H^∞-Control of Markovian Jump Systems and Solutions to Associated Piecewise-Deterministic Differential Games *

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1 Introduction

Systems prone to failure or abrupt structural changes can be modeled by the introduction an auxiliary stochastic process, generally described as a Markov chain, the outcome of which determines the structure according to which (out of a finite number of alternatives) the system behaves at each point in time. In the case of linear systems, this manifests itself as the system matrices depending on the current state of a Markov chain, leading to what is called *jump linear systems*. This class of systems has attracted much attention for a long time, with some of the representative books and papers being [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11]. The optimal control problem for jump linear systems (with or without additive white noise inputs) has been solved completely when the state of the Markov chain is available ([12] [13] [14]). There has also been a major effort in studying the solution to the adaptive jump linear quadratic Gaussian optimal control, where the state of the Markov chain is not measured perfectly ([15] [16] [17] [18] [19]).

The worst-case design problem for jump linear systems has been studied earlier in [20], where it was assumed that the controller has access to both the system state x and the form process θ and the disturbance is restricted to time functions only. These results actually follow from a game theoretic approach, and using a result on Nash equilibria of differential games with jump linear dynamics [21]. The results obtained in [21] cover also nonlinear systems and different modes of equilibria for the case when both players have complete information of x and θ. More recently, these results have been extended to sampled state information, and relationship with "H^∞-optimal" control of piecewise-deterministic nonlinear systems has been established [22]. Additional results on piecewise-deterministic nonzero-sum differential games, especially from the viewpoint of overtaking feedback equilibria, can be found in [23].

In this paper, we revisit the worst-case controller design problem for jump

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linear systems, and obtain a complete set of solutions not only for the state feedback case, but also for the case when imperfect state measurements are available to the controller. This can be viewed as an H^∞-optimal control problem [24], where now the system and cost matrices depend on the outcome of a Markov chain. Solving this problem is equivalent to solving (for upper value of) a class of parametrized soft-constrained zero-sum differential games with piecewise deterministic linear dynamics, under a quadratic cost function. In this game, Player 1 (P1), which is the controller, strives to minimize the (expected) cost function based on an exact knowledge of the history of the state of the Markov process and a partial (noise corrupted) measurement on the system state. On the other hand, Player 2 (P2), the disturbance, tries to maximize the same (expected) cost function using p! rogressively measurable strategies . As indicated earlier, we are primarily interested in the upper value of this zero-sum game, and especially in the question of whether the upper value is less than or equal to zero. A set of sufficient conditions, as well as a set of necessary conditions, are given to ensure satisfaction of this bound.

The balance of the paper is as follows. The next section, Section 2, provides a precise formulation of the problem under study, in both finite and infinite horizons. Section 3 presents results for the perfect state measurements case, which are subsequently extended to imperfect state measurements in Section 4. Section 5 contains three illustrative numerical examples, and the paper ends with the concluding remarks of Section 6 and an Appendix.

2 Problem Formulation

The jump linear system under consideration is described by:

$$\dot{x} = A(t,\theta(t))x + B(t,\theta(t))u + D(t,\theta(t))w; \quad x(t_0) = x_0 \tag{1}$$

$$y = C(t,\theta(t))x + E(t,\theta(t))w \tag{2}$$

where x is the n-dimensional system state vector; u is the p-dimensional input, determined by the strategy of P1 (the controller); w is the q-dimensional input, determined by the strategy of P2 (the disturbance); y is the m-dimensional output vector; θ is a finite state Markov chain defined on the state space $\mathcal{S} = \{1,\dots,s\}$ with the infinitesimal generator matrix

$$\Lambda = (\lambda_{ij}(t))_{s\times s},$$

and a positive initial distribution $\pi_0 := [\pi_{01},\dots,\pi_{0s}]$ (i.e., $\pi_{0i} > 0\ \forall i \in \mathcal{S}$). The underlying probability space is the triple $(\Omega,\mathcal{F},\boldsymbol{P})$. The initial condition x_0 is unknown and is determined by the strategy of P2. Here $\theta(t)$ is the so-called *form* or *indicant process*, which determines the current form of the system at time t. The system state x, inputs u, y, and w each belong to appropriate $(\mathcal{L}^2)$ Hilbert spaces $\mathcal{H}_x$, $\mathcal{H}_u$, $\mathcal{H}_y$ and $\mathcal{H}_w$ respectively, defined on the time interval

$[t_0, t_f]$. Let $\boldsymbol{E}$ denote the expectation with respect to the underlying probability space.

The input u is generated by a strategy μ_I of P1, according to

$$u(t) = \mu_I(t, y_{[t_0,t]}, \theta_{[t_0,t]}) \tag{3}$$

where $\mu_I : [t_0, t_f] \times \mathcal{H}_y \times \Omega \longrightarrow \mathcal{H}_u$ is piecewise continuous in t and Lipschitz continuous in y and measurable in θ, further satisfying the given causality condition. Let us denote the class of all admissible strategies for P1 by $\mathcal{M}_I$.

The initial condition x_0 and the input w are generated by a strategy $\delta := (\delta_0, \nu)$ of P2, according to:

$$\begin{aligned} x_0 &= \delta_0(\theta(t_0)) && (4)\\ w(t) &= \nu(t, x_{[t_0,t]}, \theta_{[t_0,t]}) && (5) \end{aligned}$$

where $\delta_0 : \boldsymbol{\mathcal{S}} \longrightarrow \mathcal{R}^n$, and $\nu : [t_0, t_f] \times \mathcal{H}_x \times \Omega \longrightarrow \mathcal{H}_w$ is piecewise continuous in t and Lipschitz continuous in x and measurable in θ, further satisfying the given causality condition. Let us denote the class of all admissible δ by $\mathcal{D}$.

Associated with this system is the finite-horizon quadratic performance index, parametrized by $\gamma > 0$:

$$\begin{aligned} J_\gamma(\mu_I, \delta) &= \boldsymbol{E}\{|x(t_f)|^2_{Q_f(\theta(t_f))} + \int_{t_0}^{t_f} (|x(t)|^2_{Q(t,\theta(t))} + |u(t)|^2_{R(t,\theta(t))} \\ &\quad -\gamma^2 |w(t)|^2)\, dt - \gamma^2 |x_0|^2_{Q_0(\theta(t_0))}\} \qquad (6)\\ &\quad Q_f(.) \geq 0, \quad Q(.,.) \geq 0, \quad R(.,.) > 0, \quad Q_0(.) > 0 \end{aligned}$$

where P1 chooses his strategy μ_I to minimize the expected cost function (6), and P2 chooses the strategy δ to maximize the same expected cost function. The upper value of the game is determined by

$$\bar{V} = \inf_{\mu_I \in \mathcal{M}_I} \sup_{\delta \in \mathcal{D}} J_\gamma(\mu_I, \delta) \tag{7}$$

As in standard H^∞-control [24], we will study a parametrization of the solution to this problem in terms of γ.

Both players are assumed to have access to the past state of the form process θ. The case when past values of θ are not completely available to the controller leads to a dual control problem, which is much more difficult (if not impossible) to solve. In practice, the form process is usually available to the controller with a small delay. Hence, the formulation here should provide a good approximation to what arises in practice.

It should be noted that the upper value of the game, $\bar{V}$, is bounded below by 0, which can be ensured for P2 by choosing x_0 and $w(t)$ to be zero. Because of the linear-quadratic nature of the problem, the upper value of the game will be infinite if $\bar{V} > 0$. Hence, the case of particular interest to us is $\bar{V} = 0$. In view of this, let us introduce a set Γ_I:

$$\Gamma_I := \{\gamma > 0 : \quad \forall \gamma' > \gamma, \bar{V} = 0\} \tag{8}$$

and a threshold γ_I^*:

$$\gamma_I^* := \inf \Gamma_I. \tag{9}$$

Note that γ_I^* is then the "smallest" value of γ such that the zero-sum differential game admits a bounded upper value.

This threshold is of particular relevance to H^∞-optimal control due to the following $\mathcal{L}_2$ type gain:

$$\inf_{\mu_I \in \mathcal{M}_I} \sup_{\delta \in \mathcal{D}} \frac{J(\mu_I, \delta)^{1/2}}{(\|w\|^2 + \boldsymbol{E}\{|x_0|^2_{Q_0(\theta(t_0))}\})^{1/2}} = \gamma_I^*$$

where

$$J(\mu_I, \delta) = \boldsymbol{E}\{|x(t_f)|^2_{Q_f(\theta(t_f))} + \int_{t_0}^{t_f} (|x(t)|^2_{Q(t,\theta(t))} + |u(t)|^2_{R(t,\theta(t))})\, dt\}$$

and $\|.\|$ denotes the norm:

$$\left[\boldsymbol{E}\{\int_{t_0}^{t_f} |.|^2\, dt\}\right]^{1/2}.$$

The methodology developed in this paper can actually be applied to more general class of problems where the system dynamics further admits a known bias input term $c(t, \theta(t))$:

$$\dot{x} \;\; = \;\; A(t,\theta(t))x + B(t,\theta(t))u + D(t,\theta(t))w + c(t,\theta(t))$$

and the general cost structure:

$$\begin{aligned} J_\gamma(\mu_I, \nu) = \boldsymbol{E}\{|x(t_f)|^2_{Q_f(\theta(t_f))} + \int_{t_0}^{t_f} (|x(t)|^2_{Q(t,\theta(t))} + 2x(t)'P(t,\theta(t))u(t) \\ + |u(t)|^2_{R(t,\theta(t))} + 2x(t)'p(t,\theta(t)) - \gamma^2 |w(t)|^2)\, dt - \gamma^2 |x_0|^2_{Q_0(\theta(t_0))}\} \end{aligned}$$

with additional a cross term and a linear term. This general class of problems addresses a wider practical issues, such as target tracking problem. But, we will only present the solution to the specific class of problems formulated by (1)–(6) to illustrate the basic idea in solving these problems without extra notational burdens.

We now make the following four basic assumptions for the problem formulated above:

Assumption 1 $A(t,i)$, $Q(t,i)$, $B(t,i)$, $D(t,i)$, $C(t,i)$, $E(t,i)$, $R(t,i)$ are piecewise continuous in t for each $i \in \mathcal{S}$.

Assumption 2 Matrix functions $R(t,i)$ and $Q_0(i)$ are positive definite for all $t \in [t_0, t_f]$ and $i \in \mathcal{S}$.

Assumption 3 The Markov chain θ is irreducible.

Assumption 4 The matrices $D(t,i)$ and $E(t,i)$ satisfy the conditions $D(t,i)$ $E'(t,i) = 0$ and $N(t,i) := E(t,i)E'(t,i) > 0$ for all $t \in [t_0, t_f]$ and $i \in \mathcal{S}$.

To study the infinite horizon case (i.e. as $t_f \to \infty$ and $t_0 \to -\infty$, as well as when $t_f = \infty$ and $t_0 = -\infty$), we take A, B, D, C, E, Q, R, Λ to be time-invariant, $Q_f(.) = 0$, π_0 be the stationary distribution of the Markov chain, and further require $x(t) \to 0$ as $t \to -\infty$. Obviously, the weighting matrix $Q_0(.)$ does not play any role here since $x(-\infty) = 0$. In this case, P2's strategy only dictates the choice of the input $w(t)$.

As the counterparts of (8) and (9), let us introduce, respectively, the set

$$\Gamma_{I\infty} := \{\gamma > 0 : \quad \forall \gamma' > \gamma, \bar{V} = 0\}, \tag{10}$$

and the threshold

$$\gamma^*_{I\infty} := \inf \Gamma_{I\infty}. \tag{11}$$

Our objective, for both the finite and infinite horizon cases, is to obtain upper and lower bounds on γ^*_I and $\gamma^*_{I\infty}$, and to derive strategies for P1 that will lead to finite (i.e. zero) upper values for the game. We first study in the next section the perfect state measurements case, that is when $y(t) \equiv x(t)$. Subsequently, in Section 4, we study the original differential game, as formulated above, with imperfect state measurements, which is directly related to the H^∞-optimal control problem for jump linear systems.

