\}\}\}$$

We will first obtain an expression for the quantity:

$$J_b \;:=\; \tilde{\boldsymbol{E}}\{\exp\{\frac{1}{2}|\varepsilon(0)|^2_{\Sigma_0^{-1}} - \frac{1}{2}|e(t_f)|^2_{\tilde{\Sigma}(t_f)^{-1}-\theta Q_f}\}|\boldsymbol{Y}_0^{t_f}\}$$

Toward that end, we first derive a differential equation for e in terms of v_t:

$$\begin{aligned} de \;&=\; (\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})e\, dt + (B - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(B + \theta\tilde{\Sigma}P))\tilde{u}\, dt + (L - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1} \\ &\quad \cdot(\tilde{\Sigma}C' + L))N^{-1}E\, dv_t + (G - LN^{-1}E)\, dv_t \end{aligned}$$

Note that the processes $\{Ev_t\}_{0\le t\le t_f}$ and $\{(G - LN^{-1}E)v_t\}_{0\le t\le t_f}$ are independent, since $(G - LN^{-1}E)E' = 0$. Then, we can decompose e into $e = \hat{e} + \tilde{e}$, where

$$\begin{aligned} d\hat{e} \;&=\; (\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})\hat{e}\, dt + (B - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(B + \theta\tilde{\Sigma}P))\tilde{u}\, dt + (L - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1} \\ &\quad \cdot(\tilde{\Sigma}C' + L))N^{-1}E\, dv_t \\ d\tilde{e} \;&=\; (\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})\tilde{e}\, dt + (G - LN^{-1}E)\, dv_t; \quad \tilde{e}(0) = \varepsilon(0) \end{aligned}$$

The process $\hat{e}$ belongs to $\boldsymbol{Y}_0^{t_f}$ and the process $\tilde{e}$ is independent of $\boldsymbol{Y}_0^{t_f}$. Therefore, the process $\hat{e}$ is the conditional expectation of the process e given $\boldsymbol{Y}_0^{t_f}$. The conditional distribution of the vector $[\tilde{e}(0)'\ e(t_f)']'$, given the measurement sigma-field $\boldsymbol{Y}_0^{t_f}$, is Gaussian with mean and covariance:

$$\begin{bmatrix} 0 \\ \hat{e}(t_f) \end{bmatrix}; \quad \begin{bmatrix} \Sigma_0 & D(t_f)' \\ D(t_f) & \Phi(t_f) \end{bmatrix} =: \Lambda, \qquad \text{respectively,}$$

where D satisfies:

$$\dot{D} = (\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})D; \quad D(0) = \Sigma_0$$

and Φ is the solution to the following Lyapunov differential equation:

$$\dot{\Phi} = (\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})\Phi + \Phi(\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})' + \tilde{M}; \quad \Phi(0) = \Sigma_0.$$

Hence, J_b can be evaluated as:

$$
\begin{aligned}
J_b &= \int_{\boldsymbol{R}^{2n}} \frac{1}{(2\pi)^n(\det(\Lambda))^{1/2}} \exp\{\frac{1}{2}|\tilde{e}(0)|^2_{\Sigma_0^{-1}} - \frac{1}{2}|\tilde{e}(t_f)+\hat{e}(t_f)|^2_{\tilde{\Sigma}(t_f)^{-1}-\theta Q_f} \\
&\quad -\frac{1}{2}\left|\left[\begin{array}{c}\tilde{e}(0)\\ \tilde{e}(t_f)\end{array}\right]\right|^2_{\Lambda^{-1}}\}\, d\tilde{e}(0)\, d\tilde{e}(t_f) = \int_{\boldsymbol{R}^{2n}} \frac{1}{(2\pi)^n(\det(\Lambda))^{1/2}} \exp\{-\frac{1}{2}\left|\left[\begin{array}{c}\tilde{e}(0)\\ \tilde{e}(t_f)\end{array}\right]\right. \\
&\quad \left. +\tilde{\Lambda}\left[\begin{array}{c}0\\ (\tilde{\Sigma}^{-1}-\theta Q_f)\hat{e}(t_f)\end{array}\right]\right|^2_{\tilde{\Lambda}^{-1}}\}\, d\tilde{e}(0)\, d\tilde{e}(t_f) = \sqrt{\frac{\det(\tilde{\Lambda})}{\det(\Lambda)}}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\Lambda} &= \left.\left[\begin{array}{cc}\Sigma_0^{-1}D'\tilde{\Phi}^{-1}D\Sigma_0^{-1} & -\Sigma_0^{-1}D'\tilde{\Phi}^{-1}\\ -\tilde{\Phi}^{-1}D\Sigma_0^{-1} & \tilde{\Phi}^{-1}+\tilde{\Sigma}^{-1}-\theta Q_f\end{array}\right]^{-1}\right|_{t=t_f} \\
\tilde{\Phi} &= \Phi - D\Sigma_0^{-1}D'
\end{aligned}
$$

Thus, the cost function can be written as follows:

$$
\begin{aligned}
J_{I\theta} &= |\bar{x}_0|^2_{\tilde{\Psi}} + \int_0^{t_f} \mathrm{Tr}(\tilde{\Psi}(\tilde{\Sigma}C'+L)N^{-1}(C\tilde{\Sigma}+L') + \frac{1}{\theta}\tilde{\Sigma}^{-1}(\tilde{M}+\tilde{\Sigma}C'N^{-1}C\tilde{\Sigma})\, dt \\
&\quad +\frac{1}{\theta}\ln(\det(\tilde{\Lambda})) - \frac{1}{\theta}\ln(\det(\Lambda)) + \frac{2}{\theta}\ln\{\boldsymbol{E}\{\exp\{\frac{\theta}{2}\int_0^{t_f} |\tilde{u}|^2_R\, dt\}\}\} \\
&\geq |\bar{x}_0|^2_{\tilde{\Psi}} + \int_0^{t_f} \mathrm{Tr}(\tilde{\Psi}(\tilde{\Sigma}C'+L)N^{-1}(C\tilde{\Sigma}+L') + \frac{1}{\theta}\tilde{\Sigma}^{-1}(\tilde{M}+\tilde{\Sigma}C'N^{-1}C\tilde{\Sigma})\, dt \\
&\quad +\frac{2}{\theta}\ln(J_b) \qquad (57)
\end{aligned}
$$

The controller (51) achieves the lower bound above, and hence is optimal. It is easy to see that the controller (51) is also conditionally optimal.

Some straightforward algebraic manipulations lead to the conclusion that the lower bound in (57) is indeed the same as (52).

This completes the proof. □

Remark 1 *We observe that the optimal controller obtained for the LEQG problem is precisely the central controller for the H^∞-optimal control problem [2]. The preceding Theorem also subsumes the main result of [5] as a special case, and obtains it under less restrictive conditions.* □

Remark 2 *Theorem 2 also holds when $\Sigma_0 \geq 0$, if we restrict the set of admissible controllers to be the set of linear controllers. This generalization can be proved via a standard perturbation analysis, by first replacing Σ_0 by $\Sigma_0+\rho I$, $\rho>0$, and then letting $\rho \downarrow 0$.* □

4.2 The Infinite-Horizon Problem

We now study the infinite-horizon case, where the cost function is taken to be the time-average of (44):

$$J_{I\theta}(\mu_I) = \lim_{t_f\to\infty} \frac{2}{\theta t_f}\ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2}\left(\int_0^{t_f}(x'Qx+2x'Pu+u'Ru)\, dt\right)\right]\right\}\right\} \qquad (58)$$

Introduce two GAREs:

$$\bar{A}'\tilde{Z}+\tilde{Z}\bar{A}-\tilde{Z}\bar{S}\tilde{Z}+\bar{Q}=0 \qquad (59)$$

and

$$\tilde{\Sigma}\tilde{A}' + \tilde{A}\tilde{\Sigma} - \tilde{\Sigma}\tilde{R}\tilde{\Sigma} + \tilde{M} = 0 \tag{60}$$

Define the quantity:

$$\begin{aligned}\theta_I^* \;:=\; & \sup\{\theta \in \boldsymbol{R} : \text{the GAREs (59) and (60) admit minimal positive definite} \\ & \text{solutions } \tilde{Z} \text{ and } \tilde{\Sigma}, \text{ respectively, and that the matrix } I - \theta\tilde{\Sigma}\tilde{Z} \\ & \text{has only positive eigenvalues. } \}\end{aligned} \tag{61}$$

For $\theta < \theta_I^*$, we introduce the filter:

$$d\check{x} \;=\; (A + \theta\tilde{\Sigma}Q)\check{x}\,dt + (B + \theta\tilde{\Sigma}P)u\,dt + (\tilde{\Sigma}C' + L)N^{-1}(dy - C\check{x}\,dt) \tag{62}$$

with the initial state $\check{x}(0) = \bar{x}_0$. Let $\hat{x} := (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\check{x}$; then it is not difficult to show, as in the finite horizon case, that $\hat{x}$ is generated by the following differential equation:

$$\begin{aligned}d\hat{x} \;=\; & (\bar{A} - \bar{S}\tilde{Z})\hat{x}\,dt + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(B + \theta\tilde{\Sigma}P)\tilde{u}\,dt + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1} \\ & \cdot(\tilde{\Sigma}C' + L)N^{-1}(dy - (C + \theta L'\tilde{Z})\hat{x}\,dt)\end{aligned} \tag{63}$$

where $\tilde{u} = u + R^{-1}(B'\tilde{Z} + P')\hat{x}$.

Suppose $\Sigma_0 > 0$, but $\Sigma_0 \leq \tilde{\Sigma}$; then, the solution of GRDE (46) converges to $\tilde{\Sigma}$ exponentially as $t \to \infty$.

We will define the set of admissible controllers, $\mathcal{M}_I$, to be all mappings $\mu_I : [0, \infty) \times \mathcal{H}_y \to \mathcal{H}_u$ that are admissible for every finite-horizon problem with initial time 0 and final time $t_f > 0$, for all $t_f \in \boldsymbol{R}^+$. Then, we have the following counterpart of Theorem 2:

Theorem 3 *Consider the general LEQG problem described by (43) and (58). Let $\Sigma_0 > 0$, $R > 0$, $Q - PR^{-1}P' \geq 0$ and $N > 0$, and assume that the pairs (A, B) and (A, G) are controllable, and the pairs (A, C) and (A, Q) are observable. For each $\theta < \theta_I^*$, if $\Sigma_0 \leq \tilde{\Sigma}$, then the optimal controller is given by*

$$u_{It}^* = \mu_I^*(t, y_{[0,t]}) \equiv -R^{-1}(B'\tilde{Z} + P')(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\check{x} = -R^{-1}(B'\tilde{Z} + P')\hat{x} \tag{64}$$

where $\check{x}$ is generated by the filter (62), or equivalently, $\hat{x}$ is generated by filter (63). The optimal cost can be written as:

$$\begin{aligned}J_{I\theta}^* \;&=\; \inf_{\mu_I \in \mathcal{M}_I} J_{I\theta}(\mu_I) \\ &=\; Tr(\tilde{\Sigma}Q + (\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L')\tilde{Z}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1})\end{aligned} \tag{65}$$

Proof By Theorem 2, we have the following relationships, for any admissible controller:

$$\begin{aligned}&\frac{2}{\theta t_f}\ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2}\left(\int_0^{t_f}(x'Qx + u'u)\,dt\right)\right]\right\}\right\} \geq \frac{1}{t_f}(\bar{x}_0'\tilde{Z}^{t_f}(0)(I - \theta\Sigma_0\tilde{Z}^{t_f}(0))^{-1}\bar{x}_0 \\ &\quad -\frac{1}{\theta}\ln\{\det(I - \theta\tilde{\Sigma}^{t_f}(t_f)Q_f)\} + \int_0^{t_f}\mathrm{Tr}(\tilde{\Sigma}^{t_f}Q + (\tilde{\Sigma}^{t_f}C' + L)N^{-1}(C\tilde{\Sigma}^{t_f} + L')\tilde{Z}^{t_f} \\ &\quad\quad \cdot(I - \theta\tilde{\Sigma}^{t_f}\tilde{Z}^{t_f})^{-1})\,dt)\end{aligned}$$

where $\tilde{Z}^{t_f}$ and $\tilde{\Sigma}^{t_f}$ are the solutions to GRDEs (45) and (46), respectively, on the time interval $[0, t_f]$, in the former case with $Q_f = 0$. Hence,

$$J_{I\theta}(\mu_I) \geq \mathrm{Tr}(\tilde{\Sigma}Q + (\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L')\tilde{Z}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}) = J^*_{I\theta}$$

since, $\tilde{Z}^{t_f}$ converges to $\tilde{Z}$ as $t_f \to \infty$ and $\tilde{\Sigma}^t_f$ converges to $\tilde{\Sigma}$ as $t \to \infty$ exponentially.

