H^∞-Optimal Control of Singularly Perturbed Systems with Sampled-State Measurements*

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1 Introduction

One of the important recent developments in control theory has been the recognition of the close relationship that exists between H^∞-optimal control problems, (originally formulated in the frequency domain [1] [2], and then extended to state space formulations [3] [4] [5] [6] [7] [8]) and a class of linear-quadratic differential games [9] [10] [11] [12] [13], which has not only led to simpler derivations of existing results on the former, but also enabled us to develop worst-case (H^∞-optimal) controllers under various information patterns, such as (in addition to perfect and imperfect state measurements) delayed state and sampled state measurements [14] [15]. An up-to-date coverage of this relationship and the derivation of H^∞-optimal controllers under different information patterns can be found in the recent book [16], which also contains an extensive list of references on the topic.

It is by now well-known that in both finite- and infinite-horizon formulations ,under sampled-data state measurements, one can come arbitrarily close to the H^∞-optimal performance level by designing a dynamic controller, of the same order as that of the plant (which may also include the dynamics of the disturbance, if any), whose construction involves the solution of a parametrized differential or algebraic Riccati equation, subject to the condition that the system can be stabilized to attain the performance level on each sampling interval. The problem we address in this paper is the possible order reduction of this controller when the plant exhibits open-loop time-scale separation. In state space, such systems are commonly modeled using the mathematical framework of singular perturbations, with a small parameter, say ϵ, determining the degree of separation between the "slow" and "fast" modes of the system [17]. In this framework, our objective may be rephrased as one of obtaining "approximate" controllers which do not depend on the singular perturbation parameter ϵ, and proving that these approximate controllers can be used "reliably" on the original system when $\epsilon > 0$ is sufficiently small.

In an earlier paper, [18], we have already initiated a study on this prob-

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lem when the controller has access to full state information, where we use the framework of differential games. One of our results in this context has been that the H^∞-optimal performance does not show continuity at $\epsilon = 0$, in the sense that the performance of the original system as $\epsilon \to 0$ is not necessarily the same as the H^∞-optimal performance of the reduced-order (slow) system, even if the fast subsystem is stable. We have actually proven that the former is upper bounded by the maximum of the H^∞-optimal performances of the "slow" and "fast" subsystems, appropriately defined. We have also constructed composite controllers (from parametrized solutions of slow and fast games), independently of ϵ, which guarantee a desired achievable performance level for the full-order plant when ϵ is sufficiently small. In another paper, [19], we studied the same problem for the case when full state measurements are not available, which is the well-known four-block problem. In this case the composite controller exhibits a performance loss due to the lack of knowledge of the small singular perturbation parameter, which was not the case under perfect state measurements.

In the present paper, we develop counterparts of the results of [18] for the sampled-data measurements case. Some of our conclusions are similar in spirit to those of [18] (though details of proofs are much more involved), while some others are quite different. For a brief preview of our main results, let us introduce the quantity $\gamma^*_{SD}(\epsilon)$ to denote the H^∞-optimum performance of the full-order system under sampled-data state measurements, and the quantities γ_{SDs} and γ_{of} to denote the H^∞-optimum performances of (appropriately defined) reduced slow subsystems under sampled-data state measurements and fast subsystems under open-loop state measurements, respectively. We first show that $\gamma^*_{SD}(\epsilon = 0)$ is bounded above by $\max\{\gamma_{SDs}, \gamma_{of}\}$. We then obtain a "slow" controller, designed based on the slow subsystem only (though with the knowledge of γ_{of}) and independent of the singular perturbation parameter, under which the associated differential game has a bounded upper value, to achieve any desired achievable H^∞-performance level for the full-order problem.

The problem of designing controllers for singularly perturbed systems subject to unknown disturbances has been studied before in the literature, notably in papers [20], [21], [22], where the objective has been to obtain composite controllers that guarantee stability of the overall (possibly nonlinear) system. The main approach of the authors in these papers has involved the construction of appropriate Lyapunov functions, in terms of which a class of stabilizing controllers has been characterized. However, no optimality properties have been associated with these controllers, which is our main concern in this paper. Yet another paper that deals with uncertain (linear) systems which exhibit time-scale separation is [23], which obtains a two-frequency-scale decomposition for H^∞-disk problems, but does not address the issue of optimal controller design. There are a number of papers that have studied the problem of sampled-data H^∞-optimal control for regular systems; see, for example, [14] [15] [24] [25].

The balance of the present paper is organized as follows. In the next section (Section 2) we formulate the singularly perturbed H^∞-optimal control problem with sampled-data state measurements, and identify the associated linear-quadratic differential game. We also provide in that section the solution to the full-order problem, for both finite and infinite horizons. Then, we identify appropriate slow and fast subsystems, and define the associated differential games. In Section 3, we construct the "slow" sampled-data controller, and study its optimality as well as its robustness to order reduction. Section 4 presents some numerical results to illustrate the theory, and Section 5 provides a discussion on some immediate extensions. The paper ends with three Appendices, which provide the details of some of the derivations given in the main body of the paper.

2 Problem Formulation

The system under consideration, with slow and fast dynamics, is described in the standard "singularly perturbed" form by

$$\left\{ \begin{array}{rcll} \dot{x}_1 & = & A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u + D_1(t)w; & x_1(0) = 0 \\ \epsilon\dot{x}_2 & = & A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u + D_2(t)w; & x_2(0) = 0 \end{array} \right. \tag{1}$$

where $x' := (x_1', x_2')$ is the n-dimensional state vector, with x_1 of dimension n_1 and x_2 of dimension $n_2 := n - n_1$; u is the control input, and w is the disturbance, each belonging to appropriate ($\mathcal{L}^2$) Hilbert spaces $\mathcal{H}_x$, $\mathcal{H}_u$ and $\mathcal{H}_w$, respectively, defined on the time interval $[0, t_f]$. The control input u is generated by a sampled-data control policy μ_{SD}, according to

$$u(t) = \mu_{SD}(t, x(t_k),\, x(t_{k-1}),\, \ldots,\, x(t_1),\, x_0) \quad t_k \le t < t_{k+1} \tag{2}$$

where $0 < t_1 < \cdots t_K = t_f$ are the sampling times, $\mu(\cdot)$ is piecewise continuous in t and measurable in $x_0, x(t_1), \ldots, x(t_K)$. Let us denote the class of all these controllers by $\mathcal{M}_{SD}$. With this system, we associate the standard quadratic performance index

$$\begin{aligned} L(u,w) &= |x(t_f)|^2_{Q_f} + \int_0^{t_f} (|x(t)|^2_{Q(t)} + |u(t)|^2)\, dt \\ &\equiv |x(t_f)|^2_{Q_f} + \|x\|^2_Q + \|u\|^2; \quad Q_f \ge 0, \quad Q(.) \ge 0 \end{aligned} \tag{3}$$

where Q_f will show dependence on $\epsilon > 0$, as to be clarified later. Let us also introduce the notation $J(\mu_{SD}, w)$ to denote $L(u,w)$, with u given by (2). The H^∞-optimal control problem is the minimization of the quantity

$$\sup_{w \in \mathcal{H}_w} \{J(\mu_{SD}, w)\}^{1/2} / \|w\| \tag{4}$$

over all permissible controllers μ_{SD}, or if a minimum does not exist to find a controller μ_{SD} that will ensure a performance within a given neighborhood of the infimum of (4). Let us denote this infimum by $\gamma^*_{SD}(\epsilon)$, i.e.

$$\inf_{\mu_{SD}\in\mathcal{M}_{SD}} \sup_{w\in\mathcal{H}_w} \{J(\mu_{SD}, w)\}^{1/2}/\|w\| = \gamma^*_{SD}(\epsilon) \tag{5}$$

where we explicitly show the dependence of γ^*_{SD} on the singular perturbation parameter $\epsilon > 0$.

For each $\epsilon > 0$, we can associate a soft-constrained linear-quadratic differential game with this worst-case design problem (see [16]), which has the cost function

$$L_\gamma(u, w) = L(u, w) - \gamma^2\|w\|^2 \tag{6}$$

The performance level $\gamma^*_{SD}(\epsilon)$ in (5) is then the "smallest" value of $\gamma \geq 0$ under which the differential game with state equation (1) and cost function (6) has a bounded upper value, when u is chosen according to (2). We also know that for every fixed $\epsilon > 0$, and for each $\gamma > \gamma^*_{SD}(\epsilon)$, this differential game admits a saddle-point solution, with the saddle-point controller μ^*_{SD} being a linear feedback law that depends only on the most recent sampled value of the state (see [16], chapter 4).

Even though the problem formulated above has been solved completely for every $\epsilon > 0$, the computation of $\gamma^*_{SD}(\epsilon)$ and a corresponding H^∞-optimal or suboptimal controller for small values of $\epsilon > 0$ present serious difficulties, due to numerical stiffness. To remedy this, we pose in this paper the question of whether $\gamma^*_{SD}(\epsilon)$ and the H^∞-optimal controller can be determined for small value of $\epsilon > 0$ by solving well-behaved ϵ-independent smaller-order problems, as in the case of the singularly perturbed linear-quadratic regulator problem ([26]). Another question of interest is whether the fast subsystem dynamics can be neglected completely in the design of such controllers, and if so under what conditions.

Before studying these questions, we first present the solution to the full-order problem, under four basic assumptions:

Assumption 1 Q_f and $Q(.)$ in (3) are partitioned as

$$Q_f = \begin{bmatrix} Q_{f11} & \epsilon Q_{f12} \\ \epsilon Q_{f21} & \epsilon Q_{f22} \end{bmatrix}; \qquad Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$$

where in each case the 11-block is of dimension $n_1 \times n_1$, and the 22-block of dimension $n_2 \times n_2$.

Assumption 2 $A_{ij}(t)$, $Q_{ij}(t)$, $B_i(t)$, $D_i(t)$ $\quad (i = 1, 2 \;\; j = 1, 2)$ are continuously differentiable, and $\|A_{ij}\|$, $\|Q_{ij}\|$, $\|B_i\|$, $\|D_i\|$, $\|\dot{A}_{ij}\|$, $\|\dot{Q}_{ij}\|$, $\|\dot{B}_i\|$, $\|\dot{D}_i\|$ $(i = 1, 2 \;\; j = 1, 2)$ are bounded on $t \in [0, t_f]$.

Assumption 3 $A_{22}(t)$ and $Q_{22}(t)$ are invertible for all $t \in [0, t_f]$.

Assumption 4 $T_m := \min_{1\leq k\leq K} t_k - t_{k-1}$ is positive and ϵ-free.

Let us further introduce the notation $A(t)$ and $A_\epsilon(t)$, to denote the partitioned matrices

$$A(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}; \qquad A_\epsilon(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \frac{1}{\epsilon}A_{21}(t) & \frac{1}{\epsilon}A_{22}(t) \end{bmatrix}$$

where we take $\epsilon > 0$. Similarly, we introduce the partitioned matrices

$$B(t) := \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}; \qquad B_\epsilon(t) := \begin{bmatrix} B_1(t) \\ \frac{1}{\epsilon}B_2(t) \end{bmatrix}$$

$$D(t) := \begin{bmatrix} D_1(t) \\ D_2(t) \end{bmatrix}; \qquad D_\epsilon(t) := \begin{bmatrix} D_1(t) \\ \frac{1}{\epsilon}D_2(t) \end{bmatrix}$$

and define for each $\epsilon > 0$,

$$S(t;\gamma) = B(t)B'(t) - \frac{1}{\gamma^2}D(t)D'(t)$$
$$S_\epsilon(t;\gamma) = B_\epsilon(t)B'_\epsilon(t) - \frac{1}{\gamma^2}D_\epsilon(t)D'_\epsilon(t)$$

with the ij-th block of $S(t;\gamma)$ denoted by $S_{ij}(t;\gamma)$, $i,j = 1,2$.