3 Perfect State Measurements

The finite-horizon case

In this subsection, we study the problem for finite horizon and under perfect state measurements, i.e., $C(t,\theta(t)) \equiv I$ and $E(t,\theta(t)) \equiv 0$, under Assumptions 1–2. The strategy space for P1 is then denoted by $\mathcal{M}$, which consists of strategies $\mu(t, x(t), \theta(t))$ that are piecewise continuous in t and Lipschitz continuous in x, and measurable in θ. Furthermore, we denote the counterparts of (8) and (9) in this case by Γ and γ^*, respectively.

Let $\gamma > 0$ be fixed. Then it follows from the results of Section 5 of [21] that if the following set of coupled backward generalized Riccati differential equations

(GRDE's):

$$\dot{Z}_i + A'(t,i)Z_i + Z_iA(t,i) - Z_i(B(t,i)R^{-1}(t,i)B'(t,i) - \frac{1}{\gamma^2}D(t,i)D'(t,i))$$

$$\cdot Z_i + Q(t,i) + \sum_{j=1}^{s} \lambda_{ij}(t)Z_j = 0; \quad Z_i(t_f) = Q_f(i) \quad i \in \mathcal{S} \qquad (12)$$

admits nonnegative definite solutions $Z_i(t),\ i = 1,\ldots,s$, on $[t_0,t_f]$, and furthermore

$$\gamma^2 Q_0(i) - Z_i(t_0) \geq 0, \qquad i = 1,\ldots,s, \qquad (13)$$

then the game has a finite value, which is zero. This implies that $\gamma \geq \gamma^*$.

We will now show that the existence of a set of nonnegative definite solutions to (12) along with (13) is also necessary for $\gamma > \gamma^*$ (i.e., for the upper value to be bounded). Toward this end, we first introduce the matrix-valued function $Z(t,\theta(t))$ on $[t_0,t_f] \times \mathcal{S}$, defined as:

$$Z(t,\theta(t)) \quad := \quad Z_i(t) \qquad \text{if} \quad \theta(t) = i$$

when $Z_i(t),\ i = 1,\ldots,s$, exist on $[t_0,t_f]$. Next, we introduce the notion of a conjugate point for the set (12).

Definition 1 *The set of coupled GRDE's (12) is said to have a conjugate point at T, where $T \in [t_0,t_f)$, if $Z_i(t),\ i = 1,\ldots,s$, exists on $(T,t_f]$ and, for some $i \in \mathcal{S}$, one of the eigenvalues of $Z_i(t)$ goes to $+\infty$ as $t \downarrow T$.*

Now, we can state (and prove) the following theorem:

Theorem 1 *Consider the piecewise-deterministic soft-constrained zero-sum differential game of this subsection. Let Assumptions 1–2 hold. Then, $\gamma^* < +\infty$, and for $\gamma > \gamma^*$, the GRDE's (12) admit nonnegative definite solutions $Z_i(t)$, $i = 1,\ldots,s$, on $[t_0,t_f]$, satisfying (13), and hence the upper value $\bar{V}$ of the zero-sum game is 0. For $\gamma > \gamma^*$, a minimax strategy for P1 that attains this upper value is:*

$$u^*_\gamma(t) = \mu^*_\gamma(t,x(t),\theta(t)) = -R^{-1}(t,\theta(t))B'(t,\theta(t))Z(t,\theta(t))x(t). \qquad (14)$$

Furthermore, for $\gamma \leq \gamma^*$, condition (13) is violated for at least one $i \in \mathcal{S}$ and the upper value of the game is $+\infty$.

Proof In view of the discussions preceding the theorem, we need only show that if condition (13) is violated, then the upper value of the game is $+\infty$.

Choose a $\gamma > 0$ such that condition (13) is violated. Then, either the set of GRDE's (12) admits solutions on $[t_0,t_f]$, but the matrix $\gamma^2 Q_0(i) - Z_i(t_0)$ has a negative eigenvalue for at least one i, or the set (12) has a conjugate point at T

for some $T \in [t_0, t_f)$. If we have the former case, then by conditioning (6) first on $\theta(t_0) = i$, and realizing that μ has x_0 as its argument,

$$\inf_{\mu \in \mathcal{M}} \sup_{(\delta_0,\nu)\in\mathcal{D}} J_\gamma(\mu, \delta_0, \nu) = \sup_{x_0 \in \mathcal{R}^n} \boldsymbol{E}\{|x_0|^2_{Z_i(t_0)} - \gamma^2 |x_0|^2_{Q_0(i)}\}$$

which directly leads to the necessity of condition (13) for the boundedness (by zero) of the upper value.

The proof for the latter case is more involved. Again, choose a $\gamma > 0$ such that the set of GRDE's (12) does not admit a solution on $[t_0, t_f]$. Since the solution of (12) is always bounded below by 0 for each $i \in \mathcal{S}$ whenever it exists, the set of GRDE's (12) must have a conjugate point at T for some $T \in [t_0, t_f)$. Let $i_0 \in \mathcal{S}$ be the particular index associated with this conjugate point, as in Definition 1. Then, there exists a vector $x^* \in \mathcal{R}^n$ and a sequence $\{T_k\}_{k=0}^{\infty}$ such that $t_f = T_0 > T_1 > \cdots$, $\lim_{k\to\infty} T_k = T$ and

$$|x^*|^2_{Z_{i_0}(T_k)} \to +\infty \qquad \text{as} \quad k \to \infty.$$

Suppose that there is a strategy $\mu \in \mathcal{M}$ for P1 that guarantees a finite upper value $\bar{V}$. Since μ is locally Lipschitz continuous in x, there exists an integer p_0 such that, for each $k = p_0, p_0+1, \ldots$, there is a corresponding vector $x_0^k \in \mathcal{R}^n$ such that

$$\dot{x} = A(t, i_0)x + B(t, i_0)\mu; \qquad x(t_0) = x_0^k$$

leads to $x(T_k) = \left(|x^*|^2_{Z_{i_0}(T_k)}\right)^{-1/4} x^*$.

For any integer $k \geq p_0$, the maximum value of the game with P1 using strategy μ and system starting at x_0^k is bounded below by $J_\gamma(\mu, x_0^k, \nu_k)$, where $\nu_k \in \mathcal{D}$ is defined by

$$\begin{aligned} w(t) &= 0; \qquad t \in [t_0, T_k) \\ w(t) &= \frac{1}{\gamma^2} D'(t, \theta(t)) Z(t, \theta(t)) x(t); \qquad t \in [T_k, t_f] \end{aligned}$$

Then,

$$\sup_{\delta \in \mathcal{D}} J_\gamma(\mu, \delta) \geq \left(|x^*|^2_{Z_{i_0}(T_k)}\right)^{1/2} \boldsymbol{P}(r_{[t_0,T_k]} \equiv i_0) - \gamma^2 \boldsymbol{E}\{|x_0^k|^2_{Q_0(\theta(t_0))}\}$$

As $k \to \infty$ the left-hand-side (LHS) should be bounded by $\bar{V}$, while the right-hand-side (RHS) converges to $+\infty$. This leads to a contradiction. Therefore, there is no strategy $\mu \in \mathcal{M}$ that can guarantee a finite cost for P1.

This completes the proof of the Theorem. □

Remark 1 Let $\bar{\gamma}_i^* > 0$ be the H^∞-optimal performance level of the deterministic linear system with coefficient matrices $\{A_i + (1/2)\lambda_{ii} I, B_i, D_i, Q_i, R_i\}$, $i = 1, \ldots, s$. Then, we have $\gamma^* \geq \bar{\gamma}_i^*$, $i = 1, \ldots, s$. □

The infinite-horizon case

We now turn to the infinite-horizon case, where we take $t_0 = 0$, $t_f \to \infty$ (as well as the case when $t_f = \infty$), the system matrices A, B, D, Q, R, Λ to be time-invariant, $Q_f(.) = 0$, and π_0 be the stationary distribution of the Markov chain.

Let $\bar{V}$ denote the upper value of the game, and introduce the set (as the counterpart of (10))

$$\Gamma_\infty := \{\gamma > 0 : \quad \forall \gamma' > \gamma, \bar{V} = 0\} \tag{15}$$

Further introduce the counterpart of (11) for the perfect state measurements case:

$$\gamma_\infty^* := \inf \Gamma_\infty . \tag{16}$$

Introduce the following set of coupled generalized algebraic Riccati equations (GARE's):

$$\begin{aligned} A'(i)Z_i + Z_i A(i) - Z_i(B(i)R^{-1}(i)B'(i) - \frac{1}{\gamma^2} D(i)D'(i))Z_i & \\ +Q(i) + \sum_{j=1}^{s} \lambda_{ij} Z_j = 0; \quad i = 1, \ldots, s & \end{aligned} \tag{17}$$

and introduce the condition:

$$\gamma^2 Q_0(i) - Z_i \geq 0; \qquad i = 1, \ldots, s. \tag{18}$$

Let $Z(\theta(t))$ be defined as:

$$Z(\theta(t)) \quad := \quad Z_i \qquad\qquad \text{if} \quad \theta(t) = i$$

when Z_i, $i = 1, \ldots, s$, exist.

The following two assumptions will be needed for the ensuing analysis.

Assumption 5 The pair $(A(\theta(t)), B(\theta(t)))$ is stochastically stabilizable ([9]).

Assumption 6 The pair $(A(i), Q(i))$ is observable for each $i \in \mathcal{S}$.

Then, we have the following counterpart of Theorem 1.

Theorem 2 *Consider the piecewise-deterministic soft-constrained zero-sum differential game with perfect state measurements in the infinite-horizon case, as defined in this subsection. Let Assumptions 2, 3, 5 and 6 hold. Then,* $\gamma_\infty^* < +\infty$, *and for any* $\gamma > \gamma_\infty^*$, *there exist a set of minimal positive definite solutions* Z_i, $i = 1, \ldots, s$, *to GARE's (17), which further satisfy the condition (18), and a strategy* $\mu_{\gamma\infty}^*$ *for P1 that guarantees the zero upper value:*

$$u_{\gamma\infty}^*(t) = \mu_{\gamma\infty}^*(t, x(t), \theta(t)) = -R^{-1}(\theta(t))B'(\theta(t))Z(\theta(t))\, x(t), \tag{19}$$

The jump linear system driven by this control, and without any disturbance,

$$\dot{x} \quad = \quad \left(A(\theta(t)) - B(\theta(t))R^{-1}(\theta(t))B'(\theta(t))Z(\theta(t))\right)x(t)\,, \tag{20}$$

is mean-square stable, i. e., $\lim_{t\to\infty} \boldsymbol{E}\{|x(t)|^2\} = 0$.

Furthermore, for almost all $\gamma > \gamma_\infty^*$, *the jump linear system driven by both the optimal control and the optimal disturbance,*

$$\begin{aligned}\dot{x} \quad = \quad & \big(A(\theta(t)) - (B(\theta(t))R^{-1}(\theta(t))B'(\theta(t)) \\ & -\frac{1}{\gamma^2}D(\theta(t))D'(\theta(t)))Z(\theta(t))\big)\,x(t)\end{aligned} \tag{21}$$

is also mean-square stable.

For $\gamma < \gamma_\infty^*$, *on the other hand, either condition (18) is not satisfied, or the set of GARE's (17) does not admit nonnegative definite solutions, and in both cases the upper value of the game is* $+\infty$.

Proof First, we will show that for any $\gamma > 0$ such that the set of GARE's (17) does not admit any nonnegative definite solutions, the upper value of the game is $+\infty$.