To prove the theorem, it is sufficient to show that the controller defined by (64) and (62), or equivalently, the one given by (64) and (63), achieves a performance level that is equal to $J^*_{I\theta}$ given by (65).

Let

$$\begin{aligned} J^{t_f}_{I\theta}(\mu_I) &:= \frac{2}{\theta}\ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2}\left(\int_0^{t_f}(x'Qx + u'u)\,dt\right)\right]\right\}\right\} \\ \bar{J}^{t_f}_{I\theta}(\mu_I) &:= \frac{2}{\theta}\ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2}\left(\int_0^{t_f}(x'Qx + u'u)\,dt + x(t_f)'\tilde{Z}x(t_f)\right)\right]\right\}\right\} \end{aligned}$$

Then, clearly $\bar{J}^{t_f}_{I\theta}(\mu_I) \geq J^{t_f}_{I\theta}(\mu_I)$, and

$$J_{I\theta}(\mu_I) = \lim_{t_f\to\infty}\frac{1}{t_f}J^{t_f}_{I\theta}(\mu_I)$$

Hence, we have

$$J_{I\theta}(\mu_I) \leq \lim_{t_f\to\infty}\frac{1}{t_f}\bar{J}^{t_f}_{I\theta}(\mu_I)$$

By the proof of Theorem 2, and in particular by (55), we have the identity:

$$\begin{aligned} \bar{J}^{t_f}_{I\theta}(\mu_I) &= |\bar{x}_0|^2_{\tilde{\Psi}} + \int_0^{t_f}\mathrm{Tr}(\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L') + \frac{1}{\theta}\tilde{\Sigma}^{-1}(\tilde{M} + \tilde{\Sigma}C' \\ &\quad \cdot N^{-1}C\tilde{\Sigma})\,dt + \frac{2}{\theta}\ln\{\boldsymbol{E}\{\exp\{\frac{1}{2}|\varepsilon(0)|^2_{\tilde{\Sigma}^{-1}} - \frac{1}{2}|e(t_f)|^2_{\tilde{\Sigma}^{-1}-\theta\tilde{Z}} \\ &\quad + \frac{\theta}{2}\int_0^{t_f}|\tilde{u}|^2_R\,dt\}\zeta(t_f)\}\} \end{aligned} \tag{66}$$

where $\varepsilon := x - \check{x}$, $e := x - \hat{x}$, $\tilde{u} := u + R^{-1}(B'\tilde{Z} + P')\hat{x}$, $\tilde{\Psi} := \tilde{Z}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}$ and ζ is defined by:

$$\begin{aligned} \zeta &= \exp\{\int_0^t(\varepsilon'\tilde{\Sigma}^{-1}G - e'\tilde{\Sigma}^{-1}(\tilde{\Sigma}C' + L)N^{-1}E\,dw_t \\ &\qquad - \frac{1}{2}\int_0^t|G'\tilde{\Sigma}^{-1}\varepsilon - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}e|^2\,dt\} \end{aligned}$$

It is clear that, under the controller μ^*_I, as defined by (64) and (62) (as well as (63)), the process ζ is a martingale on $[0, t_f]$. Introduce a change of probability:

$$\frac{d\tilde{\boldsymbol{P}}}{d\boldsymbol{P}} = \zeta(t_f)$$

Then, under the new measure,

$$\begin{aligned} \bar{J}^{t_f}_{I\theta}(\mu^*_I) &= |\bar{x}_0|^2_{\tilde{\Psi}} + \int_0^{t_f}\mathrm{Tr}(\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L') + \frac{1}{\theta}\tilde{\Sigma}^{-1}(\tilde{M} + \tilde{\Sigma}C' \\ &\quad N^{-1}C\tilde{\Sigma})\,dt + \frac{2}{\theta}\ln\{\tilde{\boldsymbol{E}}\{\exp\{\frac{1}{2}|\varepsilon(0)|^2_{\tilde{\Sigma}^{-1}} - \frac{1}{2}|e(t_f)|^2_{\tilde{\Sigma}^{-1}-\theta\tilde{Z}}\}\}\} \end{aligned}$$

This leads to the inequality:

$$J_{I\theta}(\mu^*_I) \leq \lim_{t_f\to\infty}\frac{1}{t_f}\bar{J}^{t_f}_{I\theta}(\mu^*_I) \leq J^*_{I\theta}$$

from which the theorem follows. □

Remark 3 *The Theorem can be generalized to the case when $\Sigma_0 \geq 0$, if we restrict the set of admissible controllers to linear controllers, in view of Remark 2 and the independence of the solution from the initial condition.* □

Before concluding this subsection, we write down the full-order solution (optimal controller) to the original problem formulated in Section 2, as a special case of the one given in Theorem 3 above.

First we introduce some additional notation:

$$C_\epsilon := \begin{bmatrix} C_{11} & C_{12} \\ \epsilon^{1/2}C_{21} & C_{22} \end{bmatrix}; \quad E_\epsilon := \begin{bmatrix} E_1 \\ \epsilon^{1/2}E_2 \end{bmatrix}; \quad N_\epsilon := E_\epsilon E_\epsilon'$$

$$R_\epsilon(\theta) := C_\epsilon' N_\epsilon^{-1} C_\epsilon - \theta Q =: \begin{bmatrix} R_{\epsilon 11} & \frac{1}{\sqrt{\epsilon}}R_{\epsilon 12} \\ \frac{1}{\sqrt{\epsilon}}R_{\epsilon 21} & \frac{1}{\epsilon}R_{\epsilon 22} \end{bmatrix}; \quad R_{12}(\theta) = R_{21}(\theta)' := C_1' N^{-1} C_2;$$

$$R_{22}(\theta) := C_2 N^{-1} C_2; \quad R_{11}(\theta) := C_1' N^{-1} C_1 - \theta Q_{11}; \quad R(\theta) := \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

Let us assume that the pairs (A_ϵ, B_ϵ) and (A_ϵ, G_ϵ) are controllable, and the pairs (A_ϵ, Q) and (A_ϵ, C_ϵ) are observable for every $\epsilon > 0$. Then the controller is given by:

$$u_{I\epsilon}^*(t) = \mu_{I\epsilon}^*(t, y_{[0,t]}) = -B_\epsilon' \tilde{Z}(\epsilon)\hat{x}_t \tag{67}$$

$$d\hat{x}_t = (A_\epsilon - S_\epsilon \tilde{Z}(\epsilon))\hat{x}_t\,dt + (I - \theta\tilde{\Sigma}(\epsilon)\tilde{Z}(\epsilon))^{-1}\tilde{\Sigma}(\epsilon)C_\epsilon' N_\epsilon^{-1}(dy_t - C_\epsilon \hat{x}_t\,dt) \tag{68}$$

where $\hat{x}(0) = (I - \theta\tilde{\Sigma}(\epsilon)\tilde{Z}(\epsilon))^{-1}\bar{x}_0$.[7] Here $\tilde{Z}(\epsilon)$ is the minimal positive-definite solution to the GARE (10), and $\tilde{\Sigma}(\epsilon)$ is the minimal positive-definite solution to

$$A_\epsilon\tilde{\Sigma} + \tilde{\Sigma}A_\epsilon' - \tilde{\Sigma}R_\epsilon\tilde{\Sigma} + G_\epsilon G_\epsilon' = 0\,, \tag{69}$$

with

$$R_\epsilon := C_\epsilon' N_\epsilon^{-1} C_\epsilon - \theta Q\,. \tag{70}$$

This is a valid solution for all $\theta < \theta_I^*(\epsilon)$, provided that $\Sigma_0 \leq \tilde{\Sigma}(\epsilon)$, where

$$\begin{aligned} \theta_I^*(\epsilon) \quad := \quad \sup\{\theta \in \boldsymbol{R} : & \text{ the GAREs (10) and (69) admit minimal positive} \\ & \text{definite solutions, and that the matrix} \\ & I - \theta\tilde{\Sigma}(\epsilon)\tilde{Z}(\epsilon) \text{has only positive eigenvalues}\}\,, \end{aligned} \tag{71}$$

For all $\theta < \theta_I^*(\epsilon)$ the optimal cost is:

$$J_{I\theta}^*(\epsilon) = \mathrm{Tr}(\tilde{\Sigma}(\epsilon)Q + \tilde{\Sigma}(\epsilon)C_\epsilon' N_\epsilon^{-1} C_\epsilon \tilde{\Sigma}(\epsilon)\tilde{Z}(\epsilon)(I - \theta\tilde{\Sigma}(\epsilon)\tilde{Z}(\epsilon))^{-1}) \tag{72}$$

Remark 4 *The condition $\Sigma_0 \leq \tilde{\Sigma}(\epsilon)$ is essential for the existence of an optimal time-invariant control policy. Accordingly, we will assume henceforth that Σ_0 is less than or equal to (in matrix sense) the solution to GARE (69) at $\theta = 0$ for sufficiently small $\epsilon > 0$.*

---

[7] It follows from a result of [2] that $\hat{r}_t$ of (4.2) in [5] is precisely $(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\hat{x}_t$, where $\hat{x}_t$ is generated by (68).

5 Imperfect State Measurements : Model Simplification

We now turn to model simplification for the noisy state measurements case, with (43) replaced by (1)-(2), with $\alpha = 1/2$, and cost taken as given by (3) (instead of (58)). As in the perfect state measurements case discussed in Section 3, we first decompose the system into slow and fast subsystems.