We know from the existing theory on linear-quadratic differential games (see [27] Chapter 4 pp. 105, [16]) that for each $\epsilon > 0$, there exists a $\hat{\gamma}_{SD}(\epsilon) \geq 0$ such that for all $\gamma > \hat{\gamma}_{SD}(\epsilon)$ the zero-sum differential game described by (1) and (6) has a bounded upper value (which in this case is equal to *zero*, because the initial states are zero), as well as a saddle point. For all such γ, the following generalized Riccati differential equations do not have conjugate points in the given time intervals:

$$\dot{\tilde{Z}} + A'_\epsilon\tilde{Z} + \tilde{Z}A_\epsilon - \tilde{Z}S_\epsilon\tilde{Z} + Q = 0; \qquad \tilde{Z}(t_f) = Q_f \quad t \in [0, t_f] \tag{7}$$

$$\begin{gathered} \dot{\tilde{P}}_k + A'_\epsilon\tilde{P}_k + \tilde{P}_kA_\epsilon + \tilde{P}_k\tfrac{1}{\gamma^2}D_\epsilon D'_\epsilon\tilde{P}_k + Q = 0 \\ \tilde{P}_k(t_k) = \tilde{Z}(t_k) \quad t \in [t_{k-1}, t_k], \quad k = K, K-1, \dots, 1 \end{gathered} \tag{8}$$

and a saddle-point controller is given by

$$\begin{gathered} u^*_{SD}(t) = \mu^*_{SD}(t, x(t)) = -B'_\epsilon\tilde{Z}(t)\Phi_\epsilon(t, t_k)x(t_k) \\ t_k \leq t < t_{k+1}, \quad k = 0, 1, \dots, K \end{gathered} \tag{9}$$

where Φ_ϵ is the state transition function associated with the matrix

$$\tilde{F}(t) := A_\epsilon - [B_\epsilon B'_\epsilon - \frac{1}{\gamma^2}D_\epsilon D'_\epsilon]\tilde{Z}(t) \tag{10}$$

For $\gamma < \hat{\gamma}_{SD}(\epsilon)$, on the other hand, at least one of the generalized Riccati equations (7)–(8) has a conjugate point in the corresponding open interval and the game has infinite upper value under sampled-data information. The level $\hat{\gamma}_{SD}(\epsilon)$ is indeed the H^{∞}-optimal performance level (5).

Since GRDEs of the type above (7) and (8) have been studied extensively in a recent paper [18], we will apply those results directly without further study of the full-order GRDEs here.

To obtain the ϵ-free solution, we now introduce the slow and fast subsystems as in [18].

The slow subsystems and the associated soft-constrained games

To obtain the slow dynamics associated with (1), we let $\epsilon = 0$ and solve for x_2 (to be denoted $\bar{x}_2$) in terms of $x_1 =: x_s$, $u =: u_s$, $w =: w_s$, and under the working Assumption 3:

$$\bar{x}_2 = -A_{22}^{-1}(A_{21}x_s + B_2 u_s + D_2 w_s) \tag{11}$$

We also introduce the transformation (12) given below (as in [18], [19])

$$\begin{aligned} \hat{w}_s \;&=\; (\gamma^2 I - D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} D_2)^{1/2}[w_s - \frac{1}{\gamma^2} D_2'(A_{22}Q_{22}^{-1}A_{22}' \\ &\quad -\frac{1}{\gamma^2} D_2 D_2')^{-1}(B_2 u_s + (A_{21} - A_{22}Q_{22}^{-1}Q_{21})x_s)] \end{aligned} \tag{12}$$

which is valid under the condition

$$\gamma^2 I - D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} D_2 > 0 \tag{13}$$

Note that the dependence of $\hat{w}_s$ on u_s is allowed here since we are interested in the upper value of the game. Using this transformation in (1), (11) and (6), we arrive at the following LQ structure:

$$\dot{x}_s = A^{\square} x_s + B^{\square} u_s + D^{\square} \hat{w}_s; \quad x_s(0) = 0 \tag{14}$$

$$L_{\gamma s} = |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f} (|x_s|^2_{Q^{\square}} + 2x_s' P^{\square} u_s + |u_s|^2_{R^{\square}} - |\hat{w}_s|^2)\, dt \tag{15}$$

where

$$\begin{aligned} A^{\square}(\gamma) \;&:=\; A_{11} - A_{12}Q_{22}^{-1}Q_{21} - (A_{12}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_1 D_2')(A_{22}Q_{22}^{-1}A_{22}' \\ &\quad -\frac{1}{\gamma^2}D_2 D_2')^{-1}(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) \end{aligned} \tag{16}$$

$$B^{\square}(\gamma) \;:=\; B_1 - (A_{12}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_1 D_2')(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2 D_2')^{-1} B_2 \tag{17}$$

$$D^{\square}(\gamma) \;:=\; (D_1 - A_{12}A_{22}^{-1}D_2)(\gamma^2 I - D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} D_2)^{-1/2} \tag{18}$$

$$Q^{\square}(\gamma) \;:=\; Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (A_{21}' - Q_{12}Q_{22}^{-1}A_{22}')(A_{22}Q_{22}^{-1}A_{22}'$$

$$-\frac{1}{\gamma^2}D_2D_2')^{-1}(A_{21}-A_{22}Q_{22}^{-1}Q_{21}) \tag{19}$$

$$P^\square(\gamma) \quad := \quad (A_{21}'-Q_{12}Q_{22}^{-1}A_{22}')(A_{22}Q_{22}^{-1}A_{22}'-\frac{1}{\gamma^2}D_2D_2')^{-1}B_2 \tag{20}$$

$$R^\square(\gamma) \quad := \quad I+B_2'(A_{22}Q_{22}^{-1}A_{22}'-\frac{1}{\gamma^2}D_2D_2')^{-1}B_2 \tag{21}$$

The above is the equivalence form of the representation in [18] pp.35–36.

The GRDE associated with the above transformed game is[1]

$$\dot{Z}_s+\hat{A}_0'Z_s+Z_s\hat{A}_0-Z_s\hat{S}_0Z_s+\hat{Q}=0; \qquad Z_s(t_f)=Q_{f11} \tag{22}$$

where we have introduced

$$\hat{A}_0 := A^\square - B^\square R^{\square -1}P^{\square\prime} \qquad \hat{S}_0 := B^\square R^{\square -1}B^{\square\prime} - D^\square D^{\square\prime}$$
$$\hat{Q} := Q^\square - P^\square R^{\square -1}P^{\square\prime}$$

It is shown in [18] pp.36–37 that $\hat{A}$, $\hat{S}_0$ and $\hat{Q}$ can be rewritten as

$$\begin{aligned}
\hat{A}_0 &= A_{11}-A_{12}Q_{22}^{-1}Q_{21}-(S_{12}+A_{12}Q_{22}^{-1}A_{22}')(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1} \\
&\quad \cdot(A_{21}-A_{22}Q_{22}^{-1}Q_{21}) && (23)\\
\hat{S}_0 &= S_{11}+A_{12}Q_{22}^{-1}A_{12}'-(S_{12}+A_{12}Q_{22}^{-1}A_{22}')(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1} \\
&\quad \cdot(S_{21}+A_{22}Q_{22}^{-1}A_{12}') && (24)\\
\hat{Q} &= Q_{11}-Q_{12}Q_{22}^{-1}Q_{21}+(A_{21}'-Q_{12}Q_{22}^{-1}A_{22}')(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1} \\
&\quad \cdot(A_{21}-A_{22}Q_{22}^{-1}Q_{21}) && (25)
\end{aligned}$$

Now, we introduce the following set:

$$\hat{\Gamma}_s \quad := \quad \{\gamma'>0 : \forall\gamma\geq\gamma',\ (13) \text{ is satisfied and } (22) \text{ has a bounded nonnegative definite solution, } Z_{s\gamma} \text{ over } [0,t_f]\} \tag{26}$$

and further define

$$\hat{\gamma}_s := \inf\{\gamma\in\hat{\Gamma}_s\} \tag{27}$$

On each of the sampling interval, the system is controlled by an open-loop control law. The "open-loop" GRDE associated with the kth sampling interval, where $k=1,2,\ldots,K$, is

$$\dot{P}_{sk}+A^{\square\prime}P_{sk}+P_{sk}A^\square+P_{sk}D^\square D^{\square\prime}P_{sk}+Q^\square=0; \quad P_{sk}(t_k)=Z_{s\gamma}(t_k) \tag{28}$$

We introduce the following sets, for $k=K,K-1,\ldots,1$:

$$\hat{\Gamma}_{sk} \quad := \quad \{\gamma'\in\hat{\Gamma}_s : \forall\gamma\geq\gamma',\ (13) \text{ is satisfied and } (28) \text{ admits a bounded nonnegative definite solution } P_{s\gamma k} \text{ over } [t_{k-1},t_k]\}$$

[1]Under condition (13), $Q^\square > 0$.

and define

$$\gamma_{sk} := \inf\{\gamma \in \hat{\Gamma}_{sk}\}, \quad k = K, K-1, \ldots, 1 \tag{29}$$

Now, we further define

$$\gamma_{SDs} := \max\{\gamma_{sK}, \ldots, \gamma_{s1}\} \tag{30}$$

We will see this γ_{SDs} plays an important role in our problem.

The fast subsystem and the associated soft-constrained game

Let $x_f := x_2 - \bar{x}_2$, $u_f := u - u_s$, $w_f := w - w_s$ and $\tau = \frac{t'-t}{\epsilon}$ where we take t to be frozen, and t' to vary on the same scale as t. We define the fast subsystem and the associated cost (as in the standard regulator problem; see [26]) by

$$\frac{d}{d\tau} x_f^t = A_{22}(t) x_f^t + B_2(t) u_f^t + D_2(t) w_f^t; \quad x_f^t(0) = x_f(t) \tag{31}$$

$$L_{\gamma f}^t = \int_0^\infty (|x_f^t|^2_{Q_{22}(t)} + |u_f^t|^2 - \gamma^2 |w_f^t|^2)\, d\tau \tag{32}$$

The "feedback" GARE associated with this infinite-horizon game, for each t, is

$$A_{22}'(t) Z_f + Z_f A_{22}(t) + Q_{22}(t) - Z_f S_{22}(t) Z_f = 0 \tag{33}$$

and the "open-loop" GARE is

$$A_{22}'(t) Z_{of} + Z_{of} A_{22}(t) + Q_{22}(t) + Z_{of} \frac{1}{\gamma^2} D_2(t) D_2(t)' Z_{of} = 0 \tag{34}$$

We now let γ_f^t denote the minimax disturbance attenuation bound for the H^∞-optimal control problem defined by (31)–(32) under closed-loop information, and γ_{of}^t denote the same under open-loop information,[2] where we must have $\gamma_{of}^t \geq \gamma_f^t$. For every $\gamma > \gamma_f^t$, let $Z_{f\gamma}(t)$ be the minimal positive definite solution for (33) (and, for every $\gamma > \gamma_{of}^t$, let $Z_{of\gamma}(t)$ be the minimal positive solution to (34)). We now define

$$\gamma_f := \sup_{t \in [0, t_f]} \gamma_f^t; \quad \gamma_{of} := \sup_{t \in [0, t_f]} \gamma_{of}^t$$

Then, for every $\gamma > \gamma_{of}$, the GAREs (33) and (34) both admit positive definite solutions for all $t \in [0, t_f]$. Let

$$\bar{\gamma}_{SD} := \max\{\gamma_{of}, \gamma_{SDs}\} \tag{35}$$

We will shortly see that this value plays an important role in our problem.

[2] To ensure that $\gamma_{of}^t < \infty$, it will be sufficient to take $A_{22}(t)$ to be Hurwitz.