Fix any $\gamma > 0$ such that the set of GARE's (17) does not admit any nonnegative definite solutions. Consider the following set of coupled GRDE's indexed by $T > 0$:

$$\begin{aligned}&\dot{Z}_i^T + A'(i)Z_i^T + Z_iA(i) - Z_i^T(B(i)R^{-1}(i)B'(i) - \frac{1}{\gamma^2}D(i)D'(i))Z_i^T \\ &\qquad + Q(i) + \sum_{j=1}^{s}\lambda_{ij}Z_j^T = 0 \quad Z_i^T(T) = 0 \quad i = 1,\ldots,s.\end{aligned} \tag{22}$$

Denote the solutions to the above GRDE's by $Z_i^T(t)$, $i = 1,\ldots,s$. The following relationship holds for all $i \in \mathcal{S}$:

$$Z_i^{T_1}(t) \geq Z_i^{T_2}(t) \qquad \forall\;\; 0 \leq t \leq T_2 \leq T_1.$$

Since the set of GARE's (17) does not admit any nonnegative definite solutions, there exists a sequence of strictly increasing positive scalars $\{T_k\}_{k=0}^\infty$ and an index $i_0 \in \mathcal{S}$ such that the maximum eigenvalue of $Z_{i_0}^{T_k}(0)$ goes to ∞ as $k \to \infty$. Without loss of generality, we can assume there exists an $x^* \in \mathcal{R}^n$ such that

$$\lim_{k\to\infty} |x^*|^2_{Z_{i_0}^{T_k}(0)} = +\infty.$$

(Such an x^* must exist for some subsequence of $\{T_k\}_{k=0}^\infty$. In the proof below we work directly with that subsequence.)

For any integer k, the maximum value of the game with P1 using strategy μ and system starting at x^* is bounded below by $J_\gamma(\mu, x^*, \nu_k)$, where $\nu_k \in \mathcal{D}$ is defined by

$$
\begin{aligned}
w(t) &= 0; \qquad (T_k, \infty) \\
w(t) &= \frac{1}{\gamma^2} D'(\theta(t)) Z^{T_k}(t, \theta(t)) x(t); \qquad t \in [0, T_k]
\end{aligned}
$$

where $Z^{T_k}(t, \theta(t))$ is defined on $[0, T_k]$ by

$$
Z^{T_k}(t, \theta(t)) = Z_i^{T_k}(t); \qquad \text{if} \qquad \theta(t) = i
$$

Then,

$$
\sup_{\delta \in \mathcal{D}} J_\gamma(\mu_\infty, \delta) \quad \geq \quad (|x^*|^2_{Z^{T_k}_{i_0}(0)}) \pi_{0 i_0} - \gamma^2 \boldsymbol{E}\{|x^*|^2_{Q_0(\theta(0))}\}
$$

As $k \to \infty$ RHS converges to $+\infty$. Therefore, there is no strategy $\mu \in \mathcal{M}$ that can guarantee a finite cost for P1. This implies that $\gamma \leq \gamma^*_\infty$.

For the case where (17) admits a nonnegative definite solution but the condition (18) is violated, again first condition (6) on $\theta(t_0) = i_0$, where $i_0 \in \mathcal{S}$ is such that $\gamma^2 Q_0(i_0) - Z_{i_0}$ has at least one negative eigenvalue, and note that

$$
\inf_{\mu \in \mathcal{M}} \sup_{(\delta_0, \nu) \in \mathcal{D}} J_\gamma = \sup_{x_0 \in \mathcal{R}^n} \boldsymbol{E}\{|x_0|^2_{Z_{i_0}} - \gamma^2 |x_0|^2_{Q_0(i_0)}\}
$$

which directly leads to the conclusion that the upper value is $+\infty$.

Next, we fix a $\gamma > 0$ such that the set of GARE's (17) admits nonnegative definite solutions $\bar{Z}_i$, $i = 1, \ldots, s$. Then the set of GRDE's (22) must admit nonnegative definite solutions $Z_i^T(t)$, $i \in \mathcal{S}$, on $[0, T]$ for any $T > 0$, which further satisfy the following bound:

$$
Z_i^T(t) \leq \bar{Z}_i; \qquad \forall i \in \mathcal{S},\ \forall T > 0,\ \forall t \in [0, T].
$$

By the monotonicity of the $Z_i^T(t)$'s, we can define

$$
Z_i \quad := \quad \lim_{T \to \infty} Z_i^T(0); \qquad \forall i \in \mathcal{S}.
$$

Hence, Z_i, $i \in \mathcal{S}$, satisfy the set of GARE's (17). Obviously, $Z_i \leq \bar{Z}_i$, $i = 1 \ldots, s$. Thus, Z_i, $i = 1, \ldots, s$ is the set of minimal nonnegative definite solutions to the GARE's (17).

For each $i \in \mathcal{S}$, (17) can be rewritten in the following form:

$$
\begin{aligned}
(A'(i) + \frac{1}{2}\lambda_{ii} I) Z_i + Z_i (A(i) + \frac{1}{2}\lambda_{ii} I) - Z_i (B(i) R^{-1}(i) B'(i) \\
- \frac{1}{\gamma^2} D(i) D'(i)) Z_i + Q(i) + \sum_{\substack{j=1 \\ j \neq i}}^{s} \lambda_{ij} Z_j = 0
\end{aligned}
$$

By Assumption 6, we have the observability of the pair $(A(i)+\frac{1}{2}\lambda_{ii}I, Q(i)$ $+\sum_{\substack{j=1\\j\neq i}}^{s}\lambda_{ij}Z_j)$, $i=1,\ldots,s$. This implies, by the existing theory on the solution to GARE's [24], that $Z_i>0$, $i=1,\ldots,s$. Hence, $Z_i>0$, $i=1,\ldots,s$, is the set of minimal positive definite solutions to the GARE's (17).

For this value of γ, a "completion of squares" yields the following equivalent expression for J_γ:

$$\begin{aligned} J_\gamma(\mu,\delta) &= \boldsymbol{E}\{\int_0^T (|u(t)+R^{-1}(\theta(t))B'(\theta(t))Z(\theta(t))x(t)|^2_{R(\theta(t))} \\ &\quad -\gamma^2|w(t)-\frac{1}{\gamma^2}D'(\theta(t))Z(\theta(t))x(t)|^2)\,dt \\ &\quad +|x_0|^2_{Z(\theta(0))-\gamma^2Q_0(\theta(0))}-|x(T)|^2_{Z(\theta(t))}\}, \end{aligned}$$

for all $T>0$. Hence, if further the condition (18) is satisfied, the strategy μ^*_∞ is minimax, and it guarantees a cost 0 for P1. This further implies that $\gamma\ge\gamma^*_\infty$.

Obviously, the set of GARE's (17) admits nonnegative definite solutions also at $\gamma=\infty$ under Assumption 5. Hence, we have the threshold $\gamma^*_\infty<+\infty$.

Thus, we have shown that γ^*_∞ is the infimum of γ's such that the set of GARE's (17) admits positive definite solutions, which further satisfy condition (18). Rewrite the set of GARE's (17) in the following form:

$$\begin{aligned} &(A(i)-B(i)R^{-1}(i)B'(i)Z_i)'Z_i+Z_i(A(i)-B(i)R^{-1}(i)B'(i)Z_i)+Z_i(B(i) \\ &\quad \cdot R^{-1}(i)B'(i)+\frac{1}{\gamma^2}D(i)D'(i))Z_i+Q(i)+\sum_{j=1}^{s}\lambda_{ij}Z_j=0; \quad i=1,\ldots,s. \end{aligned}$$

Since the pairs $(A(i)-B(i)R^{-1}(i)B'(i)Z_i, Z_i(B(i)R^{-1}(i)B'(i)+(1/\gamma^2)D(i)$ $D'(i))Z_i+Q(i))$, $i=1,\ldots,s$ are observable, the mean-square stability of the jump linear system (20) readily follows.

To complete the proof of the Theorem, we only need to show that, for almost all $\gamma>\gamma^*_\infty$, the set of GARE's (17) admits nonnegative definite solutions Z_i, $i=1,\ldots,s$, such that the jump linear system (21) is mean-square stable.

Denote Z_i by $Z_{\gamma i}$ to emphasize its dependence on the parameter γ. Since $Z_{\gamma i}$ is nonincreasing in γ for each $i\in\mathcal{S}$, the derivative $\frac{d}{d\gamma}Z_{\gamma i}$ exists for all $i\in\mathcal{S}$ almost everywhere in the interval (γ^*_∞,∞).

Fix any $\gamma>\gamma^*_\infty$ such that $\frac{d}{d\gamma}Z_{\gamma i}$ exists for all $i\in\mathcal{S}$. Let

$$-\Delta_{\gamma i}:=\frac{d}{d\gamma}Z_{\gamma i};\qquad \forall i\in\mathcal{S}.$$

Then, $\Delta_{\gamma i} \geq 0$ for each $i \in \mathcal{S}$, and they further satisfy the following coupled Lyapunov equations:

$$A'_{F\gamma}(i)\Delta_{\gamma i} + \Delta_{\gamma i}A_{F\gamma}(i) + \frac{2}{\gamma^3}Z_{\gamma i}D(i)D'(i)Z_{\gamma i} + \sum_{j=1}^{s}\lambda_{ij}\Delta_{\gamma j} = 0; \quad i \in \mathcal{S},$$

where

$$A_{F\gamma}(i) := A(i) - (B(i)R^{-1}(i)B'(i) - \frac{1}{\gamma^2}D(i)D'(i))Z_{\gamma i}; \quad i = 1,\ldots,s.$$

This, coupled with the mean-square stability of the jump linear system (20), implies the mean-square stability of the jump linear system (21).

This completes the proof of the theorem. □

Remark 2 In the proof of the above theorem, Assumption 3 may be relaxed to the condition that the initial distribution π_0 has only positive elements. □

The above theorem further implies (with some additional work) the following stronger conclusion on the mean square stability of the closed-loop system:

Corollary 1 *Consider the piecewise-deterministic soft-constrained zero-sum differential game covered by Theorem 2. For any $\gamma > \gamma^*_\infty$, there exists a set of positive definite solutions $\bar{Z}_i$, $i = 1,\ldots,s$, to the following GARI's:*

$$\begin{aligned} A'(i)\bar{Z}_i + \bar{Z}_iA(i) - \bar{Z}_i(B(i)R^{-1}(i)B'(i) - \frac{1}{\gamma^2}D(i)D'(i))\bar{Z}_i \\ +Q(i) + \sum_{j=1}^{s}\lambda_{ij}\bar{Z}_j < 0 \quad i = 1,\ldots,s \end{aligned} \tag{23}$$

such that the jump linear systems

$$\dot{x} \;=\; (A(\theta(t)) - B(\theta(t))R^{-1}(\theta(t))B'(\theta(t))\bar{Z}(\theta(t)))x(t) \tag{24}$$

and

$$\begin{aligned} \dot{x} \;=\; & (A(\theta(t)) - (B(\theta(t))R^{-1}(\theta(t))B'(\theta(t)) \\ & -\frac{1}{\gamma^2}D(\theta(t))D'(\theta(t)))\bar{Z}(\theta(t)))x(t) \end{aligned} \tag{25}$$

are mean-square stable. The strategy $\bar{\mu}^*_{\gamma\infty}$ defined by:

$$\bar{\mu}^*_{\gamma\infty}(t, x(t), \theta(t)) = -R^{-1}(\theta(t))B'(\theta(t))\bar{Z}(\theta(t))x(t). \tag{26}$$

guarantees the zero upper value for P1.

