H^∞-Optimal Control for Singularly Perturbed Systems

Part I: Perfect State Measurements*†

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Design of composite controllers for singularly perturbed systems under the H^∞ criterion requires a much different methodology than that available for the singularly perturbed regulator problem.

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Abstract

We study the H^∞-optimal control of singularly perturbed linear systems under perfect state measurements, for both finite and infinite horizons. Using a differential game theoretic approach, we show that as the singular perturbation parameter ϵ approaches zero, the optimal disturbance attenuation level for the full-order system under a quadratic performance index converges to a value that is bounded above by the maximum of the optimal disturbance attenuation levels for the *slow* and *fast* subsystems under appropriate "slow" and "fast" quadratic cost functions. Furthermore, we construct a composite controller based on the solution of the slow and fast games, which guarantees a desired achievable performance level for the full-order plant, as ϵ approaches zero. A "slow" controller, however, is not generally robust in this sense, but still under some conditions, which are delineated in the paper, the fast dynamics can be totally ignored. The paper also studies optimality when the controller includes a feedforward term in the disturbance, and presents some numerical examples to illustrate the theoretical results.

Key Words— Singular perturbations, H^∞-optimal control, Feedback control, Feedforward control, Differential games, Optimal regulators, Boundary-layer system.

1 INTRODUCTION

One of the important recent developments in control theory has been the recognition of the close relationship that exists between H^∞-optimal control problems and a class of linear-quadratic differential games, which has not only led to simpler derivations of existing results on the former, but also enabled us to develop worst-case (H^∞-optimal) controllers under various information patterns, such as (in addition to perfect and imperfect state measurements) delayed state and sampled state measurements. An up-to-date coverage of this relationship and the derivation of H^∞-optimal controllers under different information patterns can be found in the recent book Başar and Bernhard (1991), which also contains an extensive list of references on the topic.

This paper makes further use of the relationship between H^∞-optimal control and differential games in studying the issue of worst-case controller design for systems with fast and slow dynamics—systems commonly modeled using the mathematical framework of "singular perturbations". The main objective here is to obtain "approximate" controllers independently of the "small" singular perturbation parameter, say ϵ, and to prove that the approximate controllers can be used "reliably" on the original system when ϵ is sufficiently small. It is, of course, well-known that in the absence of the disturbance, a composite design based on two separate designs for the slow and fast subsystems performs remarkably well in the linear-quadratic (singularly perturbed) regulator problem, with the approximation to the optimal cost being $O(\epsilon^2)$ (Chow and Kokotović, 1976). Another appealing feature of the composite controller (in the regulator problem) is that it avoids the difficulty of dealing with stiff differential equations.

One natural question here is whether the counterpart of the "singularly perturbed regulator theory" can be developed for H^∞-optimal control, or equivalently for the related class of linear-quadratic zero-sum differential games, by going through similar steps. One of the messages of this paper is that this is not necessarily so, and one has to develop a separate theory by going through some new lines of reasoning, which has indeed been accomplished in the paper. To make the above statement somewhat more precise, let $\gamma^*(\epsilon)$ denote the H^∞-optimum performance of the full-order system under perfect state measurements, and γ_s and γ_f denote the H^∞-optimum performances of (appropriately defined) reduced slow and fast subsystems, respectively. Our first result is that it is not necessarily true that $\gamma^*(\epsilon = 0) = \gamma_s$, and an H^∞-norm bounding controller for the reduced-order system that guarantees a performance (for the reduced system) in the neighborhood of γ_s could lead to a catastrophic performance for the full-order system, no matter how small ϵ is. In terms of the soft-constrained game associated with the full-order system, such a feedback controller (which depends only on the slow state, by construction) may lead to an unbounded value. This is quite in contrast with the limiting properties of feedback controllers in *disturbance-free* optimal control problems with singularly perturbed dynamics, but is not that surprising in view of some existing results on singularly perturbed Nash games (Gardner, Jr. and Cruz, Jr., 1978), where it is known that the well-posedness (or, continuity at $\epsilon = 0$) depends on the consistency of the information structure between the full and reduced-order problems. Zero-sum differential games fall somewhere between optimal control problems and nonzero-sum differential games, and there the saddle-point value shows continuity whenever it exists for both the full and reduced-order problems, while the condition of existence depends on the consistency of the underlying information structures. But this is precisely what determines the optimal level of performance and the corresponding minimax controller in H^∞-optimization, which therefore requires a careful study of the dependence of conjugate points in zero-sum differential games on the small (singular perturbation) parameter ϵ. This paper undertakes such a study and provides a complete analysis of the singularly perturbed H^∞-optimal control problem under perfect state measurements. It is proven that $\gamma^*(\epsilon = 0) \leq \max\{\gamma_s, \gamma_f\}$, and that a composite controller exists under which such a performance bound is attained "approximately".

Design of controllers for singularly perturbed systems subject to unknown disturbances has been studied before in the literature, notably in papers Garofalo and Leitmann (1988); Corless, Leitmann, and Ryan (1990); Garofalo and Leitmann (1990), where the objective has been to obtain composite controllers that guarantee stability of the overall (possibly nonlinear) system. The main approach of the authors in these papers has involved the construction of appropriate Liapunov functions, in terms of which a class of stabilizing controllers has been characterized. However, no optimality properties have been associated with these controllers, which is our main concern in this paper. Yet another paper that deals with uncertain (linear) systems which exhibit time-scale separation is Luse and Ball (1989), which obtains a two-frequency-scale decomposition for H^∞-disk problems, but does not address the issue of optimal controller design.

The present paper is organized as follows. In the next section (Section 2) we formulate the singularly perturbed H^∞-optimal control problem and the associated linear-quadratic differential game, both under perfect state measurements. We also provide in that section the solution to the full-order problem, for both finite and infinite horizons. In Section 3, we identify the slow and fast subsystems and the associated differential games. There are in fact **two** "slow" games, differing only in the information available to the controller, both of which are relevant to the problem at hand. The saddle-point solutions to these "slow" games as well as to the "fast" game are also presented in Section 3, in both finite and infinite horizons. In Section 4, where the main results are presented, we introduce the composite controller and prove optimality properties of the composite and slow controllers in the context of the original H^∞-optimal control problem for both the finite and infinite horizon cases, and obtain precise (tight) performance bounds attained by them. Section 5 presents some numerical results to illustrate the theory, and Section 6 provides a discussion on extensions to other information patterns; some of this work has already been completed and will appear elsewhere, and some is currently underway. The paper ends with four Appendices, which provide the details of some of the derivations given in the main body of the paper.

2 PROBLEM FORMULATION

The system under consideration, with slow and fast dynamics, is described in the standard "singularly perturbed" form by

$$\begin{cases} \dot{x}_1 &= A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u + D_1(t)w; \quad x_1(0) = 0 \\ \epsilon\dot{x}_2 &= A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u + D_2(t)w; \quad x_2(0) = 0 \end{cases} \tag{1}$$

where $x' := (x_1', x_2')$ is the n-dimensional state vector, with x_1 of dimension n_1 and x_2 of dimension $n_2 := n - n_1$; u is the control input, and w is the disturbance, each belonging to appropriate ($\mathcal{L}^2$) Hilbert spaces $\mathcal{H}_x$, $\mathcal{H}_u$ and $\mathcal{H}_w$, respectively, defined on the time interval $[0, t_f]$. The control input u is generated by a closed-loop control policy μ, according to

$$u(t) = \mu(t, x_{[0,t]}) \tag{2}$$

where $\mu : [0, t_f] \times \mathcal{H}_x \to \mathcal{H}_u$ is piecewise continuous in t and Lipschitz continuous in x, further satisfying the given causality condition. Let us denote class of all these controllers by $\mathcal{M}$. With this system, we associate the standard quadratic performance index:

$$\begin{aligned} L(u, w) &= |x(t_f)|^2_{Q_f} + \int_0^{t_f} (|x(t)|^2_{Q(t)} + |u(t)|^2)\, dt \\ &\equiv |x(t_f)|^2_{Q_f} + \|x\|^2_Q + \|u\|^2; \quad Q_f \geq 0, \quad Q(.) \geq 0 \end{aligned} \tag{3}$$

where Q_f will show dependence on $\epsilon > 0$, as to be clarified later. Let us also introduce the notation $J(\mu, w)$ to denote $L(u, w)$, with u given by (2). The H^∞-optimal control problem is the minimization of the quantity

$$\sup_{w \in \mathcal{H}_w} \{J(\mu, w)\}^{1/2} / \|w\| \tag{4}$$

over all permissible controllers μ, and in the case a minimum does not exist, the derivation of a controller μ that will assure a performance within a given neighborhood of the infimum of (4). Let us denote this infimum by $\gamma^*(\epsilon)$, i. e.

$$\inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{H}_w} \{J(\mu, w)\}^{1/2} / \|w\| = \gamma^*(\epsilon) \tag{5}$$

where we explicitly show the dependence of γ^* on the singular perturbation parameter $\epsilon > 0$.

For each $\epsilon > 0$, we can associate a soft-constrained linear-quadratic differential game with this worst-case design problem (see Başar and Bernhard, 1991), which has the cost function

$$L_\gamma(u, w) = L(u, w) - \gamma^2 \|w\|^2 \tag{6}$$

The performance level $\gamma^*(\epsilon)$ in (5) is then the "smallest" value of $\gamma \geq 0$ under which the differential game with state equation (1) and cost function (6) has a bounded upper value, when u is chosen according to (2). We also know that for every fixed $\epsilon > 0$, and for each $\gamma > \gamma^*(\epsilon)$, this differential game admits a saddle-point solution, with the saddle-point controller μ^* being a linear feedback law (see Başar and Bernhard, 1991, Chapter 4).

Even though the problem formulated above has been solved completely for every $\epsilon > 0$, the computation of $\gamma^*(\epsilon)$ and that of a corresponding H^∞-optimal or suboptimal controller for small values of $\epsilon > 0$ present serious difficulties, due to numerical stiffness. To remedy this, we pose in this paper the question of whether $\gamma^*(\epsilon)$ and the H^∞-optimal controller can be determined, for small values of $\epsilon > 0$, by solving well-behaved ϵ-independent smaller-order problems, as in the case of the singularly perturbed linear-quadratic regulator problem (Chow and Kokotović, 1976). Another question of interest is whether the fast subsystem dynamics can be neglected completely in the design of such controllers, and if so under what conditions.

Before studying these questions, we first present below the solution to the full-order problem, under three basic assumptions:

A1 Q_f and $Q(.)$ in (3) are partitioned as

$$Q_f = \begin{bmatrix} Q_{f11} & \epsilon Q_{f12} \\ \epsilon Q_{f21} & \epsilon Q_{f22} \end{bmatrix}; \qquad Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$$

where in each case the 11-block is of dimension $n_1 \times n_1$, and the 22-block is of dimension $n_2 \times n_2$.

A2 The matrix functions $A_{ij}(t)$, $Q_{ij}(t)$, $B_i(t)$, $D_i(t)$ $\quad (i = 1, 2 \;\; j = 1, 2)$ are continuously differentiable in $t \geq 0$.

A3 The matrices $A_{22}(t)$ and $Q_{22}(t)$ are invertible for all $t \in [0, t_f]$.

Let us further introduce the notation $A(t)$ and $A_\epsilon(t)$, to denote the partitioned matrices

$$A(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}; \qquad A_\epsilon(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \frac{1}{\epsilon}A_{21}(t) & \frac{1}{\epsilon}A_{22}(t) \end{bmatrix} \tag{7}$$

where we take $\epsilon > 0$. Similarly, we introduce the partitioned matrices

$$B(t) := \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}; \qquad B_\epsilon(t) := \begin{bmatrix} B_1(t) \\ \frac{1}{\epsilon}B_2(t) \end{bmatrix} \tag{8}$$

$$D(t) := \begin{bmatrix} D_1(t) \\ D_2(t) \end{bmatrix}; \qquad D_\epsilon(t) := \begin{bmatrix} D_1(t) \\ \frac{1}{\epsilon}D_2(t) \end{bmatrix} \tag{9}$$

and define

$$S(t;\gamma) := B(t)B'(t) - \frac{1}{\gamma^2}D(t)D'(t); \qquad S_\epsilon(t;\gamma) := B_\epsilon(t)B_\epsilon'(t) - \frac{1}{\gamma^2}D_\epsilon(t)D_\epsilon'(t) \tag{10}$$

with the ij-th block of $S(t;\gamma)$ denoted henceforth by $S_{ij}(t;\gamma)$, $i, j = 1, 2$.