The infinite-horizon case

We now turn to the infinite-horizon case (i. e., as $t_f \to \infty$, as well as when $t_f = \infty$). We take A, B, D, Q to be time-invariant, and $Q_f = 0$. Furthermore, we assume that (A_ϵ, B_ϵ) is controllable, and (A_ϵ, Q) is observable. Because of the nature of the infinite-horizon game, the GRDEs in (8) will be identical, except for the length of the corresponding sampling interval. We let

$$T := \sup_{k=0,1,\ldots} t_{k+1} - t_k \tag{36}$$

Then, for each $\epsilon > 0$, such that $\exists \hat{\gamma}_{SD\infty} > 0$, $\forall \gamma > \hat{\gamma}_{SD\infty}$, the following GARE and GRDE have positive definite solutions:

$$A'_\epsilon \tilde{Z}_\infty + \tilde{Z}_\infty A_\epsilon - \tilde{Z}_\infty S_\epsilon \tilde{Z}_\infty + Q = 0 \tag{37}$$

$$\dot{\tilde{P}} + A'_\epsilon \tilde{P} + \tilde{P} A_\epsilon + \tilde{P} \frac{1}{\gamma^2} D_\epsilon D'_\epsilon \tilde{P} + Q = 0; \quad \tilde{P}(T) = \tilde{Z}_\infty, \quad t \in [0, T] \tag{38}$$

and hence the infinite-horizon soft-constrained game has a finite value. The upper value is achieved by the controller

$$u^*_{SD}(t) = \mu^{\infty}_{SD}(t, x) = -B'_\epsilon \tilde{Z}_\infty(\epsilon) \Phi(t, t_k) x(t_k), \quad t_k \le t < t_{k+1} \tag{39}$$

where Φ is the state transition function associated with the matrix

$$\tilde{F}_\infty := A_\epsilon - (B_\epsilon B'_\epsilon - \frac{1}{\gamma^2} D_\epsilon D'_\epsilon) \tilde{Z}_\infty \tag{40}$$

which is Hurwitz for all $\gamma > \hat{\gamma}_{SD\infty}$. For $\gamma < \hat{\gamma}_{SD\infty}$, either the GARE does not have a nonnegative definite solution or the GRDE has at least one conjugate point in $[0, T]$, and the game has infinite upper value. The level $\hat{\gamma}_{SD\infty}$ is again the one defined in (5).

Again, since GARE and GRDE of the type above (37) and (38) have been extensively studied in [18], we will not further discuss their direct derivation here.

As in the finite-horizon case, we study the derivation of (approximate) ϵ-free solutions, by solving the slow and fast subgames separately. By following steps similar to those in the finite-horizon case, we first decompose the system into slow and fast subsystems. The slow game is described by

$$\dot{x}_s = A^{\square} x_s + B^{\square} u_s + D^{\square} \hat{w}_s; \quad x_s(0) = 0 \tag{41}$$

$$L_{\gamma s} = \int_0^{\infty} (|x_s|^2_{Q^{\square}} + 2x'_s P^{\square} u_s + |u_s|^2_{R^{\square}} - |\hat{w}_s|^2)\, dt \tag{42}$$

under the condition (13) and the transformation (12), where $A^{\square}$, $B^{\square}$, $D^{\square}$, $Q^{\square}$, $P^{\square}$ and $R^{\square}$ are as defined before, with the only difference being that they are now time-invariant. The associated GARE is

$$\hat{A}'_0 \tilde{Z}_s + \tilde{Z}_s \hat{A}_0 - \tilde{Z}_s \hat{S}_0 \tilde{Z}_s + \hat{Q} = 0 \tag{43}$$

where $\hat{A}_0$, $\hat{S}_0$ and $\hat{Q}$ are defined as before, but now they are time-invariant.

Now, we introduce the set

$$\hat{\Gamma}_{s\infty} := \{\gamma' > 0 : \forall\gamma \geq \gamma',\ (13) \text{ is satisfied and } (43) \text{ has a minimal positive definite solution, } Z_{s\gamma}, \text{ and } \hat{A}_0 - \hat{S}_0 Z_{s\gamma} \text{ is Hurwitz. } \} \quad (44)$$

and further define[3]

$$\hat{\gamma}_{s\infty} := \inf\{\gamma \in \hat{\Gamma}_{s\infty}\} \quad (45)$$

For every $\gamma > \hat{\gamma}_{s\infty}$, let $Z_{s\gamma}$ be the minimal positive definite solution to (43).

To study the system behavior in between the sampling intervals, we consider only the sampling interval with the longest duration, T.

The GARE associated with it is:

$$\dot{P}_{sT} + A^{\square\prime} P_{sT} + P_{sT} A^{\square} + P_{sT} D^{\square} D^{\square\prime} P_{sT} + Q^{\square} = 0; \quad P_{sT}(T) = Z_{s\gamma} \quad (46)$$

Similar to the finite-horizon case, we introduce the following set:

$$\hat{\Gamma}_{sT} \quad := \quad \{\gamma' \in \hat{\Gamma}_{s\infty} : \forall\gamma \geq \gamma',\ (13) \text{ is satisfied and } (46) \text{ has a bounded nonnegative definite solution } P_{s\gamma T} \text{ over } [0, T]\}$$

and define

$$\gamma_{SDs\infty} := \inf\{\gamma \in \hat{\Gamma}_{sT}\}. \quad (47)$$

The fast part of the system is the same as in the finite-horizon case, where the coefficient matrices are now constants. The fast game is described by (31)–(32), and the GARE is the same as (33), and the "open-loop" GARE is the same as (34). We will use $\gamma_{f\infty}$ to denote the minimax disturbance attenuation bound (and $\gamma_{of\infty}$ for the open-loop case), and let $Z_{f\gamma}$ denote the minimal positive definite solution to (33) (and $Z_{of\gamma}$ for the open-loop case). We define, similar to (35),

$$\bar{\gamma}_{SD\infty} := \max\{\gamma_{of\infty}, \gamma_{SDs\infty}\} \quad (48)$$

This quantity will also play an important role in our analysis in the next section.

3 Main Results

Before we present the main results, we first introduce some useful matrices, defined in [18]:

$$U \quad \equiv \quad U_1 Z_{s\gamma} + U_2$$

[3] Again, for $\hat{\gamma}_{s\infty}$ to be finite, it will be sufficient to have the pair $(A^{\square}, B^{\square})$ controllable at $\gamma = \infty$ (see [16]), which is equivalent to having the pair $A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2)$ controllable.

$$
\begin{aligned}
V &\equiv V_1 Z_{s\gamma} + V_2 \\
U_1 &:= Q_{22}^{-1}A_{12}' - Q_{22}^{-1}A_{22}'(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(S_{21}+A_{22}Q_{22}^{-1}A_{12}') \\
U_2 &:= Q_{22}^{-1}Q_{21} + Q_{22}^{-1}A_{22}'(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(A_{21}-A_{22}Q_{22}^{-1}Q_{21}) \\
V_1 &:= -(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(S_{21}+A_{22}Q_{22}^{-1}A_{12}') \\
V_2 &:= (S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(A_{21}-A_{22}Q_{22}^{-1}Q_{21})
\end{aligned}
$$

We now introduce the ϵ-free sampled-data controller we will use in the full-order system to achieve a desired performance level:

$$
u_{SD}(t) = \mu_{SD}(t, x_{1\,k-1}) = -(B_1' Z_{s\gamma} + B_2' V)x_1^*(t), \quad t \in [t_{k-1}, t_k] \tag{49}
$$

where x_1^* is the solution to

$$
\dot{x}_1^* = (\hat{A}_0 - \hat{S}_0 Z_{s\gamma})x_1^*, \quad x_1^*(t_{k-1}) = x_{1\,k-1} \quad t \in [t_{k-1}, t_k] \tag{50}
$$

We introduce the maximizing disturbance corresponding to this control as

$$
w_{SD}(t) = \frac{1}{\gamma^2}(D_1' Z_{s\gamma} + D_2' V)x_1^*(t), \quad t \in [t_{k-1}, t_k] \quad k = K, \ldots, 1 \tag{51}
$$

The corresponding trajectory for x_2, obtained by setting $\epsilon = 0$ and letting u and w in (11) be the ones defined in (49) and (51), is

$$
x_2^* = -U x_1^* \tag{52}
$$

It is not difficult to see that, on each sampling interval $[t_{k-1}, t_k]$ and under the initial condition $x_1^*(t_{k-1}) = x_{1\,k-1}$, the x_1^* generated by (50) is actually the optimal trajectory for the slow subgame under closed-loop information (the terminal cost term in the cost function being $|x_s|^2_{Z_{s\gamma}(t_k)}$) with the optimal control and corresponding maximizing disturbance being u_{SD} and w_{SD}, respectively, (actually, they are particular representations of the optimal control and corresponding disturbance along the optimal trajectory), and x_2^* also coincides with $\bar{x}_2$, introduced in (11), generated by x_1^*, u_{SD} and w_{SD} (see Appendix A). Let x^* denote $(x_1^{*\prime}\ x_2^{*\prime})'$.

The finite-horizon case

Theorem 1 *For the singularly perturbed system (1)–(6), let Assumptions 1–4 be satisfied, $A_{22}(t)$ be Hurwitz for each $t \in [0, t_f]$, and the following condition hold:*

$Q_{f22} \le Z_{f\gamma}(t_f)$, where $Z_{f\gamma}(t_f)$ is the solution to (33) at $t = t_f$ with γ fixed.

Then,

1. *$\gamma_{SD}^*(\epsilon) \le \bar{\gamma}_{SD}$, asymptotically as $\epsilon \to 0$, where $\bar{\gamma}_{SD}$, as defined in (35), is finite.*

2. *$\forall \gamma > \bar{\gamma}_{SD}$, $\exists \epsilon_\gamma > 0$ such that $\forall \epsilon \in [0, \epsilon_\gamma)$, the set of equations (7)–(8) admits nonnegative definite solutions, and consequently, the full-order game (1)–(6) has a finite value.*

3. *$\forall \gamma > \bar{\gamma}_{SD}$, if we apply the controller defined in (49), μ_{SD}, to the system, then $\exists \epsilon'_\gamma > 0$ such that $\forall \epsilon \in [0, \epsilon'_\gamma)$, the disturbance attenuation level γ is attained for the full-order system.*

Proof We first note that under Assumption 3, and with A_{22} being Hurwitz for every fixed $t \in [0, t_f]$, the fast subsystem is open-loop stable and hence $\gamma_{of} < \infty$. Then we have $\bar{\gamma}_{SD}$ finite. Fix $\gamma > \bar{\gamma}_{SD}$,[4] by Theorem 2 in [18],[5] (7) does not have a conjugate point on $[0, t_f]$ for small ϵ, and the solution can be approximated by

$$\tilde{Z} = \left[\begin{array}{cc} Z_{s\gamma}(t) + O(\epsilon) & \epsilon(Z'_c + Z_{cb}(\tau)') + O(\epsilon^2) \\ \epsilon(Z_c + Z_{cb}(\tau)) + O(\epsilon^2) & \epsilon(Z_{f\gamma}(t) + Z_{fb}(\tau)) + O(\epsilon^2) \end{array} \right] \quad (53)$$

for all $t \in [0, t_f]$, where $Z_c := Z_{f\gamma}(t)U + V$, $Z_{cb}(\tau)$ and $Z_{fb}(\tau)$ are the boundary layer terms and as $\tau \to -\infty$ they converge to 0 exponentially in the τ time scale.

On each sampling interval $[t_{k-1}, t_k]$, we have $Z_{of\gamma}(t_k) \geq Z_{f\gamma}(t_k)$ and $\gamma > \max\{\gamma_{sk}, \gamma_{of}\}$, and hence each "open-loop" GRDE in (8) does not have a conjugate point on its corresponding time interval for small ϵ (see [18] Theorem 2). Thus, results 1) and 2) are proved.

To prove 3), we will use a dynamic programming argument to show that the control u_{SD} actually achieves the γ performance level, i.e.,

$$\sup_{w \in \mathcal{H}_w} L_\gamma(u_{SD}, w) = 0.$$

First, we write out the dynamic programming equation, adapted to the sampling intervals over which u_{SD} was defined:

$$\begin{aligned} &\sup_{w_{[0,t_f]}} L_\gamma(u_{SD}, w) \\ &= \sup_{w_{[0,t_1]}} \{ \int_0^{t_1} (|x|_Q^2 + |u_{SD}|^2 - \gamma^2 |w|^2)\, dt + \sup_{w_{[t_1,t_2]}} \{ \cdots \\ &\qquad + \sup_{w_{[t_{K-1},t_f]}} \{ |x(t_f)|_{Q_f}^2 + \int_{t_{K-1}}^{t_f} (|x|_Q^2 + |u_{SD}|^2 - \gamma^2 |w|^2)\, dt \} \cdots \} \end{aligned}$$

In this connection let us introduce

$$L_K(w) \;\; = \;\; |x(t_f)|_{Q_f}^2 + \int_{t_{K-1}}^{t_f} (|x(t)|_{Q(t)}^2 + |u_{SD}|^2 - \gamma^2 |w|^2)\, dt \quad (54)$$

[4] Since $\gamma > \gamma_{of}$, then the condition (13) is satisfied by Lemma 2 in [18].
[5] As discussed in [18], $\hat{\gamma}_s \geq \gamma_s$, where γ_s is defined in [18] pp.11.

$$L_K^* = \sup_{w_{[t_{K-1},t_f]}} L_K(w) \tag{55}$$

$$L_i(w) = L_{i+1}^* + \int_{t_{i-1}}^{t_i} (|x(t)|^2_{Q(t)} + |u_{SD}|^2 - \gamma^2|w|^2)\, dt \tag{56}$$

$$L_i^* = \sup_{w_{[t_{i-1},t_i]}} L_i(w)\ ; \quad i = K-1, \ldots, 1 \tag{57}$$

If we can show that L_1^* is bounded for small ϵ (by linearity of the problem, it must be zero), then it follows that the control achieves the γ bound.