Proof Fix a $\gamma > \gamma_\infty^*$. Then, by Theorem 2, there exists a $\gamma_1 \in (\gamma_\infty^*, \gamma)$ such that a set of minimal positive definite solutions $Z_{\gamma_1 i}$ to GARE's (17) exists and the jump linear system:

$$\dot{x} = (A(\theta(t)) - (B(\theta(t))R^{-1}(\theta(t))B'(\theta(t)) - \frac{1}{\gamma_1^2}D(\theta(t))D'(\theta(t)))Z_{\gamma_1}(\theta(t)))x(t)$$

is mean-square stable. Then, by an application of the Implicit Function Theorem, there exists a $\rho_0 > 0$ such that $\forall \rho \in [0, \rho_0]$, the following set of coupled GARE's:

$$\begin{aligned} &(A'(i) + 1/2\lambda_{ii}I)\bar{Z}_i + \bar{Z}_i(A(i) + 1/2\lambda_{ii}I) - \bar{Z}_i(B(i)R^{-1}(i)B'(i) \\ &-\frac{1}{\gamma_1^2}D(i)D'(i))\bar{Z}_i + Q(i) + \sum_{\substack{j=1 \\ j\neq i}}^{s}(1+\rho)\lambda_{ij}\bar{Z}_j = 0 \quad i = 1, \dots, s \end{aligned} \tag{27}$$

admits minimal positive definite solutions $\bar{Z}_{\gamma_1 \rho i}$, $i = 1, \dots, s$ (see the Appendix for a proof).

This further implies that $\forall \rho \in [0, \rho_0]$, the following set of coupled GARE's:

$$\begin{aligned} &(A'(i) + 1/2\lambda_{ii}I)\bar{Z}_i + \bar{Z}_i(A(i) + 1/2\lambda_{ii}I) - \bar{Z}_i(B(i)R^{-1}(i)B'(i) \\ &-\frac{1}{\gamma^2}D(i)D'(i))\bar{Z}_i + Q(i) + \sum_{\substack{j=1 \\ j\neq i}}^{s}(1+\rho)\lambda_{ij}\bar{Z}_j = 0 \quad i = 1, \dots, s \end{aligned}$$

admits minimal positive definite solutions $\bar{Z}_{\gamma\rho i}$, $i = 1, \dots, s$. Furthermore, $\bar{Z}_{\gamma\rho i}$ is nondecreasing in ρ for each $i \in \mathcal{S}$. Hence, the derivative $\frac{d}{d\rho}\bar{Z}_{\gamma\rho i}$ exists for all $i \in \mathcal{S}$ almost everywhere on $[0, \rho_0]$.

Fix a $\rho \in (0, \rho_0]$ such that $\frac{d}{d\rho}\bar{Z}_{\gamma\rho i}$ exists for all $i \in \mathcal{S}$. Obviously, $\bar{Z}_{\gamma\rho i}$, $i = 1, \dots, s$, satisfy the GARI's (23). Let

$$\bar{\Delta}_{\gamma\rho i} := \frac{d}{d\rho}\bar{Z}_{\gamma\rho i}; \qquad \forall i \in \mathcal{S}.$$

Then, $\bar{\Delta}_{\gamma\rho i} \geq 0$ for each $i \in \mathcal{S}$, and they further satisfy the following coupled Lyapunov equations, for $i = 1, \dots, s$:

$$\bar{A}'_{F\gamma}(i)\bar{\Delta}_{\gamma\rho i} + \bar{\Delta}_{\gamma\rho i}\bar{A}_{F\gamma}(i) + \sum_{\substack{j=1 \\ j\neq i}}^{s}(\lambda_{ij}\bar{Z}_j + \rho\lambda_{ij}\bar{\Delta}_{\gamma\rho j}) + \sum_{j=1}^{s}\lambda_{ij}\bar{\Delta}_{\gamma\rho j} = 0,$$

where

$$\bar{A}_{F\gamma}(i) := A(i) - (B(i)R^{-1}(i)B'(i) - \frac{1}{\gamma^2}D(i)D'(i))\bar{Z}_{\gamma\rho i}; \quad i = 1, \dots, s.$$

This implies the mean-square stability of the jump linear system (25).

Note that $\gamma_1^2 Q_0(i) \geq Z_{\gamma_1 i}$ since $\gamma_1 > \gamma_\infty^*$. Then, by choosing ρ sufficiently small, the solutions $\bar{Z}_{\gamma\rho i}$, $i = 1, \ldots, s$, can be made to satisfy condition (18). For this value of ρ, by an argument similar to that used in the proof of Theorem (2), we have that the jump linear system (24) is mean-square stable. The optimality of the strategy $\bar{\mu}_{\gamma\infty}^*$ follows easily from a "completion of squares" argument as in the case of the preceding theorem, and this completes the proof of the corollary. □

Remark 3 Let $\bar{\gamma}_{i\infty}^* > 0$ be the H^∞-optimal performance level of the deterministic linear system with coefficient matrices $\{A_i + (1/2)\lambda_{ii} I, B_i, D_i, Q_i, R_i\}$, $i = 1, \ldots, s$. Then, we have $\gamma_\infty^* \geq \bar{\gamma}_{i\infty}^*$, $i = 1, \ldots, s$. □

4 Imperfect State Measurements

The finite-horizon case

Before presenting the solution to the imperfect state measurements case, as formulated in Section 2, we introduce a set of coupled generalized Riccati differential inequalities (GRDI's) defined in forward time:

$$\dot{\Theta}_i + A'(t,i)\Theta_i + \Theta_i A(t,i) + \Theta_i D(t,i) D'(t,i)\Theta_i - C'(t,i) N^{-1}(t,i) C(t,i) + \frac{1}{\gamma^2} Q(t,i) + \sum_{j=1}^{s} \lambda_{ij}(t)\Theta_j \leq 0; \quad \Theta_i(t_0) = Q_0(i) \quad i = 1, \ldots, s \tag{28}$$

Furthermore, we introduce a set $\hat{\Gamma}_I \subset \mathcal{R}$ by:

$$\begin{aligned}\hat{\Gamma}_I := \{\gamma > 0 : \ & \text{the set of coupled GRDE's (12) admits a set of} \\ & \text{nonnegative definite solutions } Z_i(t),\ i = 1, \ldots, s, \text{ on } [t_0, t_f], \\ & \text{and the set of coupled GRDI's (28) admits a set of nonnegative} \\ & \text{definite solutions } \Theta_i(t),\ i = 1, \ldots, s, \text{ on } [t_0, t_f], \text{ such that} \\ & \gamma^2 \Theta_i(t) - Z_i(t) > 0 \text{ for all } t \in [t_0, t_f] \text{ and } i \in \mathcal{S}.\}\end{aligned} \tag{29}$$

Let $\hat{\gamma}_I$ be the infimum of the set Γ_I:

$$\hat{\gamma}_I := \inf \hat{\Gamma}_I \tag{30}$$

Let $\Theta(t, \theta(t))$ be defined on $[t_0, t_f]$ as:

$$\Theta(t, \theta(t)) \quad := \quad \Theta_i(t) \qquad\qquad \text{if } \theta(t) = i$$

when $\Theta_i(t)$, $i = 1, \ldots, s$, exist on $[t_0, t_f]$.

Then, we have the following sufficiency result:

Theorem 3 *Consider the piecewise-deterministic soft-constrained zero-sum differential game formulated by (1)–(7), and let Assumptions 1–4 hold. If $\hat{\gamma}_I < +\infty$, then for $\gamma > \hat{\gamma}_I$, the upper value $\bar{V}$ of the zero-sum game is 0, and a minimax strategy for P1 that attains this upper value is $\mu^*_{I\gamma}$:*

$$u^*_{I\gamma}(t) = \mu^*_{I\gamma}(t, y_{[t_0,t]}, \theta_{[t_0,t]}) = -R^{-1}(t,\theta(t))B'(t,\theta(t))Z(t,\theta(t))\,\hat{x}(t) \tag{31}$$

where $\hat{x}$ is generated by the filter (observer):

$$\begin{aligned}\dot{\hat{x}} \;=\;& (A(t,\theta(t)) - (B(t,\theta(t))R^{-1}(t,\theta(t))B'(t,\theta(t)) - \frac{1}{\gamma^2}D(t,\theta(t)) \\ & \cdot D'(t,\theta(t)))Z(t,\theta(t)))\hat{x} + \gamma^2(\gamma^2\Theta(t,\theta(t)) - Z(t,\theta(t)))^{-1}C'(t,\theta(t)) \\ & \cdot N^{-1}(t,\theta(t))(y - C(t,\theta(t))\,\hat{x}); \qquad \hat{x}(t_0) = 0 \end{aligned} \tag{32}$$

Hence, we have the relationship $\gamma^*_I \leq \hat{\gamma}_I$.

Proof Fix a $\gamma > \hat{\gamma}_I$. Substitution of the minimax strategy $\mu^*_{I\gamma}$ into the original system (1) and the cost function (6), leads to the following system and cost function, in terms of $\xi := [x', \hat{x}']'$:

$$\begin{aligned}\dot{\xi} =& \begin{bmatrix} A(t,\theta(t)) & -B(t,\theta(t))K(t,\theta(t)) \\ G(t,\theta(t))C(t,\theta(t)) & F(t,\theta(t)) \end{bmatrix}\xi \\ & + \begin{bmatrix} D(t,\theta(t)) \\ \gamma^2\Xi(t,\theta(t))C'(t,\theta(t))N^{-1}(t,\theta(t))E(t,\theta(t)) \end{bmatrix} w \\ =:& \; \bar{A}(t,\theta(t))\xi + \bar{D}(t,\theta(t))w \end{aligned} \tag{33}$$

$$\begin{aligned} J_\gamma(\mu^*_{I\gamma},\delta) = \boldsymbol{E}\{ & \int_{t_0}^{t_f} (\xi(t)' \begin{bmatrix} Q(t,\theta(t)) & 0 \\ 0 & K'(t,\theta(t))R(t,\theta(t))K(t,\theta(t)) \end{bmatrix} \xi(t) \\ & -\gamma^2 w'(t)w(t))\,dt + |\xi(t_f)|^2_{\begin{bmatrix} Q_f(\theta(t_f)) & 0 \\ 0 & 0 \end{bmatrix}} \\ & -\gamma^2|\xi(t_0)|^2_{\begin{bmatrix} Q_0(\theta(t_0)) & 0 \\ 0 & 0 \end{bmatrix}} \end{aligned} \tag{34}$$

where

$$\begin{aligned} \Xi(t,\theta(t)) &:= \gamma^2\Theta(t,\theta(t)) - Z(t,\theta(t)) \\ K(t,\theta(t)) &:= R^{-1}(t,\theta(t))B'(t,\theta(t))Z(t,\theta(t)) \\ G(t,\theta(t)) &:= \gamma^2\Xi^{-1}(t,\theta(t))C'(t,\theta(t))N^{-1}(t,\theta(t)) \\ F(t,\theta(t)) &:= A(t,\theta(t)) - (B(t,\theta(t))R^{-1}(t,\theta(t))B'(t,\theta(t)) \\ & \quad -\frac{1}{\gamma^2}D(t,\theta(t))D'(t,\theta(t)))Z(t,\theta(t)) - G(t,\theta(t))C(t,\theta(t)) \end{aligned}$$