We know from the existing theory on linear-quadratic differential games (see Başar and Olsder, 1982; Başar and Bernhard, 1991) that for each $\epsilon > 0$, there exists a $\hat{\gamma}(\epsilon) \geq 0$ such that for all $\gamma > \hat{\gamma}(\epsilon)$ the zero-sum differential game described by (1) and (6) has a bounded upper value (which in this case is equal to *zero*, because the initial states are zero), as well as a saddle point, with a saddle-point controller being the feedback law

$$u^*(t) = \mu^*(t, x(t)) = -B_\epsilon'\tilde{Z}(t;\epsilon)x(t), \quad t \geq 0 \tag{11}$$

where $\tilde{Z}(t;\epsilon)$ is the unique bounded nonnegative definite solution of the generalized matrix Riccati differential equation (GRDE)

$$\dot{\tilde{Z}} + A'_\epsilon \tilde{Z} + \tilde{Z} A_\epsilon - \tilde{Z} S_\epsilon \tilde{Z} + Q = 0; \qquad \tilde{Z}(t_f) = Q_f \tag{12}$$

For $\gamma < \hat{\gamma}(\epsilon)$, on the other hand, this GRDE has a conjugate point in the open interval $(0, t_f)$, which says that it has finite escape, and furthermore the associated soft-constrained game has unbounded upper value. The level $\hat{\gamma}(\epsilon)$ is indeed the H^∞-optimal performance $\gamma^*(\epsilon)$ given by (5).

Now, to study the behavior of the solution of (12) for small values of $\epsilon > 0$, we first partition $\tilde{Z}$ as follows (in a way quite analogous to the partitioning in the standard regulator problem, and consistent with the given partitioning on Q_f):

$$\tilde{Z} := \begin{bmatrix} \tilde{Z}_{11} & \epsilon \tilde{Z}_{12} \\ \epsilon \tilde{Z}_{21} & \epsilon \tilde{Z}_{22} \end{bmatrix} \tag{13}$$

where $\tilde{Z}_{12} = \tilde{Z}'_{21}$. Then, by substitution of this structure into (12), it is a simple matter to see that $\tilde{Z}_{11}$, $\tilde{Z}_{12}$ and $\tilde{Z}_{22}$ satisfy the following matrix differential equations:

$$\begin{aligned}
&\dot{\tilde{Z}}_{11} + A'_{11}\tilde{Z}_{11} + A'_{21}\tilde{Z}'_{12} + \tilde{Z}_{11}A_{11} + \tilde{Z}_{12}A_{21} + Q_{11} - \tilde{Z}_{11}S_{11}\tilde{Z}_{11} \\
&\quad - \tilde{Z}_{12}S_{21}\tilde{Z}_{11} - \tilde{Z}_{11}S_{12}\tilde{Z}'_{12} - \tilde{Z}_{12}S_{22}\tilde{Z}'_{12} = 0; \quad \tilde{Z}_{11}(t_f) = Q_{f11} &(14)\\
&\epsilon\dot{\tilde{Z}}_{12} + \epsilon A'_{11}\tilde{Z}_{12} + A'_{21}\tilde{Z}_{22} + \tilde{Z}_{11}A_{12} + \tilde{Z}_{12}A_{22} + Q_{12} - \epsilon\tilde{Z}_{11}S_{11}\tilde{Z}_{12} \\
&\quad - \epsilon\tilde{Z}_{12}S_{21}\tilde{Z}_{12} - \tilde{Z}_{11}S_{12}\tilde{Z}_{22} - \tilde{Z}_{12}S_{22}\tilde{Z}_{22} = 0; \quad \tilde{Z}_{12}(t_f) = Q_{f12} &(15)\\
&\epsilon\dot{\tilde{Z}}_{22} + \epsilon A'_{12}\tilde{Z}_{12} + A'_{22}\tilde{Z}_{22} + \epsilon\tilde{Z}'_{12}A_{12} + \tilde{Z}_{22}A_{22} + Q_{22} - \epsilon^2\tilde{Z}'_{12}S_{11}\tilde{Z}_{12} \\
&\quad - \epsilon\tilde{Z}_{22}S_{21}\tilde{Z}_{12} - \epsilon\tilde{Z}'_{12}S_{12}\tilde{Z}_{22} - \tilde{Z}_{22}S_{22}\tilde{Z}_{22} = 0; \quad \tilde{Z}_{22}(t_f) = Q_{f22} &(16)
\end{aligned}$$

Let Z_{11}, Z_{12} and Z_{22} denote the limiting solutions to above equations in the open interval $(0, t_f)$, as $\epsilon \downarrow 0$[1]. Then Z_{11}, Z_{12} and Z_{22} are given by the following differential and algebraic equations, which are obtained by setting $\epsilon = 0$ in (14) - (16):

$$\begin{aligned}
&\dot{Z}_{11} + A'_{11}Z_{11} + A'_{21}Z'_{12} + Z_{11}A_{11} + Z_{12}A_{21} + Q_{11} - Z_{11}S_{11}Z_{11} \\
&\quad - Z_{12}S_{21}Z_{11} - Z_{11}S_{12}Z'_{12} - Z_{12}S_{22}Z'_{12} = 0; \quad Z_{11}(t_f) = Q_{f11} &(17)\\
&A'_{21}Z_{22} + Z_{11}A_{12} + Z_{12}A_{22} + Q_{12} - Z_{11}S_{12}Z_{22} - Z_{12}S_{22}Z_{22} = 0 &(18)\\
&A'_{22}Z_{22} + Z_{22}A_{22} + Q_{22} - Z_{22}S_{22}Z_{22} = 0 &(19)
\end{aligned}$$

Since, as $t \to t_f$, the solution Z_{12} and Z_{22} above will not, in general, satisfy the given terminal conditions (in (15) and (16), respectively), we have to introduce a boundary layer correction term (see Kokotović and Yackel, 1972). Toward this end, introduce the fast time

[1] The existence of such limits will be shown later in Section (see, Theorem 2).

variable $\tau = \frac{t-t_f}{\epsilon}$, and rewrite (15) and (16) on this time scale, as $\epsilon \downarrow 0$:

$$\begin{aligned}\frac{d}{d\tau}L_{12}(\tau) &= -L_{12}(\tau)(A_{22}(t_f) - S_{22}(t_f)L_{22}(\tau)) - (A'_{21}(t_f) - Z_{11}(t_f)S_{12}(t_f))L_{22}(\tau) \\ &\quad -Z_{11}(t_f)A_{12}(t_f) - Q_{12}(t_f); \quad L_{12}(0) = Q_{f12} \qquad (20)\\ \frac{d}{d\tau}L_{22}(\tau) &= -A'_{22}(t_f)L_{22}(\tau) - L_{22}(\tau)A_{22}(t_f) - Q_{22}(t_f) \\ &\quad +L_{22}(\tau)S_{22}(t_f)L_{22}(\tau); \quad L_{22}(0) = Q_{f22} \qquad (21)\end{aligned}$$

Define

$$\begin{aligned}Z'_{cb}(\tau) &:= L_{12}(\tau) - Z_{12}(t_f) \qquad (22)\\ Z_{fb}(\tau) &:= L_{22}(\tau) - Z_{22}(t_f) \qquad (23)\end{aligned}$$

Then the solution of (14)–(16) will be approximated by

$$\begin{aligned}\tilde{Z}_{11}(t) &= Z_{11}(t) + O(\epsilon) \\ \tilde{Z}_{12}(t) &= Z_{12}(t) + Z'_{cb}(\tfrac{t-t_f}{\epsilon}) + O(\epsilon) \qquad \forall t \in [0, t_f] \qquad (24)\\ \tilde{Z}_{22}(t) &= Z_{22}(t) + Z_{fb}(\tfrac{t-t_f}{\epsilon}) + O(\epsilon)\end{aligned}$$

Conditions under which this approximation is valid will be delineated later in Theorem 2 of Section .

Of course, the preceding analysis is valid provided that γ is larger than $\sup\{\hat{\gamma}(\epsilon), 0 < \epsilon \leq \epsilon_0\}$, where ϵ_0 is some prechosen small (positive) scalar.

To study the infinite horizon case (i.e. as $t_f \to \infty$, as well as when $t_f = \infty$), we take A, B, D, Q to be time-invariant, and $Q_f = 0$. Furthermore, we assume that (A_ϵ, B_ϵ) is controllable, and (A_ϵ, Q) is observable for every $\epsilon > 0$. Then, for each $\epsilon > 0$, there exists a $\hat{\gamma}_\infty(\epsilon)$ such that, for all $\gamma > \hat{\gamma}_\infty(\epsilon)$, the infinite-horizon soft-constrained game has a finite upper value, achieved by the time-invariant feedback controller

$$u^*(t) = \mu^\infty(x) = -B'_\epsilon \tilde{Z}_\infty(\epsilon)x(t), \quad t \geq 0 \qquad (25)$$

where $\tilde{Z}_\infty > 0$ is the minimal positive definite solution of the generalized algebraic Riccati equation (GARE)

$$A'_\epsilon \tilde{Z}_\infty + \tilde{Z}_\infty A_\epsilon - \tilde{Z}_\infty S_\epsilon \tilde{Z}_\infty + Q = 0 \qquad (26)$$

The level $\hat{\gamma}_\infty(\epsilon)$ is again the H^∞-optimal performance level $(\gamma^*_\infty(\epsilon))$ of the infinite-horizon disturbance attenuation problem, and for $\gamma < \hat{\gamma}_\infty(\epsilon)$, the soft-constrained game has infinite upper value.

As in the finite-horizon case, we now substitute the structure (13) into (26), to arrive at the following coupled algebraic equations for $\tilde{Z}_{\infty 11}$, $\tilde{Z}_{\infty 12}$ and $\tilde{Z}_{\infty 22}$:

$$\begin{aligned}&A'_{11}\tilde{Z}_{\infty 11} + A'_{21}\tilde{Z}'_{\infty 12} + \tilde{Z}_{\infty 11}A_{11} + \tilde{Z}_{\infty 12}A_{21} + Q_{11} - \tilde{Z}_{\infty 11}S_{11}\tilde{Z}_{\infty 11} \\ &\quad -\tilde{Z}_{\infty 12}S_{21}\tilde{Z}_{\infty 11} - \tilde{Z}_{\infty 11}S_{12}\tilde{Z}'_{\infty 12} - \tilde{Z}_{\infty 12}S_{22}\tilde{Z}'_{\infty 12} = 0 \qquad (27)\\ &\epsilon A'_{11}\tilde{Z}_{\infty 12} + A'_{21}\tilde{Z}_{\infty 22} + \tilde{Z}_{\infty 11}A_{12} + \tilde{Z}_{\infty 12}A_{22} + Q_{12} - \epsilon\tilde{Z}_{\infty 11}S_{11}\tilde{Z}_{\infty 12} \\ &\quad -\epsilon\tilde{Z}_{\infty 12}S_{21}\tilde{Z}_{\infty 12} - \tilde{Z}_{\infty 11}S_{12}\tilde{Z}_{\infty 22} - \tilde{Z}_{\infty 12}S_{22}\tilde{Z}_{\infty 22} = 0 \qquad (28)\\ &\epsilon A'_{12}\tilde{Z}_{\infty 12} + A'_{22}\tilde{Z}_{\infty 22} + \epsilon\tilde{Z}'_{\infty 12}A_{12} + \tilde{Z}_{\infty 22}A_{22} + Q_{22} - \epsilon^2\tilde{Z}'_{\infty 12}S_{11}\tilde{Z}_{\infty 12} \\ &\quad -\epsilon\tilde{Z}_{\infty 22}S_{21}\tilde{Z}_{\infty 12} - \epsilon\tilde{Z}'_{\infty 12}S_{12}\tilde{Z}_{\infty 22} - \tilde{Z}_{\infty 22}S_{22}\tilde{Z}_{\infty 22} = 0 \qquad (29)\end{aligned}$$

Letting $Z_{\infty 11}$, $Z_{\infty 12}$ and $Z_{\infty 22}$ denote the limiting solutions to the above equations as $\epsilon \downarrow 0$,[2] we obtain (by setting $\epsilon = 0$ in (27)–(29)) that $Z_{\infty 11}$, $Z_{\infty 12}$ and $Z_{\infty 22}$ satisfy the following coupled matrix equations:

$$\begin{aligned} &A'_{11}Z_{\infty 11} + A'_{21}Z_{\infty 21} + Z_{\infty 11}A_{11} + Z_{\infty 12}A_{21} + Q_{11} - Z_{\infty 11}S_{11}Z_{\infty 11} \\ &\quad -Z_{\infty 12}S_{21}Z_{\infty 11} - Z_{\infty 11}S_{12}Z_{\infty 21} - Z_{\infty 12}S_{22}Z_{\infty 21} = 0 \qquad (30) \\ &A'_{21}Z_{\infty 22} + Z_{\infty 11}A_{12} + Z_{\infty 12}A_{22} + Q_{12} - Z_{\infty 11}S_{12}Z_{\infty 22} - Z_{\infty 12}S_{22}Z_{\infty 22} = 0 \qquad (31) \\ &A'_{22}Z_{\infty 22} + Z_{\infty 22}A_{22} + Q_{22} - Z_{\infty 22}S_{22}Z_{\infty 22} = 0 \qquad (32) \end{aligned}$$

Thus completing the analysis of the direct solution to the full-order problem as $\epsilon \downarrow 0$, we now turn, in the next two sections, to the original goal of this paper — which is the derivation of the approximate solution based on a time-scale decomposition. First we identify, in Section 3, the slow and fast subsystems associated with the original problem, and obtain the solutions of two separate H^∞-optimal control problems, one defined on the slow time scale and the other one on the fast time scale.