5.1 Time-Scale Decomposition

Slow subsystem

The slow subsystem is obtained by letting $dx_2 = 0$ and solving for $x_2\,dt$ (to be denoted $\bar{x}_2\,dt$) in terms of x_1, u, dw_t and under the working assumption A2:

$$\bar{x}_2\,dt = -A_{22}^{-1}(\sqrt{\epsilon}A_{21}x_1\,dt + B_2u\,dt + \sqrt{\epsilon}G_2\,dw_t) \tag{73}$$

Using this in (1)-(2) and denoting $y_s := y$, we obtain:

$$\begin{aligned} dx_1 &= ((A_{11} + O(\sqrt{\epsilon})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u)\,dt + (G_1 + O(\sqrt{\epsilon})\,dw_t; \\ &\qquad x_1(0) = x_{10} \end{aligned} \tag{74}$$

$$\begin{aligned} dy_{s1} &= ((C_{11} + O(\sqrt{\epsilon}))x_s - C_{12}A_{22}^{-1}B_2u)\,dt + (E_1 + \sqrt{\epsilon}C_{12}A_{22}^{-1}G_2)\,dw_t; \\ &\qquad y_1(0) = 0 \end{aligned} \tag{75}$$

$$\begin{aligned} dy_{s2} &= (\sqrt{\epsilon}(C_{21} - C_{22}A_{22}^{-1}A_{21})x_s - C_{22}A_{22}^{-1}B_2u)\,dt + \sqrt{\epsilon}(E_2 \\ &\quad -C_{22}A_{22}^{-1}G_2)\,dw_t; \quad y_2(0) = 0 \end{aligned} \tag{76}$$

For each $\epsilon > 0$, measurements (75) and (76) are equivalent to the vector measurement $\tilde{y}$, defined by

$$\tilde{y}_s := \begin{bmatrix} y_{s1} \\ \frac{1}{\sqrt{\epsilon}}\,y_{s2} \end{bmatrix} + \begin{bmatrix} C_{12} \\ \frac{1}{\sqrt{\epsilon}}\,C_{22} \end{bmatrix} A_{22}^{-1}B_2u \tag{77}$$

Now set $\epsilon = 0$ in (74) and (77) to arrive at the slow subproblem in terms of $x_s := x_1$, w_t, u and $\tilde{y}_s$, with $\bar{x}_2 = -A_{22}^{-1}B_2u$, reduced (slow) dynamics:

$$dx_s = (A_{11}x_s + B_0u)\,dt + G_1\,dw_t; \quad x_s(0) = x_{10}, \tag{78}$$

slow measurements:

$$d\tilde{y}_s = C_0x_s\,dt + E^{\square}\,dw_t; \quad \tilde{y}_s(0) = 0, \tag{79}$$

and reduced (slow) cost function:

$$\begin{aligned} J_{Is\theta}(\mu_s) &= \lim_{t_f\to\infty} \frac{2}{\theta t_f} \ln\{\boldsymbol{E}\{\exp[\frac{\theta}{2}\int_0^{t_f}(|x_s|^2_{Q_{11}} - 2x_s'Q_{12}A_{22}^{-1}B_2u \\ &\qquad +|u|^2_{I+B_2'A_{22}^{-1}{}'Q_{22}A_{22}^{-1}B_2})\,dt]\}\} \end{aligned} \tag{80}$$

where

$$B_0 := B_1 - A_{12}A_{22}^{-1}B_2; \quad C_0 := C_1 - C_2A_{22}^{-1}A_{21}; \quad E^{\square} := E - C_2A_{22}^{-1}G_2$$
$$C_1 := \begin{bmatrix} C_{11}' & C_{21}' \end{bmatrix}'; \quad C_2 := \begin{bmatrix} 0 & C_{22}' \end{bmatrix}'$$

Note that even though we started with independent system and measurement noises, the slow LEQG involves correlated noises and a coupling term between the state and the control in the exponent of the cost. But, thanks to the general solution presented in the previous section for this class of LEQG problems, we can now solve this slow LEQG problem. By Theorem 3, it admits an optimal solution if

(i) the following GARE admits a (minimal) positive-definite solution, $Z_{s\theta}$:

$$\bar{A}_s' Z_{s\theta} + Z_{s\theta}\bar{A}_s - Z_{s\theta}\bar{S}_s Z_{s\theta} + \bar{Q}_s = 0 \tag{81}$$

where

$$\begin{aligned}
\bar{A}_s &:= A_{11} + B_0(I + B_2' A_{22}^{-1\prime} Q_{22} A_{22}^{-1} B_2)^{-1} B_2' A_{22}^{-1\prime} Q_{21} \\
\bar{S}_s(\theta) &:= B_0(I + B_2' A_{22}^{-1\prime} Q_{22} A_{22}^{-1} B_2)^{-1} B_0' - \theta G_1 G_1' \\
\bar{Q}_s &:= Q_{11} - Q_{12} A_{22}^{-1} B_2 (I + B_2' A_{22}^{-1\prime} Q_{22} A_{22}^{-1} B_2)^{-1} B_2' A_{22}^{-1\prime} Q_{21}
\end{aligned}$$

(ii) the following GARE admits a (minimal) positive-definite solution, $\Sigma_{s\theta}$:

$$\tilde{A}_s \Sigma_{s\theta} + \Sigma_{s\theta}\tilde{A}_s' - \Sigma_{s\theta}\tilde{R}_s\Sigma_{s\theta} + \tilde{M}_s = 0 \tag{82}$$

where

$$\begin{aligned}
\tilde{A}_s &:= A_{11} - G_1 E^{\square\prime} N^{\square -1} C_0; \quad \tilde{R}_s := C_0' N^{\square -1} C_0 - \theta Q_{11} \\
\tilde{M}_s &:= G_1 {G_1}' - G_1 E^{\square\prime} N^{\square -1} E^{\square} {G_1}'; \quad N^{\square} := E^{\square} E^{\square\prime}
\end{aligned}$$

and

(iii) the solutions to (81) and (82) satisfy the spectral radius condition:

$$I - \theta\Sigma_{s\theta} Z_{s\theta} \quad \text{has only positive eigenvalues.} \tag{83}$$

Hence, let us introduce the quantity:

$$\begin{aligned}
\theta_{Is} := \sup\{\theta \in \boldsymbol{R} : &\text{ the GAREs (81) and (82) admit minimal positive definite} \\
&\text{solutions, and further satisfy (83).}\}
\end{aligned} \tag{84}$$

For $\theta < \theta_{Is}$, the slow LEQG problem admits an optimal controller, given by:

$$\tilde{u}_{Is}^* = \tilde{\mu}_{Is}^*(t, \tilde{y}_{s\,[0,t]}) = -(I + B_2' A_{22}^{-1\prime} Q_{22} A_{22}^{-1} B_2)^{-1}(B_0' Z_{s\theta} - B_2' A_{22}^{-1\prime} Q_{21})\hat{x}_s \tag{85}$$

$$\begin{aligned}
d\hat{x}_s = {} & (A_{11} + \theta G_1 {G_1}' Z_{s\theta})\hat{x}_s\, dt + B_0 \tilde{u}_{Is}^*\, dt + (I - \theta\Sigma_{s\theta} Z_{s\theta})^{-1}(\Sigma_{s\theta} C_0' + G_1 E^{\square\prime}) N^{\square -1} \\
& \cdot(d\tilde{y}_s - C_0 \hat{x}_s\, dt - E^{\square}\theta {G_1}' Z_{s\theta}\hat{x}_s\, dt); \quad \hat{x}_s(0) = (I - \theta\Sigma_{s\theta} Z_{s\theta})^{-1}\bar{x}_{10}
\end{aligned} \tag{86}$$

Fast subsystem

To obtain the fast subsystem, we let $x_f := x_2 - \bar{x}_2 = x_2 + A_{22}^{-1} B_2 u_{Is}^*$, $u_f := u - u_s$, $y_f := y - y_s$ and $\tau = \frac{t'-t}{\epsilon}$, where we take t to be frozen, and t' to vary on the same time scale as t. In terms of the equivalent measurements:

$$\tilde{y}_f := \left[\begin{array}{cc} \frac{1}{\sqrt{\epsilon}} y_{f1}' & \frac{1}{\epsilon} y_{f2}' \end{array}\right] \tag{87}$$

we define the fast subsystem and the associated cost, respectively, by:

$$dx_f^t = (A_{22}x_f^t + B_2 u_f^t)\, d\tau + G_2 dw_\tau;\quad x_f^t(0) = x_f \tag{88}$$

$$d\tilde{y}_f^t = C_2 x_f^t\, d\tau + E dw_\tau;\quad y_f^t(0) = 0 \tag{89}$$

$$J_{If\theta}^t(\mu_f^t) = \boldsymbol{E}\left(\int_0^\infty (|x_f^t|_{Q_{22}}^2 + |u_f^t|^2)\, d\tau\right) \tag{90}$$

This is a risk-neutral LQG problem, which does not depend on the parameter θ. It admits an optimal controller:

$$u_{If}^{t*}(\tau) = \mu_{If}^{t*}(\tau, \tilde{y}_{f(-\infty,\tau]}^t) = -B_2' Z_f \hat{x}_f^t(\tau) \tag{91}$$

$$d\hat{x}_f^t = (A_{22} - S_{22}Z_f)\hat{x}_f^t\, d\tau + \Sigma_f C_2' N^{-1}(d\tilde{y}_f^t - C_2\hat{x}_f^t\, d\tau) \tag{92}$$

where Z_f and Σ_f are the unique nonnegative definite solutions to the AREs:[8]

$$A_{22}' Z_f + Z_f A_{22} + Q_{22} - Z_f S_{22} Z_f = 0 \tag{93}$$

and

$$A_{22}\Sigma_f + \Sigma_f A_{22}' + G_2 G_2' - \Sigma_f R_{22}\Sigma_f = 0\,, \tag{94}$$

respectively. Transforming the control policy μ_{If}^{t*} back to the t time scale, we obtain the fast controller:

$$u_{If}^* = \mu_{If}^* = \mu_f^{t*}(x_f^t(0)) = -B_2' Z_f \hat{x}_f \tag{95}$$

$$\epsilon d\hat{x}_f = (A_{22} - S_{22}Z_f)x_f\, dt + \Sigma_f C_2' N^{-1}(\begin{bmatrix} \sqrt{\epsilon} y_{f1} \\ y_{f2} \end{bmatrix} - C_2 \hat{x}_f\, dt);\quad \hat{x}_f(0) = x_f(0) \tag{96}$$

Also, we introduce two Lyapunov equations, for the case when A_{22} is Hurwitz:

$$A_{22}' Z_{of} + Z_{of} A_{22} + Q_{22} = 0 \tag{97}$$

$$A_{22}\Sigma_{of} + \Sigma_{of} A_{22}' + G_2 G_2' = 0 \tag{98}$$

which, as we shall see shortly, are relevant to the problem under consideration.