We first consider stage K. Let $x(t_{K-1}) = x_{K-1}$. Introduce a new state variable:

$$\tilde{x} := x - x^* \tag{58}$$

It is shown in Appendix B that $\tilde{x}$ satisfies the singularly perturbed differential equation

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \epsilon\dot{\tilde{x}}_2 \end{bmatrix} = A(t)\tilde{x} + D(t)w - D(t)w_{SD}(t) - \begin{bmatrix} 0 \\ O(\epsilon) \end{bmatrix} x_1^* \tag{59}$$

Furthermore, we can rewrite the cost $L_K(w)$ as

$$\begin{aligned} L_K(w) &= |x(t_f)|^2_{Q_f} + \int_{t_{K-1}}^{t_f} (|x(t)|^2_{Q(t)} + |u_{SD}(t)|^2 - \gamma^2|w(t)|^2)\, dt \\ &= |x^*(t_f)|^2_{Q_f} + 2\tilde{x}'(t_f)Q_f x^*(t_f) + |\tilde{x}(t_f)|^2_{Q_f} + \int_{t_{K-1}}^{t_f} (|\tilde{x}(t)|^2_{Q(t)} \\ &\quad +2\tilde{x}'(t)Q(t)x^*(t) + |x^*(t)|^2_{Q(t)} + |u_{SD}(t)|^2 - \gamma^2|w(t)|^2 \\ &\quad +\gamma^2|w_{SD}(t)|^2 - \gamma^2|w_{SD}(t)|^2)\, dt \end{aligned}$$

Since x^*, u_{SD} and w_{SD} coincide with the solution of the slow subsystem under perfect state measurements, we have that

$$\begin{aligned} |x_1^*(t_{K-1})|^2_{Z_{s\gamma}(t_{K-1})} &= |x_1^*(t_f)|^2_{Q_{f11}} + \int_{t_{K-1}}^{t_f} (|x^*(t)|^2_{Q(t)} \\ &\quad +|u_{SD}(t)|^2 - \gamma^2|w_{SD}(t)|^2)\, dt \end{aligned} \tag{60}$$

Now, using the structure of Q_f and a "completion of squares" argument, we arrive at the following expression:[6]

$$\begin{aligned} L_K^* &= |x_{1\,K-1}|^2_{Z_{s\gamma}(t_{K-1})+O(\epsilon)} + |\tilde{x}(t_{K-1})|^2_{P_\epsilon(t_{K-1})} \\ &\quad +2Y_\epsilon'(t_{K-1})\tilde{x}(t_{K-1}) + m(t_{K-1}) \end{aligned}$$

where $P_\epsilon(t)$, $Y_\epsilon(t)$ and $m(t)$ are the solutions to

$$\dot{P}_\epsilon + A_\epsilon' P_\epsilon + P_\epsilon A_\epsilon + P_\epsilon \tfrac{1}{\gamma^2} D_\epsilon D_\epsilon' P_\epsilon + Q = 0; \qquad P_\epsilon(t_f) = Q_f \tag{61}$$

[6] The term $|x_{1\,K-1}|^2_{O(\epsilon)}$ is due to $O(\epsilon)$ terms in the terminal cost term $|x^*(t_f)|^2_{Q_f}$.

$$\dot{Y}_\epsilon + (A'_\epsilon + P_\epsilon \tfrac{1}{\gamma^2} D_\epsilon D'_\epsilon) Y_\epsilon - P_\epsilon D_\epsilon w_{SD} - P_\epsilon \begin{bmatrix} 0 \\ O(1) \end{bmatrix} x_1^* + Q x^* = 0 \tag{62}$$

$$Y_\epsilon(t_f) = Q_f x^*(t_f)$$

$$\dot{m} = -\tfrac{1}{\gamma^2} |Y_\epsilon - \begin{bmatrix} Z_{s\gamma} \\ \epsilon V \end{bmatrix} x_1^*|^2_{D_\epsilon D'_\epsilon} + 2Y'_\epsilon \begin{bmatrix} 0 \\ O(1) \end{bmatrix} x_1^*; \quad m(t_f) = 0 \tag{63}$$

on the time interval $[t_{K-1}, t_f]$. Since $\gamma > \max\{\gamma_{of}, \gamma_{sK}\}$, by Theorem 2 in [18], P_ϵ exists for small ϵ and can be approximated by

$$P_\epsilon = \begin{bmatrix} P_{s\gamma}(t) + O(\epsilon) & \epsilon(P'_c(t) + P'_{cb}(\tau)) + O(\epsilon^2) \\ \epsilon(P_c(t) + P_{cb}(\tau)) + O(\epsilon^2) & \epsilon(Z_{of\gamma}(t) + P_{fb}(\tau)) + O(\epsilon^2) \end{bmatrix} \tag{64}$$

where $\tau := \frac{t-t_f}{\epsilon}$, $P_{cb}(\tau)$ and $P_{fb}(\tau)$ are the boundary correction terms and they converge to zero exponentially in τ time scale.

Next, we introduce $Z_\epsilon(t)$, which is the solution to the following GRDE:

$$\dot{Z}_\epsilon + \hat{A}'_\epsilon Z_\epsilon + Z_\epsilon \hat{A}_\epsilon + Z_\epsilon \frac{1}{\gamma^2} D_\epsilon D'_\epsilon Z_\epsilon + \hat{Q}_w = 0; \quad Z_\epsilon(t_f) = Q_f \ \ t \in [0, t_f] \tag{65}$$

where

$$\hat{A}_\epsilon := A_\epsilon - B_\epsilon B' \begin{bmatrix} Z_{s\gamma} & 0 \\ V & 0 \end{bmatrix} \tag{66}$$

$$\hat{Q}_w := Q + \begin{bmatrix} Z_{s\gamma} & V' \\ 0 & 0 \end{bmatrix} BB' \begin{bmatrix} Z_{s\gamma} & 0 \\ V & 0 \end{bmatrix} \tag{67}$$

We can see that this GRDE is exactly the same equation as the GRDE we got in the perfect state measurements case when we apply the slow controller in the full-order system. So, by Theorem 2 in [18], the Equation (65) has a nonnegative-definite solution on $[0, t_f]$ for small ϵ, which can be approximated by

$$Z_\epsilon = \begin{bmatrix} Z_{s\gamma}(t) + O(\epsilon) & \epsilon(Z'_{oc} + Z_{cb}(\tau)') + O(\epsilon^2) \\ \epsilon(Z_{oc} + Z_{cb}(\tau)) + O(\epsilon^2) & \epsilon(Z_{of\gamma}(t) + Z_{fb}(\tau)) + O(\epsilon^2) \end{bmatrix} \tag{68}$$

where $Z_{oc} := Z_{of\gamma} U + V$, the $Z_{cb}(\tau)$ and $Z_{fb}(\tau)$ are still the boundary layer correction terms and they converge to zero exponentially in the τ time scale.

Next we state the following identity, which will be used shortly:

$$\begin{aligned} Z_\epsilon x^* - \begin{bmatrix} Z_{s\gamma} \\ \epsilon V \end{bmatrix} x_1^* &= (Z_\epsilon - \begin{bmatrix} Z_{s\gamma} & 0 \\ \epsilon V & 0 \end{bmatrix}) x^* \\ &= \begin{bmatrix} O(\epsilon) & O(\epsilon) \\ \epsilon(Z_{of\gamma} U + Z_{cb}) + O(\epsilon^2) & \epsilon(Z_{of\gamma} + Z_{fb}) + O(\epsilon^2) \end{bmatrix} x^* \\ &= \begin{bmatrix} O(\epsilon) \\ \epsilon(Z_{cb} - Z_{fb} U) + O(\epsilon^2) \end{bmatrix} x_1^* \end{aligned} \tag{69}$$

Start with the following structure (where the partition is the same as x), which defines the variables Y_1 and Y_2:

$$Y_\epsilon - Z_\epsilon x^* := \begin{bmatrix} Y_1 \\ \epsilon Y_2 \end{bmatrix} \tag{70}$$

and note that $(Y_1', Y_2')'$ thus defined satisfies the following differential equation (the detailed derivation can be found in Appendix C):

$$\begin{aligned} \begin{bmatrix} \dot{Y}_1 \\ \epsilon \dot{Y}_2 \end{bmatrix} &= -\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} - \begin{bmatrix} O(\epsilon) \\ O(\epsilon) \end{bmatrix} x_1^* \\ &\quad + \begin{bmatrix} O(1) \\ O(\epsilon) \end{bmatrix} (Z_{cb} - Z_{fb}U) x_1^*; \quad \begin{bmatrix} Y_1(t_f) \\ Y_2(t_f) \end{bmatrix} = 0 \end{aligned} \tag{71}$$

where

$$\begin{aligned} F_{11} &:= A_{11}' + P_{s\gamma}\frac{1}{\gamma^2} D_1 D_1' + (P_c' + P_{cb}')\frac{1}{\gamma^2} D_2 D_1' + O(\epsilon) \\ F_{12} &:= A_{21}' + P_{s\gamma}\frac{1}{\gamma^2} D_1 D_2' + (P_c' + P_{cb}')\frac{1}{\gamma^2} D_2 D_2' + O(\epsilon) \\ F_{21} &:= A_{12}' + (Z_{of\gamma} + P_{fb})\frac{1}{\gamma^2} D_2 D_1' + O(\epsilon) \\ F_{22} &:= A_{22}' + (Z_{of\gamma} + P_{fb})\frac{1}{\gamma^2} D_2 D_2' + O(\epsilon) \end{aligned}$$

This is a singularly perturbed system. Let t_a be the time by which the boundary layer terms Z_{cb}, Z_{fb}, P_{cb} and P_{fb} becomes of $O(\epsilon)$. Obviously, $t_a - t_f = O(\epsilon \ln \epsilon)$. Since $A_{22}' + Z_{of\gamma}\frac{1}{\gamma^2} D_2 D_2'$ is Hurwitz for every fixed t and its eigenvalues are uniformly bounded away from 0 by $-c$ for some $c \in \mathcal{R}^+$ on $[0, t_f]$ (where c is ϵ-free), we have $A_{22}' + (Z_{of\gamma} + P_{fb})\frac{1}{\gamma^2} D_2 D_2' + O(\epsilon)$ also Hurwitz, with its eigenvalues are bounded away from 0 by $-c/2$ uniformly after a boundary layer $[t_b, t_f]$, where now, $t_b - t_f = O(\epsilon)$. We choose ϵ small enough to make $t_{K-1} < t_a < t_b$ (under Assumption 4). By the approximation results in singular perturbation theory (see [17] Chapter 5 Theorem 6.1), in $[t_{K-1}, t_b]$, the state transition function for Y_1 and Y_2 is of $O(1)$.

In $[t_b, t_f]$, the growth of Y_1 and Y_2 can be bounded by

$$\begin{aligned} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &\le \int_0^{O(\epsilon)} \exp(\frac{1}{\epsilon} O(1) t)\, O(1)\, dt \\ &= O(\epsilon) \end{aligned}$$

In $[t_a, t_b]$, the growth of Y_1 and Y_2 can be bounded by

$$\begin{aligned} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &\le \int_{O(\epsilon)}^{O(\epsilon \ln \epsilon)} O(1) \cdot O(1)\, dt + O(\epsilon) \cdot O(1) \\ &= o(\epsilon^{1/2}) \end{aligned}$$

Then, in $[t_{K-1}, t_a]$, the growth can be bounded by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \le \int_{t_{K-1}}^{t_a} O(1) \cdot O(\epsilon)\, dt + o(\epsilon^{1/2}) \cdot O(1) = o(\epsilon^{1/2})$$

Hence, we actually have

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = o(\epsilon^{1/2}); \quad \forall t \in [t_{K-1}, t_f]$$

Then, we can further deduce that

$$Y_\epsilon = \begin{bmatrix} Z_{s\gamma} \\ \epsilon(V + Z_{cb} - Z_{fb} U) \end{bmatrix} x_1^* + \begin{bmatrix} o(\epsilon^{1/2}) \\ o(\epsilon^{3/2}) \end{bmatrix} \tag{72}$$

Now, it is clear that

$$\begin{aligned} m(t_{K-1}) &= \int_{t_{K-1}}^{t_f} \dot{m}\, dt \\ &\le \int_{t_{K-1}}^{t_a} O(\epsilon)\, dt + \int_{t_a}^{t_f} (O(\epsilon) + b(\tau))\, dt \\ &= O(\epsilon) \end{aligned}$$

where $b(\tau)$ is some term that is exponentially decaying in the τ time scale.