3 A TIME-SCALE DECOMPOSITION

The Slow Subsystem and the Associated Soft-Constrained Game

To obtain the slow dynamics associated with (1), we let $\epsilon = 0$ and solve for x_2 (to be denoted $\bar{x}_2$) in terms of $x_1 =: x_s$, $u =: u_s$, $w =: w_s$, and under the working assumption A3:

$$\bar{x}_2 = -A_{22}^{-1}(A_{21}x_s + B_2u_s + D_2w_s) \qquad (33)$$

Using this in the first equation of (1), we obtain the reduced-order (slow) dynamics:

$$\dot{x}_s = A_0x_s + B_0u_s + D_0w_s \qquad (34)$$

where

$$\begin{aligned} A_0 &:= A_{11} - A_{12}A_{22}^{-1}A_{21} \qquad (35) \\ B_0 &:= B_1 - A_{12}A_{22}^{-1}B_2 \qquad (36) \\ D_0 &:= D_1 - A_{12}A_{22}^{-1}D_2 \qquad (37) \end{aligned}$$

Using (33) also in the cost function (3) leads to the reduced (slow) cost (with $x_1 = x_s$):

$$L_s = |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f} (|x_s|^2_{Q_{11}} + x'_sQ_{12}\bar{x}_2 + \bar{x}'_2Q_{21}x_s + |\bar{x}_2|^2_{Q_{22}} + |u_s|^2)\, dt \qquad (38)$$

In view of our earlier discussion, the H^∞-optimal control problem with state equation (34) and cost function (38) is closely related to the zero-sum differential game with the same state equation and with cost function

$$L_{\gamma s} = L_s - \gamma^2\|w_s\|^2 \qquad (39)$$

[2]The existence of such limits will be verified later in Section (see, Theorem 1).

and the quantity of interest is the upper value of this game. But note that this differential game (henceforth called the "slow game") is structurally somewhat different from the "full" game discussed in the previous section, because the cost function (38) now includes "cross terms" between the state x_s, control u_s, and disturbance w_s. The presence of these cross terms makes it necessary to distinguish between two different types of information structures: one where the control is allowed to depend only on the state (and not directly on the disturbance), and another where the control is allowed to depend on both (the current values of) the state and the disturbance. We will refer to a controller of the former type as *state feedback* (SF) controller, and to one of the latter type as *full information* (FI) controller. As we shall see shortly, these controllers require different existence conditions, and hence lead to different optimal values for γ — of course the one associated with the SF controller being more restrictive than the one obtained under the FI controller. In the absence of the cross terms, however, no such distinction exists (see Başar and Bernhard, 1991, Chapter 7).

We now study the solution to the slow differential game under these two controllers (or information structures) separately, since both will be needed in the analysis to follow in Section 4. Let us first consider the case of the FI controller, which essentially involves a disturbance feedforward term. Compatible with the given information structure, we introduce the transformations

$$\begin{aligned}\tilde{u}_s &= (I + B_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} B_2)^{1/2} [u_s + (I + B_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} B_2)^{-1} \\ &\quad \cdot B_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} ((A_{21} - A_{22} Q_{22}^{-1} Q_{21}) x_s + D_2 w_s)] \qquad (40)\\ \tilde{w}_s &= (\gamma^2 I - D_2' (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} D_2)^{1/2} \\ &\quad \cdot [w_s + (\gamma^2 I - D_2' (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} D_2)^{-1} \\ &\quad \cdot D_2' (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} (A_{22} Q_{22}^{-1} Q_{21} - A_{21}) x_s] \qquad (41)\end{aligned}$$

where the latter one is valid under the condition

$$\gamma^2 I - D_2' (A_{22} Q_{22}^{-1} A_{22}' + B_2 B_2')^{-1} D_2 > 0 \qquad (42)$$

which can equivalently be written as

$$S_{22}(\gamma) + A_{22} Q_{22}^{-1} A_{22}' > 0 \qquad (43)$$

where

$$S_{22}(\gamma) := B_2 B_2' - \frac{1}{\gamma^2} D_2 D_2'. \qquad (44)$$

If we use (40) and (41) in (34) and (39), we arrive at the following standard LQ differential game, which has no cross terms between state, control and disturbance in the cost (see Appendix A for details of the underlying manipulations):

$$\dot{x}_s = \tilde{A}_0 x_s + \tilde{B}_0 \tilde{u}_s + \tilde{D}_0 \tilde{w}_s; \quad x_s(0) = 0 \qquad (45)$$

$$L_{\gamma s} = |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f} (|x_s|^2_{\tilde{Q}} + |\tilde{u}_s|^2 - |\tilde{w}_s|^2)\, dt \qquad (46)$$

The coefficient matrices above are explicit functions of the parameter γ, and are written as:

$$\begin{aligned}
\tilde{A}_0(\gamma) &= A_{11} - A_{12}Q_{22}^{-1}Q_{21} - (S_{12} + A_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) && (47)\\
\tilde{B}_0(\gamma) &= B_0(I + B'_2 A'_{22}{}^{-1}Q_{22}A_{22}^{-1}B_2)^{-1/2} && (48)\\
\tilde{D}_0(\gamma) &= (D_0 - B_0B'_2(A_{22}Q_{22}^{-1}A'_{22} + B_2B'_2)^{-1}D_2) \\
&\quad \cdot(\gamma^2 I - D'_2(A_{22}Q_{22}^{-1}A'_{22} + B_2B'_2)^{-1}D_2)^{-1/2} && (49)\\
\tilde{Q}(\gamma) &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (A'_{21} - Q_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) && (50)
\end{aligned}$$

For each $\gamma \geq 0$, we associate the following generalized RDE with this standard LQ differential game:

$$\dot{\tilde{Z}}_s + \tilde{A}'_0\tilde{Z}_s + \tilde{Z}_s\tilde{A}_0 - \tilde{Z}_s(\tilde{B}_0\tilde{B}'_0 - \tilde{D}_0\tilde{D}'_0)\tilde{Z}_s + \tilde{Q} = 0; \qquad \tilde{Z}_s(t_f) = Q_{f11}. \tag{51}$$

Let us introduce

$$S_0 := \tilde{B}_0\tilde{B}'_0 - \tilde{D}_0\tilde{D}'_0 \tag{52}$$

which can be rewritten in terms of the original system matrices as follows (see Appendix A for details):

$$\begin{aligned}
S_0 &= S_{11} + A_{12}Q_{22}^{-1}A'_{12} - (S_{12} + A_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot(S_{21} + A_{22}Q_{22}^{-1}A'_{12}) && (53)
\end{aligned}$$

In view of this , let us introduce the set:

$$\begin{aligned}
\tilde{\Gamma}_s &:= \{\gamma' > 0 : \forall\gamma \geq \gamma', S_{22}(\gamma) + A_{22}Q_{22}^{-1}A'_{22} > 0 \text{ and (51) has a bounded} \\
&\qquad \text{nonnegative definite solution over } [0, t_f].\} && (54)
\end{aligned}$$

and further define

$$\gamma_s := \inf\{\gamma \in \tilde{\Gamma}_s\} \tag{55}$$

Then, it follows from the analysis of Başar and Bernhard (1991), Chapter 8, that the transformed game with cost function $L_{\gamma s}$ has a bounded upper value[3] if $\gamma > \gamma_s$, and only if $\gamma \geq \gamma_s$. For $\gamma > \gamma_s$, let $Z_{s\gamma}$ be the unique nonnegative definite solution of (51). Then, there exist strongly time consistent (feedback) saddle-point policies for the transformed game, given by

$$\begin{aligned}
\tilde{u}^*_{s\gamma} &= \tilde{\mu}^*_{s\gamma}(t, x_s(t)) = -\tilde{B}'_0 Z_{s\gamma}x_s(t) && (56)\\
\tilde{w}^*_{s\gamma} &= \tilde{\nu}^*_{s\gamma}(t, x_s(t)) = \tilde{D}'_0 Z_{s\gamma}x_s(t) && (57)
\end{aligned}$$

[3]Note that under condition (43), $\tilde{Q}(\gamma) \geq 0$.

Applying the inverse transformation of (41) to (57), we arrive (after some matrix manipulations, details of which can be found in Appendix A) at the following expression:

$$\begin{aligned} w_{s\gamma}^* &= \nu_{s\gamma}^*(t, x_s(t)) = \frac{1}{\gamma^2}(D_1' Z_{s\gamma} - D_2'(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\ &\quad -(A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_s(t) \end{aligned} \tag{58}$$

The inverse transformation of (40) applied to (56) yields:

$$\begin{aligned} u_{sf\gamma}^* &= \mu_{sf\gamma}^*(t, x_s(t), w_s(t)) = -(I + B_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} B_2)^{-1}(B_0' Z_{s\gamma} \\ &\quad -B_2' A_{22}'^{-1} Q_{22} A_{22}^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}))x_s(t) \\ &\quad -B_2'(B_2 B_2' + A_{22}Q_{22}^{-1}A_{22}')^{-1} D_2 w_s \end{aligned} \tag{59}$$

which is a controller that achieves the finite upper value (which is zero) in the slow FI game. Note that this controller involves a feedforward term in w_s, as expected. If, however, we wish to obtain one which depends only on the state, we simply substitute (58) into (59), to arrive at the following expression:

$$\begin{aligned} u_{s\gamma}^* &= \mu_{s\gamma}^*(t, x_s(t)) = (-B_1' Z_{s\gamma} + B_2'(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\ &\quad -(A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_s(t) \end{aligned} \tag{60}$$

By the "ordered interchangeability" property of multiple saddle point equilibria (see Başar and Olsder, 1982), $\mu_{s\gamma}^*$ is also in equilibrium with $\nu_{s\gamma}^*$, but not for all $\gamma > \gamma_s$. In other words, there will exist some $\gamma \in \tilde{\Gamma}_s$ for which under $\mu_{s\gamma}^*$ the upper value of the game will be infinite, even though it is finite (actually zero) if $\mu_{sf\gamma}^*$ given by (59) is used. We will shortly see that the pair $(\mu_{s\gamma}^*, \nu_{s\gamma}^*)$ in fact provides a saddle-point solution to the SF slow game, but under a more restrictive condition (than (42)). Toward this end we now study the upper value of the slow game under the SF information pattern.

To bring $L_{\gamma s}$ to standard form, we now introduce the transformations given below:[4]

$$\begin{aligned} \hat{w}_s &= (\gamma^2 I - D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} D_2)^{1/2}[w_s - (\gamma^2 I - D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} D_2)^{-1} \\ &\quad \cdot D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1}(B_2 u_s + (A_{21} - A_{22}Q_{22}^{-1}Q_{21})x_s)] \end{aligned} \tag{61}$$

$$\begin{aligned} \hat{u}_s &= (I + B_2'(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2 D_2')^{-1}B_2)^{1/2} \\ &\quad \cdot [u_s + (I + B_2'(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2 D_2')^{-1}B_2)^{-1} \\ &\quad \cdot B_2'(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2 D_2')^{-1}(A_{21} - A_{22}Q_{22}^{-1}Q_{21})x_s] \end{aligned} \tag{62}$$

which are valid under the condition:

$$\begin{aligned} &\gamma^2 I - D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} D_2 > 0 \\ \Leftrightarrow\quad & A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2 D_2' > 0 \end{aligned} \tag{63}$$

[4]Note that the dependence of $\hat{w}_s$ on u_s is allowed here, because we are interested in the upper value of the game.