5.2 Performance of Suboptimal Controllers

We now address the performance evaluation under the slow controller and the composite controller (to be defined), when applied to the full-order system, and the resulting degree of suboptimality with respect to the optimal performance of the full-order system. Toward this end, we first simplify the slow controller (85)–(86) using some straightforward algebraic manipulations of the type that can be found in [14] (derived in the context of the corresponding H^∞-optimal control problem):

$$\tilde{u}_{Is}^* = \tilde{\mu}_{Is}^*(t, \tilde{y}_{s[0,t]}) = -(B_1' Z_{s\theta} + B_2' V)\hat{x}_s \tag{99}$$

$$\begin{aligned} d\hat{x}_s = {} & (\bar{A}_s - \bar{S}_s Z_{s\theta})\hat{x}_s\, dt + (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}(\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}(d\tilde{y}_s \\ & -(C_1 - C_2 A_{22}^{-1}(A_{21} + \theta G_2 G_1' Z_{s\theta}))\hat{x}_s\, dt);\quad \hat{x}_s(0) = (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} \end{aligned} \tag{100}$$

[8] As well known, these AREs admit positive definite solutions if (A_{22}, B_2) is controllable and (A_{22}, C_2) is observable.

where

$$
\begin{aligned}
Y &:= Y_1\Sigma_{s\theta}+Y_2; \quad V := V_1 Z_{s\theta}+V_2 \\
Y_1 &:= -(R_{22}+A_{22}'(G_2G_2)^{-1}A_{22})^{-1}(R_{21}+A_{22}'(G_2G_2)^{-1}A_{21}) && (101)\\
Y_2 &:= -(R_{22}+A_{22}'(G_2G_2)^{-1}A_{22})^{-1}A_{22}'(G_2G_2')^{-1}G_2G_1 && (102)\\
V_1 &:= -(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(S_{21}+A_{22}Q_{22}^{-1}A_{12}') && (103)\\
V_2 &:= -(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}A_{22}Q_{22}^{-1}Q_{21} && (104)
\end{aligned}
$$

Using (77) in (100), we obtain the following alternative form for the slow controller:

$$
\begin{aligned}
u_{Is}^* &= \mu_{Is}^*(t,y_{[0,t]}) = -(B_1'Z_{s\theta}+B_2'V)\hat{x}_s^s && (105)\\
d\hat{x}_s^s &= (\bar{A}_s-\bar{S}_sZ_{s\theta})\hat{x}_s^s\,dt+(I-\theta\Sigma_{s\theta}Z_{s\theta})^{-1}(\Sigma_{s\theta}C_1'+Y'C_2')N^{-1}(\begin{bmatrix} dy_1' & \frac{1}{\sqrt{\epsilon}}dy_2' \end{bmatrix}'\\
&\quad -(C_{2\epsilon}A_{22}^{-1}B_2(B_1'Z_{s\theta}+B_2'V)+C_1-C_2A_{22}^{-1}(A_{21}+\theta G_2G_1'Z_{s\theta}))\hat{x}_s^s\,dt);\\
&\quad \hat{x}_s^s(0)=(I-\theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} && (106)
\end{aligned}
$$

Before deriving an expression for the composite controller, let us introduce the notation:

$$
\begin{aligned}
X &:= X_1\Sigma_{s\theta}+X_2; \quad U := U_1Z_{s\theta}+U_2\\
X_1 &:= (G_2G_2)^{-1}A_{21}-(G_2G_2)^{-1}A_{22}(R_{22}+A_{22}'(G_2G_2)^{-1}A_{22})^{-1}\\
&\quad (R_{21}+A_{22}'(G_2G_2)^{-1}A_{21}) && (107)\\
X_2 &:= (G_2G_2)^{-1}G_2G_1'-(G_2G_2)^{-1}A_{22}(R_{22}+A_{22}'(G_2G_2)^{-1}A_{22})^{-1}\\
&\quad A_{22}'(G_2G_2')^{-1}G_2G_1 && (108)\\
U_1 &:= Q_{22}^{-1}A_{12}'-Q_{22}^{-1}A_{22}'(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(S_{21}+A_{22}Q_{22}^{-1}A_{12}') && (109)\\
U_2 &:= Q_{22}^{-1}Q_{21}-Q_{22}^{-1}A_{22}'(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}A_{22}Q_{22}^{-1}Q_{21} && (110)
\end{aligned}
$$

Then, combining (95) and (105), we arrive at the composite controller, expressed as:

$$
\begin{aligned}
u_{Ic}^* &= \mu_{Ic}^*(t,y_{[0,t]}) := \mu_{Is}^*(t,y_{[0,t]})+\mu_{If}^*(t,y_{[0,t]})\\
&= -(B_1'Z_{s\theta}+B_2'V)\hat{x}_s^c-B_2'Z_f\hat{x}_f^c && (111)
\end{aligned}
$$

where a differential equation representation for $\hat{x}_s^c$ can be obtained by using (77) in (100):

$$
\begin{aligned}
d\hat{x}_s^c &= (\bar{A}_s-\bar{S}_sZ_{s\theta})\hat{x}_s^c\,dt+(I-\theta\Sigma_{s\theta}Z_{s\theta})^{-1}(\Sigma_{s\theta}C_1'+Y'C_2')N^{-1}\\
&\quad (\begin{bmatrix} dy_1' & \frac{1}{\sqrt{\epsilon}}dy_2' \end{bmatrix}'-(C_{2\epsilon}A_{22}^{-1}B_2(B_1'Z_{s\theta}+B_2'V)+C_1-C_2A_{22}^{-1}(A_{21}+\theta G_2G_1'Z_{s\theta}))\hat{x}_s^c\,dt\\
&\quad -C_{2\epsilon}A_{22}^{-1}B_2B_2'Z_f\hat{x}_f^c); \quad \hat{x}_s^c(0)=(I-\theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} && (112)
\end{aligned}
$$

To obtain the differential equation governing $\hat{x}_f^c$, we let

$$
\begin{bmatrix} \sqrt{\epsilon}dy_{f1} \\ dy_{f2} \end{bmatrix} = \begin{bmatrix} \sqrt{\epsilon}dy_1 \\ dy_2 \end{bmatrix} - \sqrt{\epsilon}(C_{2\epsilon}A_{22}^{-1}B_2(B_1'Z_{s\theta}+B_2'V)+C_1 -C_2A_{22}^{-1}(A_{21}+\theta G_2G_1'Z_{s\theta}))\hat{x}_s^s\,dt)
$$

and

$$
\bar{x}_2 = -A_{22}^{-1}B_2u_{Is}^* = -U\hat{x}_s^c
$$

and substitute the above into (96), to arrive at:

$$\epsilon d\hat{x}_f^c = (A_{22} - S_{22}Z_f)\hat{x}_f^c\,dt + \Sigma_f C_2' N^{-1}(\left[\ \sqrt{\epsilon}y_1' \quad y_2'\ \right] - \sqrt{\epsilon}(C_{2\epsilon}A_{22}^{-1}B_2(B_1'Z_{s\theta} + B_2'V) \\ + C_1 - C_2A_{22}^{-1}(A_{21} + \theta G_2G_1'Z_{s\theta}))\hat{x}_s^c\,dt - C_2\hat{x}_f^c\,dt); \quad \hat{x}_f^c(0) = \bar{x}_{20} + U\hat{x}_s^c(0) \tag{113}$$

The main results of this section are now given in Theorem 4 below, which provides expressions for the performances of the full-order system under full-order, slow and composite controllers, and establishes their asymptotic optimality (as $\epsilon \to 0$). We note that the condition of Remark 4 is guaranteed in this case by the conditions that $\Sigma_{011} < \Sigma_{s\theta}$ at $\theta = 0$, and $\Sigma_{022} < \Sigma_f$.

Theorem 4 *For the singularly perturbed system (1)-(2) with $\alpha = 1/2$, and under cost function (3):*

1. *For each $\epsilon > 0$, if (A_ϵ, B_ϵ) and (A_ϵ, G_ϵ) are controllable, (A_ϵ, C_ϵ) and (A_ϵ, Q) are observable, and N_ϵ is invertible, then $\forall\theta < \theta_{I\infty}^*(\epsilon)$, the optimal cost for the full-order LEQG problem can be written as:*

$$J_{I\theta}^*(\epsilon) = Tr(\tilde{\Sigma}Q + (\tilde{\Pi}^{-1} + \theta(\tilde{Z} - \theta\tilde{Z}\tilde{\Sigma}\tilde{Z}))^{-1}((I - \theta\tilde{Z}\tilde{\Sigma})Q(I - \theta\tilde{\Sigma}\tilde{Z}) + \tilde{Z}B_\epsilon B_\epsilon'\tilde{Z})) \tag{114}$$

   where the matrix $\tilde{\Pi}$ is the unique positive definite solution to the following Lyapunov equation:

$$(A_\epsilon - S_\epsilon\tilde{Z})\tilde{\Pi} + \tilde{\Pi}(A_\epsilon' - \tilde{Z}S_\epsilon) + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon(I - \theta\tilde{Z}\tilde{\Sigma})^{-1} = 0 \tag{115}$$

2. *Let assumptions A1 and A2 be satisfied, the pairs (A_{11}, B_0), $(A_{11}, G_1G_1' - G_1G_2'(G_2G_2')^{-1}G_2G_1')$ and (A_{22}, B_2) be controllable, and the pairs (A_{11}, C_0), $(A_{11}, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ and (A_{22}, C_2) be observable. Let $\theta_I^*(\epsilon)$ be as defined by (71). Then,*

   i. $\lim_{\epsilon\to 0^+}\theta_I^*(\epsilon) = \theta_{Is}$.

   ii. *$\forall\theta < \theta_{Is}$, there exists $\epsilon_\theta > 0$, such that $\forall 0 < \epsilon \le \epsilon_\theta$, the GAREs (10) and (69) admit minimal positive definite solutions, which can be approximated by*

$$\tilde{Z}(\epsilon) = \begin{bmatrix} Z_{s\theta} + O(\sqrt{\epsilon}) & \epsilon(Z_fU + V)' + O(\epsilon^{3/2}) \\ \epsilon(Z_fU + V) + O(\epsilon^{3/2}) & \epsilon Z_f + O(\epsilon^{3/2}) \end{bmatrix} \tag{116}$$

   and

$$\tilde{\Sigma}(\epsilon) = \begin{bmatrix} \Sigma_{s\theta} + O(\sqrt{\epsilon}) & \sqrt{\epsilon}(\Sigma_fX + Y)' + O(\epsilon) \\ \sqrt{\epsilon}(\Sigma_fX + Y) + O(\epsilon) & \Sigma_f + O(\sqrt{\epsilon}) \end{bmatrix} \tag{117}$$

   Furthermore, $I - \theta\tilde{\Sigma}(\epsilon)\tilde{Z}(\epsilon)$ has only positive eigenvalues.

   iii. *$\forall\theta < \theta_{Is}$, there exists $\tilde{\epsilon}_\theta \in (0, \epsilon_\theta]$, such that $\forall 0 < \epsilon \le \tilde{\epsilon}_\theta$, the Lyapunov equation (115) admits a positive definite solution, which can be approximated by*

$$\tilde{\Pi}(\epsilon) = \begin{bmatrix} I & 0 \\ -U & I \end{bmatrix}\begin{bmatrix} \Pi_{s\theta} + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Pi_f + O(\sqrt{\epsilon}) \end{bmatrix}\begin{bmatrix} I & -U' \\ 0 & I \end{bmatrix} \tag{118}$$

   where $\Pi_{s\theta}$ is the unique positive definite solution to the following Lyapunov equation:

$$(\bar{A}_s - \bar{S}_sZ_{s\theta})\Pi_{s\theta} + \Pi_{s\theta}(\bar{A}_s' - Z_{s\theta}\bar{S}_s) + (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}((\Sigma_{s\theta}C_1' + Y'C_2') \\ \cdot N^{-1}(C_1\Sigma_{s\theta} + C_2Y) + X'G_2G_2'X)(I - \theta Z_{s\theta}\Sigma_{s\theta})^{-1} = 0 \tag{119}$$

   and Π_f is the unique positive definite solution to the following Lyapunov equation:

$$(A_{22} - B_2B_2'Z_f)\Pi_f + \Pi_f(A_{22}' - Z_fB_2B_2') + \Sigma_fC_2'N^{-1}C_2\Sigma_f = 0. \tag{120}$$

iv. $\forall \theta < \theta_{Is}$, $\forall 0 < \epsilon \leq \tilde{\epsilon}_\theta$, *the optimal cost for the full-order LEQG problem can be approximated by:*

$$\begin{aligned} J^*_{I\theta}(\epsilon) = \; & Tr(\Sigma_{s\theta} Q_{11} + (\Pi_{s\theta}^{-1} + \theta(Z_{s\theta} - \theta Z_{s\theta}\Sigma_{s\theta} Z_{s\theta}))^{-1}((Z_{s\theta} B_1 + V' B_2) \\ & (B_1' Z_{s\theta} + B_2' V) + (I - \theta Z_{s\theta}\Sigma_{s\theta}) Q_{11} (I - \theta \Sigma_{s\theta} Z_{s\theta}) - (I - \theta Z_{s\theta}\Sigma_{s\theta}) Q_{12} U \\ & - U' Q_{21}(I - \theta\Sigma_{s\theta} Z_{s\theta}) + U' Q_{22} U) + \Sigma_f Q_{22} \\ & + \Pi_f (Q_{22} + Z_f B_2 B_2' Z_f)) + O(\sqrt{\epsilon}) \end{aligned} \tag{121}$$

v. $\forall \theta < \theta_{Is}$, *if the composite controller* $\mu^*_{Ic\theta}$ *is applied to the system, then* $\exists \epsilon'_\theta > 0$ *such that* $\forall \epsilon \in [0, \epsilon'_\theta)$,

$$J^c_{I\theta} := J_{I\theta}(\mu^*_{Ic\theta}) = J^*_{I\theta}(\epsilon) + O(\sqrt{\epsilon}) \tag{122}$$

vi. $\forall \theta < \theta_{Is}$, *if, in addition, the matrix* A_{22} *is Hurwitz, and the slow controller* $\mu^*_{Is\theta}$ *is applied to the system, then* $\exists \hat{\epsilon}_\theta > 0$ *such that* $\forall \epsilon \in [0, \hat{\epsilon}_\theta)$,

$$\begin{aligned} J^s_{I\theta} := J_{I\theta}(\mu^*_{Is\theta}) = J^*_{I\theta}(\epsilon) + \; & Tr(\Sigma_{of} Q_{22} - \Sigma_f Q_{22} \\ & - \Pi_f(Q_{22} + Z_f B_2 B_2' Z_f)) + O(\sqrt{\epsilon}) \end{aligned} \tag{123}$$