Since the terminal condition and driving terms in (62) are linear functions of $x_{1\,K-1}$, Y_ϵ is also a linear function of $x_{1\,K-1}$ and consequently $\dot{m}(t)$ and $m(t_{K-1})$ are quadratic functions of it. Thus,

$$m(t_{K-1}) = x'_{1\,K-1} O(\epsilon) x_{1\,K-1}$$

Since $x^*(t_{K-1}) = [x'_{1\,K-1}, -x'_{1\,K-1} U'(t_{K-1})]'$ and $x_1(t_{K-1}) = x_{1\,K-1}$, we have

$$\tilde{x}(t_{K-1}) = \begin{bmatrix} 0 \\ x_{2\,K-1} + U(t_{K-1}) x_{1\,K-1} \end{bmatrix}$$

Consequently,

$$\begin{aligned} L_K^* = {} & |x_{1\,K-1}|^2_{Z_{s\gamma}(t_{K-1}) + O(\epsilon)} + |\tilde{x}(t_{K-1})|^2_{P_\epsilon(t_{K-1})} + 2Y'_\epsilon(t_{K-1})\tilde{x}(t_{K-1}) \\ = {} & |x_{1\,K-1}|^2_{Z_{s\gamma}(t_{K-1}) + O(\epsilon)} + |x_{2\,K-1} + U(t_{K-1}) x_{1\,K-1}|^2_{\epsilon Z_{of\gamma}(t_{K-1}) + O(\epsilon^2)} \\ & + 2x'_{1\,K-1}(\epsilon V(t_{K-1}) + o(\epsilon^{3/2}))'(x_{2\,K-1} + U(t_{K-1}) x_{1\,K-1}) \\ = {} & x'_{K-1} \left. \begin{bmatrix} Z_{s\gamma} + O(\epsilon) & \epsilon Z'_{oc} + o(\epsilon^{3/2}) \\ \epsilon Z_{oc} + o(\epsilon^{3/2}) & \epsilon Z_{of\gamma}(t) + O(\epsilon^2) \end{bmatrix} \right|_{t = t_{K-1}} x_{K-1} \end{aligned} \tag{73}$$

Now, we consider the next stage, $K-1$. The arguments used for the analysis of stage K are equally valid here, with the only difference being that we now have to watch for the effect of the additional terminal cost perturbation term, which is

$$x'_{K-1} \begin{bmatrix} O(\epsilon) & o(\epsilon^{3/2}) \\ o(\epsilon^{3/2}) & O(\epsilon^2) \end{bmatrix} x_{K-1},$$

The solution to (61), P_ϵ, will still exist, and will have the same form as in (64) for small ϵ (since the solution of ϵ-free lower order problems depend continuously on the terminal condition). Due to the perturbation term, the terminal conditions for Y_1 and Y_2 will not be zero as in stage K, but they will be $o(\epsilon^{1/2})$. Note that, in this stage, there is no longer a boundary layer term for Z_ϵ. Thus, it is easy to see that Y_1 and Y_2 will still be of order $o(\epsilon^{1/2})$ in $[t_{K-2}, t_{K-1}]$, and consequently the approximation in (72) still holds.

Also, the perturbation term will not affect the approximation for the terms containing only x^* because the state transition function for x^* is of $O(1)$. Then, L^*_{K-1} will be of the same approximation form as L^*_K but the evaluation time is at t_{K-2}.

Since there is only a finite number of sampling intervals, we can find $\epsilon'_\gamma > 0$ such that $\forall \epsilon \in [0, \epsilon'_\gamma)$, L^*_1 is bounded (and L^*_1 have the approximation form as in (73)). This completes the proof of 3). □

The infinite-horizon case

To guarantee that $\bar{\gamma}_{SD\infty}$ is finite (although the theorem to be presented below is true also when it is infinite), we make two additional assumptions:

Assumption 5 $(A_{11} - A_{12}A_{22}^{-1}A_{21}, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ is observable.

Assumption 6 $(A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2)$ is controllable and A_{22} is Hurwitz.

Theorem 2 *Consider the singularly perturbed system (1)–(6), with $t_f = \infty$, $Q_f = 0$ and A, B, D, Q time-invariant. If Assumptions 1–6 hold, then*

1. *$\gamma^*_{SD\infty}(\epsilon) \to \bar{\gamma}_{SD\infty}$ asymptotically as $\epsilon \to 0^+$, where $\bar{\gamma}_{SD\infty}$, defined in (48), is finite.*

2. *$\forall \gamma > \bar{\gamma}_{SD\infty}$, $\exists \epsilon_\gamma > 0$ such that $\forall \epsilon \in [0, \epsilon_\gamma)$, equation (37) admits a positive definite solution and (38) admits a nonnegative definite solution over the interval $[0, T]$, and consequently, the game has a finite upper value.*

3. *$\forall \gamma > \bar{\gamma}_{SD\infty}$, if we apply the controller μ_{SD} defined in (49) to the system, then $\exists \epsilon'_\gamma > 0$ such that, $\forall \epsilon \in [0, \epsilon'_\gamma)$, the disturbance attenuation level γ is attained for the full-order system.*

Proof We first note that under the assumptions stated, the slow game has a saddle-point solution for sufficiently large values of γ, since at $\gamma = \infty$ the problem becomes a "regular" LQR problems admitting stabilizing optimal controller. This shows that $\hat{\gamma}_{s\infty}$ is finite. Also, since the fast subsystem is open-loop stable, $\gamma_{of\infty} < \infty$. These implies that $\bar{\gamma}_{SD\infty}$ is finite. Now fix $\gamma > \bar{\gamma}_{SD\infty}$. By arguments similar to those used in the finite-horizon case, we can easily deduce that 2) is true, and $\limsup_{\epsilon\to 0^+} \gamma^*_{SD\infty}(\epsilon) \leq \bar{\gamma}_{SD\infty}$. For any $\gamma < \bar{\gamma}_{SD\infty}$, by the Theorem 2 of [28] and its proof, either the GARE (37) doesn't admit any nonnegative definite solutions or the GRDE (38) has at least one conjugate point in the interval $[0, T]$. Hence, 1) follows.

For 3), we first note here that the controller is the same as in the finite-horizon case. Since we have already shown validity of the result when t_f is finite, it will be sufficient here to find a $\epsilon'_\gamma > 0$ that works (uniformly) when $t_f \to \infty$. Toward this end, we first note that since $\hat{A}_0 - \hat{S}_0 Z_{s\gamma}$ is Hurwitz, we can find a constant matrix $\Delta > 0$ such that

$$(\hat{A}_0 - \hat{S}_0 Z_{s\gamma})'\Delta + \Delta(\hat{A}_0 - \hat{S}_0 Z_{s\gamma}) < 0.$$

Then, we have

$$|x_1^*(T)|^2_\Delta < \alpha |x_1^*(0)|^2_\Delta; \quad \text{for some } \alpha \in (0,1)$$

Consider a sample interval of length T (which we may as well assume to be $[0, T]$ due to the time-invariant nature), with the "cost to go" being

$$x(T)' \begin{bmatrix} Z_{s\gamma} + \delta\Delta & \epsilon(U'Z_{of\gamma} + V') \\ \epsilon(Z_{of\gamma}U + V) & \epsilon(Z_{of\gamma} + \delta I_{n_2}) \end{bmatrix} x(T) := |x(T)|^2_\Xi \tag{74}$$

where δ is some small positive real number. We introduce the cost function to be

$$L_T(w) := |x(T)|^2_\Xi + \int_0^T (|x(t)|^2_{Q(t)} + |u_{SD}|^2 - \gamma^2 |w|^2)\, dt$$

By following a line of reasoning similar to that in the finite-horizon case, we first introduce the new set of state variables $\tilde{x} := x - x^*$. By Appendix B, $\tilde{x}$ satisfies the following dynamics:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \epsilon\dot{\tilde{x}}_2 \end{bmatrix} = A(t)\tilde{x} + D(t)w - D(t)w_{SD}(t) - \begin{bmatrix} 0 \\ \epsilon U(\hat{A}_0 - \hat{S}_0 Z_{s\gamma}) \end{bmatrix} x_1^*$$

Then, we can rewrite the L_T into the following form:

$$\begin{aligned} L_T(w) \quad = \quad & |x^*(T)|^2_\Xi + 2\tilde{x}'(T)\Xi x^*(T) + |\tilde{x}(T)|^2_\Xi + \int_0^T (|\tilde{x}(t)|^2_{Q(t)} \\ & + 2\tilde{x}'(t)Q(t)x^*(t) + |x^*(t)|^2_{Q(t)} + |u_{SD}(t)|^2 - \gamma^2|w(t)|^2 \\ & + \gamma^2|w_{SD}(t)|^2 - \gamma^2|w_{SD}(t)|^2)\, dt \end{aligned}$$

By following the same arguments as in the finite-horizon case, we deduce that those terms containing only x^* in L_T can be approximated by

$$\begin{aligned} & |x_1^*(0)|^2_{Z_{s\gamma}+O(\epsilon)+O(\epsilon\delta)} + |x_1^*(T)|^2_{\delta\Delta} \\ \leq \; & |x_1^*(0)|^2_{Z_{s\gamma}+O(\epsilon)+O(\epsilon\delta)} + \alpha|x_1^*(0)|^2_{\delta\Delta} \\ = \; & |x_1^*(0)|^2_{Z_{s\gamma}+\alpha\delta\Delta+O(\epsilon)+O(\epsilon\delta)} \end{aligned}$$

By using a "completion of squares" argument, we can compute that

$$L_K^* = |x_1(0)|^2_{Z_{s\gamma}+\alpha\delta\Delta+O(\epsilon)+O(\epsilon\delta)} + |\tilde{x}(0)|^2_{P_\epsilon(0)} + 2Y_\epsilon'(0)\tilde{x}(0) + m(0)$$

where $P_\epsilon(t)$, $Y_\epsilon(t)$ and $m(t)$ are again the solutions to

$$\dot{P}_\epsilon + A_\epsilon' P_\epsilon + P_\epsilon A_\epsilon + P_\epsilon \tfrac{1}{\gamma^2} D_\epsilon D_\epsilon' P_\epsilon + Q = 0; \quad P_\epsilon(T) = \Xi \tag{75}$$

$$\begin{aligned} & \dot{Y}_\epsilon + (A_\epsilon' + P_\epsilon \tfrac{1}{\gamma^2} D_\epsilon D_\epsilon')Y_\epsilon - P_\epsilon D_\epsilon w_{SD} - P_\epsilon \begin{bmatrix} 0 \\ O(1) \end{bmatrix} x_1^* + Qx^* = 0 \\ & Y_\epsilon(T) = \Xi x^*(T) \end{aligned} \tag{76}$$

$$\dot{m} = -\tfrac{1}{\gamma^2}|Y_\epsilon - \begin{bmatrix} Z_{s\gamma} \\ \epsilon V \end{bmatrix} x_1^*|^2_{D_\epsilon D_\epsilon'} + 2Y_\epsilon' \begin{bmatrix} 0 \\ O(1) \end{bmatrix} x_1^*; \quad m(T) = 0 \tag{77}$$

on the time interval $[0, T]$. Since $\gamma > \max\{\gamma_{SDs\infty}, \gamma_{of\infty}\}$ and the solution to the ϵ-free lower-order GRDE depends continuously on terminal condition, we can choose δ small enough (but still ϵ-free) to make P_ϵ exist for small ϵ. Then P_ϵ will have the same approximation form as in (64).