These transformations essentially "complete the squares" in $L_{\gamma s}$ so as to cancel out the cross terms in x_s, u_s and w_s. The transformed game then is

$$\dot{x}_s = \hat{A}_0 x_s + \hat{B}_0 \hat{u}_s + \hat{D}_0 \hat{w}_s; \quad x_s(0) = 0 \tag{64}$$

$$L_{\gamma s} = |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f} (|x_s|^2_{\hat{Q}} + |\hat{u}_s|^2 - |\hat{w}_s|^2)\, dt \tag{65}$$

where the coefficient matrices are (see Appendix A for details of the derivations)

$$\begin{aligned}
\hat{A}_0(\gamma) &= A_{11} - A_{12}Q_{22}^{-1}Q_{21} - (S_{12} + A_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot (A_{21} - A_{22}Q_{22}^{-1}Q_{21}) && (66)\\
\hat{B}_0(\gamma) &= (B_0 + \frac{1}{\gamma^2}D_0 D'_2 (A_{22}Q_{22}^{-1}A'_{22} - \frac{1}{\gamma^2}D_2 D'_2)^{-1} B_2) \\
&\quad \cdot (I + B'_2 (A_{22}Q_{22}^{-1}A'_{22} - \frac{1}{\gamma^2}D_2 D'_2)^{-1} B_2)^{-1/2} && (67)\\
\hat{D}_0(\gamma) &= D_0(\gamma^2 I - D'_2 {A'_{22}}^{-1} Q_{22} A_{22}^{-1} D_2)^{-1/2} && (68)\\
\hat{Q}(\gamma) &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (A'_{21} - Q_{12}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\
&\quad \cdot (A_{21} - A_{22}Q_{22}^{-1}Q_{21}) && (69)
\end{aligned}$$

The GRDE associated with this differential game is

$$\dot{\hat{Z}}_s + \hat{A}'_0 \hat{Z}_s + \hat{Z}_s \hat{A}_0 - \hat{Z}_s \hat{S}_0 \hat{Z}_s + \hat{Q} = 0; \qquad \hat{Z}_s(t_f) = Q_{f11} \tag{70}$$

where

$$\hat{S}_0 := \hat{B}_0 \hat{B}'_0 - \hat{D}_0 \hat{D}'_0 \tag{71}$$

Some matrix manipulations, details of which are again included in Appendix A, show that

$$\hat{A}_0 \equiv \tilde{A}_0 \quad \hat{Q} \equiv \tilde{Q} \quad \hat{S}_0 \equiv S_0$$

where $\tilde{A}_0$, $\tilde{Q}$, S_0 were defined earlier by (47), (50) and (53), respectively, and hence that the GRDE (70) is identical with the GRDE (51). It can further be shown that condition (63) implies condition (43), and hence that the set

$$\begin{aligned}
\hat{\Gamma}_s \;:=\; &\{\gamma' > 0 : \forall \gamma \geq \gamma',\ (63) \text{ holds and } (70) \text{ has a bounded} \\
&\qquad \text{nonnegative definite solution over } [0, t_f].\} && (72)
\end{aligned}$$

is included in $\tilde{\Gamma}_s$. Now, define

$$\hat{\gamma}_s := \inf\{\gamma \in \hat{\Gamma}_s\} \tag{73}$$

which is clearly not smaller than γ_s, defined by (55). Following the earlier reasoning, the slow differential game with the SF information pattern has a bounded upper value if $\gamma > \hat{\gamma}_s$, and only if $\gamma \geq \hat{\gamma}_s$. For $\gamma > \hat{\gamma}_s$, the game admits a strongly time consistent (feedback) saddle-point solution, given by the policies (for the transformed game):

$$\hat{u}^*_{s\gamma} = \hat{\mu}^*_{s\gamma}(t, x_s(t)) = -\hat{B}'_0 Z_{s\gamma} x_s(t) \tag{74}$$

$$\hat{w}^*_{s\gamma} = \hat{\nu}^*_{s\gamma}(t, x_s(t)) = \hat{D}'_0 Z_{s\gamma} x_s(t) \tag{75}$$

where $Z_{s\gamma}$ is the unique nonnegative definite solution of (51) (or equivalently (70)). Now applying the inverse transformation of (62) to (74), we obtain precisely the policy (60). This shows that for the slow game with SF controllers, (60) leads to a finite upper value, provided that $\gamma > \hat{\gamma}_s$. In general, $\hat{\gamma}_s > \gamma_s$, and hence one natural question is whether one could have a correction term, depending only on the state, added on to (60), which would improve the performance from $\hat{\gamma}_s$ to γ_s. The answer to this lies in the analysis of the "fast game" which is studied next, and is provided fully in Section .

The Fast Subsystem and the Associated Soft-Constrained Game

Let $x_f := x_2 - \bar{x}_2$, $u_f := u - u_s$, $w_f := w - w_s$ and $\tau = \frac{t'-t}{\epsilon}$ where we take t to be frozen, and t' to vary on the same scale as t. We define the fast subsystem and the associated cost (as in the standard regulator problem; see Chow and Kokotović, 1976) by:

$$\frac{d}{d\tau} x^t_f = A_{22}(t) x^t_f + B_2(t) u^t_f + D_2(t) w^t_f; \quad x^t_f(0) = x_f(t) \tag{76}$$

$$L^t_{\gamma f} = \int_0^\infty (|x^t_f|^2_{Q_{22}(t)} + |u^t_f|^2 - \gamma^2 |w^t_f|^2)\, d\tau \tag{77}$$

The GARE associated with this infinite-horizon game, for each t, is:

$$A'_{22}(t) Z_f + Z_f A_{22}(t) + Q_{22}(t) - Z_f S_{22}(t) Z_f = 0 \tag{78}$$

We now let γ^t_f denote the minimax disturbance attenuation bound for the H^∞-optimal control problem defined by (76)–(77) under closed-loop information[5], and γ^t_{of} denote the same under open-loop information, where we must have $\gamma^t_{of} \geq \gamma^t_f$ (see Başar and Bernhard, 1991). For every $\gamma > \gamma^t_f$, let $Z_{f\gamma}(t)$ be the minimal positive definite solution for (78). Then, for any $\gamma > \gamma^t_f$, the feedback controller that attains the finite upper value of the fast game is

$$\begin{aligned} u^{t*}_{f\gamma}(\tau) &= \mu^{t*}_{f\gamma}(x^t_f(\tau)) = -B'_2(t) Z_{f\gamma}(t) x^t_f(\tau) \\ \Rightarrow \quad \mu^*_{f\gamma}(t) &= \mu^{t*}_{f\gamma}(x^t_f(0)) = -B'_2(t) Z_{f\gamma}(t) x_f(t) \\ &= -B'_2(t) Z_{f\gamma}(t)(x_2(t) - \bar{x}_2(t)) \end{aligned} \tag{79}$$

Substitute (33), (60) and (58) into (79), to obtain

$$\begin{aligned} \mu^*_{f\gamma}(t, x(t)) = &-B'_2 Z_{f\gamma} x_2 - B'_2 Z_{f\gamma} Q^{-1}_{22} (A'_{12} Z_{s\gamma} + Q_{21} - A'_{22}(S_{22} + A_{22} Q^{-1}_{22} A'_{22})^{-1} \\ &\cdot((S_{21} + A_{22} Q^{-1}_{22} A'_{12}) Z_{s\gamma} - (A_{21} - A_{22} Q^{-1}_{22} Q_{21})))x_1(t) \end{aligned} \tag{80}$$

[5]To ensure that $\gamma^t_f < \infty$, it will be sufficient to take the pair $(A_{22}(t), B_2(t))$ to be controllable.

We now define

$$\gamma_f := \sup_{t\in[0,t_f]} \gamma_f^t \tag{81}$$

$$\gamma_{of} := \sup_{t\in[0,t_f]} \gamma_{of}^t \tag{82}$$

Then, for every $\gamma > \gamma_f$, the GARE (78) admits a positive definite solution for all $t \in [0, t_f]$. Let

$$\bar{\gamma} := \max\{\gamma_f, \gamma_s\} \tag{83}$$

We will shortly see that this value plays an important role in our problem.

The Infinite-Horizon Case

We now turn to the infinite horizon case. Let A, B, D and Q be time invariant and Q_f be zero. By following steps similar to those in the finite horizon case, we first decompose the system into slow and fast subsystems. After an appropriate transformation, the slow game with FI controller is described by

$$\dot{x}_s = \tilde{A}_0 x_s + \tilde{B}_0 \tilde{u}_s + \tilde{D}_0 \tilde{w}_s; \quad x_s(0) = 0 \tag{84}$$

$$L_{\gamma s} = \int_0^\infty (|x_s|^2_{\tilde{Q}} + |\tilde{u}_s|^2 - |\tilde{w}_s|^2)\, dt \tag{85}$$

where $\tilde{A}_0$, $\tilde{B}_0$, $\tilde{D}_0$ and $\tilde{Q}$ are as defined before, with the only difference being that they are now time invariant. The associated GARE is

$$\tilde{A}_0' \tilde{Z}_s + \tilde{Z}_s \tilde{A}_0 - \tilde{Z}_s S_0 \tilde{Z}_s + \tilde{Q} = 0 \tag{86}$$

Let us define the following set as the counterpart of (54):

$$\begin{aligned}\tilde{\Gamma}_{s\infty} \;:=\; & \{\gamma' > 0 : \forall\gamma \geq \gamma', S_{22} + A_{22}Q_{22}^{-1}A_{22}' > 0, \text{(86) has a nonnegative} \\ & \qquad \text{definite solution } Z_{s\gamma}, \text{ and } \tilde{A}_0'(\gamma) - S_0(\gamma)Z_{s\gamma} \text{ is Hurwitz.}\}\end{aligned}$$

and further define

$$\gamma_{s\infty} := \inf\{\gamma \in \tilde{\Gamma}_{s\infty}\}. \tag{87}$$

For every $\gamma > \gamma_{s\infty}$[6], let $Z_{s\gamma}$ be the minimal positive definite solution to (86). Then, for each $\gamma > \gamma_{s\infty}$, the FI controller that attains the upper value is the time-invariant version of (56), and the maximizing disturbance with respect to this control is the time-invariant version of (57). Then we can solve for $\mu^*_{s\gamma}$ and $\nu^*_{s\gamma}$, which are the time-invariant versions of (58) and

[6]Again, for $\gamma_{s\infty}$ to be finite, it will be sufficient to have the pair $(\tilde{A}_0, \tilde{B}_0)$ controllable at $\gamma = \infty$, which is equivalent to having the pair (A_0, B_0) controllable.

(59), respectively.[7] To obtain the corresponding SF controller, we substitute $\nu^*_{s\gamma}$ into $\mu^*_{sf\gamma}$, to arrive at an expression for $\mu^*_{s\gamma}$, which is the time-invariant version of (60). Note that this SF controller yields a finite upper value if

$$\gamma > \hat{\gamma}_{s\infty} := \inf\{\gamma \in \hat{\Gamma}_{s\infty}\}$$

where

$$\hat{\Gamma}_{s\infty} \;:=\; \{\gamma' \geq \gamma_{s\infty} : \forall\gamma > \gamma',\ (63)\text{ holds.}\}$$

The fast part of the system is the same as in the finite-horizon case, where the coefficient matrices are now constants. The fast game is described by (76)–(77), and the GARE is the same as (78). We will use $\gamma_{f\infty}$ to denote the minimax disturbance attenuation bound (and $\gamma_{of\infty}$ for the open-loop case), and let $Z_{f\gamma}$ denote the minimal positive definite solution to (78). Then, the minimizing controller is given by (80). As the counterpart of (83), we define

$$\bar{\gamma}_{\infty} := \max\{\gamma_{f\infty}, \gamma_{s\infty}\} \tag{88}$$

This quantity will also play an important role in our analysis in the next section.

4 MAIN RESULTS

Before we present the main results, we first give three useful results, listed below as Lemma 1, Lemma 2, and Fact 1.

Lemma 1 *Consider the following GARE:*

$$A'Z + ZA + Q - Z(BB' - \frac{1}{\gamma^2}DD')Z = 0 \tag{89}$$

where the matrices A, Q, B and D are taken to be quite general, apart from the requirement that $Q > 0$ and (A, B) is controllable. Let $0 \leq \gamma_m < \infty$ be the infimum of all γ's under which this GARE admits a nonnegative definite solution. Then, for $\gamma > \gamma_m$,

$$BB' - \frac{1}{\gamma^2}DD' + AQ^{-1}A' > 0 \tag{90}$$

Proof Since (A, Q) is observable, the nonnegative definite solution of the GARE is actually positive definite (see Başar and Bernhard, 1991, Chapter 4, Theorem 4.8). Let Z be the minimal positive definite solution of (89). It is well-known that, under the conditions specified, the matrix $A - (BB' - \frac{1}{\gamma^2}DD')Z$ is Hurwitz. Let $S := BB' - \frac{1}{\gamma^2}DD'$. Then, using standard matrix manipulations and the GARE (89), we obtain:

$$\begin{aligned}
&S + AQ^{-1}A' = S - Z^{-1}(-ZA)Q^{-1}A' \\
&= S - Z^{-1}(A'Z + Q - ZSZ)Q^{-1}A' = S - Z^{-1}A' + SZQ^{-1}A' - Z^{-1}A'ZQ^{-1}A' \\
&= (S - Z^{-1}A')(I + ZQ^{-1}A') = Z^{-1}(ZS - A')ZQ^{-1}(Q + A'Z)Z^{-1} \\
&= Z^{-1}(ZS - A')ZQ^{-1}Z(SZ - A)Z^{-1} > 0
\end{aligned}$$

[7] Note that here $\nu^*_{s\gamma}$ is maximizing under $\mu^*_{s\gamma}$, but $\mu^*_{s\gamma}$ is not necessarily minimizing under $\nu^*_{s\gamma}$; see Başar and Bernhard, 1991, pp. 94–96.