To complete the proof of the theorem, we only need to show that

$$\sup_{\delta\in\mathcal{D}} J_\gamma(\mu_{I\gamma},\delta) \leq 0.$$

Introduce the matrix-valued function

$$\Sigma(t,i) := \begin{bmatrix} \gamma^2\Theta(t,i) & -\Xi(t,i) \\ -\Xi(t,i) & \Xi(t,i) \end{bmatrix}; \qquad t \in [t_0, t_f],\ i \in \mathcal{S}, \tag{35}$$

and denote $\Sigma(t,i)$ by $\Sigma_i(t)$, for $i \in \mathcal{S}$. It can be shown, by extensive but straightforward algebraic manipulations, that the matrices Σ_i, $i = 1, \ldots, s$, satisfy the following differential inequalities:

$$\dot{\Sigma}_i + \Sigma_i \bar{A}(t,i) + \bar{A}'(t,i)\Sigma_i + \frac{1}{\gamma^2}\Sigma_i \bar{D}(t,i)\bar{D}'(t,i)\Sigma_i + \sum_{j=1}^{s} \lambda_{ij}\Sigma_j$$
$$+ \begin{bmatrix} Q(t,i) & 0 \\ 0 & Z(t,i)B(t,i)R^{-1}(t,i)B'(t,i)Z(t,i) \end{bmatrix} \leq 0; \quad i \in \mathcal{S}. \tag{36}$$

Let $\mathcal{A}$ be the extended generator [25] associated with the jump linear system (33). Then, we have

$$\mathcal{A}(\xi'(t)\Sigma(t,\theta(t)=i)\xi(t)) = \xi'(t)\dot{\Sigma}(t,i)\xi(t) + \sum_{j=1}^{s}\lambda_{ij}\xi'(t)\Sigma(t,j)\xi(t)$$
$$+2\xi'(t)\Sigma(t,i)(\bar{A}(t,i)\xi + \bar{D}(t,i)w).$$

Thus, by a "completion of squares" argument, the cost function $J_\gamma(\mu^*_{I\gamma}, \delta)$ can be bounded above by zero, for any strategy δ of P2:

$$J_\gamma(\mu^*_{I\gamma},\delta) = J_\gamma(\mu^*_{I\gamma},\delta) + \boldsymbol{E}\{\xi'(t_0)\Sigma(t_0,\theta(t_0))\xi(t_0) - \xi'(t_f)\Sigma(t_f,\theta(t_f))\xi(t_f)$$
$$+ \int_{t_0}^{t_f} \mathcal{A}(\xi'(t)\Sigma(t,\theta(t))\xi(t))\,dt\} = \boldsymbol{E}\{\int_{t_0}^{t_f} -\gamma^2|w(t) - \frac{1}{\gamma^2}\bar{D}'\Sigma(t,\theta(t))$$
$$\cdot\xi(t)|^2\,dt + |\xi(t_f)|^2_{\begin{bmatrix} Q_f(\theta(t_f)) & 0 \\ 0 & 0 \end{bmatrix}} - \gamma^2|\xi(t_0)|^2_{\begin{bmatrix} Q_0(\theta(t_0)) & 0 \\ 0 & 0 \end{bmatrix}}$$
$$+\xi'(t_0)\Sigma(t_0,\theta(t_0))\xi(t_0) - \xi'(t_f)\Sigma(t_f,\theta(t_f))\xi(t_f)\}$$
$$\leq \boldsymbol{E}\{-|\xi(t_f)|^2_{\begin{bmatrix} \Xi(t_f,\theta(t_f)) & -\Xi(t_f,\theta(t_f)) \\ -\Xi(t_f,\theta(t_f)) & \Xi(t_f,\theta(t_f)) \end{bmatrix}}$$
$$+\gamma^2|\xi(t_0)|^2_{\begin{bmatrix} 0 & -\Xi(t_0,\theta(t_0)) \\ -\Xi(t_0,\theta(t_0)) & \Xi(t_0,\theta(t_0)) \end{bmatrix}}\} \leq 0$$

The last inequality follows from the fact that $\hat{x}(t_0) = 0$. It, in turn, implies that the proposed strategy $\mu^*_{I\gamma}$ guarantees the upper value 0 for P1, and hence it is minimax. This completes the proof of the theorem. $\square$

Remark 4 Let $\bar{\gamma}^*_{Ii} > 0$ be the finite-horizon (imperfect-state) H^∞-optimal performance level of the deterministic linear system with coefficient matrices $\{A_i + (1/2)\lambda_{ii}I, B_i, D_i, C_i, E_i, Q_i, R_i\}$, $i = 1, \ldots, s$. Then, we have $\gamma^*_I \geq \bar{\gamma}^*_{Ii}$, $i = 1, \ldots, s$. $\square$

Theorem 3 provides only a sufficient condition for a minimax strategy for P1 when $\gamma > \gamma_I^*$. In other words, it leaves open the possibility that $\hat{\gamma}_I$ might be strictly larger than γ_I^*. In the next theorem, we present a necessary condition for the upper value of the game to be 0.

Theorem 4 *Consider the piecewise-deterministic soft-constrained zero-sum differential game formulated by (1)–(7), and let Assumptions 1–4 hold. For any $\gamma > \gamma_I^*$, if there exists a minimax strategy μ_I that guarantees a cost of 0 for P1, then the set of coupled GRDE's (12) admits nonnegative definite solutions $Z_i(t)$, $i = 1, \dots, s$, on $[t_0, t_f]$. Furthermore, for each $T \in [t_0, t_f]$, the following set of coupled backward GRDE's:*

$$\begin{aligned} &\dot{\Theta}_i^T + A'(t,i)\Theta_i^T + \Theta_i^T A(t,i) + \Theta_i^T D(t,i)D'(t,i)\Theta_i^T - C'(t,i)N^{-1}(t,i) \\ &\quad \cdot C(t,i) + \frac{1}{\gamma^2}Q(t,i) + \sum_{j=1}^{s} \lambda_{ij}(t)\Theta_j^T = 0; \quad \Theta_i^T(T) = \frac{1}{\gamma^2}Z_i(T) \ i \in \mathcal{S} \end{aligned} \tag{37}$$

admits symmetric solutions $\Theta_i^T(t)$, $i = 1, \dots, s$, on $[t_0, T]$ such that

$$\Theta_i^T(t_0) < Q_0(i); \qquad i = 1, \dots, s$$

Proof Fix a $\gamma > \gamma_I^*$; then the set of coupled GRDE's (12) must admit nonnegative definite solutions on $[t_0, t_f]$, since this is a necessary condition for the game to have upper value 0 when P1 has access to full state information.

Note that the minimax strategy μ_I must satisfy the following condition:

$$\mu_I(t, y_{[t_0,t]}, \theta_{[t_0,t]}) = 0, \qquad \text{if} \quad y_{[t_0,t]} \equiv 0.$$

Otherwise, P2 can simply choose $x_0 = 0$ and $w(t) \equiv 0$ to achieve a positive cost.

By the assumption of the theorem, we must have

$$\sup_{\delta \in \mathcal{D}} J_\gamma(\mu_I, \delta) \leq 0.$$

Fix any $T \in [t_0, t_f]$, and introduce a subclass, $\mathcal{D}_1$, of strategies for P2 that satisfy the following condition:

$$\begin{aligned} w(t) &= \frac{1}{\gamma^2}D'(t,\theta(t))Z(t,\theta(t))x(t); \qquad t \in [T, t_f] \\ w(t) &= -E'(t,\theta(t))N^{-1}(t,\theta(t))C(t,\theta(t))x(t) \\ &\quad + D'(t,\theta(t))v(t, x_{[t_0,t]}, \theta_{[t_0,t]}); \quad t \in [t_0, T] \end{aligned}$$

where x_0 and v are to be determined later. Obviously, any strategy in $\mathcal{D}_1$ results in $y_{[t_0,T]} \equiv 0$, and thus $u_{[t_0,T]} \equiv 0$ when $u(t)$ is generated by the strategy μ_I.

Fix any $\delta_1 \in \mathcal{D}_1$. Then, we have

$$J_\gamma(\mu_I, \delta_1) = \boldsymbol{E}\{\int_{t_0}^{T} (|x(t)|^2_{Q(t,\theta(t)) - \gamma^2 C'(t,\theta(t)) N^{-1}(t,\theta(t)) C(t,\theta(t))}$$
$$-\gamma^2 |v(t)|^2_{D'(t,\theta(t)) D(t,\theta(t))}) \, dt - \gamma^2 |x_0|^2_{Q_0(\theta(t_0))} + |x(t_f)|^2_{Q_f(\theta(t_f))}$$
$$+ \int_T^{t_f} (|x(t)|^2_{Q(t,\theta(t)) - (1/\gamma^2) Z(t,\theta(t)) D'(t,\theta(t)) D(t,\theta(t)) Z(t,\theta(t))} +$$
$$|u(t)|^2_{R(t,\theta(t))}) \, dt\} = \boldsymbol{E}\{\int_{t_0}^{T} (|x(t)|^2_{Q(t,\theta(t)) - \gamma^2 C'(t,\theta(t)) N^{-1}(t,\theta(t)) C(t,\theta(t))}$$
$$-\gamma^2 |v(t)|^2_{D'(t,\theta(t)) D(t,\theta(t))}) \, dt - \gamma^2 |x_0|^2_{Q_0(\theta(t_0))} + |x(T)|^2_{Z(T,\theta(T))}$$
$$+ \int_T^{t_f} |u(t) + B'(t,\theta(t)) Z(t,\theta(t)) x(t)|^2_{R(t,\theta(t))} \, dt\}$$
$$\geq \boldsymbol{E}\{\int_{t_0}^{T} (|x(t)|^2_{Q(t,\theta(t)) - \gamma^2 C'(t,\theta(t)) N^{-1}(t,\theta(t)) C(t,\theta(t))}$$
$$-\gamma^2 |v(t)|^2_{D'(t,\theta(t)) D(t,\theta(t))}) \, dt - \gamma^2 |x_0|^2_{Q_0(\theta(t_0))} + |x(T)|^2_{Z(T,\theta(T))}\}$$

where the second equality follows from a "completion of squares" argument.

The following inequalities then follow:

$$\begin{aligned} 0 \quad \geq \quad & \sup_{\delta \in \mathcal{D}} J_\gamma(\mu_I, \delta) \geq \sup_{\delta_1 \in \mathcal{D}_1} J_\gamma(\mu_I, \delta_1) \geq \sup_{\delta_1 \in \mathcal{D}_1} \boldsymbol{E}\{|x(T)|^2_{Z(T,\theta(T))} \\ & -\gamma^2 |x_0|^2_{Q_0(\theta(t_0))} + \int_{t_0}^{T} (|x(t)|^2_{Q(t,\theta(t)) - \gamma^2 C'(t,\theta(t)) N^{-1}(t,\theta(t)) C(t,\theta(t))} \\ & -\gamma^2 |v(t)|^2_{D'(t,\theta(t)) D(t,\theta(t))}) \, dt\} \end{aligned}$$

Note that the solution $\Theta_i^T(t)$, $i = 1, \ldots, s$, to GRDE's (37) cannot explode to $-\infty$ on $[t_0, T]$, because if they do then the supremum in the last inequality would be $+\infty$, which leads to a clear contradiction.

When the GRDE's (37) admit symmetric solutions on $[t_0, t_f]$, the following inequality holds (again by completion of squares):

$$\begin{aligned} 0 \quad \geq \quad & \sup_{\delta_1 \in \mathcal{D}_1} \boldsymbol{E}\{\int_{t_0}^{T} -\gamma^2 |D(t,\theta(t))(v(t) + \Theta^T(t,\theta(t)) x(t))|^2) \, dt \\ & -\gamma^2 |x_0|^2_{Q_0(\theta(t_0)) - \Theta^T(t_0,\theta(t_0))}\}. \end{aligned}$$

Then, necessarily, $Q_0(i) \geq \Theta_i^T(t_0)$ for each $i \in \mathcal{S}$. The strict inequality follows because the above holds for all $\gamma > \gamma_I^*$.

This completes the proof of the theorem. $\square$

The infinite-horizon case

Now, we turn to study the infinite-horizon case as formulated in Section 2.