Let Z_ϵ be the solution to the following GARE:

$$\hat{A}_\epsilon' Z_\epsilon + Z_\epsilon \hat{A}_\epsilon + Z_\epsilon \frac{1}{\gamma^2} D_\epsilon D_\epsilon' Z_\epsilon + \hat{Q}_w = 0 \tag{78}$$

where $\hat{A}_\epsilon$ and $\hat{Q}_w$ are defined in (66) and (67). By the same argument as in finite-horizon case, Z_ϵ exists, and it can be approximated by

$$Z_\epsilon = \begin{bmatrix} Z_{s\gamma}(t) + O(\epsilon) & \epsilon(U'Z_{of\gamma} + V') + O(\epsilon^2) \\ \epsilon(Z_{of\gamma}U + V) + O(\epsilon^2) & \epsilon Z_{of\gamma}(t) + O(\epsilon^2) \end{bmatrix} \tag{79}$$

Note here Z_ϵ does not have any boundary layer terms. Thus, we have

$$Z_\epsilon x^* - \begin{bmatrix} Z_{s\gamma} \\ \epsilon V \end{bmatrix} x_1^* \; = \; O(\epsilon)$$

Next we define Y_1 and Y_2 as in the finite-horizon case (see (70)), and follow a similar reasoning, to show that

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = O(\epsilon) + O(\delta); \quad \forall t \in [0, T]$$

Consequently, we have the following:

$$\begin{aligned} Y_\epsilon &= \begin{bmatrix} Z_{s\gamma} \\ \epsilon V \end{bmatrix} x_1^* + \begin{bmatrix} O(\epsilon) + O(\delta) \\ O(\epsilon^2) + O(\epsilon\delta) \end{bmatrix} \\ m(0) &= O(\epsilon^2) + O(\epsilon\delta) + O(\delta^2) \end{aligned}$$

Hence, by choosing ϵ small enough so that the boundary terms of P_ϵ die out at $t = 0$, we arrive at

$$\begin{aligned} L_T^* &= x'(0) \begin{bmatrix} Z_{s\gamma} + \alpha\delta\Delta + O(\epsilon) + O(\delta^2) & \epsilon Z'_{oc} + O(\epsilon^2) + O(\epsilon\delta) \\ \epsilon Z_{oc} + O(\epsilon^2) + O(\epsilon\delta) & \epsilon Z_{of\gamma} + O(\epsilon^2) \end{bmatrix} x(0) \\ &:= |x(0)|^2_{\tilde{\Xi}} \end{aligned}$$

Now, we first choose a small enough $\delta_0 > 0$ so that $O(\delta^2) \ll O(\delta)$; then we find a small enough $\epsilon'_\gamma > 0$ so that $O(\epsilon) \ll O(\delta)$, $O(\epsilon\delta) \ll O(\delta)$ and $O(\epsilon^2) \ll O(\epsilon\delta)$. Then, $\forall \epsilon \in (0, \epsilon'_\gamma]$, $\Xi > \tilde{\Xi}$.

Noticing the following two facts:

1. We can obtain the above results for every $T' \in [T_m, T]$, the quantity δ_0 can be chosen uniformly on $[T_m, T]$, and the dependence of ϵ'_γ on $T' \in [T_m, T]$ is continuous.

2. If the terminal cost is decreased, the maximal cost for a given controller is not increased.

Thus, we deduce that, we can find a $\epsilon'_\gamma > 0$ that works for all $T' \in [T_m, T]$. Then, $\forall \epsilon \in (0, \epsilon'_\gamma]$, the infinite horizon game (1)–(6) has a bounded upper value , which is bounded by Ξ. □

4 Examples

We present here some numerical results for the infinite-horizon case with uniform sampling. As stressed earlier, the two quantities $\gamma_{SDs\infty}$ and $\gamma_{of\infty}$ play important roles in the computation of an approximate value for $\gamma^*_{SD\infty}(\epsilon)$. In the examples below, we will also compare the effectiveness of the full-order optimal sampled-data controller with that of the "slow" sampled-data controller introduced in (49) and (50) by plotting the system response (under these controllers) to various disturbance inputs.

Example 1 Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ 3 \end{bmatrix} w \tag{80}$$

$$L_\gamma = \int_0^\infty (2x_1^2 + 2x_1x_2 + 3x_2^2 + |u|^2 - \gamma^2|w|^2)\, dt \tag{81}$$

where we take the uniform sampling rate to be

$$T = 0.1$$

Then, the two critical performance levels are:

$$\gamma_{of\infty} = 2.5981; \quad \gamma_{SDs\infty} = 2.8574,$$

which show that $\bar{\gamma}_{SD\infty} = 2.8574$. We can also compute the minimax disturbance attenuation level $\gamma^*_{SD\infty}(\epsilon)$ of system (80)–(81) for different values of ϵ:

$\gamma^*_{SD\infty}(\epsilon)$	1.3311	2.0883	2.8127	2.8536
ϵ	1	0.1	0.01	0.001

We can see that as $\epsilon \to 0$, $\gamma^*_{SD\infty}(\epsilon) \to \bar{\gamma}_{SD\infty}$. Now, we choose $\gamma = 3 > \bar{\gamma}_{SD\infty}$, and design the "slow" sampled-data controller based on this value of γ:

$$Z_{s\gamma} = 2.1379; \quad \mu_{SD} = -1.1618x_1^*$$

where

$$\dot{x}_1^* \quad = \quad -2.6890x_1^*; \quad x_1^*(t_k) = x_1(t_k) \quad t \in [t_k, t_{k+1}) \quad k = 0, 1, \ldots$$

Now, fix $\epsilon = 0.001$, and design the optimal sampled-data controller $\mu^*_{SD\infty}$ for the full-order system:

$$\mu^*_{SD\infty} = -3.3917\hat{x}_1 - 1.0697\hat{x}_2$$

where

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -4.6930 & -9.6106 \\ -7512.2 & -3604.3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}; \quad \hat{x}_i(t_k) = x_i(t_k)$$
$$t \in [t_k, t_{k+1}) \quad k = 0, 1, \ldots$$

In the next three figures, we show some simulation results on the response of the system to three different disturbance inputs, under the μ_{SD} and $\mu^*_{SD\infty}$ given above, with $\epsilon = 0.001$. The disturbances are plotted in Figures 1(a), 2(a) and 3(a), which are constant, exponentially decaying, and sinusoidal, respectively; Figures 1(b) and (c), 2(b) and (c), 3(b) and (c), respectively, show plots of the corresponding full-order sampled-data control inputs and two components of the state responses to the controllers. Figures 1(d) and (e), 2(d) and (e), 3(d) and (e), respectively, do the same for the "slow" sampled-data controller. Note that the full-order controller results in a better state response, but a larger and more oscillatory control signal. $\diamond$

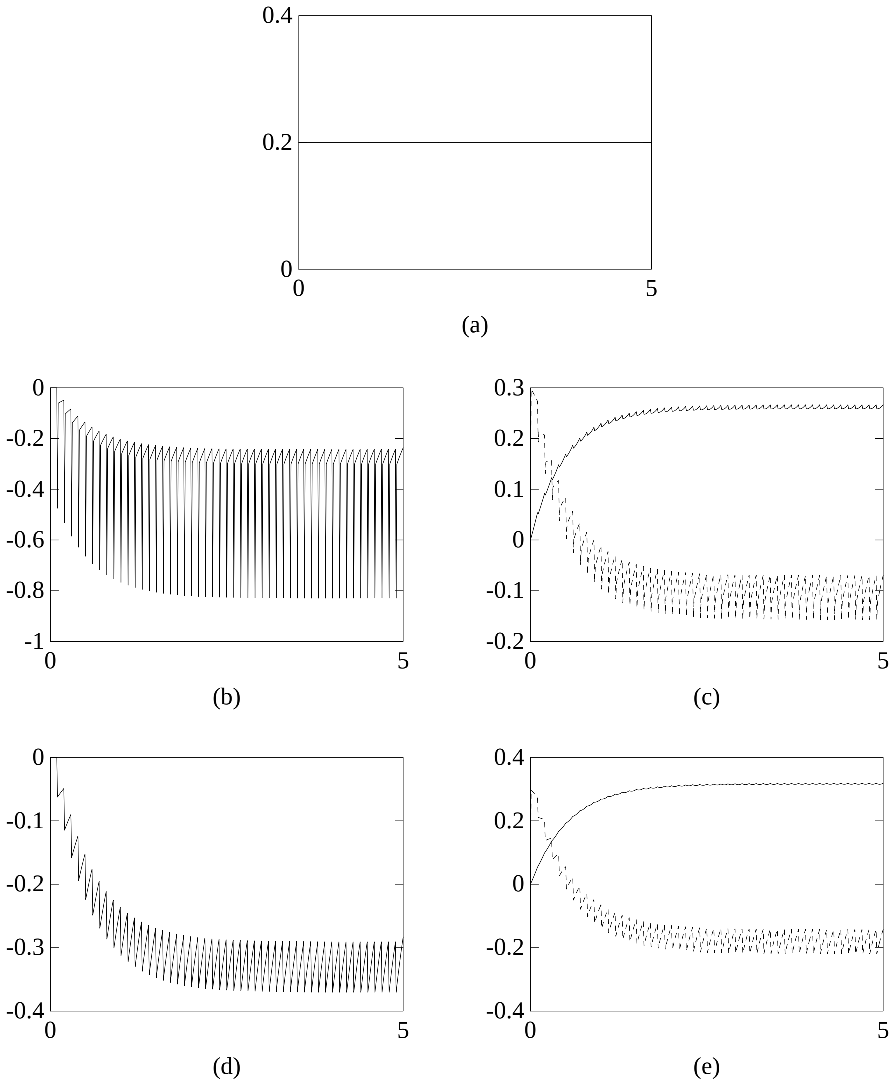

Figure 1: Input/output signals for Example 1: Constant disturbance (a) disturbance input; (b) full-order controller; (c) state responses to full-order controller; (d) slow controller; (e) state responses to slow controller

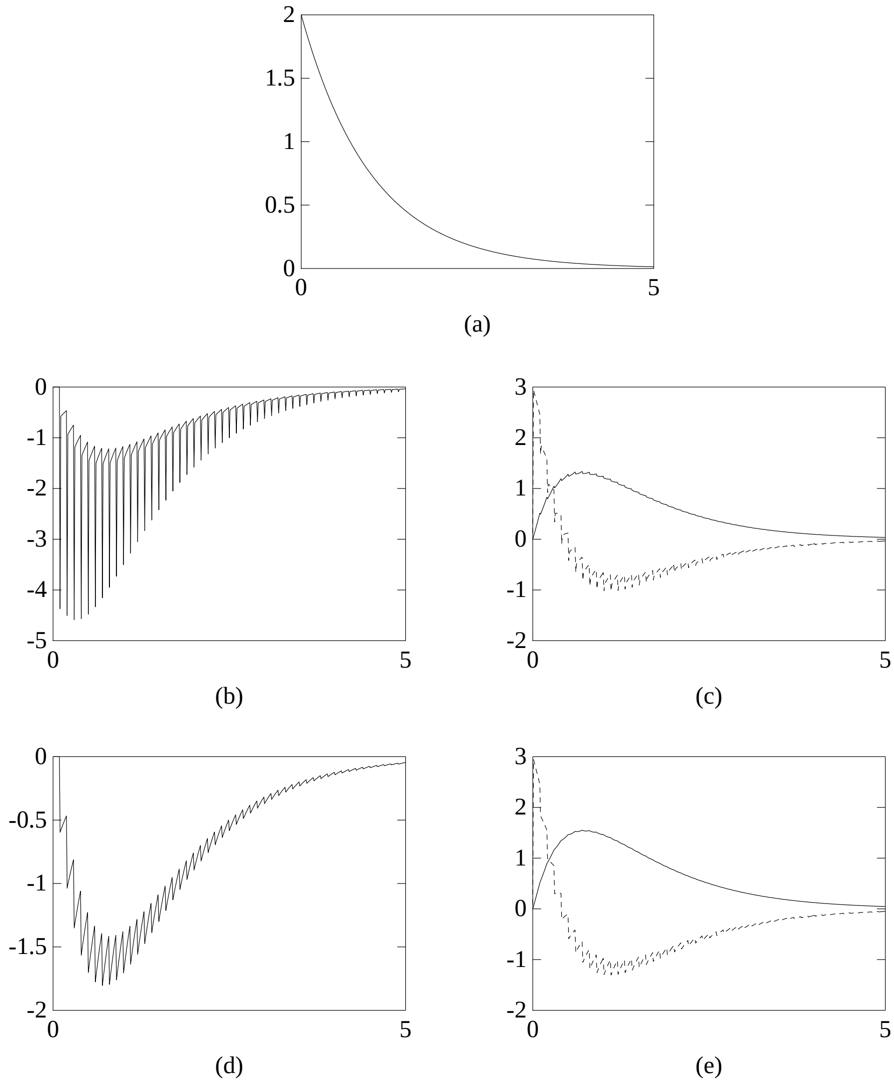

Figure 2: Input/output signals for Example 1: Exponentially decaying disturbance
(a) disturbance input; (b) full-order controller; (c) state responses to full-order controller; (d) slow controller; (e) state responses to slow controller