$\square$

Lemma 2 *Under assumptions A1 and A3, if $\gamma > \gamma_f$ (or $\gamma > \gamma_{f\infty}$, in the infinite horizon case), then $\tilde{Q}$ defined by (50) is nonnegative definite.*

Proof Since $\gamma > \gamma_f$ ($\gamma > \gamma_{f\infty}$) and $Q_{22} > 0$, it follows from Lemma 1 that $S_{22} + A_{22}Q_{22}^{-1}A'_{22} > 0$. Since nonnegative definiteness of Q implies $Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \geq 0$, the result follows. $\square$

Next we recall a well-known fact:

Fact 1 *Let A, $P \geq 0$ and $Q \geq 0$ be three square matrices of the same dimensions. Then if (A, Q) is observable, so are $(A, Q + P)$ and $(A + KP_s, Q + P)$, the latter for all K of appropriate dimensions, where $P'_sP_s = P$.* $\square$

The Composite Controller

We now introduce, for both finite and infinite horizons, the composite controller:

$$\mu^*_{c\gamma}(t, x) = \mu^*_{s\gamma}(t, x) + \mu^*_{f\gamma}(t, x) \tag{91}$$

where $\mu^*_{s\gamma}$ and $\mu^*_{f\gamma}$ were defined by (60) and (80) respectively, and $\gamma > \bar{\gamma}$ (or $\bar{\gamma}_\infty$, in the infinite horizon case). After some manipulations, this composite controller can be rewritten as

$$\mu^*_{c\gamma}(t, x) = -B' \begin{bmatrix} Z_{s\gamma} & 0 \\ Z_c & Z_{f\gamma} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{92}$$

where

$$\begin{aligned} Z_c &:= Z_{f\gamma}Q_{22}^{-1}(A'_{12}Z_{s\gamma} + Q_{21}) - (I + Z_{f\gamma}Q_{22}^{-1}A'_{22})(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1} \\ &\quad \cdot((S_{21} + A_{22}Q_{22}^{-1}A'_{12})Z_{s\gamma} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})) && (93) \\ &\equiv Z_{f\gamma}U + V && (94) \\ &\equiv Z_{f\gamma}(U_1Z_{s\gamma} + U_2) + V_1Z_{s\gamma} + V_2 && (95) \end{aligned}$$

and

$$\begin{aligned} U_1 &:= Q_{22}^{-1}A'_{12} - Q_{22}^{-1}A'_{22}(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1}(S_{21} + A_{22}Q_{22}^{-1}A'_{12}) && (96) \\ U_2 &:= Q_{22}^{-1}Q_{21} + Q_{22}^{-1}A'_{22}(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1}(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) && (97) \\ V_1 &:= -(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1}(S_{21} + A_{22}Q_{22}^{-1}A'_{12}) && (98) \\ V_2 &:= (S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1}(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) && (99) \end{aligned}$$

Now, we are in a position to present our main results, first for the infinite horizon and subsequently for the finite horizon cases.

The Infinite-Horizon Case

To guarantee that $\bar{\gamma}_\infty$ is finite (although the Theorem to be presented below is true also when it is infinite), we make two additional assumptions:

Assumptions

A4 $(A_0, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ is observable.

A5 (A_0, B_0) and (A_{22}, B_2) are controllable.

Theorem 1 *Consider the singularly perturbed system (1)–(6), with $t_f = \infty$, $Q_f = 0$ and A, B, D, Q time-invariant. If assumptions A1–A5 hold, then*

1. *$\gamma^*(\epsilon) \leq \bar{\gamma}_\infty$, asymptotically as $\epsilon \to 0$, where $\bar{\gamma}_\infty$, as defined in (88), is finite.*

2. *$\forall \gamma > \bar{\gamma}_\infty$, $\exists \epsilon_\gamma > 0$ such that $\forall \epsilon \in [0, \epsilon_\gamma)$, the GARE (26) admits a positive definite solution, and consequently, the game has a finite upper value. Furthermore, the minimal such solution can be approximated by*

$$\tilde{Z} = \begin{bmatrix} Z_{s\gamma} + O(\epsilon) & \epsilon Z_c' + O(\epsilon^2) \\ \epsilon Z_c + O(\epsilon^2) & \epsilon Z_{f\gamma} + O(\epsilon^2) \end{bmatrix} \tag{100}$$

3. *$\forall \gamma > \bar{\gamma}_\infty$, if we apply the composite controller $\mu_{c\gamma}^*$ to the system, then $\exists \epsilon_\gamma' > 0$ such that, $\forall \epsilon \in [0, \epsilon_\gamma')$, the disturbance attenuation level γ is attained for the full-order system.*

4. *$\forall \gamma > \max\{\gamma_{s\infty}, \gamma_{of\infty}\} = \max\{\hat{\gamma}_{s\infty}, \gamma_{of\infty}\}$, if we only apply the slow controller $\mu_{s\gamma}^*$ to the system, then $\exists \epsilon_\gamma'' > 0$ such that, $\forall \epsilon \in [0, \epsilon_\gamma'')$, the disturbance attenuation level γ is attained for the full-order system.*

Proof We first note that under assumptions A3, A4 and A5, both slow (with either SF or FI controllers) and fast games have saddle-point solutions for sufficiently large values of γ, since at $\gamma = \infty$ both problems become "regular" LQR problems admitting stabilizing optimal controllers. This shows that both $\gamma_{s\infty}$ and $\gamma_{f\infty}$ are finite, implying that $\bar{\gamma}_\infty$ is also finite. Now fix $\gamma > \bar{\gamma}_\infty$. Under assumptions A1–A4, we know from Lemma 1 that $S_{22} + A_{22}Q_{22}^{-1}A_{22}' > 0$. Then, by Lemma 2, $\tilde{Q} \geq 0$, and also by the nonnegative definiteness of Q, $Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \geq 0$. Since

$$\begin{aligned} Q_s &:= \begin{bmatrix} I & -A_{21}'A_{22}'^{-1} \end{bmatrix} \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} \begin{bmatrix} I \\ -A_{22}^{-1}A_{21} \end{bmatrix} \\ &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (Q_{12}Q_{22}^{-1} - A_{21}'A_{22}'^{-1})Q_{22}(Q_{22}^{-1}Q_{21} - A_{22}^{-1}A_{21}), \end{aligned}$$

by Fact 1, (A_0, Q_s) is observable. Also, we can rewrite (47) as follows:

$$\tilde{A}_0 = A_0 - \begin{bmatrix} B_0 & D_0 \end{bmatrix} \begin{bmatrix} B_2' \\ -\frac{1}{\gamma^2}D_2' \end{bmatrix} (S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1}(A_{21} - A_{22}Q_{22}^{-1}Q_{21})$$

By Fact 1, $(\tilde{A}_0, \tilde{Q})$ is observable. From the observability of pairs $(\tilde{A}_0, \tilde{Q})$ and (A_{22}, Q_{22}), we conclude that $Z_{s\gamma} > 0$, $Z_{f\gamma} > 0$ (see Başar and Bernhard, 1991, Chapter 4, Theorem 4.8). Furthermore, $A_{22} - S_{22}Z_{f\gamma}$ is Hurwitz for $\gamma > \gamma_{f\infty}$, and by definition $\tilde{A}_0 - S_0 Z_{s\gamma}$ is Hurwitz, for $\gamma > \gamma_{s\infty}$. From the observability of (A_{22}, Q_{22}), we can deduce that, for ϵ sufficiently small, the pair (A_ϵ, Q) is also observable (see Kokotović, Khalil, and O'Reilly, 1986, Corollary 6.1 p.79). Then the game (1)–(6) has a bounded upper value if (26) admits a positive definite solution, while the upper value is unbounded if there is no nonnegative definite solution. We show in Appendix C that the set of coupled matrix equations (30)–(32) admits a solution $Z_{11} = Z_{s\gamma}$, $Z_{12} = Z_c'$ and $Z_{22} = Z_{f\gamma}$. We now use the Implicit Function Theorem to complete the proofs of parts 1) and 2). Let $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$ be the vector forms of $\tilde{Z}_{11}$, $\tilde{Z}_{12}$ and $\tilde{Z}_{22}$ respectively. Let us write equation (27)–(29) as $\tilde{\xi}(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = 0$. We already know that $\tilde{\xi}(0, K_1, K_2, K_3) = 0$, where K_1, K_2 and K_3 are the vector forms of $Z_{s\gamma}$, Z_c and $Z_{f\gamma}$. Let

$$\tilde{\Xi}_{ij} = \frac{\partial \tilde{\xi}_i}{\partial \tilde{K}_j}\Big|_{(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = (0, K_1, K_2, K_3)} \quad i = 1,2,3 \quad j = 1,2,3.$$

Then

$$\begin{aligned}
&\tilde{\Xi}_{31} = 0; \quad \tilde{\Xi}_{32} = 0 \\
&\tilde{\Xi}_{33} = I_{n_2} \otimes (A_{22} - S_{22}Z_{f\gamma}) + (A_{22}' - Z_{f\gamma}S_{22}) \otimes I_{n_2} \\
&\tilde{\Xi}_{22} = (A_{22}' - Z_{f\gamma}S_{22}) \otimes I_{n_1}
\end{aligned}$$

where $\otimes$ refers to the Kronecker product. Furthermore, $\tilde{\Xi}_{33}$ and $\tilde{\Xi}_{22}$ are invertible. Solving for $\tilde{Z}_{12}$ from (28) we obtain

$$\begin{aligned}
\tilde{Z}_{12} \;=\; & -\tilde{Z}_{11}(A_{12} - S_{12}\tilde{Z}_{22})(A_{22} - S_{22}\tilde{Z}_{22})^{-1} \\
& -(Q_{12} + A_{21}'\tilde{Z}_{22})(A_{22} - S_{22}\tilde{Z}_{22})^{-1} + O(\epsilon)
\end{aligned} \tag{101}$$

and its substitution into (27) leads to an equation in terms of ϵ, $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$, which we denote as $\xi_1(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = 0$. Let

$$\Xi_{1j} = \frac{\partial \xi_1}{\partial \tilde{K}_j}\Big|_{(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = (0, K_1, K_2, K_3)} \qquad j = 1,2,3;$$

then

$$\begin{aligned}
&\Xi_{12} = 0 \\
&\Xi_{11} = I_{n_1} \otimes R + R' \otimes I_{n_1} \;,
\end{aligned} \tag{102}$$

where $R = \tilde{A}_0 - S_0 Z_{s\gamma}$ (see Appendix C). Hence,

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \tilde{\Xi}_{21} & \tilde{\Xi}_{22} & \tilde{\Xi}_{23} \\ \tilde{\Xi}_{31} & \tilde{\Xi}_{32} & \tilde{\Xi}_{33} \end{bmatrix} = \begin{bmatrix} \Xi_{11} & 0 & * \\ * & \tilde{\Xi}_{22} & * \\ 0 & 0 & \tilde{\Xi}_{33} \end{bmatrix} \text{ is nonsingular}$$

where $*$ refers to any constant matrix. This implies that

$$\begin{bmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \tilde{\Xi}_{13} \\ \tilde{\Xi}_{21} & \tilde{\Xi}_{22} & \tilde{\Xi}_{23} \\ \tilde{\Xi}_{31} & \tilde{\Xi}_{32} & \tilde{\Xi}_{33} \end{bmatrix} \text{ is also nonsingular}$$

Then, by the Implicit Function Theorem, $\exists \epsilon_1 > 0$ such that $\forall \epsilon \in [0, \epsilon_1)$, there exist $\tilde{Z}_{11}(\epsilon)$, $\tilde{Z}_{12}(\epsilon)$ and $\tilde{Z}_{22}(\epsilon)$ that solve (27)–(29) and $\tilde{Z}_{11}(\epsilon) = Z_{s\gamma} + O(\epsilon)$, $\tilde{Z}_{12}(\epsilon) = Z_c' + O(\epsilon)$ and $\tilde{Z}_{22}(\epsilon) = Z_{f\gamma} + O(\epsilon)$. Since $Z_{s\gamma} > 0$ $Z_{f\gamma} > 0$ and $Z_{s\gamma}$, $Z_{f\gamma}$ are the minimal positive definite solutions for (86), (78), $\exists \epsilon_\gamma$, $0 < \epsilon_\gamma \leq \epsilon_1$, such that $\forall \epsilon \in [0, \epsilon_\gamma)$, the minimal positive definite solution[8] for (27)–(29) is

$$\tilde{Z}(\epsilon) = \begin{bmatrix} \tilde{Z}_{11}(\epsilon) & \epsilon \tilde{Z}_{12}(\epsilon) \\ \epsilon \tilde{Z}_{21}(\epsilon) & \epsilon \tilde{Z}_{22}(\epsilon) \end{bmatrix} > 0$$