The following results are the counterparts of those in the finite-horizon case. Introduce a set of coupled generalized algebraic Riccati inequalities (GARI's):

$$A'(i)\Theta_i + \Theta_i A(i) + \Theta_i D(i)D'(i)\Theta_i - C'(i)N^{-1}(i)C(i) + \frac{1}{\gamma^2}Q(i) + \sum_{j=1}^{s}\lambda_{ij}\Theta_j \leq 0 \quad i = 1,\ldots,s \tag{38}$$

Define a set $\hat{\Gamma}_{I\infty} \subset \mathcal{R}$ by:

$$\hat{\Gamma}_{I\infty} := \{\gamma > 0 : \text{ the set of coupled GARE's (17) admits a set of nonnegative definite solutions } Z_i,\ i = 1,\ldots,s, \text{ and the set of coupled GARI's (38) admits a set of nonnegative definite solutions } \Theta_i,\ i = 1,\ldots,s, \text{ such that } \gamma^2\Theta_i - Z_i > 0 \text{ for all } i \in \mathcal{S}.\} \tag{39}$$

Furthermore, let $\hat{\gamma}_{I\infty}$ be defined as:

$$\hat{\gamma}_{I\infty} := \inf \hat{\Gamma}_{I\infty}, \tag{40}$$

and $\Theta(\theta(t))$ be defined as:

$$\Theta(\theta(t)) \quad := \quad \Theta_i \qquad\qquad \text{if} \quad \theta(t) = i$$

whenever Θ_i $i = 1,\ldots,s$, exist.

Then, we have the following sufficiency result as the counterpart of Theorem 3:

Theorem 5 *Consider the piecewise-deterministic soft-constrained zero-sum differential game formulated by (1)–(7) in the infinite-horizon case, with $t_f \to \infty$ and $t_0 \to -\infty$, as well as when $t_f = \infty$ and $t_0 = -\infty$, and with system matrices A, B, D, C, E, Q, R, Λ being time-invariant, $Q_f(.) = 0$, and $x(t) \to 0$ as $t \to -\infty$. Let Assumptions 2–6 hold. If $\hat{\gamma}_{I\infty} < +\infty$, then for $\gamma > \hat{\gamma}_{I\infty}$ the upper value, $\bar{V}$, of the zero-sum game is 0, and a minimax strategy for P1 that attains this upper value is $\mu^*_{I\gamma\infty}$:*

$$u^*_{I\gamma\infty}(t) = \mu^*_{I\gamma\infty}(t, y_{(-\infty,t]}, \theta_{(-\infty,t]}) = -R^{-1}(\theta(t))B(\theta(t))Z(\theta(t))\,\hat{x}(t) \tag{41}$$

where $\hat{x}$ is generated by the filter (observer):

$$\dot{\hat{x}} = (A(\theta(t)) - (B(\theta(t))R^{-1}(\theta(t))B'(\theta(t)) - \frac{1}{\gamma^2}D(\theta(t))D'(\theta(t)))Z(\theta(t)))\hat{x} + \gamma^2(\gamma^2\Theta(\theta(t)) - Z(\theta(t)))^{-1}C'(\theta(t))N^{-1}(\theta(t))(y - C(\theta(t))\hat{x}) \tag{42}$$

with initial condition $\hat{x}(-\infty) = 0$. Hence, we have the relationship $\gamma^*_{I\infty} \leq \hat{\gamma}_{I\infty}$.

Proof Fix a $\gamma > \hat{\gamma}_{I\infty}$. Substitution of the minimax strategy $\mu^*_{I\gamma\infty}$ into the original system (1) and the cost function (6), leads to system dynamics and cost function that are the time-invariant versions of (33) and (34), expressed in terms of $\xi := [x', \hat{x}']'$.

Introduce a function $\Sigma(i)$, $i \in \mathcal{S}$, as the time-invariant counterpart of (35). Then, the matrices Σ_i, $i = 1, \ldots, s$, satisfy the following GARI's:

$$\Sigma_i \bar{A}(i) + \bar{A}'(i)\Sigma_i + \frac{1}{\gamma^2}\Sigma_i \bar{D}(i)\bar{D}'(i)\Sigma_i + \sum_{j=1}^{s} \lambda_{ij}\Sigma_j$$
$$+ \begin{bmatrix} Q(i) & 0 \\ 0 & Z(i)B(i)R^{-1}(i)B'(i)Z(i) \end{bmatrix} \leq 0; \qquad i = 1, \ldots, s. \tag{43}$$

An application of the "completion of squares" argument similar to that in the finite-horizon case completes the proof of this theorem. □

Remark 5 Let $\bar{\gamma}^*_{Ii\infty} > 0$ be the infinite-horizon (imperfect-state) H^∞-optimal performance level of the deterministic linear system with coefficient matrices $\{A_i + (1/2)\lambda_{ii} I, B_i, D_i, C_i, E_i, Q_i, R_i\}$, $i = 1, \ldots, s$. Then, we have $\gamma^*_{I\infty} \geq \bar{\gamma}^*_{Ii\infty}$, $i = 1, \ldots, s$. □

A set of necessary conditions for $\gamma > \gamma^*_{I\infty}$ is stated below as the counterpart of Theorem 4.

Theorem 6 *Consider the piecewise-deterministic soft-constrained zero-sum differential game covered by Theorem 5. Let Assumptions 2–6 hold, and assume that $\gamma^*_I < +\infty$. For any $\gamma > \gamma^*_I$, if there exists a minimax strategy $\mu_{I\infty}$ that guarantees a cost of 0 for P1, then the set of coupled GARE's (17) admits nonnegative definite solutions Z_i, $i = 1, \ldots, s$. Furthermore, the following set of coupled backward GRDE's:*

$$\dot{\Theta}^\infty_i + A'(i)\Theta^\infty_i + \Theta^\infty_i A(i) + \Theta^\infty_i D(i)D'(i)\Theta^\infty_i - C'(i)N^{-1}(i)C(i)$$
$$+ \frac{1}{\gamma^2}Q(i) + \sum_{j=1}^{s} \lambda_{ij}\Theta^\infty_j = 0; \quad \Theta^\infty_i(0) = \frac{1}{\gamma^2}Z_i \quad i = 1, \ldots, s \tag{44}$$

admits symmetric solutions $\Theta^\infty_i(t)$, $i = 1, \ldots, s$, on $(-\infty, 0]$.

Proof Fix a $\gamma > \gamma^*_{I\infty}$; then the set of coupled GARE's (17) must admit nonnegative definite solutions, since this is a necessary condition for the game to have upper value 0 when P1 has access to full state information.

Note that the minimax strategy $\mu_{I\infty}$ must satisfy the following condition:

$$\mu_{I\infty}(t, y_{(-\infty,t]}, \theta_{(-\infty,t]}) = 0; \quad \text{if} \quad y_{(-\infty,t]} \equiv 0.$$

By the hypothesis of the theorem, we must have

$$\sup_{\delta \in \mathcal{D}} J_\gamma(\mu_I, \delta) \leq 0.$$

Take a subclass, $\mathcal{D}_1$, of strategies for P2 that satisfy the following condition on $(-\infty, 0]$:

$$w(t) = -E'(\theta(t))N^{-1}(\theta(t))C(\theta(t))x(t) + D(\theta(t))v(t, x_{(-\infty,t]}, \theta_{(-\infty,t]}).$$

Obviously, any strategy in $\mathcal{D}_1$ results in $y_{(-\infty,0]} \equiv 0$, and thus $u_{(-\infty,0]} \equiv 0$ when $u(t)$ is generated by the strategy μ_I.

Fix any $\delta_1 \in \mathcal{D}_1$; then we have

$$\begin{aligned}
J_\gamma(\mu_I, \delta_1) = \boldsymbol{E}\{ &\int_{-\infty}^{0} (|x(t)|^2_{Q(\theta(t))-\gamma^2 C'(\theta(t))N^{-1}(\theta(t))C(\theta(t))} \\
&-\gamma^2 |v(t)|^2_{D'(\theta(t))D(\theta(t))})\, dt + \int_0^\infty (|x(t)|^2_{Q(\theta(t))} + |u(t)|^2_{R(\theta(t))} \\
&-\gamma^2 |w(t)|^2)\, dt\} \geq \boldsymbol{E}\{ \int_{-\infty}^{0} (|x(t)|^2_{Q(\theta(t))-\gamma^2 C'(\theta(t))N^{-1}(\theta(t))C(\theta(t))} \\
&-\gamma^2 |v(t)|^2_{D'(\theta(t))D(\theta(t))})\, dt + |x(0)|^2_{Z(\theta(0))}\}
\end{aligned}$$

where the inequality follows from a "completion of squares" argument.

Hence,

$$\begin{aligned}
0 \quad \geq \quad \sup_{\delta_1 \in \mathcal{D}_1} \boldsymbol{E}\{ &\int_{-\infty}^{0} (|x(t)|^2_{Q(\theta(t))-\gamma^2 C'(\theta(t))N^{-1}(\theta(t))C(\theta(t))} \\
&-\gamma^2 |v(t)|^2_{D'(\theta(t))D(\theta(t))})\, dt + |x(0)|^2_{Z(\theta(0))}\}
\end{aligned}$$

Note that the solution $\Theta_i^\infty(t)$, $i = 1, \ldots, s$, to GRDE's (44) cannot explode to $-\infty$ on $[t, 0]$ for any $t < 0$, because if they do, then the supremum in the last inequality above must be $+\infty$, which is a contradiction. Hence, GRDE's (44) must admit solutions on $(-\infty, 0]$.

This completes the proof of the theorem. □

Remark 6 The well-known results for the deterministic H^∞-optimal control problem [24] can be obtained as a special case of the results in this paper by choosing $\mathcal{S}$ to be a singleton. Note that when $\mathcal{S}$ is a singleton there is no gap between the sufficiency and necessity conditions of Theorems 3 and 4, respectively (Theorems 5 and 6, in the infinite-horizon case). □

Remark 7 *Computational issues.* As shown in Theorem 5, the solution to the H^∞-control problem for jump linear systems involves the solution of a set of coupled GARE's and a set of coupled GARI's. The former can be solved using iteratively any standard Riccati solver, in view of the observation that the solution to (17) can be obtained as

$$Z_i = \lim_{k\to\infty} Z_i^{(k)}, \qquad i \in \mathcal{S},$$

where $Z_i^{(k)}$, $k = 0, 1, \ldots$, are recursively generated from:

$$(A'(i) + \frac{1}{2}\lambda_{ii} I) Z_i^{(k+1)} + Z_i^{(k+1)}(A(i) + \frac{1}{2}\lambda_{ii} I) - Z_i^{(k+1)}(B(i)R^{-1}(i)B'(i)$$
$$-\frac{1}{\gamma^2} D(i)D'(i))Z_i^{(k+1)} + Q(i) = -\sum_{\substack{j=1 \\ j\neq i}}^{s} \lambda_{ij} Z_j^{(k)}; \quad Z_i^{(0)} = 0 \quad i \in \mathcal{S}.$$

Note that the sequence thus generated is monotonically nondecreasing. For the latter, on the other hand, i.e. (38), along with the spectral radius condition $\gamma^2\Theta_i - Z_i > 0$, $i \in \mathcal{S}$, it is possible to obtain an equivalent linear matrix inequality (LMI), and thereby use an existing software package for LMI's [26]. □

We note that, as $\gamma \to \infty$, the filters (32) and (42) do not yield the conditional mean estimators for the case when the disturbances are white Gaussian noises. [1] This observation is in contrast with the fact that the deterministic H^∞-optimal controller yields the LQG controller as $\gamma \to +\infty$.

5 Examples

In this section, we consider three numerical examples in the infinite-horizon case to illustrate the theoretical findings of the previous two sections.