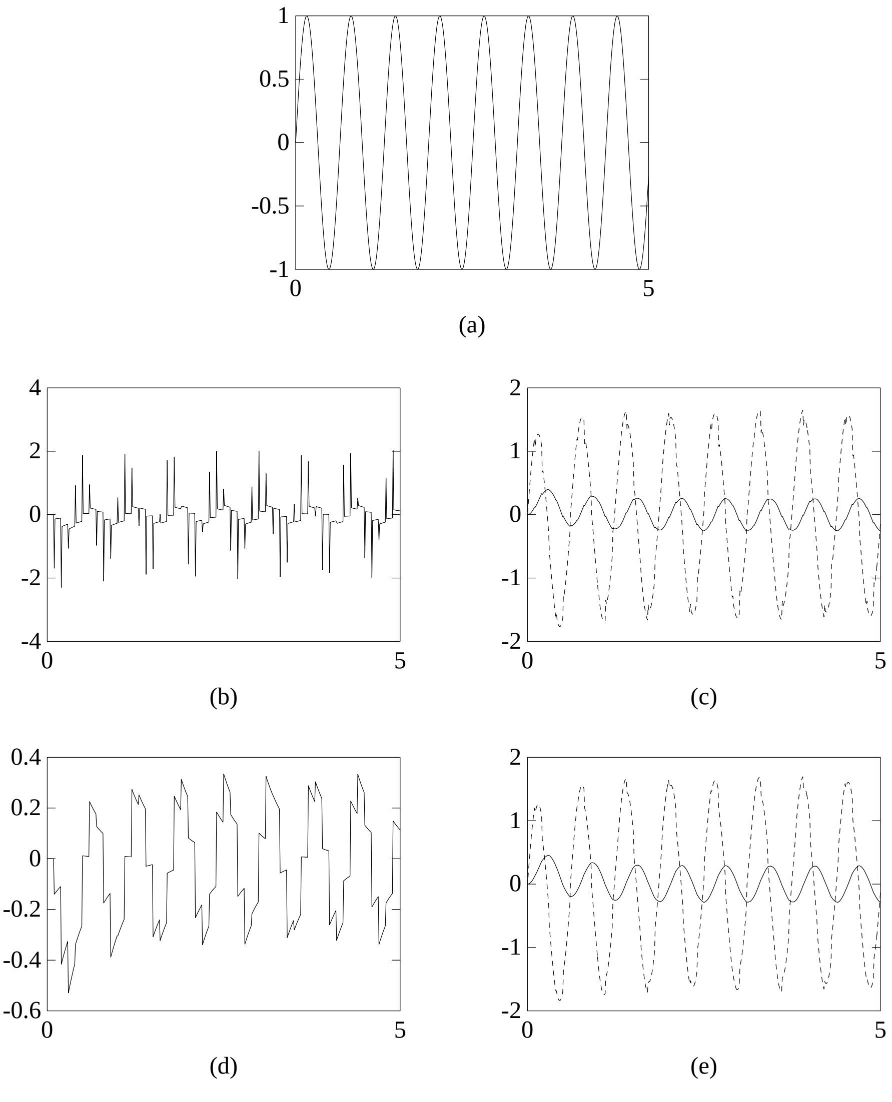

Figure 3: Input/output signals for Example 1: Sinusoidal disturbance (a) disturbance input; (b) full-order controller; (c) state responses to full-order controller; (d) slow controller; (e) state responses to slow controller

Example 2 Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w \tag{82}$$

$$L_\gamma = \int_0^\infty (2x_1^2 + 2x_1x_2 + x_2^2 + |u|^2 - \gamma^2 |w|^2)\, dt \tag{83}$$

where the uniform sampling rate is again taken to be $T = 0.1$. The two critical performance levels are:

$$\gamma_{of\infty} = 0.49941; \quad \gamma_{SDs\infty} = 1.0084.$$

Note that this depicts a different feature than Example 1, since $\gamma_{s\infty} > \gamma_{of\infty}$, whereas in Example 1 it was the opposite. Again in this case, we have $\bar{\gamma}_{SD\infty} = \gamma_{SDs\infty} = 1.0084$. We can also compute the minimax disturbance attenuation level $\gamma^*_{SD\infty}(\epsilon)$ of the system (82)–(83) for different values of ϵ,

$\gamma^*_{SD\infty}(\epsilon)$	0.75253	0.92789	1.0001	1.0080
ϵ	1	0.1	0.01	0.001

Note that as $\epsilon \to 0$, $\gamma^*_{SD\infty}(\epsilon) \to \bar{\gamma}_{SD\infty}$. Now, we choose $\gamma = 1.01 > \bar{\gamma}_{SD\infty}$, and design the "slow" sampled-data controller accordingly:

$$Z_{s\gamma} = 1.7836; \quad \mu_{SD} = -4.9973 x_1^*$$

where

$$\dot{x}_1^* = -4.7678 x_1^*; \quad x_1^*(t_k) = x_1(t_k) \quad t \in [t_k, t_{k+1}) \quad k = 0, 1, \ldots$$

Now, fix $\epsilon = 0.001$, and design the optimal sampled-data controller $\mu^*_{SD\infty}$ for the full-order system

$$\mu^*_{SD\infty} = -4.9746 \hat{x}_1 - 0.25303 \hat{x}_2$$

where

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -4.8210 & -0.74061 \\ 153.60 & -2006.4 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}; \quad \hat{x}_i(t_k) = x_i(t_k)$$
$$t \in [t_k, t_{k+1}) \quad k = 0, 1, \ldots$$

We again present some simulation results, given in Figures 4, 5 and 6. This time the "slow" controller results in a slightly larger and more oscillatory control signal, worse response for the fast state variable, but a better response for the slow state variable. This is mainly due to the fact that the fast subsystem is less sensitive to noise than the slow subsystem. $\diamond$

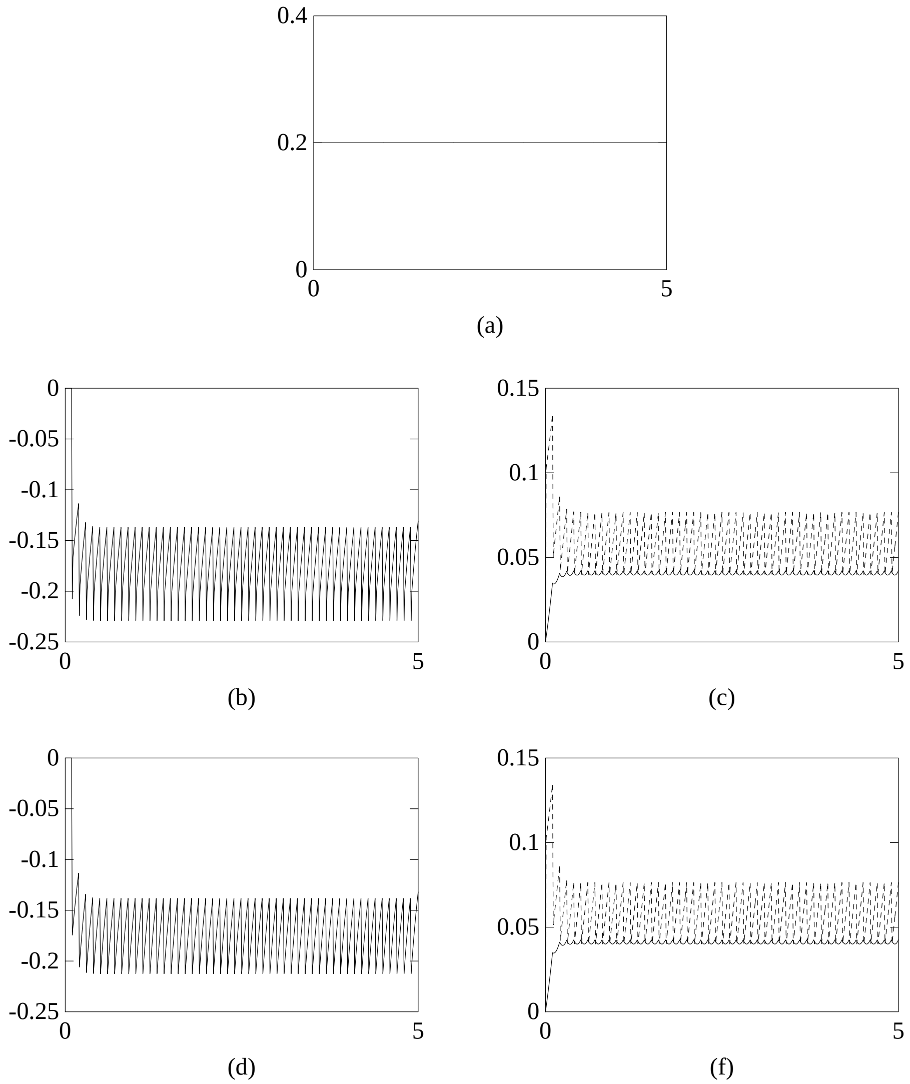

Figure 4: Input/output signals for Example 2: Constant disturbance (a) disturbance input; (b) full-order controller; (c) state responses to full-order controller; (d) slow controller; (e) state responses to slow controller

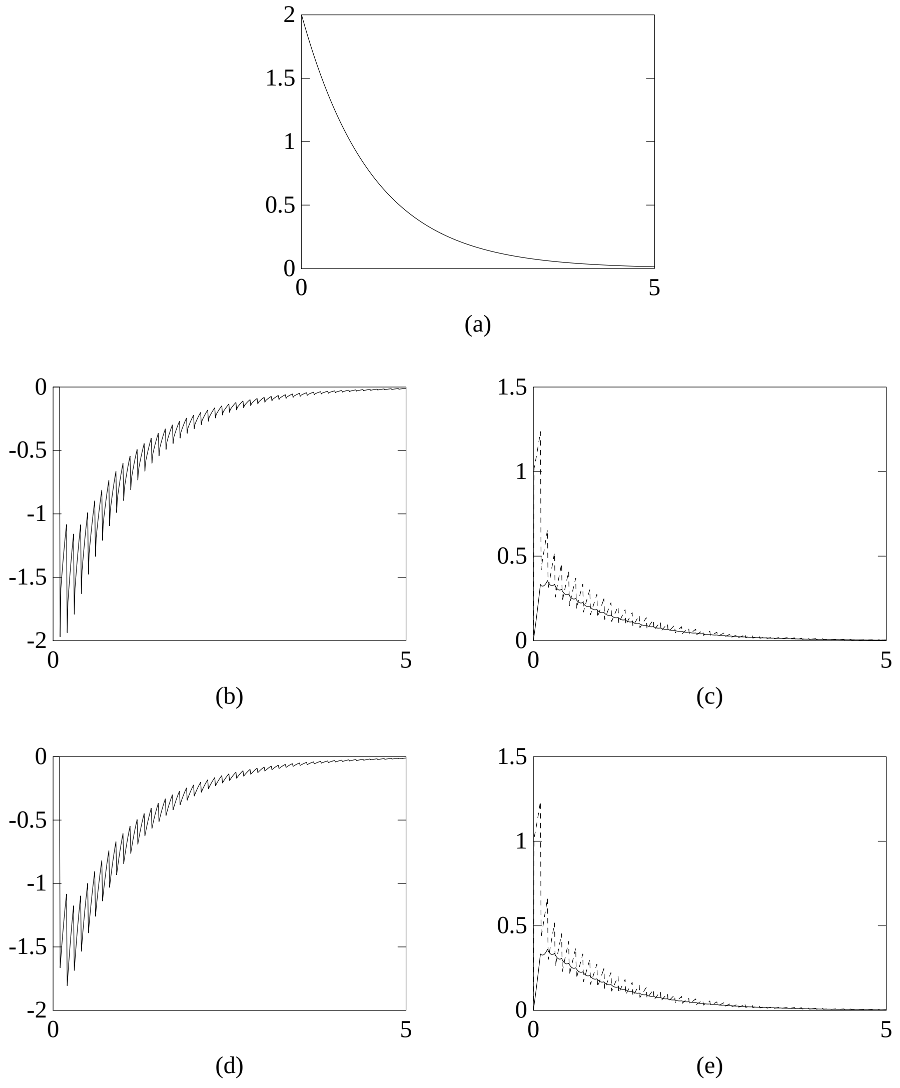

Figure 5: Input/output signals for Example 2: Exponentially decaying disturbance
(a) disturbance input; (b) full-order controller; (c) state responses to full-order controller; (d) slow controller; (e) state responses to slow controller