So far, we have proved parts 1) and 2). For part 3), we use $\mu_{c\gamma}^*$ in system (1)–(6):

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} - B_1 B_1' Z_{s\gamma} - B_1 B_2' Z_c & A_{12} - B_1 B_2' Z_{f\gamma} \\ A_{21} - B_2 B_1' Z_{s\gamma} - B_2 B_2' Z_c & A_{22} - B_2 B_2' Z_{f\gamma} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w \quad (103) \\ L_\gamma &= \int_0^\infty (|x|_Q^2 - \gamma^2 |w|^2)\, dt \quad (104) \end{aligned}$$

The ARE associated with this maximization problem is:

$$\bar{A}_\epsilon' \tilde{W} + \tilde{W} \bar{A}_\epsilon + \frac{1}{\gamma^2} \tilde{W} D_\epsilon D_\epsilon' \tilde{W} + \bar{Q} = 0 \quad (105)$$

where

$$\begin{aligned} \bar{A}_\epsilon &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \frac{1}{\epsilon}\bar{A}_{21} & \frac{1}{\epsilon}\bar{A}_{22} \end{bmatrix} && (106) \\ \bar{A}_{11} &= A_{11} - B_1 B_1' Z_{s\gamma} - B_1 B_2' Z_c && (107) \\ \bar{A}_{12} &= A_{12} - B_1 B_2' Z_{f\gamma} && (108) \\ \bar{A}_{21} &= A_{21} - B_2 B_1' Z_{s\gamma} - B_2 B_2' Z_c && (109) \\ \bar{A}_{22} &= A_{22} - B_2 B_2' Z_{f\gamma} && (110) \\ \bar{Q} &= Q + \begin{bmatrix} Z_{s\gamma} & Z_c' \\ 0 & Z_{f\gamma} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1' & B_2' \end{bmatrix} \begin{bmatrix} Z_{s\gamma} & 0 \\ Z_c & Z_{f\gamma} \end{bmatrix} && (111) \\ &= \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} && (112) \end{aligned}$$

[8]The positive definiteness follows from $Z_{s\gamma} > 0$, $Z_{f\gamma} > 0$ and the following well-known result: Let

$$Z := \begin{bmatrix} Q & P \\ P' & R \end{bmatrix}$$

where Q and R are positive definite matrices. Then $Z > 0$ if and only if $Q - PR^{-1}P' > 0$.

By Fact 1, the observability of (A_ϵ, Q) implies the observability of $(\bar{A}_\epsilon, \bar{Q})$. Then the maximization problem (103)–(104) has a finite cost if (105) admits a positive definite solution, and the cost is unbounded if (105) does not have a nonnegative definite solution. Suppose that $\tilde{W}$ is in the form:

$$\tilde{W} = \begin{bmatrix} \tilde{W}_{11} & \epsilon\tilde{W}_{12} \\ \epsilon\tilde{W}_{21} & \epsilon\tilde{W}_{22} \end{bmatrix}$$

where $\tilde{W}_{12} = \tilde{W}_{21}'$. Expanding (105) around $\epsilon = 0$ we obtain:

$$\begin{aligned}
&\bar{A}_{11}'\tilde{W}_{11} + \bar{A}_{21}'\tilde{W}_{12}' + \tilde{W}_{11}\bar{A}_{11} + \tilde{W}_{12}\bar{A}_{21} + \bar{Q}_{11} + \frac{1}{\gamma^2}(\tilde{W}_{11}D_1D_1'\tilde{W}_{11} \\
&\quad + \tilde{W}_{12}D_2D_1'\tilde{W}_{11} + \tilde{W}_{11}D_1D_2'\tilde{W}_{12}' + \tilde{W}_{12}D_2D_2'\tilde{W}_{12}') = 0 && (113) \\
&\epsilon\bar{A}_{11}'\tilde{W}_{12} + \bar{A}_{21}'\tilde{W}_{22} + \tilde{W}_{11}\bar{A}_{12} + \tilde{W}_{12}\bar{A}_{22} + \bar{Q}_{12} + \frac{1}{\gamma^2}(\epsilon\tilde{W}_{11}D_1D_1'\tilde{W}_{12} \\
&\quad + \epsilon\tilde{W}_{12}D_2D_1'\tilde{W}_{12} + \tilde{W}_{11}D_1D_2'\tilde{W}_{22} + \tilde{W}_{12}D_2D_2'\tilde{W}_{22}) = 0 && (114) \\
&\epsilon\bar{A}_{12}'\tilde{W}_{12} + \bar{A}_{22}'\tilde{W}_{22} + \epsilon\tilde{W}_{12}'\bar{A}_{12} + \tilde{W}_{22}\bar{A}_{22} + \bar{Q}_{22} + \frac{1}{\gamma^2}(\epsilon^2\tilde{W}_{12}'D_1D_1'\tilde{W}_{12} \\
&\quad + \epsilon\tilde{W}_{22}D_2D_1'\tilde{W}_{12} + \epsilon\tilde{W}_{12}'D_1D_2'\tilde{W}_{22} + \tilde{W}_{22}D_2D_2'\tilde{W}_{22}) = 0 && (115)
\end{aligned}$$

It is straightforward to verify that when $\epsilon = 0$, $\tilde{W}_{11} = Z_{s\gamma}$, $\tilde{W}_{12} = Z_c'$ and $\tilde{W}_{22} = Z_{f\gamma}$ satisfy (113)–(115). Let $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$ be the vector forms of $\tilde{W}_{11}$, $\tilde{W}_{12}$ and $\tilde{W}_{22}$ respectively. Viewing equation (113)–(115) as the vector equation $\tilde{\eta}(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = 0$, we let K_1, K_2 and K_3 be the vector forms of $Z_{s\gamma}$, Z_c and $Z_{f\gamma}$, and introduce

$$\tilde{\Xi}_{ij} = \frac{\partial\tilde{\eta}_i}{\partial\tilde{K}_j}\Big|_{(\epsilon,\tilde{K}_1,\tilde{K}_2,\tilde{K}_3)=(0,K_1,K_2,K_3)} \quad i = 1,2,3 \quad j = 1,2,3$$

Then

$$\begin{aligned}
&\tilde{\Xi}_{31} = 0; \quad \tilde{\Xi}_{32} = 0 \\
&\tilde{\Xi}_{33} = I_{n_2} \otimes (A_{22} - S_{22}Z_{f\gamma}) + (A_{22}' - Z_{f\gamma}S_{22}) \otimes I_{n_2} \\
&\tilde{\Xi}_{22} = (A_{22}' - Z_{f\gamma}S_{22}) \otimes I_{n_1}
\end{aligned}$$

which shows that $\tilde{\Xi}_{33}$ and $\tilde{\Xi}_{22}$ are nonsingular. Solving for $\tilde{W}_{12}$ from equation (114) we obtain:

$$\begin{aligned}
\tilde{W}_{12} &= -\tilde{W}_{11}(\bar{A}_{12} + \frac{1}{\gamma^2}D_1D_2'\tilde{W}_{22})(\bar{A}_{22} + \frac{1}{\gamma^2}D_2D_2'\tilde{W}_{22})^{-1} \\
&\quad -(\bar{Q}_{12} + \bar{A}_{21}'\tilde{W}_{22})(\bar{A}_{22} + \frac{1}{\gamma^2}D_2D_2'\tilde{W}_{22})^{-1} + O(\epsilon) \qquad (116)
\end{aligned}$$

Substitution of this into (113) leads to an equation in terms of ϵ, $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$, which we write as $\eta_1(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = 0$. Let

$$\Xi_{1j} = \frac{\partial\eta_1}{\partial\tilde{K}_j}\Big|_{(\epsilon,\tilde{K}_1,\tilde{K}_2,\tilde{K}_3)=(0,K_1,K_2,K_3)} \qquad j = 1,2,3.$$

Then, we have the structure:

$$\Xi_{12} = 0$$
$$\Xi_{11} = I_{n_1} \otimes R + R' \otimes I_{n_1} \tag{117}$$

where we can readily see, from the analysis of Appendix C, that $R = \tilde{A}_0 - S_0 Z_{s\gamma}$. Then Ξ_{11} is invertible by the stability of R, and thus 3) is established by the Implicit Function Theorem.

For part 4), we first compare (98) and (99) with (60), to obtain $\mu^*_{s\gamma} = -(B_1' Z_{s\gamma} + B_2' V)x_s(t)$. Substituting $\mu^*_{s\gamma}$ into system (1)–(6), we arrive at the maximization problem:

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - B_1 B_1' Z_{s\gamma} - B_1 B_2' V & A_{12} \\ A_{21} - B_2 B_1' Z_{s\gamma} - B_2 B_2' V & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w \tag{118}$$

$$J = \int_0^\infty (|x|^2_{\hat{Q}} - \gamma^2 |w|^2)\, dt \longrightarrow \text{maximize} \tag{119}$$

The associated ARE is

$$\hat{A}_\epsilon' \tilde{Y} + \tilde{Y} \hat{A}_\epsilon + \frac{1}{\gamma^2} \tilde{Y} D_\epsilon D_\epsilon' \tilde{Y} + \hat{Q} = 0 \tag{120}$$

where

$$\hat{A}_\epsilon = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \frac{1}{\epsilon}\hat{A}_{21} & \frac{1}{\epsilon}\hat{A}_{22} \end{bmatrix} \tag{121}$$

$$\hat{A}_{11} = A_{11} - B_1 B_1' Z_{s\gamma} - B_1 B_2' V \tag{122}$$

$$\hat{A}_{12} = A_{12} \tag{123}$$

$$\hat{A}_{21} = A_{21} - B_2 B_1' Z_{s\gamma} - B_2 B_2' V \tag{124}$$

$$\hat{A}_{22} = A_{22} \tag{125}$$

$$\hat{Q} = Q + \begin{bmatrix} Z_{s\gamma} & V' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1' & B_2' \end{bmatrix} \begin{bmatrix} Z_{s\gamma} & 0 \\ V & 0 \end{bmatrix} \tag{126}$$

$$= \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix} \tag{127}$$

By Fact 1, the observability of (A_ϵ, Q) implies the observability of $(\hat{A}_\epsilon, \hat{Q})$. Then the optimization problem (118)–(119) has a finite cost if (120) admits a positive definite solution and the cost is not bounded if (120) does not have a nonnegative definite solution. Also, $\gamma > \gamma_{of\infty}$ implies that $-\frac{1}{\gamma^2} D_2 D_2' + A_{22} Q_{22}^{-1} A_{22}' > 0$ by Lemma 1, and consequently $S_{22} + A_{22} Q_{22}^{-1} A_{22}' > 0$. Thus, the equality $\max\{\gamma_{s\infty}, \gamma_{of\infty}\} = \max\{\hat{\gamma}_{s\infty}, \gamma_{of\infty}\}$ holds by the definitions of $\gamma_{s\infty}$ and $\hat{\gamma}_{s\infty}$. Starting again with the form:

$$\tilde{Y} = \begin{bmatrix} \tilde{Y}_{11} & \epsilon \tilde{Y}_{12} \\ \epsilon \tilde{Y}_{21} & \epsilon \tilde{Y}_{22} \end{bmatrix}$$

where $\tilde{Y}_{12} = \tilde{Y}'_{21}$, we expand (120) around $\epsilon = 0$:

$$\hat{A}'_{11}\tilde{Y}_{11} + \hat{A}'_{21}\tilde{Y}'_{12} + \tilde{Y}_{11}\hat{A}_{11} + \tilde{Y}_{12}\hat{A}_{21} + \hat{Q}_{11} + \frac{1}{\gamma^2}(\tilde{Y}_{11}D_1D'_1\tilde{Y}_{11}$$
$$+\tilde{Y}_{12}D_2D'_1\tilde{Y}_{11} + \tilde{Y}_{11}D_1D'_2\tilde{Y}'_{12} + \tilde{Y}_{12}D_2D'_2\tilde{Y}'_{12}) = 0 \quad (128)$$
$$\epsilon\hat{A}'_{11}\tilde{Y}_{12} + \hat{A}'_{21}\tilde{Y}_{22} + \tilde{Y}_{11}\hat{A}_{12} + \tilde{Y}_{12}\hat{A}_{22} + \hat{Q}_{12} + \frac{1}{\gamma^2}(\epsilon\tilde{Y}_{11}D_1D'_1\tilde{Y}_{12}$$
$$+\epsilon\tilde{Y}_{12}D_2D'_1\tilde{Y}_{12} + \tilde{Y}_{11}D_1D'_2\tilde{Y}_{22} + \tilde{Y}_{12}D_2D'_2\tilde{Y}_{22}) = 0 \quad (129)$$
$$\epsilon\hat{A}'_{12}\tilde{Y}_{12} + \hat{A}'_{22}\tilde{Y}_{22} + \epsilon\tilde{Y}'_{12}\hat{A}_{12} + \tilde{Y}_{22}\hat{A}_{22} + \hat{Q}_{22} + \frac{1}{\gamma^2}(\epsilon^2\tilde{Y}'_{12}D_1D'_1\tilde{Y}_{12}$$
$$+\epsilon\tilde{Y}_{22}D_2D'_1\tilde{Y}_{12} + \epsilon\tilde{Y}'_{12}D_1D'_2\tilde{Y}_{22} + \tilde{Y}_{22}D_2D'_2\tilde{Y}_{22}) = 0 \quad (130)$$