Example 1

Take a scalar piecewise deterministic system with a two-state Markov chain, where the system matrices are specified as follows:

$$A(1) = 1; \quad B(1) = 1; \quad D(1) = \begin{bmatrix} 1 & 0 \end{bmatrix}; \quad C(1) = 1;$$
$$E(1) = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad Q(1) = 1; \quad R(1) = 1; \quad Q_0(1) = 100;$$
$$A(2) = -2; \quad B(2) = -1; \quad D(2) = \begin{bmatrix} 1/2 & 0 \end{bmatrix}; \quad C(2) = -1;$$
$$E(2) = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad Q(2) = 1; \quad R(2) = 1; \quad Q_0(2) = 100.$$

[1] When the disturbances are white Gaussian noises, the conditional mean estimator is given by a standard Kalman filter whose gain is determined from the solution of a piecewise deterministic forward Riccati differential equation.

The infinitesimal generator of the Markov chain is given as

$$\Lambda = \begin{bmatrix} -6 & 6 \\ 1 & -1 \end{bmatrix}.$$

Consider first the perfect state measurements case. (In this case, the matrices $C(1)$, $C(2)$, $E(1)$ and $E(2)$ are irrelevant.) Using a particular search algorithm, we find that $\gamma_\infty^* = 0.7154$.

The quantities $\bar{\gamma}_{1\infty}^*$ and $\bar{\gamma}_{2\infty}^*$, on the other hand, are found to be

$$\bar{\gamma}_{1\infty}^* = 0.4472 \qquad \bar{\gamma}_{2\infty}^* = 0.1836.$$

Clearly, γ_∞^* is larger than both $\bar{\gamma}_{1\infty}^*$ and $\bar{\gamma}_{2\infty}^*$, which corroborates the statement of Remark 3.

Let $\gamma_{1\infty}^*$ denote the optimal performance level of system 1 under perfect state measurements without any switching, and $\gamma_{2\infty}^*$ denote the same for system 2. Then, these quantities can be computed to be

$$\gamma_{1\infty}^* = 1 \qquad \gamma_{2\infty}^* = 0.2236.$$

Note that here γ_∞^* lies between $\gamma_{1\infty}^*$ and $\gamma_{2\infty}^*$, but this is not a general rule (as we shall observe a different pattern in the next example).

Choose the desired performance level as $\gamma = 0.75$; then,

$$Z_1 = 1.061; \quad Z_2 = 0.3949.$$

A minimax strategy for P1 (or equivalently a controller that guarantees the performance level γ) is therefore

$$\begin{aligned} \mu_\infty^*(x(t), \theta(t)) &= -1.061x \qquad \theta(t) = 1 \\ &= 0.3949x \qquad \theta(t) = 2 \end{aligned}$$

Next, we consider the same system under imperfect state measurements. Using a particular search algorithm (that involves iterative Riccati and LMI solvers), we compute $\hat{\gamma}_{I\infty} = 0.8004$. Obviously, we have the relationship $0.8004 \geq \gamma_{I\infty}^* \geq 0.7154$, where the latter value is the threshold level above which the set of GARE's (17) admits positive solutions.

The quantities $\bar{\gamma}_{I1\infty}^*$ and $\bar{\gamma}_{I2\infty}^*$ are found to be

$$\bar{\gamma}_{I1\infty}^* = 0.4495 \qquad \bar{\gamma}_{I2\infty}^* = 0.1944,$$

which shows that $\gamma_{I\infty}^*$ is larger than both $\bar{\gamma}_{I1\infty}^*$ and $\bar{\gamma}_{I2\infty}^*$ — an observation that corroborates the statement of Remark 5.

Let $\gamma_{I1\infty}^*$ denote the optimal performance level of system 1 under imperfect state measurements without any switching, and $\gamma_{I2\infty}^*$ denote the same for system 2. Then, these quantities can be computed to be

$$\gamma_{I1\infty}^* = 2.732 \qquad \gamma_{I2\infty}^* = 0.22425,$$

where again $\hat{\gamma}_{I\infty}$ lies in between these two values.

Choose a desired performance level $\gamma = 0.85$; then,

$$Z_1 = 0.8495; \quad Z_2 = 0.3535; \quad \Theta_1 = 2.309; \quad \Theta_2 = 0.5723.$$

A minimax strategy for P1 (or equivalently a controller that guarantees the performance level γ) is

$$\begin{aligned} \mu^*_{I\infty}(y_{(-\infty,t]}, \theta_{(-\infty,t]}) &= -0.8495\hat{x} \qquad \theta(t) = 1 \\ &= 0.3535\hat{x} \qquad \theta(t) = 2 \end{aligned}$$

where $\hat{x}$ is generated by:

$$\begin{aligned} \dot{\hat{x}} &= 0.4438\hat{x} + 0.8825y; \qquad \theta(t) = 1 \\ &= -14.28\hat{x} - 12.05y; \qquad \theta(t) = 2 \end{aligned}$$

Example 2

We now have a two-dimensional piecewise deterministic system with a two-state Markov chain. The system matrices are chosen as

$$A(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad D(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$
$$C(1) = \begin{bmatrix} 2 & 1 \end{bmatrix}; \quad E(1) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix};$$
$$Q(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad R(1) = 1; \quad Q_0(1) = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix};$$
$$A(2) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}; \quad B(2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad D(2) = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix};$$
$$C(2) = \begin{bmatrix} -1 & 2 \end{bmatrix}; \quad E(2) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix};$$
$$Q(2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad R(2) = 1; \quad Q_0(2) = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix},$$

and infinitesimal generator of the Markov chain is taken to be

$$\Lambda = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}.$$

In the perfect state measurements case, using a particular search algorithm, we find that $\gamma^*_\infty = 2.736$. It is again strictly larger than the quantities $\bar{\gamma}^*_{1\infty} = 0.7559$ and $\bar{\gamma}^*_{2\infty} = 2.052$.

The quantities $\gamma^*_{1\infty}$ and $\gamma^*_{2\infty}$ are computed to be

$$\gamma^*_{1\infty} = 1.414 \qquad \gamma^*_{2\infty} = 2.384.$$

Note that here γ_∞^* is larger than both $\gamma_{1\infty}^*$ and $\gamma_{2\infty}^*$, which is different from the observation made in the previous example.

Pick the desired performance level as $\gamma = 2.8$; then,

$$Z_1 = \begin{bmatrix} 26.98 & 11.38 \\ 11.38 & 6.289 \end{bmatrix}; \quad Z_2 = \begin{bmatrix} 33.35 & 10.14 \\ 10.14 & 4.368 \end{bmatrix}.$$

A minimax strategy for P1 (guaranteeing the chosen performance level γ) is

$$\begin{aligned} \mu_\infty^*(x(t), \theta(t)) &= \begin{bmatrix} -11.38 & -6.289 \end{bmatrix} x \qquad \theta(t) = 1 \\ &= \begin{bmatrix} -23.20 & -5.774 \end{bmatrix} x \qquad \theta(t) = 2 \end{aligned}$$

In the imperfect state measurements case, using a particular search algorithm (that involves iterative Riccati and LMI solvers), the quantity $\hat{\gamma}_{I\infty}$ is found to be $\hat{\gamma}_{I\infty} = 6.094$. Obviously, $6.094 \geq \gamma_{I\infty}^* \geq 2.736$, where again the latter value is the threshold level above which the set of GARE's (17) admits positive solutions.

The quantities $\bar{\gamma}_{I1\infty}^*$ and $\bar{\gamma}_{I2\infty}^*$ are

$$\bar{\gamma}_{I1\infty}^* = 0.7559 \qquad \bar{\gamma}_{I2\infty}^* = 4.072,$$

and by Remark 5, we should have $\gamma_{I\infty}^* \geq 4.072$.

Furthermore, the quantities $\gamma_{I1\infty}^*$ and $\gamma_{I2\infty}^*$ are:

$$\gamma_{I1\infty}^* = 2.0242 \qquad \gamma_{I2\infty}^* = 6.110.$$

Choose the desired performance level as $\gamma = 6.5$; then,

$$Z_1 = \begin{bmatrix} 5.331 & 2.060 \\ 2.060 & 1.912 \end{bmatrix}; \quad Z_2 = \begin{bmatrix} 6.155 & 1.477 \\ 1.477 & 1.361 \end{bmatrix};$$

$$\Theta_1 = \begin{bmatrix} 0.8288 & -1.104 \\ -1.104 & 1.979 \end{bmatrix}; \quad \Theta_2 = \begin{bmatrix} 0.6572 & -0.4838 \\ -0.4838 & 0.5781 \end{bmatrix}.$$

A minimax strategy for P1 (guaranteeing the performance level γ) is

$$\begin{aligned} \mu_{I\infty}^*(y_{(-\infty,t]}, \theta_{(-\infty,t]}) &= \begin{bmatrix} -2.060 & -1.912 \end{bmatrix} \hat{x} \qquad \theta(t) = 1 \\ &= \begin{bmatrix} -4.677 & -0.1165 \end{bmatrix} \hat{x} \qquad \theta(t) = 2 \end{aligned}$$

where $\hat{x}$ is generated by:

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} -336.7 & -167.4 \\ -203.8 & -102.8 \end{bmatrix} \hat{x} + \begin{bmatrix} 168.4 \\ 100.9 \end{bmatrix} y; \qquad \theta(t) = 1 \\ &= \begin{bmatrix} 45.14 & -97.99 \\ 56.74 & -99.17 \end{bmatrix} \hat{x} + \begin{bmatrix} 48.49 \\ 49.74 \end{bmatrix} y; \qquad \theta(t) = 2 \end{aligned}$$

Example 3

We now have a three-dimensional piecewise deterministic system with a three-state Markov chain. The system matrices are in this case

$$A(1) = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -0.5 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad D(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix};$$

$$C(1) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}'; \quad E(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}'; \quad Q(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$B(1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad R(1) = 1; \quad Q_0(1) = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix};$$

$$A(2) = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}; \quad D(2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & 2 & -3 & 0 \end{bmatrix};$$

$$C(2) = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}'; \quad E(2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}'; \quad Q(2) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

$$B(2) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}; \quad R(2) = 1; \quad Q_0(2) = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix};$$

$$A(3) = \begin{bmatrix} 0 & -1 & -2 \\ 3 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}; \quad D(3) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{bmatrix};$$

$$C(3) = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}'; \quad E(3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}'; \quad Q(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$B(3) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}; \quad R(3) = 1; \quad Q_0(3) = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix},$$

and the infinitesimal generator of the Markov chain is taken to be

$$\Lambda = \begin{bmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

In the perfect state measurements case, using a particular search algorithm, we find that $\gamma_\infty^* = 24.55$. It is again strictly larger than the quantities $\bar{\gamma}_{1\infty}^* = 1.464$, $\bar{\gamma}_{2\infty}^* = 4.754$ and $\bar{\gamma}_{3\infty}^* = 1.848$.

The quantities $\gamma_{1\infty}^*$, $\gamma_{2\infty}^*$ and $\gamma_{3\infty}^*$ are found to be

$$\gamma_{1\infty}^* = 3.113 \qquad \gamma_{2\infty}^* = 10.26 \qquad \gamma_{3\infty}^* = 1.959.$$

Note that here γ_∞^* is larger than all the three threshold levels, which is different from the observation made in Example 1, but in line with the observation made in Example 2.

Pick a desired performance level $\gamma = 25$; then, a minimax strategy for P1 (guaranteeing the performance level γ) is

$$\begin{aligned}
\mu_\infty^*(x(t), \theta(t)) &= \begin{bmatrix} 2.448 & -2.774 & -7.456 \end{bmatrix} x \qquad \theta(t) = 1 \\
&= \begin{bmatrix} -2.726 & -9.017 & -9.080 \end{bmatrix} x \qquad \theta(t) = 2 \\
&= \begin{bmatrix} -5.302 & -4.286 & 0.4695 \end{bmatrix} x \qquad \theta(t) = 3
\end{aligned}$$

In the imperfect state measurements case, using a particular search algorithm (that involves iterative Riccati and LMI solvers), the quantity $\hat{\gamma}_{I\infty}$ is found to be $\hat{\gamma}_{I\infty} = 28.87$. Obviously, $28.87 \geq \gamma_{I\infty}^* > \gamma_\infty^* = 24.49$, where again the latter value is the threshold above which the set of GARE's (17) admits positive solutions.