Proof We first substitute the optimal controller (67)-(68) into the full-order system, for any $\theta < \theta^*_{Is}(\epsilon)$, to obtain the following control-free LEQG problem in terms of $x^e := [x' \; \hat{x}']'$ and w:

$$\begin{aligned} dx^e = \; & \begin{bmatrix} A_\epsilon & -B_\epsilon B_\epsilon' \tilde{Z} \\ (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma} C_\epsilon' N_\epsilon^{-1} C_\epsilon & A_\epsilon - S_\epsilon \tilde{Z} - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma} C_\epsilon' N_\epsilon^{-1} C_\epsilon \end{bmatrix} \\ & \cdot x^e \, dt + \begin{bmatrix} G_\epsilon \\ (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma} C_\epsilon' N_\epsilon^{-1} E_\epsilon \end{bmatrix} dw_t \\ := \; & F^e_\epsilon x^e \, dt + G^e_\epsilon \, dw_t \end{aligned} \tag{124}$$

$$J^*_{I\theta}(\epsilon) = \lim_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp\left[\frac{\theta}{2} \int_0^{t_f} x^{e\prime} H^e x^e \, dt \right] \right\} \right\} \tag{125}$$

where

$$H^e := \begin{bmatrix} Q & 0 \\ 0 & \tilde{Z} B_\epsilon B_\epsilon' \tilde{Z} \end{bmatrix} \tag{126}$$

and $x^e(0)$ is a Gaussian random variable with mean and variance:

$$\boldsymbol{E} x^e(0) = \begin{bmatrix} \bar{x}_0 \\ (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\bar{x}_0 \end{bmatrix} =: \bar{x}^e_0; \quad \mathrm{Var}(x^e(0)) = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix} =: \Theta^e_0$$

To compute $J^*_{I\theta}(\epsilon)$ explicitly, we associate with the above system a fictitious measurement:

$$dy^e = dv_t; \qquad y^e(0) = 0 \tag{127}$$

where v_t, or y^e, is a standard Wiener process independent of the initial condition and w_t.

Then, the two GAREs associated with this problem are (from subsection 4.2):

$$F^{e\prime}_\epsilon \tilde{\Xi} + \tilde{\Xi} F^e_\epsilon + \tilde{\Xi} \theta G^e_\epsilon G^{e\prime}_\epsilon \tilde{\Xi} + H^e = 0 \tag{128}$$

and

$$F_\epsilon^e\tilde{\Theta} + \tilde{\Theta}F_\epsilon^{e\prime} + \tilde{\Theta}\theta H^e\tilde{\Theta} + G_\epsilon^e G_\epsilon^{e\prime} = 0 \tag{129}$$

It is shown in [17] that the minimal solutions to these GAREs are

$$\tilde{\Xi} = \begin{bmatrix} \tilde{Z}^{-1} & \tilde{Z}^{-1} \\ \tilde{Z}^{-1} & (\tilde{Z} - \theta\tilde{Z}\tilde{\Sigma}\tilde{Z})^{-1} + \tilde{\Delta}^{-1} \end{bmatrix}^{-1} \tag{130}$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Sigma}^{-1} & -\tilde{\Sigma}^{-1} + \theta\tilde{Z} \\ -\tilde{\Sigma}^{-1} + \theta\tilde{Z} & \tilde{\Sigma}^{-1} - \theta\tilde{Z} + \tilde{\Pi}^{-1} \end{bmatrix}^{-1} \tag{131}$$

where $\tilde{\Delta}$ is the unique positive definite solution to the Lyapunov equation:

$$(I - \theta\tilde{Z}\tilde{\Sigma})(A_\epsilon' - R_\epsilon\tilde{\Sigma})(I - \theta\tilde{Z}\tilde{\Sigma})^{-1}\tilde{\Delta} + \tilde{\Delta}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(A_\epsilon - \tilde{\Sigma}R_\epsilon)(I - \theta\tilde{\Sigma}\tilde{Z}) + \tilde{Z}B_\epsilon B_\epsilon'\tilde{Z} = 0 \tag{132}$$

Furthermore, the matrices $F_\epsilon^e + \theta G_\epsilon^e G_\epsilon^{e\prime}\tilde{\Xi}$ and $F_\epsilon^{e\prime} + \theta H^e\tilde{\Theta}$ are Hurwitz.

We note that $\frac{1}{\theta}\tilde{\Theta}^{-1}$ is the maximal solution to the GARE (128), since $F_\epsilon^e + \theta G_\epsilon^e G_\epsilon^{e\prime}\tilde{\Theta}$ is an antistable matrix. Thus, by Theorem 5 of [24], we have $\frac{1}{\theta}\tilde{\Theta}^{-1} > \tilde{\Xi}$. It is easily seen that $\tilde{\Theta} > \Theta_0^e$. Hence, we obtain

$$J_{I\theta}^* = \mathrm{Tr}(\tilde{\Theta}H^e)$$

in view of Theorem 3. Using a matrix inversion identity, we obtain (114), which is an alternative expression to (72). Hence, part 1) is proven.

The GARE (10) also arises in the singularly perturbed H^∞-optimal control problem, whose approximate solution has been studied extensively in [13]. Here, due to the factor $\sqrt{\epsilon}$ multiplying A_{21} and G_2, these two matrices do not enter the slow GARE (81), nor do they enter the zero*th* order approximation of the solution. Thus, GARE (10) admits a minimal positive definite solution, which is approximated by (116), for sufficiently small ϵ if $\theta < \theta_{Is}$.

To study the behavior of the solution of GARE (69), we first partition $\tilde{\Sigma}$ as follows (in a way consistent with the given partitioning on Σ_0):

$$\tilde{\Sigma} := \begin{bmatrix} \tilde{\Sigma}_{11} & \sqrt{\epsilon}\,\tilde{\Sigma}_{12} \\ \sqrt{\epsilon}\,\tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{bmatrix} \tag{133}$$

where $\tilde{\Sigma}_{12} = \tilde{\Sigma}_{21}'$. Then, by substituting this structure back into GARE (69), we obtain the following coupled matrix Riccati equations for the matrices $\tilde{\Sigma}_{11}$, $\tilde{\Sigma}_{12}$ and $\tilde{\Sigma}_{22}$:

$$\begin{aligned} &A_{11}\tilde{\Sigma}_{11} + \sqrt{\epsilon}A_{12}\tilde{\Sigma}_{12}' + \tilde{\Sigma}_{11}A_{11}' + \sqrt{\epsilon}\tilde{\Sigma}_{12}A_{12}' + G_1G_1' - \tilde{\Sigma}_{11}R_{\epsilon 11}\tilde{\Sigma}_{11} \\ &\quad -\tilde{\Sigma}_{12}R_{\epsilon 21}\tilde{\Sigma}_{11} - \tilde{\Sigma}_{11}R_{\epsilon 12}\tilde{\Sigma}_{12}' - \tilde{\Sigma}_{12}R_{\epsilon 22}\tilde{\Sigma}_{12}' = 0 \end{aligned} \tag{134}$$

$$\begin{aligned} &\epsilon A_{11}\tilde{\Sigma}_{12} + \sqrt{\epsilon}A_{12}\tilde{\Sigma}_{22} + \tilde{\Sigma}_{11}A_{21}' + \tilde{\Sigma}_{12}A_{22}' + G_1G_2' - \epsilon\tilde{\Sigma}_{11}R_{\epsilon 11}\tilde{\Sigma}_{12} \\ &\quad -\epsilon\tilde{\Sigma}_{12}R_{\epsilon 21}\tilde{\Sigma}_{12} - \tilde{\Sigma}_{11}R_{\epsilon 12}\tilde{\Sigma}_{22} - \tilde{\Sigma}_{12}R_{\epsilon 22}\tilde{\Sigma}_{22} = 0 \end{aligned} \tag{135}$$

$$\begin{aligned} &\epsilon A_{21}\tilde{\Sigma}_{12} + A_{22}\tilde{\Sigma}_{22} + \epsilon\tilde{\Sigma}_{12}'A_{21}' + \tilde{\Sigma}_{22}A_{22}' + G_2G_2' - \epsilon^2\tilde{\Sigma}_{12}'R_{\epsilon 11}\tilde{\Sigma}_{12} \\ &\quad -\epsilon\tilde{\Sigma}_{22}R_{\epsilon 21}\tilde{\Sigma}_{12} - \epsilon\tilde{\Sigma}_{12}'R_{\epsilon 12}\tilde{\Sigma}_{22} - \tilde{\Sigma}_{22}R_{\epsilon 22}\tilde{\Sigma}_{22} = 0 \end{aligned} \tag{136}$$

The above set of equations are the same as (2.26)–(2.28) of [14] for $\epsilon \to 0$ (except for certain obvious modifications), which permits us to apply the results of [14] to the present case. Hence, for $\theta < \theta_{Is}$, (134)–(136) admit solutions for sufficiently small ϵ, which can be approximated by

$\tilde{\Sigma}_{11} = \Sigma_{s\theta} + O(\sqrt{\epsilon})$, $\tilde{\Sigma}_{12} = X'\Sigma_f + Y' + O(\sqrt{\epsilon})$ and $\tilde{\Sigma}_{22} = \Sigma_f + O(\sqrt{\epsilon})$. Thus, the solution to (69) can be approximated by (117), for sufficiently small ϵ and for $\theta < \theta_{Is}$.

Furthermore, for $\theta < \theta_{Is}$ and sufficiently small ϵ, the matrix $I - \theta\tilde{\Sigma}(\epsilon)\tilde{Z}(\epsilon)$ can be approximated by

$$\begin{bmatrix} I - \theta\Sigma_{s\theta}Z_{s\theta} + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\sqrt{\epsilon}) & I + O(\sqrt{\epsilon}) \end{bmatrix}$$

Hence, it can have only positive eigenvalues. Thus, part 2.*ii*) is proved.