It is shown in Appendix C that for $\epsilon = 0$, $\tilde{Y}_{11} = Z_{s\gamma}$, $\tilde{Y}_{21} = Z_{of\gamma}U + V$ and $\tilde{Y}_{22} = Z_{of\gamma}$ satisfy (128)–(130), where $Z_{of\gamma}$ denotes the solution to the "open-loop" GARE of the fast subgame:

$$A'_{22}Z_f + Z_fA_{22} + Z_f\frac{1}{\gamma^2}D_2D'_2Z_f + Q_{22} = 0$$

We now show how the Implicit Function Theorem can again be employed to prove 4). Let $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$ be the vector forms of $\tilde{Y}_{11}$, $\tilde{Y}_{12}$ and $\tilde{Y}_{22}$ respectively. Writing equations (128)–(130) as $\tilde{\lambda}(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = 0$, we let K_1, K_2 and K_3 to be the vector forms of $Z_{s\gamma}$, $V' + U'Z_{of\gamma}$ and $Z_{of\gamma}$. Introducing

$$\tilde{\Xi}_{ij} = \frac{\partial\tilde{\lambda}_i}{\partial\tilde{K}_j}|_{(\epsilon,\tilde{K}_1,\tilde{K}_2,\tilde{K}_3)=(0,K_1,K_2,K_3)} \quad i = 1,2,3 \quad j = 1,2,3,$$

we arrive at

$$\tilde{\Xi}_{31} = 0; \quad \tilde{\Xi}_{32} = 0$$
$$\tilde{\Xi}_{33} = I_{n_2} \otimes (A_{22} + \frac{1}{\gamma^2}D_2D'_2Z_{f\gamma}) + (A'_{22} + \frac{1}{\gamma^2}Z_{f\gamma}D_2D'_2) \otimes I_{n_2}$$
$$\tilde{\Xi}_{22} = (A'_{22} + \frac{1}{\gamma^2}Z_{f\gamma}D_2D'_2) \otimes I_{n_1}$$

Since $\gamma > \gamma_{of\infty}$, $A_{22} + \frac{1}{\gamma^2}D_2D'_2Z_{of\gamma}$ is Hurwitz, and hence $\tilde{\Xi}_{33}$ and $\tilde{\Xi}_{22}$ are invertible. From equation (129):

$$\tilde{Y}_{12} = -\tilde{Y}_{11}(A_{12} + \frac{1}{\gamma^2}D_1D'_2\tilde{Y}_{22})(A_{22} + \frac{1}{\gamma^2}D_2D'_2\tilde{Y}_{22})^{-1}$$
$$-(Q_{12} + \hat{A}'_{21}\tilde{Y}_{22})(A_{22} + \frac{1}{\gamma^2}D_2D'_2\tilde{Y}_{22})^{-1} + O(\epsilon) \quad (131)$$

Substitution of (131) into (128) leads to an equation in terms of ϵ, $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$, which we write as $\lambda_1(\epsilon, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3) = 0$. Let

$$\Xi_{1j} = \frac{\partial\lambda_1}{\partial\tilde{K}_j}|_{(\epsilon,\tilde{K}_1,\tilde{K}_2,\tilde{K}_3)=(0,K_1,K_2,K_3)} \qquad j = 1,2,3.$$

Then, we have the structure:

$$\Xi_{12} = 0$$
$$\Xi_{11} = I_{n_1} \otimes R + R' \otimes I_{n_1} \tag{132}$$

where $R = \tilde{A}_0 - S_0 Z_{s\gamma}$. (See Appendix C for the manipulations that lead to this expression for R.) Hence,

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \tilde{\Xi}_{21} & \tilde{\Xi}_{22} & \tilde{\Xi}_{23} \\ \tilde{\Xi}_{31} & \tilde{\Xi}_{32} & \tilde{\Xi}_{33} \end{bmatrix} = \begin{bmatrix} \Xi_{11} & 0 & * \\ * & \tilde{\Xi}_{22} & * \\ 0 & 0 & \tilde{\Xi}_{33} \end{bmatrix} \text{ is nonsingular}$$

where $*$ refers to any constant matrix. Part 4) of the Theorem then follows from the Implicit Function Theorem.

□

Remark 1 *Although the Theorem above provides only an upper bound for the H^∞-optimal performance level of the full-order plant, we have actually proved recently, in Pan and Başar (1992), that this upper bound is also a lower bound for the performance level for sufficiently small ϵ. Thus, we have the following limit:*[9]

$$\lim_{\epsilon \to 0+} \gamma^*_\infty(\epsilon) = \bar{\gamma}_\infty \tag{133}$$

The Finite-Horizon Case

Theorem 2 *For the singularly perturbed system (1)–(6), let assumptions A1–A3 be satisfied, the pair $(A_{22}(t), B_2(t))$ be controllable for each $t \in [0, t_f]$, and the following condition hold:*

$Q_{f22} \le Z_{f\gamma}(t_f)$[10]*, where $Z_{f\gamma}(t_f)$ is the solution to (78) at $t = t_f$, with γ fixed.*

Then,

1. *$\gamma^*(\epsilon) \le \bar{\gamma}$, asymptotically as $\epsilon \to 0$, where $\bar{\gamma}$, as defined in (83), is finite.*

2. *$\forall \gamma > \bar{\gamma}$, $\exists \epsilon_\gamma > 0$ such that $\forall \epsilon \in [0, \epsilon_\gamma)$, the GRDE (12) admits a positive definite solution, and consequently, the game (1)–(6) has a finite value. Furthermore, the solution to GRDE (12) can be approximated by*

$$\tilde{Z} = \begin{bmatrix} Z_{s\gamma}(t) + O(\epsilon) & \epsilon(Z_c'(t) + Z_{cb}'(\tau)) + O(\epsilon^2) \\ \epsilon(Z_c(t) + Z_{cb}(\tau)) + O(\epsilon^2) & \epsilon(Z_{f\gamma}(t) + Z_{fb}(\tau)) + O(\epsilon^2) \end{bmatrix} \tag{134}$$

for all $t \in [0, t_f]$, where $Z_{cb}(\tau)$ and $Z_{fb}(\tau)$ are the solutions to (22)–(23) and as $\tau \to -\infty$ they converge to 0 exponentially in the τ time scale.

[9] This result is not included here because the arguments are quite technical, and its counterpart in the finite-horizon case still has not been established.

[10] This condition is imposed in order to make the boundary layer term $L_{22}(\tau)$ converge to $Z_{f\gamma}(t_f)$.

3. $\forall\gamma > \bar{\gamma}$, *if we apply the composite controller* $\mu^*_{c\gamma}$ *to the system, then* $\exists\epsilon'_\gamma > 0$ *such that* $\forall\epsilon \in [0, \epsilon'_\gamma)$, *the disturbance attenuation level* γ *is attained for the full-order system.*

4. $\forall\gamma > \max\{\gamma_s, \gamma_{of}\} = \max\{\hat{\gamma}_s, \gamma_{of}\}$, *if we apply only the slow controller* $\mu^*_{s\gamma}$ *to the system,* $\exists\epsilon''_\gamma > 0$, *such that* $\forall\epsilon \in [0, \epsilon''_\gamma)$, *the disturbance attenuation level* γ *is attained for the full-order system.*

Proof By a reasoning analogous to that used in the proof of Theorem 1, assumption A3, together with the controllability of (A_{22}, B_2), leads to finiteness of γ_f, and hence to that of $\bar{\gamma}$. Now, fix $\gamma > \bar{\gamma}$, under assumptions A1–A3. Then, we get $Z_{f\gamma}(t) > 0$ and that $A_{22}(t) - S_{22}(t)Z_{f\gamma}(t)$ is Hurwitz. Following a reasoning as in the infinite-horizon case, we deduce that $Z_{11} = Z_{s\gamma}(t)$, $Z_{12} = Z'_c(t)$ and $Z_{22} = Z_{f\gamma}(t)$ satisfy equations (17) –(19). We now use Theorem L in Kokotović and Yackel (1972) to prove 1) and 2). (For convenience to the reader, we include this theorem in Appendix D.)

View equation (14)–(16) as a nonlinear singularly perturbed system. Obviously, condition L1 of Theorem L is satisfied. Since $Z_{s\gamma}(t)$ is the unique solution for (51) (see Başar and Bernhard, 1991, pp.78), L3 is satisfied. Following some of the reasoning used in the proof of Theorem 1, L4 is established. We only need to show L2, which is equivalent to showing that $L_{22}(\tau)$, $L_{12}(\tau)$ exist, and $L_{22}(\tau) \to Z_{f\gamma}(t_f)$, $L_{12}(\tau) \to Z'_c(t_f)$ as $\tau \to -\infty$, and $Z_{11}(t)(V'_1 + U'_1 Z_{f\gamma}(t)) + V'_2 + U'_2 Z_{f\gamma}(t)$, $Z_{f\gamma}(t)$ are the asymptotically stable equilibria for (20)– (21) uniformly in $Z_{11}(t)$ when t_f is replaced by t, where $L_{22}(\tau)$ and $L_{12}(\tau)$ satisfy (20)–(21). From the proof of Theorem 1, we have that $Z_{11}(t)(V'_1 + U'_1 Z_{f\gamma}(t)) + V'_2 + U'_2 Z_{f\gamma}(t)$, $Z_{f\gamma}(t)$ are the equilibria for (20)– (21). Then they are the asymptotically stable equilibria because the linearized system is asymptotically stable.

We now show that $L_{22}(\tau) \to Z_{f\gamma}(t_f)$ as $\tau \to -\infty$. If $Q_{f22} = Z_{f\gamma}(t_f)$, the result is trivial. Suppose therefore that $Z_{f\gamma}(t_f) > Q_{f22}$[11]. Consider (21), which we rewrite in the form:

$$\frac{d}{d\tau}L_{22}(\tau)^{-1} - A_{22}(t_f)L_{22}(\tau)^{-1} - L_{22}(\tau)^{-1}A_{22}(t_f)' + S_{22}(t_f)$$
$$-L_{22}(\tau)^{-1}Q_{22}(t_f)L_{22}(\tau)^{-1} = 0.$$

We evaluate equation (78) at $t = t_f$ and rewrite it as:

$$-A_{22}(t_f)Z_{f\gamma}(t_f)^{-1} - Z_{f\gamma}(t_f)^{-1}A_{22}(t_f)' + S_{22}(t_f) - Z_{f\gamma}(t_f)^{-1}Q_{22}(t_f)Z_{f\gamma}(t_f)^{-1} = 0.$$

Let $\tilde{\Delta} := L_{22}(\tau)^{-1} - Z_{f\gamma}(t_f)^{-1}$. Take the difference of the above two equations and obtain,

$$\frac{d}{d\tau}\tilde{\Delta} - A_{22}(t_f)\tilde{\Delta} - \tilde{\Delta}A_{22}(t_f)' - \tilde{\Delta}Q_{22}(t_f)\tilde{\Delta} - \tilde{\Delta}Q_{22}(t_f)Z_{f\gamma}(t_f)^{-1}$$
$$-Z_{f\gamma}(t_f)^{-1}Q_{22}(t_f)\tilde{\Delta} = 0.$$

[11]It is well known that $L_{22}(\tau)$ converges to $Z_{f\gamma}(t_f)$ as $\tau \to -\infty$ if $Q_{f22} = 0$ (see Başar and Bernhard, 1991, Chapter 4 Theorem 4.8). In the cases when Q_{f22} is not strictly positive definite or it is not strictly less than $Z_{f\gamma}$ the results are still true, which follow from the fact that an increase in the terminal condition (in the sense of matrices) leads to an increase in the solution to the RDE.