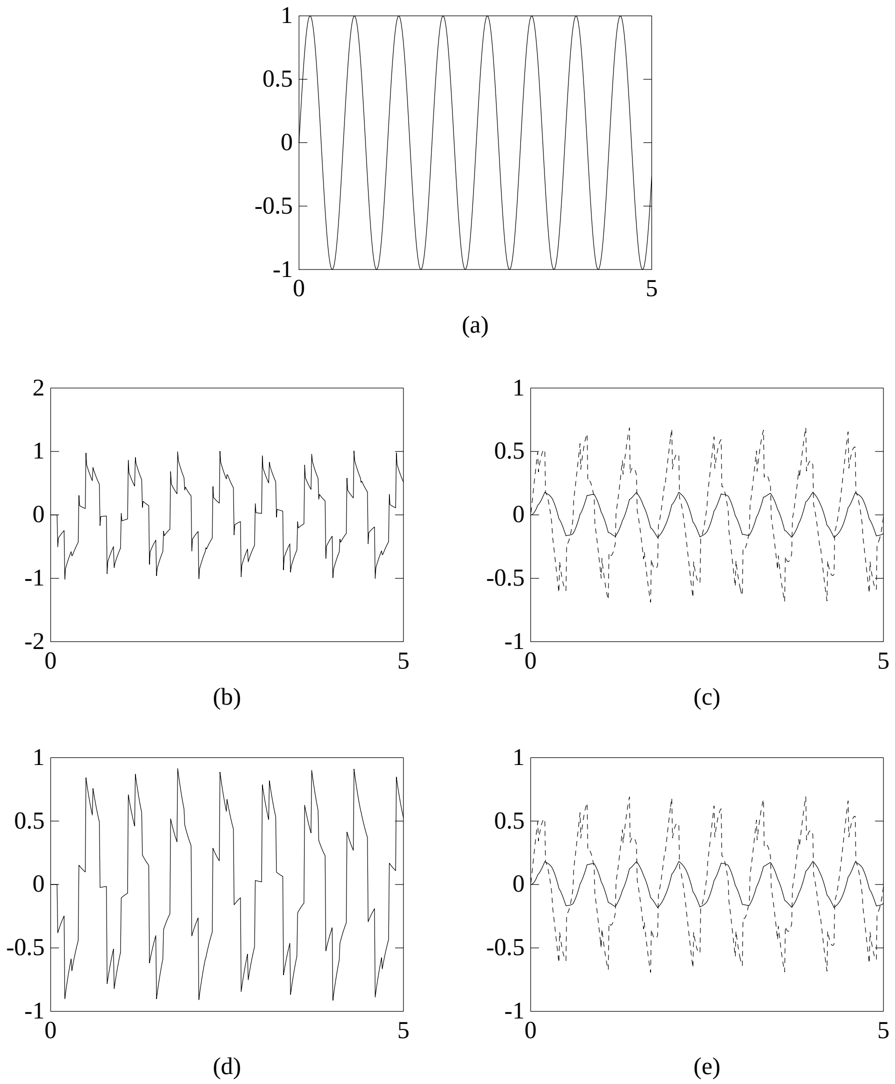

Figure 6: Input/output signals for Example 2: Sinusoidal disturbance (a) disturbance input; (b) full-order controller; (c) state responses to full-order controller; (d) slow controller; (e) state responses to slow controller

5 Conclusion

In this paper, we have provided a complete analysis of the singularly perturbed H^∞-optimal control problem with sampled-data state measurements in both finite and infinite horizons, by relating it to a class of singularly perturbed differential games. One of the main results of the paper is that one can totally neglect the fast dynamics, and still achieve a performance very close to that achieved using full-order controllers, which is quite different from the results obtained in the perfect and imperfect (nonsampled) state measurements cases. We construct a "slow" controller, based on the slow subsystem only and independent of the singular perturbation parameter, which can achieve any desired achievable performance level for the full-order system. Several numerical examples are included, which illustrate the theoretical results.

One immediate, but not trivial, extension of these results would be to the imperfect sampled-data measurements case, so as to obtain the counterparts of the results of [19] in the sampled-data measurements case. Another extension would be to obtain the counterparts of the results here for multi-time scale systems, which, however, seems to be rather routine (though cumbersome) not requiring any new conceptual developments. Another problem of interest would be the singularly perturbed version of the problem studied in [15], where we have the additional restriction that control action remains constant on each sampling interval. Again techniques developed here seem to be of direct relevance to solve this singularly perturbed digital control problem.

Appendix

A

Over a typical sampling interval $[t_{k-1}, t_k]$, consider the following game kernel for the slow subsystem:

$$L_{ck} = |x_s(t_f)|^2_{Z_{s\gamma}(t_k)} + \int_{t_{k-1}}^{t_k} (|x_s|^2_{Q^\square} + 2x_s' P^\square u_s + |u_s|^2_{R^\square} - |\hat{w}_s|^2)\, dt$$

It is easy to see, from the discussion in Section 3 of [18] about the SF information pattern for the slow subsystem, that u_{SD} given by (49) is the minimizing control input that achieves the upper value of the game under open-loop information, w_{SD} given by (51) is the corresponding maximizing disturbance under open-loop information, and x_1^* is the corresponding optimal trajectory.

With u_{SD}, w_{SD}, x_1^* given as stated, we can now compute $\bar{x}_2$:

$$\bar{x}_2 \quad = \quad -A_{22}^{-1}(A_{21}x_1^* + B_2 u_{SD} + D_2 w_{SD})$$

$$
\begin{aligned}
&= -A_{22}^{-1}(A_{21} - B_2(B_1'Z_{s\gamma} + B_2'V) + \frac{1}{\gamma^2}D_2(D_1'Z_{s\gamma} + D_2'V))x_1^* \\
&= -A_{22}^{-1}(A_{21} - S_{21}Z_{s\gamma} - S_{22}V)x_1^* = -A_{22}^{-1}(A_{22}U)x_1^* = -Ux_1^*
\end{aligned}
$$

B

We first obtain an expression for $\dot{\tilde{x}}_1$:

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \dot{x}_1 - \dot{x}_1^* \\
&= A_{11}x_1 + A_{12}x_2 + B_1u_{SD} + D_1w - (\hat{A}_0 - \hat{S}_0Z_{s\gamma})x_1^* \\
&= A_{11}x_1 + A_{12}x_2 + B_1u_{SD} + D_1w - (A_{11} - S_{11}Z_{s\gamma} - S_{12}V \\
&\qquad - A_{12}U)x_1^* \\
&= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + D_1w - D_1w_{SD}
\end{aligned}
$$

Similarly, for $\epsilon\dot{\tilde{x}}_2$:

$$
\begin{aligned}
\epsilon\dot{\tilde{x}}_2 &= \epsilon(\dot{x}_2 - \dot{x}_2^*) \\
&= A_{21}x_1 + A_{22}x_2 + B_2u_{SD} + D_2w - \epsilon(\dot{U} + U(\hat{A}_0 - \hat{S}_0Z_{s\gamma}))x_1^* \\
&= A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + B_2u_{SD} + D_2w + A_{21}x_1^* + A_{22}x_2^* - \epsilon(\dot{U} \\
&\qquad + U(\hat{A}_0 - \hat{S}_0Z_{s\gamma}))x_1^* \\
&= A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + D_2w - D_2w_{SD} - \epsilon(\dot{U} + U(\hat{A}_0 - \hat{S}_0Z_{s\gamma}))x_1^*
\end{aligned}
$$

Thus, we have

$$
\begin{bmatrix} \dot{\tilde{x}}_1 \\ \epsilon\dot{\tilde{x}}_2 \end{bmatrix} = A(t)\tilde{x} + D(t)w - D(t)w_{SD}(t) - \begin{bmatrix} 0 \\ \epsilon(\dot{U} + U(\hat{A}_0 - \hat{S}_0Z_{s\gamma})) \end{bmatrix} x_1^*
$$

Note that if the system is time-invariant then $\dot{U} = 0$.

C

The verification of (71) involves some substitutions and straightforward manipulations, as given below:

$$
\begin{aligned}
\begin{bmatrix} \dot{Y}_1 \\ \epsilon\dot{Y}_2 \end{bmatrix} &= -(A_\epsilon' + P_\epsilon\frac{1}{\gamma^2}D_\epsilon D_\epsilon')\begin{bmatrix} Y_1 \\ \epsilon Y_2 \end{bmatrix} - (A_\epsilon' + P_\epsilon\frac{1}{\gamma^2}D_\epsilon D_\epsilon')Z_\epsilon x^* \\
&\qquad + P_\epsilon D_\epsilon w_{SD} + P_\epsilon\begin{bmatrix} 0 \\ O(1) \end{bmatrix}x_1^* - Qx^* + (\hat{A}_\epsilon'Z_\epsilon + Z_\epsilon\hat{A}_\epsilon \\
&\qquad + Z_\epsilon\frac{1}{\gamma^2}D_\epsilon D_\epsilon'Z_\epsilon + \hat{Q}_w)x^* - Z_\epsilon\begin{bmatrix} \hat{A}_0 - \hat{S}_0Z_{s\gamma} \\ O(1) \end{bmatrix}x_1^* \\
&= -(A' + P_\epsilon\frac{1}{\gamma^2}D_\epsilon D')\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} - P_\epsilon\frac{1}{\gamma^2}D_\epsilon D_\epsilon'(Z_\epsilon x^* - \begin{bmatrix} Z_{s\gamma} \\ \epsilon V \end{bmatrix}x_1^*)
\end{aligned}
$$

$$
\begin{aligned}
& +P_\epsilon \begin{bmatrix} 0 \\ O(1) \end{bmatrix} x_1^* - \begin{bmatrix} Z_{s\gamma} & V' \\ 0 & 0 \end{bmatrix} BB_\epsilon'(Z_\epsilon x^* - \begin{bmatrix} Z_{s\gamma} \\ \epsilon V \end{bmatrix} x_1^*) \\
& +Z_\epsilon(\hat{A}_\epsilon + \frac{1}{\gamma^2} D_\epsilon D_\epsilon' Z_\epsilon - \begin{bmatrix} \hat{A}_0 - \hat{S}_0 Z_{s\gamma} & 0 \\ O(1) & 0 \end{bmatrix}) x^* \\
= \quad & -(A' + P_\epsilon \frac{1}{\gamma^2} D_\epsilon D') \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + P_\epsilon \begin{bmatrix} 0 \\ O(1) \end{bmatrix} x_1^* - (P_\epsilon \frac{1}{\gamma^2} D_\epsilon D' \\
& + \begin{bmatrix} Z_{s\gamma} B_1 + V' B_2 \\ 0 \end{bmatrix} B') \begin{bmatrix} O(\epsilon) \\ Z_{cb} - Z_{fb} U + O(\epsilon) \end{bmatrix} x_1^* \\
& +Z_\epsilon \begin{bmatrix} \frac{1}{\gamma^2} D_1 D_2' (Z_{cb} - Z_{fb} U) + O(\epsilon) \\ \frac{1}{\epsilon}(\frac{1}{\gamma^2} D_2 D_2' (Z_{cb} - Z_{fb} U) + O(\epsilon)) \end{bmatrix} x_1^*
\end{aligned}
$$

The above can be seen to be equivalent to (71), by writing out the boundary layer differential equations for Z_{fb} and P_{fb} (from (65) and (61)), and showing that they are actually identical.

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