Fix any $\theta > \theta_{Is}$; then, either one of the GAREs (81) and (82) does not admit any positive definite solution, or the matrix $I - \theta\Sigma_{s\theta}Z_{s\theta}$ has at least one negative eigenvalue. The former implies, in view of a result of [13], that one of the GAREs (10) or (69) does not admit any positive definite solution for sufficiently small ϵ, which further implies that $\theta > \theta_I^*(\epsilon)$. The latter, on the other hand, leads to the conclusion that the matrix $I - \theta\tilde{\Sigma}\tilde{Z}$ has at least one negative eigenvalue for sufficiently small ϵ, which again implies that $\theta > \theta_I^*(\epsilon)$. Hence, $\theta > \theta_I^*(\epsilon)$ for sufficiently small ϵ, $\forall\theta > \theta_{Is}$. Thus, part 2.*i*) is also proved.

Let $T = \begin{bmatrix} I & 0 \\ U & I \end{bmatrix}$, and $\Pi = T\tilde{\Pi}T'$. Then, premultiplying (119) by T and postmultiplying it by T' yields the following Lyapunov equation for Π:

$$\begin{aligned} T(A_\epsilon - S_\epsilon\tilde{Z})T^{-1}\Pi + \Pi T^{-1\prime}(A_\epsilon' - \tilde{Z}S_\epsilon)T' + T(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma}C_\epsilon' N_\epsilon^{-1}C_\epsilon \\ (I - \theta\tilde{Z}\tilde{\Sigma})^{-1}T' = 0 \end{aligned} \tag{137}$$

Note the following approximations for $\epsilon \in (0, \epsilon_\theta]$, which are easily obtained in view of the given approximations for $\tilde{Z}$ and $\tilde{\Sigma}$:

$$T(A_\epsilon - S_\epsilon\tilde{Z})T^{-1} = \begin{bmatrix} \bar{A}_s - \bar{S}_sZ_{s\theta} & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon}(A_{22} - B_2B_2'Z_f) + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \tag{138}$$

$$\begin{aligned} &T(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma}C_\epsilon' N_\epsilon^{-1}C_\epsilon(I - \theta\tilde{Z}\tilde{\Sigma})^{-1}T' \\ &\qquad = \begin{bmatrix} L_s + O(\sqrt{\epsilon}) & O(\frac{1}{\sqrt{\epsilon}}) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon}\Sigma_fC_2'N^{-1}C_2\Sigma_f + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \end{aligned} \tag{139}$$

$$\begin{aligned} L_s = (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}((\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}(C_1\Sigma_{s\theta} + C_2Y) + X'G_2G_2'X) \\ (I - \theta Z_{s\theta}\Sigma_{s\theta})^{-1} \end{aligned} \tag{140}$$

Suppose that Π takes the form $\begin{bmatrix} \Pi_{11} & \sqrt{\epsilon}\Pi_{12} \\ \sqrt{\epsilon}\Pi_{21} & \Pi_{22} \end{bmatrix}$, where $\Pi_{21} = \Pi_{12}'$, and substitute it into the Lyapunov equation (137) to arrive at the following equations for Π_{11}, Π_{12} and Π_{22}:

$$(\bar{A}_s - \bar{S}_sZ_{s\theta})\Pi_{11} + \Pi_{11}(\bar{A}_s - \bar{S}_sZ_{s\theta})' + L_s + O(\sqrt{\epsilon}) = 0 \tag{141}$$

$$\Pi_{11}O(1) + \Pi_{12}(A_{22} - B_2B_2'Z_f) + O(1) + O(\sqrt{\epsilon}) = 0 \tag{142}$$

$$(A_{22} - B_2B_2'Z_f)\Pi_{22} + \Pi_{22}(A_{22} - B_2B_2'Z_f)' + \Sigma_fC_2'N^{-1}C_2\Sigma_f + O(\sqrt{\epsilon}) = 0 \tag{143}$$

Then, it follows that $\Pi_{11} = \Pi_{s\theta}$, $\Pi_{22} = \Pi_f$ and some Π_{12} (which exists) solve equations (141)–(143) at $\epsilon = 0$. By an application of the implicit function theorem [25] as in the proof of Theorem 1 of [13], the solutions to (141)–(143) are approximated by $\Pi_{11} = \Pi_{s\theta} + O(\sqrt{\epsilon})$, $\Pi_{22} = \Pi_f + O(\sqrt{\epsilon})$ and $\Pi_{12} = O(1)$, for sufficiently small ϵ. This then completes the proof of part 2.*iii*).

A mere substitution of (116), (117) and (118) into (114) yields the desired result (121) (detailed algebraic manipulations can be found in [17]), which proves part 2.*iv*).

Now substitute the composite controller μ_{Ic}^* into the full-order system to arrive at an infinite-horizon control-free LEQG problem. Let

$$x_f^c := x_2 + U\hat{x}_s^c \tag{144}$$
$$\tilde{x}^{c\prime} := [x_1', \, \hat{x}_s^{c\prime} + \sqrt{\epsilon} x_f^{c\prime} X (I - \theta Z_{s\theta}\Sigma_{s\theta})^{-1}, \, x_f^{c\prime}, \, \hat{x}_f^{c\prime}] \tag{145}$$

In terms of the state variable $\tilde{x}^c$, this LEQG problem can be written as (see [17] for details):

$$\begin{aligned} d\tilde{x}^c &= \begin{bmatrix} F_{11}^c + O(\sqrt{\epsilon}) & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon}F_{22}^c + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \tilde{x}^c \, dt + \begin{bmatrix} G_1^c + O(\sqrt{\epsilon}) \\ \frac{1}{\sqrt{\epsilon}} G_2^c + O(1) \end{bmatrix} dw_t \\ &:= F_\epsilon^c \tilde{x}^c \, dt + G_\epsilon^c \, dw_t \end{aligned} \tag{146}$$

$$J_{I\theta}^c := \lim_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \int_0^{t_f} \tilde{x}^{c\prime} (H^c + O(\sqrt{\epsilon})) \tilde{x}^c \, dt \right] \right\} \right\} \tag{147}$$

where

$$F_{11}^c = \begin{bmatrix} A_{11} & -B_1B_1'Z_{s\theta} - B_1B_2'V - A_{12}U \\ (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}((\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}C_1 + X'A_{21}) & F_{cs} \end{bmatrix} \tag{148}$$

$$\begin{aligned} F_{cs} = {} & \bar{A}_s - \bar{S}_s Z_{s\theta} - (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}((\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}C_1 + X'A_{21} \\ & +\theta X'G_2G_1'Z_{s\theta}) \end{aligned} \tag{149}$$

$$F_{22}^c = \begin{bmatrix} A_{22} & -B_2B_2'Z_f \\ \Sigma_f C_2'N^{-1}C_2 & A_{22} - B_2B_2'Z_f - \Sigma_f C_2'N^{-1}C_2 \end{bmatrix} \tag{150}$$

$$G_1^c = \begin{bmatrix} G_1 \\ (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}(X'G_2 + (\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}E) \end{bmatrix} \tag{151}$$

$$G_2^c = \begin{bmatrix} G_2 \\ \Sigma_f C_2' N^{-1} E \end{bmatrix} \tag{152}$$

$$H^c = \begin{bmatrix} H_{11}^c & O(1) \\ O(1) & H_{22}^c \end{bmatrix} \tag{153}$$

$$H_{11}^c = \begin{bmatrix} Q_{11} & -Q_{12}U \\ -U'Q_{21} & (Z_{s\theta}B_1 + V'B_2)(B_1Z_{s\theta} + B_2'V) + U'Q_{22}U \end{bmatrix} \tag{154}$$

$$H_{22}^c = \begin{bmatrix} Q_{22} & 0 \\ 0 & Z_fB_2B_2'Z_f \end{bmatrix} \tag{155}$$

The initial state $\tilde{x}^c(0)$ is a Gaussian random vector with mean $\bar{x}_0^c$ and covariance Σ_0^c, given by:

$$\bar{x}_0^c := \begin{bmatrix} \bar{x}_{10} \\ (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} + O(\sqrt{\epsilon}) \\ \bar{x}_{20} + U(I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} \\ \bar{x}_{20} + U(I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} \end{bmatrix}$$

$$\Sigma_0^c := \begin{bmatrix} \begin{bmatrix} \Sigma_{011} & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \begin{bmatrix} \Sigma_{022} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

To evaluate the cost $J^c_{I\theta\infty}$, we associate a fictitious measurement (127) with this LEQG problem (as in the proof of part 1), where v_t, or y^e, is a standard Wiener process independent of the initial state and w_t.

Then, the two GAREs associated with this problem are:

$$F_\epsilon^{c\prime}\tilde{\Xi}^c + \tilde{\Xi}^c F_\epsilon^c + \tilde{\Xi}^c \theta G_\epsilon^c G_\epsilon^{c\prime} \tilde{\Xi}^c + H^c = 0 \tag{156}$$

and

$$F_\epsilon^c \tilde{\Theta}^c + \tilde{\Theta}^c F_\epsilon^{c\prime} + \tilde{\Theta}^c \theta H^c \tilde{\Theta}^c + G_\epsilon^c G_\epsilon^{c\prime} = 0 \tag{157}$$

It is shown in [17] that the minimal solutions to these GAREs exist for $\theta < \theta_{Is}$ and sufficiently small ϵ, and can be approximated by

$$\tilde{\Xi}^c = \begin{bmatrix} \tilde{\Xi}^c_{11} + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\epsilon) & \epsilon\tilde{\Xi}^c_{22} + O(\epsilon^{3/2}) \end{bmatrix} \tag{158}$$

$$\tilde{\Xi}^c_{11} = \begin{bmatrix} Z_{s\theta}^{-1} & Z_{s\theta}^{-1} \\ Z_{s\theta}^{-1} & (Z_{s\theta} - \theta Z_{s\theta}\Sigma_{s\theta}Z_{s\theta})^{-1} + \Delta_{s\theta}^{-1} \end{bmatrix}^{-1} \tag{159}$$

$$\tilde{\Xi}^c_{22} = \begin{bmatrix} Z_f + \Delta_f & -\Delta_f \\ -\Delta_f & \Delta_f \end{bmatrix} \tag{160}$$

$$\begin{aligned}\Delta_{s\theta}(I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}(\tilde{A}_s - \Sigma_{s\theta}\tilde{R}_s)(I - \theta\Sigma_{s\theta}Z_{s\theta}) + (I - \theta Z_{s\theta}\Sigma_{s\theta})(\tilde{A}_s' - \tilde{R}_s\Sigma_{s\theta})(I \\ -\theta Z_{s\theta}\Sigma_{s\theta})^{-1}\Delta_{s\theta} + (Z_{s\theta}B_1 + V'B_2)(B_1 Z_{s\theta} + B_2'V) + U'Q_{22}U = 0\end{aligned} \tag{161}$$

$$\Delta_f(A_{22} - \Sigma_f R_{22}) + \Delta_f(A_{22}' - R_{22}\Sigma_f) + Z_f B_2 B_2' Z_f = 0 \tag{162}$$

$$\tilde{\Theta}^c = \begin{bmatrix} \tilde{\Theta}^c_{11} + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \tilde{\Theta}^c_{22} + O(\sqrt{\epsilon}) \end{bmatrix} \tag{163}$$

$$\tilde{\Theta}^c_{11} = \begin{bmatrix} \Sigma_{s\theta}^{-1} & \theta Z_{s\theta} - \Sigma_{s\theta}^{-1} \\ \theta Z_{s\theta} - \Sigma_{s\theta}^{-1} & \Sigma_{s\theta}^{-1} - \theta Z_{s\theta} + \Pi_{s\theta}^{-1} \end{bmatrix}^{-1} \tag{164}$$

$$\tilde{\Theta}^c_{22} = \begin{bmatrix} \Sigma_f + \Pi & \Pi_f \\ \Pi_f & \Pi_f \end{bmatrix} \tag{165}$$

Furthermore, the matrices $F_\epsilon^c + \theta G_\epsilon^c G_\epsilon^{c\prime}\tilde{\Xi}^c$ and $F_\epsilon^{c\prime} + \theta H^c \tilde{\Theta}^c$ are Hurwitz. By Theorem 5 of [24], the matrix $I - \theta\tilde{\Theta}^c\tilde{\Xi}^c$ has only positive eigenvalues.