The quantities $\bar{\gamma}_{I1\infty}^*$, $\bar{\gamma}_{I2\infty}^*$ and $\bar{\gamma}_{I3\infty}^*$ are

$$\bar{\gamma}_{I1\infty}^* = 1.522 \qquad \bar{\gamma}_{I2\infty}^* = 6.409 \qquad \bar{\gamma}_{I3\infty}^* = 5.580\,.$$

Again, $\gamma_{I\infty}^*$ is strictly larger than these three quantities, which corroborates the statement of Remark 5.

Furthermore, the quantities $\gamma_{I1\infty}^*$, $\gamma_{I2\infty}^*$ and $\gamma_{I3\infty}^*$ are:

$$\gamma_{I1\infty}^* = 5.445 \qquad \gamma_{I2\infty}^* = 57.13 \qquad \gamma_{I3\infty}^* = 8.722.$$

Note that in this case the optimal performance level $\gamma_{I\infty}^*$ is larger than the performance levels of first and third form systems, but less than that of the second form system.

Choose a desired performance level $\gamma = 29$; then, a minimax strategy for P1 (guaranteeing the performance level γ) is

$$\begin{aligned}
\mu_{I\infty}^*(y_{(-\infty,t]}, \theta_{(-\infty,t]}) &= \begin{bmatrix} 1.719 & -2.450 & -5.922 \end{bmatrix} \hat{x} \qquad \theta(t) = 1 \\
&= \begin{bmatrix} -3.694 & -7.680 & -6.010 \end{bmatrix} \hat{x} \qquad \theta(t) = 2 \\
&= \begin{bmatrix} -4.965 & -3.945 & 0.4184 \end{bmatrix} \hat{x} \qquad \theta(t) = 3
\end{aligned}$$

where $\hat{x}$ is generated by:

$$\dot{\hat{x}} = \begin{bmatrix} -25980 & -8658 & 0 \\ -58490 & -19500 & 1.003 \\ -10850 & -3619 & -5.916 \end{bmatrix} \hat{x} + \begin{bmatrix} 8660 \\ 19500 \\ 3616 \end{bmatrix} y; \qquad \theta(t) = 1$$

$$= \begin{bmatrix} -1934 & 476.7 & 8.010 \\ 4344 & 1103 & 12.17 \\ 12760 & -3187 & 1.556 \end{bmatrix} \hat{x} + \begin{bmatrix} 483.4 \\ -1087 \\ -3190 \end{bmatrix} y; \qquad \theta(t) = 2$$

$$= \begin{bmatrix} 11780 & 8.871 & 23580 \\ 75330 & 1.016 & 150700 \\ 32950 & 1.930 & -65910 \end{bmatrix} \hat{x} + \begin{bmatrix} -5895 \\ -37660 \\ 16480 \end{bmatrix} y; \qquad \theta(t) = 3$$

Note that in this example, even though the measurement is one-dimensional, there is not much difference between the achievable performance levels of the perfect and imperfect state cases, and hence the sufficiency result of Theorem 5 is almost necessary.

6 Conclusion

In this paper, we have studied a class of soft-constrained zero-sum differential game problems for jump linear systems with perfect as well as imperfect state measurements, in both finite and infinite horizons. Sufficient, as well as some necessary, conditions are presented to ensure the existence of a minimax strategy for P1 in all cases. Under these conditions, closed-form expressions for minimax strategies for P1 are obtained and are shown to attain the upper value 0 for the game. The results obtained have immediate applications to the H^∞-optimal control problem for jump linear systems, where an $\mathcal{L}_2$ gain type inequality is guaranteed on the closed-loop system.

Some challenging extensions of these results still remain. One of these would be to remove the existing gap between necessary and sufficient conditions presented here. Another one would be the problem where the form process $\theta(t)$ is not completely available for measurement. Yet another problem would be to study model simplification in this context, when some of the transitions are rare. All these are topics currently under study.

Appendix

Here, we provide some details that were omitted in the proof of Corollary 1. Let us define vec(M), for any symmetric matrix M, to be a vector whose elements are the lower triangular elements of M, or in simple mathematical terms:

$$\mathrm{vec}(M) = \begin{bmatrix} m_{11} \\ m_{21} \\ m_{22} \\ m_{31} \\ \vdots \\ m_{nn} \end{bmatrix}; \qquad M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}.$$

Consider a jump linear system with state dynamics

$$\dot{x} = A(\theta(t))x. \tag{45}$$

This system is mean-square stable if the following set of coupled Lyapunov equations admits a unique positive definite solution

$$A(i)'M_i + M_iA(i) + I + \sum_{j=1}^{s} \lambda_{ij} M_j = 0; \qquad i \in \mathcal{S}. \tag{46}$$

The above set of coupled Lyapunov equation can equivalently be viewed as a set of algebraic equations:

$$F(A(1), \ldots, A(s), \Lambda)\vec{m} + \vec{b} = 0,$$

where $F(A(1), \ldots, A(s), \Lambda)$ denotes the Jacobian of the LHS of (46), and

$$\vec{m} := \begin{bmatrix} \mathrm{vec}(M_1) \\ \vdots \\ \mathrm{vec}(M_s) \end{bmatrix}; \quad \vec{b} = \begin{bmatrix} \mathrm{vec}(I) \\ \vdots \\ \mathrm{vec}(I) \end{bmatrix}.$$

Then, the following lemma holds:

Lemma 1 *The jump linear system is mean-square stable if and only if the matrix $F(A(1), \ldots, A(s), \Lambda)$ is Hurwitz.*

Proof First, assume that the matrix $F(A(1), \ldots, A(s), \Lambda)$ is Hurwitz. Consider the following set of coupled Lyapunov differential equations:

$$\dot{M}_i^T + A(i)'M_i^T + M_i^T A(i) + I + \sum_{j=1}^{s} \lambda_{ij} M_j^T = 0; \quad M_i^T(T) = 0, \ i \in \mathcal{S} \tag{47}$$

for any $T > 0$. Then, clearly the solutions $M_i^T(t)$, $i = 1, \ldots, s$, are monotonically nondecreasing in T. Let the vector $\vec{m}^T(t)$ be defined as

$$\vec{m}^T(t) := \begin{bmatrix} \mathrm{vec}(M_1^T(t)) \\ \vdots \\ \mathrm{vec}(M_s^T(t)) \end{bmatrix}.$$

It satisfies the following differential equation:

$$\dot{\vec{m}}^T + F(A(1), \ldots, A(s), \Lambda)\vec{m}^T + \vec{b} = 0; \qquad \vec{m}^T(T) = 0.$$

By the stability of the matrix $F(A(1), \ldots, A(s), \Lambda)$, we have

$$\lim_{T \to \infty} \vec{m}^T(0) = -F(A(1), \ldots, A(s), \Lambda)^{-1}\vec{b}.$$

Then, clearly the matrices M_i, $i = 1, \ldots, s$, such that $\vec{m} = -F(A(1), \ldots, A(s), \Lambda)^{-1}\vec{b}$ provide the unique solutions to (46) and are nonnegative definite. This implies that the jump linear system (45) is mean-square stable.

On the other hand, if the jump linear system (45) is mean-square stable, then the set of coupled Lyapunov equations (46) admits unique positive definite solutions M_i, $i = 1, \ldots, s$. In turn, the set of coupled Lyapunov differential equations (47) must admit solutions $M_i^T(t)$, $i = 1, \ldots, s$, for each $T > 0$, which further satisfy the limits:

$$\lim_{T \to \infty} M_i^T(0) = M_i.$$

The above limit must also hold for any solution to the set of differential equations (47) with a terminal condition that is positive definite and upper bounded by M_i. Now introduce the matrices $\tilde{M}_i^T$, $i = 1, \ldots, s$, which satisfy the set of differential equations (47) but with a different terminal condition:

$$\tilde{M}_i^T(T) = (1/2)M_i; \qquad i \in \mathcal{S}.$$

Suppose that the matrix $F(A(1), \ldots, A(s), \Lambda)$ is not Hurwitz. Let $\vec{p}_0$ be a nonzero vector such that the following differential equation

$$\dot{\vec{p}}^T + F(A(1), \ldots, A(s), \Lambda)\vec{p}^T = 0; \quad \vec{p}^T(T) = \vec{p}_0$$

admits a solution $\vec{p}^T(t)$ which does not converge to 0 as $T \to \infty$. Let $\vec{p}_0$ and $\vec{p}^T$ correspond (through the vec operation) to symmetric matrices P_{i0}, $i \in \mathcal{S}$, and P_i^T, $i \in \mathcal{S}$, respectively. Clearly, the matrices P_i^T, $i \in \mathcal{S}$, satisfy the following set of coupled Lyapunov differential equations:

$$\dot{P}_i^T + A(i)' P_i^T + P_i^T A(i) + \sum_{j=1}^{s} \lambda_{ij} P_j^T = 0; \qquad P_i^T(T) = P_{i0}, \quad i \in \mathcal{S}.$$

For any scalar α, the matrices $\alpha P_i^T + \tilde{M}_i^T$, $i \in \mathcal{S}$, satisfy the set of coupled Lyapunov differential equations (47) with the terminal condition:

$$\alpha P_i^T(T) + \tilde{M}_i^T(T) = \alpha P_{i0} + (1/2)M_i, \qquad i = 1, \ldots, s.$$

For sufficiently small α, we have $0 \leq \alpha P_{i0} + (1/2)M_i \leq M_i$ for each $i \in \mathcal{S}$. This then implies that $\alpha P_i^T(t) + \tilde{M}_i^T(t)$ converges to M_i as $T \to \infty$, and hence, $P_i^T(t) \to 0$ for each $i \in \mathcal{S}$. This contradicts the choice of the vector $\vec{p}_0$. Hence, the matrix $F(A(1), \ldots, A(s), \Lambda)$ is Hurwitz.

This completes the proof of the lemma. $\square$

Given the above result, we introduce the following notation:

Definition 2 The matrix $F(A(1), \ldots, A(s), \Lambda)$ is the *stability matrix* for the jump linear system

$$\dot{x} = A(\theta(t))x$$

where $\theta(t)$ is a continuous time finite state Markov chain process with generator Λ and state space $\{1, \ldots, s\}$. $\diamond$

Now, given the mean-square stability of the jump linear system

$$\dot{x} = A_{F\gamma_1}(\theta(t))x,$$

where

$$A_{F\gamma_1}(i) := A(i) - (B(i)R^{-1}(i)B'(i) - \frac{1}{\gamma_1^2}D(i)D'(i))Z_{\gamma_1}(i),$$

its stability matrix $F(A_{F\gamma_1}(1), \ldots, A_{F\gamma_1}(s), \Lambda)$ is Hurwitz. A straightforward evaluation of the Jacobian of the RHS of (27) with respect to $(\mathrm{vec}(\bar{Z}_1)', \ldots, \mathrm{vec}(\bar{Z}_s)')$ at the point $(\rho, \bar{Z}_1, \ldots, \bar{Z}_s) = (0, Z_{\gamma_1}(1), \ldots, Z_{\gamma_1}(s))$ gives exactly the matrix $F(A_{F\gamma_1}(1), \ldots, A_{F\gamma_1}(s), \Lambda)$. Thus the Implicit Function Theorem can be applied to establish the existence of a $\rho_0 > 0$, such that the set of coupled GARE's (27) admits a minimal positive definite solution.

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