Obviously $\tilde{\Theta}^c > \Theta^c_0$. Hence, by Theorem 3, $J^c_{I\theta} = \mathrm{Tr}(\tilde{\Theta}^c H^c)$. Some straightforward algebraic manipulations lead to part 2.v).

Now substitute the slow controller μ^*_{Is} into the full-order system to arrive at another control-free LEQG problem. Let

$$\begin{aligned} x_f^s &:= x_2 + U\hat{x}_s^s \qquad (166)\\ \tilde{x}^{s\prime} &:= [x_1', \ \hat{x}_s^{s\prime} + \sqrt{\epsilon}x_f^{s\prime}X(I - \theta Z_{s\theta}\Sigma_{s\theta})^{-1}, \ x_f^{s\prime}] \qquad (167)\end{aligned}$$

In terms of the state variable $\tilde{x}^s$, this LEQG problem can be written as (see again [17] for details):

$$\begin{aligned} d\tilde{x}^s &= \begin{bmatrix} F^c_{11} + O(\sqrt{\epsilon}) & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) + & \frac{1}{\epsilon}A_{22} + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \tilde{x}^s\, dt + \begin{bmatrix} G^c_1 + O(\sqrt{\epsilon}) \\ \frac{1}{\sqrt{\epsilon}}G_2 + O(1) \end{bmatrix} dw_t \\ &:= F^s_\epsilon \tilde{x}^s\, dt + G^s_\epsilon\, dw_t \qquad (168)\\ J^s_{I\theta} &:= \lim_{t_f\to\infty} \frac{2}{\theta t_f} \ln\left\{ \boldsymbol{E}\left\{ \exp\left[\frac{\theta}{2}\int_0^{t_f} \tilde{x}^{c\prime}(H^s + O(\sqrt{\epsilon}))\tilde{x}^c\, dt\right]\right\}\right\} \qquad (169)\end{aligned}$$

where

$$H^s = \begin{bmatrix} H_{11}^c & O(1) \\ O(1) & Q_{22} \end{bmatrix} \tag{170}$$

and F_{11}^c, G_1^c and H_{11}^c are as defined in (148), (151) and (154).

The initial state $\tilde{x}^s(0)$ is a Gaussian random vector with mean $\bar{x}_0^s$ and covariance Σ_0^s, given by:

$$\bar{x}_0^s := \begin{bmatrix} \bar{x}_{10} \\ (I-\theta\Sigma_\theta Z_{s\theta})^{-1}\bar{x}_{10} + O(\sqrt{\epsilon}) \\ \bar{x}_{20} + U(I-\theta\Sigma_\theta Z_{s\theta})^{-1}\bar{x}_{10} \end{bmatrix}; \quad \Sigma_0^s := \begin{bmatrix} \begin{bmatrix} \Sigma_{011} & 0 \\ 0 & 0 \end{bmatrix} + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Sigma_{022} + O(\sqrt{\epsilon}) \end{bmatrix}$$

To evaluate the cost $J_{I\theta\infty}^s$, we associate (as in the composite controller case) a fictitious measurement (127) with this LEQG problem, where v_t, or y^e, is a standard Wiener process independent of the initial state and w_t.

Then, the two GAREs associated with this problem are:

$$F_\epsilon^{s\prime}\tilde{\Xi}^s + \tilde{\Xi}^s F_\epsilon^s + \tilde{\Xi}^s\theta G_\epsilon^s G_\epsilon^{s\prime}\tilde{\Xi}^s + H^s = 0 \tag{171}$$

and

$$F_\epsilon^s\tilde{\Theta}^s + \tilde{\Theta}^s F_\epsilon^{s\prime} + \tilde{\Theta}^s\theta H^s\tilde{\Theta}^s + G_\epsilon^s G_\epsilon^{s\prime} = 0 \tag{172}$$

It is shown in [17] that, if A_{22} is Hurwitz, the minimal solutions to these GAREs exist for $\theta < \theta_{Is}$ and sufficiently small ϵ, and can be approximated by

$$\tilde{\Xi}^s = \begin{bmatrix} \tilde{\Xi}_{11}^c + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\epsilon) & \epsilon Z_{of} + O(\epsilon^{3/2}) \end{bmatrix} \tag{173}$$

$$\tilde{\Theta}^s = \begin{bmatrix} \tilde{\Theta}_{11}^c + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Sigma_{of} + O(\sqrt{\epsilon}) \end{bmatrix} \tag{174}$$

where $\tilde{\Xi}_{11}^c$ and $\tilde{\Theta}_{11}^c$ are as defined in (159) and (164). Furthermore, the matrices $F_\epsilon^s + \theta G_\epsilon^s G_\epsilon^{s\prime}\tilde{\Xi}^s$ and $F_\epsilon^{s\prime} + \theta H^s\tilde{\Theta}^s$ are Hurwitz. By Theorem 5 of [24], the matrix $I - \theta\tilde{\Theta}^s\tilde{\Xi}^s$ has only positive eigenvalues.

Hence, by Theorem 3, $J_{I\theta}^s = \mathrm{Tr}(\tilde{\Theta}^s H^s)$. Some straightforward algebraic manipulations lead to (123). This completes the proof of the Theorem. □

5.3 A Large Deviation Form

As the counterpart of the analysis of subsection 3.3, we again consider a large deviation form of the problem. The system under consideration is described by

$$\begin{cases} dx_1 &= (A_{11}x_1 + A_{12}x_2 + B_1u_t)\,dt + \xi G_1\,dw_t; \quad x_1(0) = x_{10} \\ \epsilon dx_2 &= (\epsilon^{1/2}A_{21}x_1 + A_{22}x_2 + B_2u_t)\,dt + \epsilon^{1/2}\xi G_2\,dw_t; \quad x_2(0) = x_{20} \\ dy_1 &= (C_{11}x_1 + C_{12}x_2)\,dt + \xi E_1\,dv_t; \quad y_1(0) = 0 \\ dy_2 &= (\epsilon^{1/2}C_{21}x_1 + C_{22}x_2)\,dt + \epsilon^{1/2}\xi E_2\,dv_t; \quad y_2(0) = 0 \end{cases} \tag{175}$$

and the cost function is given as

$$J_{I\theta}(\mu_I, \xi) = \lim_{t_f\to\infty}\frac{2\xi^2}{\theta t_f}\ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2\xi^2}\int_0^{t_f}(x'Qx + u'u)\,dt\right]\right\}\right\} \tag{176}$$

where the initial state x_0 is a Gaussian random variable with mean $\bar{x}_0$ and variance $\xi^2\Sigma_0$, and ξ is a small scalar parameter to be varied. We will again study the solution as the parameter $\xi \to 0$. This problem is equivalent to the one considered in Section 4, if we introduce the following substitutions:

$$\theta \leftarrow \frac{\theta}{\xi^2}; \quad G_\epsilon \leftarrow \xi G_\epsilon; \quad E_\epsilon \leftarrow \xi E_\epsilon; \quad \Sigma_0 \leftarrow \xi^2 \Sigma_0 \tag{177}$$

Define the quantity $\theta_I^*(\epsilon)$ exactly as in (71). Then, for any $\theta < \theta_I^*(\epsilon)$, the problem admits an optimal controller, given by (67) and (68).

The optimal solution to the full-order problem again depends on the value of ϵ explicitly. To obtain ϵ-free solutions, we decompose the system into slow and fast subsystems. The slow subsystem can be obtained in the same way as before. Under the substitution law (177), the slow LEQG problem is again described by (78), (79) and (80). This leads to a definition of θ_{Is} exactly as in (84). For any $\theta < \theta_{Is}$, the optimal controller for the slow subproblem is as in (85) and (86). The fast subsystem is again (88), (89) and (90), under the substitution law (177). Thus, the fast controller is exactly the same as (95)–(96). Hence, we can form the slow and composite controllers μ_{Is}^* and μ_{Ic}^* as before. The slow controller μ_{Is}^* is as in (105) and (106), and the composite controller μ_{Ic}^* is as in (111), (112) and (113), of course now parametrized also by ξ.

Then, all this leads to the following Corollary to Theorem 4.

Corollary 2 *For the ξ-parametrized singularly perturbed system (175) under the cost function (176):*

1. *For each $\epsilon > 0$, if the pairs (A_ϵ, B_ϵ) and (A_ϵ, G_ϵ) are controllable, the pairs (A_ϵ, C_ϵ) and (A_ϵ, Q) are observable, and the matrix N_ϵ is invertible, then, $\forall \theta < \theta_I^*(\epsilon)$, the optimal cost for the full-order LEQG problem can be written as:*

$$J_{I\theta}^*(\epsilon;\xi) = O(\xi^2) \tag{178}$$

2. *Let assumption A2 hold, the pairs (A_{11}, B_0), $(A_{11}, G_1G_1' - G_1G_2'(G_2G_2')^{-1}G_2G_1')$ and (A_{22}, B_2) be controllable, and the pairs (A_{11}, C_0), $(A_{11}, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ and (A_{22}, C_2) be observable. Then,*

   i. $\lim_{\epsilon\to 0^+} \theta_I^*(\epsilon) = \theta_{Is}$.

   ii. $\forall \theta < \theta_{Is}$, *if the composite controller $\mu_{Ic\theta}^*$ is applied to the system, then $\exists \epsilon_\theta' > 0$ such that* $\forall \epsilon \in [0, \epsilon_\theta')$,

$$J_{I\theta}^c := J_{I\theta}(\mu_{Ic\theta}^*) = J_{I\theta}^*(\epsilon;\xi) + O(\xi^2\sqrt{\epsilon}) \tag{179}$$

   iii. $\forall \theta < \theta_{Is}$, *if, in addition, the matrix A_{22} is Hurwitz, and the slow controller $\mu_{Is\theta}^*$ is applied to the system, then $\exists \hat{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \hat{\epsilon}_\theta)$,*

$$J_{I\theta}^s := J_{I\theta}(\mu_{Is\theta}^*) = O(\xi^2) \tag{180}$$

□

6 Examples

We present here three sets of numerical results, one for perfect state measurements and two for noisy state measurements. As stressed earlier, the quantities θ_s and θ_{Is} play important roles in the computation of approximate values for $\theta^*(\epsilon)$ and $\theta_I^*(\epsilon)$.

Example 1

Consider the system and cost function:

$$\begin{bmatrix} dx_1 \\ \epsilon dx_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u\, dt + \begin{bmatrix} 1 \\ \sqrt{\epsilon} \end{bmatrix} dw_t \tag{181}$$

$$J_\theta = \lim_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (2x_1^2 + 2x_1 x_2 + x_2^2 + u'u)\, dt \right) \right] \right\} \right\} \tag{182}$$

By using a particular search algorithm, we can compute the quantity:

$$\theta_s = 1.8892$$

We next compute the maximum allowable θ-levels for the full-order system (181)–(182) for different fixed values of ϵ.

$\theta^*(\epsilon)$	1.5616	1.9842	1.9274	1.9019	1.8930	1.8903
ϵ	1	0.1	0.01	0.001	0.0001	0.00001

Note that as $\epsilon \to 0$, $\theta^*(\epsilon) \to \theta_s$.

Now, we choose $\theta = 1.6 < \theta_s$, and design the suboptimal controllers for the system based on this value of θ:

$$\mu_s^*(x_1) = -1.7633 x_1; \quad \mu_c^*(x_1, x_2) = -3.3249 x_1 - 0.61803 x_2$$

Then, we apply these controllers, μ_s^* and μ_c^*, to system (181)–(182) and obtain the corresponding performance levels J_s and J_c, respectively. These values are tabulated below along with the optimal cost levels, for different values of ϵ:

ϵ	1	0.1	0.01	0.001	0.0001	0.00001
$J^*(\epsilon)$	∞	3.0434	2.1560	1.9818	1.9350	1.9209
$J_c(\epsilon)$	∞	∞	2.1638	1.9828	1.9351	1.9209
$J_s(\epsilon)$	∞	∞	∞	2.2150	2.1366	2.1151

We also compute the cost level at $\epsilon = 0$:

$$J^*(0) = J_c(0) = 1.9146; \quad J_s(0) = 2.1056\,,$$

and observe that the composite controller asymptotically achieves the optimal performance level as $\epsilon \to 0$; however, the slow controller achieves only a suboptimal, but finite performance level asymptotically — all this being consistent with the result of Theorem 1. Also, the composite controller appears to be more robust than the slow one with respect to changes in the value of ϵ. $\diamond$

Example 2

Consider the set-up

$$\begin{bmatrix} dx_1 \\ \epsilon dx_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \sqrt{\epsilon} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u\, dt + \begin{bmatrix} 1 \\ \sqrt{\epsilon} \end{bmatrix} dw_t \tag{183}$$

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \sqrt{\epsilon} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\epsilon} \end{bmatrix} dv_t \tag{184}$$

$$J_{I\theta} = \lim_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \int_0^{t_f} (2x_1^2 + 2x_1 x_2 + x_2^2 + u'u)\, dt \right] \right\} \right\} \tag{185}$$

The maximum allowable θ-level for the slow subsystem is

$$\theta_{Is} = 1.4212$$

We can also compute the maximum allowable θ-level, $\theta_I^*(\epsilon)$, for this system for different values of ϵ:

$\theta_I^*(\epsilon)$	1.4133	1.4189	1.4204	1.4209	1.4211
ϵ	0.001	10^{-4}	10^{-5}	10^{-6}	10^{-7}

Note again that as $\epsilon \to 0$, $\theta_I^*(\epsilon) \to \theta_{Is}$.

Now, we choose $\theta = 1 < \theta_{Is}$, and design the slow and composite controllers under the corresponding cost functions:

$$\mu_{Is}^* = -\hat{x}_s^s; \quad \mu_{Ic}^* = -\hat{x}_s^c - 0.61803\hat{x}_f^c$$

where

$$d\hat{x}_s^s = -5.6833\hat{x}_s^s\, dt + \begin{bmatrix} 0.44721 & 0.84721/\sqrt{\epsilon} \end{bmatrix} \begin{bmatrix} dy_1 + 2\hat{x}_s^s\, dt \\ dy_2 + 4\hat{x}_s^s\, dt \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} d\hat{x}_s^c \\ \epsilon d\hat{x}_f^c \end{bmatrix} &= \begin{bmatrix} -5.6833 & 2.0944/\sqrt{\epsilon} \\ -3.0902\sqrt{\epsilon} & -3.4721 \end{bmatrix} \begin{bmatrix} \hat{x}_s^c \\ \hat{x}_f^c \end{bmatrix} dt \\ &\quad + \begin{bmatrix} 0.44721 & 0.84721/\sqrt{\epsilon} \\ 0 & 0.61803 \end{bmatrix} \begin{bmatrix} dy_1 + 2\hat{x}_s^c\, dt \\ dy_2 + 4\hat{x}_s^c\, dt \end{bmatrix} \end{aligned}$$

Then, we apply μ_{Is}^* and μ_{Ic}^* to system (183)–(185) and obtain the corresponding performance levels J_{Is} and J_{Ic}. They are tabulated below for different values of ϵ, along with the corresponding optimal cost levels :

ϵ	0.001	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$J_I^*(\epsilon)$	3.9236	3.7409	3.6873	3.6707	3.6655
$J_{Ic}(\epsilon)$	∞	∞	3.6921	3.6710	3.6655
$J_{Is}(\epsilon)$	∞	3.9147	3.7691	3.7361	3.7390

We can also compute the optimal cost level at $\epsilon = 0$:

$$J_I^*(0) = J_{Ic}(0) = 3.6631; \quad J_{Is}(0) = 3.7361$$

We see that the composite controller asymptotically achieves the optimal performance level as $\epsilon \to 0$. The slow controller, on the other hand, achieves a suboptimal but finite performance level asymptotically, which is again consistent with the statement of Theorem 4. We also note that in this case the composite controller is more sensitive than the slow one to changes in the value of ϵ. A possible explanation for this behavior is the following: since the quantity $J_{Is}(0)$ is very close to $J_{Ic}(0)$, this means there is little for the fast controller to do to improve the performance of the overall system. Furthermore, since the fast controller is an LQG design for the fast subsystem, the closed-loop fast subsystem under such a controller may not exhibit better H^∞ performance than the open-loop fast subsystem. $\diamond$

Example 3

Consider the following system and cost function:

$$\begin{aligned}
\begin{bmatrix} dx_1 \\ \epsilon dx_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ \sqrt{\epsilon} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u\, dt + \begin{bmatrix} 1 \\ 2\sqrt{\epsilon} \end{bmatrix} dw_t && (186) \\
\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 2\sqrt{\epsilon} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\epsilon} \end{bmatrix} dv_t && (187) \\
J_{I\theta} &= \lim_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \int_0^{t_f} (3x_1^2 + 2x_1 x_2 + 2x_2^2 + u'u)\, dt \right] \right\} \right\} && (188)
\end{aligned}$$

The maximum allowable θ-level for the slow subsystem is

$$\theta_{Is} = 6.3756\,.$$

We can also compute the maximum allowable θ-level, $\theta_I^*(\epsilon)$, for this system, for different values of ϵ:

$\theta_I^*(\epsilon)$	6.8401	6.8836	6.5390	6.4289	6.3923
ϵ	0.01	0.0015	10^{-4}	10^{-5}	10^{-6}

Note again that as $\epsilon \to 0$, $\theta_I^*(\epsilon) \to \theta_{Is}$.

Now, we choose $\theta = 5 < \theta_{Is}$, and design the slow and composite controllers under the corresponding cost functions:

$$\mu_{Is}^* = -0.87591\hat{x}_s^s; \quad \mu_{Ic}^* = -0.87591\hat{x}_s^c - 0.73205\hat{x}_f^c$$

where

$$\begin{aligned}
d\hat{x}_s^s &= -5.5564\hat{x}_s^s\, dt + \begin{bmatrix} 0.010525 & 0.24449/\sqrt{\epsilon} \end{bmatrix} \begin{bmatrix} dy_1 + 0.87591\hat{x}_s^s\, dt \\ dy_2 + 1.7518\hat{x}_s^s\, dt \end{bmatrix} \\
\begin{bmatrix} d\hat{x}_s^c \\ \epsilon d\hat{x}_f^c \end{bmatrix} &= \begin{bmatrix} -5.5564 & 0.35796/\sqrt{\epsilon} \\ -23.191\sqrt{\epsilon} & -4.8552 \end{bmatrix} \begin{bmatrix} \hat{x}_s^c \\ \hat{x}_f^c \end{bmatrix} dt \\
&\quad + \begin{bmatrix} 0.010525 & 0.24449/\sqrt{\epsilon} \\ 0 & 1.5616 \end{bmatrix} \begin{bmatrix} dy_1 + 0.87591\hat{x}_s^c\, dt \\ dy_2 + 1.7518\hat{x}_s^c\, dt \end{bmatrix}
\end{aligned}$$

Then, we apply μ^*_{Is} and μ^*_{Ic} to system (186)–(188) and obtain the corresponding performance levels J_{Is} and J_{Ic}. They are tabulated below for different values of ϵ, along with the corresponding optimal cost levels:

ϵ	0.01	0.0015	10^{-4}	10^{-5}	10^{-6}
$J^*_I(\epsilon)$	4.6326	4.0662	3.9404	3.9210	3.9157
$J_{Ic}(\epsilon)$	∞	4.1528	3.9408	3.9210	3.9157
$J_{Is}(\epsilon)$	∞	∞	4.6194	4.5808	4.5710

We can also compute the cost level at $\epsilon = 0$:

$$J^*_I(0) = J_{Ic}(0) = 3.9134; \quad J_{Is}(0) = 4.5668\,.$$

Again the composite controller asymptotically achieves the optimal performance level as $\epsilon \to 0$, and the slow controller achieves a suboptimal but finite performance level asymptotically. In contradistinction with what was observed in Example 2, however, here the composite controller is less sensitive to changes in the value of ϵ than the slow one. $\diamond$

7 Conclusion

In this paper, we have presented a model reduction technique for the LEQG problem for linear singularly perturbed systems under perfect and imperfect state measurements. We have developed a time-scale decomposition procedure that breaks the full-order problem into appropriate slow and fast lower-order subproblems, the optimal solutions of which yield slow and fast controllers. When combined in an appropriate way, these lead to a composite controller under which the optimal performance level for the full-order system is achieved asymptotically as the singular perturbation parameter $\epsilon \to 0$. It has also been shown that when the fast subsystem is open-loop stable, the slow controller can achieve asymptotically some finite (but not optimal) performance level whenever the full-order problem admits a solution. In this case there is a clear positive gap between the asymptotic performance level a slow controller can achieve and the asymptotic performance level achieved by a full-order optimal controller, and the paper has provided a clear characterization of this performance loss. This indicates that there is a definite tradeoff between controller simplicity (due to model reduction) and loss of performance. In a large deviations context, i.e. when the intensity of the noise in the system dynamics goes to zero, however, the slow controller can achieve asymptotically the performance level for the full-order system, provided that the fast subsystem is open-loop stable. Counterparts of these results in the finite-horizon case exist, and can be found in the internal report [17].

To obtain the optimal solution to the slow subsystem arrived at as a result of model reduction, it has turned out that one needs to develop a theory for the general LEQG problem with general cost structure (with cross terms between the state and control) and correlation between system and measurement noises. Since the LEQG problem has not been solved in the literature in such a full generality, we have also provided in the paper (Section 4) a clean and complete solution to this problem in both finite and infinite horizons via a different line from that of [5] which had addressed the finite-horizon case only, and under some restrictive assumptions. The solution obtained in Section 4 is precisely the central solution of the corresponding H^∞-optimal control problem [2], and the line of proof there would be useful even for the standard LEQG problem (i.e. without the cross term and with uncorrelated system and measurement noises) since it requires the least restrictive assumptions (leading to both necessary and sufficient conditions).

One possible nontrivial extension of the results of this paper would be the derivation of higher-order correction terms. The composite controller constructed in the paper achieves a performance level that is $O(\sqrt{\epsilon})$ close to the optimal one. This, however, may not be sufficient in some applications. Hence, high-order correction terms for the composite controller is of some interest. Another extension would be to the multiple time scale problems, so as to obtain the counterparts of [26] which deals with the H^∞-optimal control problem. One other challenging extension would be to the nonlinear case, under both regular [27] and singular [19] perturbations.

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