Let $\Delta(\tau') := (Z_{f\gamma}(t_f)\tilde{\Delta}Z_{f\gamma}(t_f))^{-1}$ where $\tau' = -\tau$. Then the above equation is equivalent to

$$\frac{d}{d\tau'}\Delta + A_f\Delta + \Delta A_f' - Z_{f\gamma}(t_f)^{-1}Q_{22}(t_f)Z_{f\gamma}(t_f)^{-1} = 0,$$

where $A_f = A_{22}(t_f) - S_{22}(t_f)Z_{f\gamma}(t_f)$. Now introduce the following system:

$$\frac{d}{d\tau'}y = A_f'y; \quad y(0) = y_0 \quad .$$

Obviously, this system is stable, and

$$\frac{d}{d\tau'}(y'\Delta y) = y'Z_{f\gamma}(t_f)^{-1}Q_{22}(t_f)Z_{f\gamma}(t_f)^{-1}y \quad ,$$

so that

$$y_0'e^{A_f\tau'}\Delta(\tau')e^{A_f'\tau'}y_0 = y_0'(\Delta(0) + \int_0^{\tau'} e^{A_f s}Z_{f\gamma}(t_f)^{-1}Q_{22}(t_f)Z_{f\gamma}(t_f)^{-1}e^{A_f's}\,ds\,)y_0.$$

Then, since y_0 was arbitrary,

$$\Delta(\tau') \;=\; e^{-A_f\tau'}(\Delta(0) + \int_0^{\tau'} e^{A_f s}Z_{f\gamma}(t_f)^{-1}Q_{22}(t_f)Z_{f\gamma}(t_f)^{-1}e^{A_f's}\,ds\,)e^{-A_f'\tau'}.$$

Since $Z_{f\gamma}(t_f) > Q_{f22} = L_{22}(0)$, we have $\Delta(0) > 0$, and consequently, $\Delta(\tau') > 0$. Hence $\tilde{\Delta}$ exists and is positive definite for every τ. By the stability of $A_{22}(t_f) - S_{22}Z_{f\gamma}(t_f)$, we deduce that $\Delta(\tau') \to +\infty$ as $\tau' \to +\infty$, which implies that $Z_{fb} \to 0$ as $\tau \to -\infty$ and the convergence rate is exponential in the τ time scale.

Now, we rewrite (20) as follows:

$$\begin{aligned}\frac{d}{d\tau}L_{12}(\tau) \;&=\; -L_{12}(\tau)(A_{22}(t_f) - S_{22}(t_f)Z_{f\gamma}(t_f) + S_{22}(t_f)(Z_{f\gamma}(t_f) - L_{22}(\tau)))\\ &\quad -(A_{21}'(t_f) - Z_{11}(t_f)S_{12}(t_f))L_{22}(\tau) - Z_{11}(t_f)A_{12}(t_f) - Q_{12}(t_f)\end{aligned}$$

Since $A_{22}(t_f) - S_{22}(t_f)Z_{f\gamma}(t_f)$ is Hurwitz, and $S_{22}(t_f)(Z_{f\gamma}(t_f) - L_{22}(\tau)) \to 0$ as $\tau \to -\infty$, in view of known results for stability of linear systems (see Coppel, 1965, p.70, Theorem 9), $L_{12}(\tau) \to Z_c'(t_f)$ exponentially. So L2 is satisfied. Then, by Theorem L, the solution to (14)–(16) exists for sufficiently small ϵ. Since eigenvalues $\lambda(A_{22}(t) - S_{22}(t)Z_{f\gamma}(t))$ depend continuously on t, there exists a $c > 0$ such that $\lambda(A_{22}(t) - S_{22}(t)Z_{f\gamma}(t)) \leq -c$. Then, by applying Theorem 3.1 in Chapter 1 of Kokotović, Khalil, and O'Reilly (1986), 1) and 2) follow readily.

For part 3), we first substitute $\mu_{c\gamma}^*$ into system (1)–(6), and then follow a reasoning similar to that used in the proof of Theorem 1. 3) to show (rather easily) that L1, L3, L4 are satisfied. It is only necessary to show L2 to prove 3). Introduce the boundary layer system,

$$\begin{aligned}\frac{d}{d\tau}L_{12}(\tau) \;&=\; -L_{12}(\tau)(A_{22}(t_f) - B_2(t_f)B_2'(t_f)Z_{f\gamma}(t_f) + \frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)L_{22}(\tau))\\ &\quad -(A_{21}'(t_f) - Z_{s\gamma}(t_f)S_{12}(t_f) - Z_c'(t_f)B_2(t_f)B_2'(t_f))L_{22}(\tau)\end{aligned}$$

$$
\begin{aligned}
& -Z_{s\gamma}(t_f)A_{12}(t_f) - Q_{12}(t_f) - Z_{s\gamma}(t_f)S_{12}(t_f)Z_{f\gamma}(t_f) \\
& -Z_c'(t_f)B_2(t_f)B_2'(t_f)Z_{f\gamma}(t_f); \quad L_{12}(0) = Q_{f12} \qquad (135) \\
\frac{d}{d\tau}L_{22}(\tau) \; = \; & -(A_{22}'(t_f) - Z_{f\gamma}(t_f)B_2(t_f)B_2'(t_f))L_{22}(\tau) - L_{22}(\tau)(A_{22}(t_f) \\
& -B_2(t_f)B_2'(t_f)Z_{f\gamma}(t_f) - Q_{22}(t_f) - Z_{f\gamma}(t_f)B_2(t_f)B_2'(t_f)Z_{f\gamma}(t_f) \\
& -L_{22}(\tau)\frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)L_{22}(\tau); \quad L_{22}(0) = Q_{f22} \qquad (136)
\end{aligned}
$$

It is obvious that $Z_{11}(t)(V_1' + U_1'Z_{f\gamma}(t)) + V_2' + U_2'Z_{f\gamma}(t)$, $Z_{f\gamma}(t)$ are the equilibria for the above system with t_f substituted by t. Then they are the asymptotically stable equilibria because the linearized system is asymptotically stable. Now consider (136). Since the result is trivially true when $Q_{f22} = Z_{f\gamma}(t_f)$, we only consider the case $Z_{f\gamma}(t_f) > Q_{f22}$. Let A_f be defined as before and $\tilde{\Delta}(\tau) := Z_{f\gamma}(t_f) - L_{22}(\tau)$. Subtract (136) from (78) at $t = t_f$, to arrive at

$$\frac{d}{d\tau}\tilde{\Delta} + A_f'\tilde{\Delta} + \tilde{\Delta}A_f - \tilde{\Delta}\frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)\tilde{\Delta} = 0.$$

Let $\Delta(\tau') = \tilde{\Delta}(\tau)^{-1}$, where $\tau' = -\tau$. Then

$$\frac{d}{d\tau'}\Delta + A_f\Delta + \Delta A_f' - \frac{1}{\gamma^2}D_2(t_f)D_2'(t_f) = 0$$

Define the corresponding system:

$$\frac{d}{d\tau'}y = A_f'y; \quad y(0) = y_0 \quad .$$

Obviously, this system is stable and

$$\frac{d}{d\tau'}(y'\Delta y) = y'\frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)y,$$

so that

$$y_0'e^{A_f\tau'}\Delta(\tau')e^{A_f'\tau'}y_0 = y_0'(\Delta(0) + \int_0^{\tau'} e^{A_f s}\frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)e^{A_f' s}\, ds\,)y_0.$$

Then, since y_0 was arbitrary,

$$\Delta(\tau') = e^{-A_f\tau'}(\Delta(0) + \int_0^{\tau'} e^{A_f s}\frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)e^{A_f' s}\, ds\,)e^{-A_f'\tau'}.$$

Since $Z_{f\gamma}(t_f) > Q_{f22} = L_{22}(0)$, we have $\Delta(0) > 0$, and consequently, $\Delta(\tau') > 0$. Hence, $\tilde{\Delta}$ exists and is positive definite for every τ. By the stability of $A_{22}(t_f) - S_{22}Z_{f\gamma}(t_f)$, we deduce that $\Delta(\tau') \to +\infty$ as $\tau' \to +\infty$, which implies that $L_{22}(\tau) \to Z_{f\gamma}(t_f)$ as $\tau \to -\infty$. Then, by an argument similar to that used in 1) and 2), $L_{12}(\tau) \to Z_c'(t_f)$ as $\tau \to -\infty$. Thus, L2 is established. Then 3) follows from Theorem L.

To prove part 4), we use a similar procedure. We first substitute $\mu^*_{s\gamma}$ into the system (1)–(6), and follow the reasoning of Theorem 1. 4), to show that L1, L3 and L4 are satisfied. To show L2, we introduce the boundary layer system,

$$\begin{aligned}
\frac{d}{d\tau}L_{12}(\tau) &= -L_{12}(\tau)(A_{22}(t_f)+\frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)L_{22}(\tau)) \\
&\quad -(A_{21}'(t_f)-Z_{s\gamma}(t_f)S_{12}(t_f)-V'(t_f)B_2(t_f)B_2'(t_f))L_{22}(\tau) \\
&\quad -Z_{s\gamma}(t_f)A_{12}(t_f)-Q_{12}(t_f); \quad L_{12}(0)=Q_{f12}, \qquad (137)\\
\frac{d}{d\tau}L_{22}(\tau) &= -A_{22}'(t_f)L_{22}(\tau)-L_{22}(\tau)A_{22}(t_f)-Q_{22}(t_f) \\
&\quad +L_{22}(\tau)\frac{1}{\gamma^2}D_2(t_f)D_2'(t_f)L_{22}(\tau); \quad L_{22}(0)=Q_{f22}. \qquad (138)
\end{aligned}$$

Since $Z_{of\gamma}(t_f)\geq Z_{f\gamma}(t_f)\geq Q_{f22}$, we can use a reasoning similar to that used in the proof of 3) to show that $L_{22}(\tau)\to Z_{of\gamma}(t_f)$ as $\tau\to-\infty$, and subsequently $L_{12}(\tau)\to V'+U'Z_{of\gamma}(t_f)$ as $\tau\to-\infty$. Thus, L2 is satisfied, which then establishes 4). □

5 EXAMPLES

We present in this section some numerical results for the infinite horizon case. As stressed earlier, the three quantities $\gamma_{s\infty}$, $\gamma_{f\infty}$ and $\gamma_{of\infty}$ play important roles in the computation of an approximate value for $\gamma^*_\infty(\epsilon)$. Furthermore, even though we know the order relationship $\gamma_{of\infty}\geq\gamma_{f\infty}$, there is no such relationship between $\gamma_{s\infty}$ and the other two quantities, and it is very much problem dependent as we will see shortly. In the examples below, we will also study the performances attained by the approximate controllers $\mu^*_{c\gamma}$, $\mu^*_{s\gamma}$ and $\mu^*_{sf\gamma}$, when they are applied to the original system. [12]

Example 1

Consider the following two-dimensional system and the performance index:

$$\begin{bmatrix}\dot{x}_1\\ \epsilon\dot{x}_2\end{bmatrix} = \begin{bmatrix}1&2\\2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}u+\begin{bmatrix}1\\3\end{bmatrix}w \qquad (139)$$

$$L_\gamma = \int_0^\infty(2x_1^2+2x_1x_2+x_2^2+|u|^2-\gamma^2|w|^2)\,dt \qquad (140)$$

Note that here the fast subsystem is open-loop unstable. By using a particular search algorithm, we readily compute the three basic quantities:

$$\begin{aligned}
\gamma_{s\infty} &= 2.3565\\
\gamma_{f\infty} &= 3.0000\\
\gamma_{of\infty} &= \infty
\end{aligned}$$

[12] In these examples, the relative accuracy is 0.002.

Note that here the three values satisfy the relationship $\gamma_{s\infty} < \gamma_{f\infty} < \gamma_{of\infty}$. We can also compute the minimax disturbance attenuation level $\gamma^*_\infty(\epsilon)$ of the full-order system (139)–(140) for different values of ϵ as shown in Table 1. Note that as $\epsilon \to 0$, $\gamma^*(\epsilon) \to \max\{\gamma_{s\infty}, \gamma_{f\infty}\}$.

Table 1: Optimum H^∞-performance level for the full-order system of Example 1

$\gamma^*(\epsilon)$	2.4332	2.7843	2.9724	2.9972	2.9997
ϵ	1	0.1	0.01	0.001	0.0001

Now, we choose $\gamma = 3.5 > \max\{\gamma_{s\infty}, \gamma_{f\infty}\}$, and design the optimal controller for the slow and fast subsystems based on this value of γ:

$$\begin{aligned}
Z_{s\gamma} &= 0.4822 \\
\mu^*_{s\gamma}(x_1) &= -0.31378x_1 \\
\mu^*_{sf\gamma}(x_1, w) &= -0.38576x_1 - 0.60000w \\
\nu^*_{s\gamma}(x_1) &= -0.11997x_1 \\
Z_{f\gamma} &= 15.3229 \\
\mu^*_{f\gamma}(x_1, x_2) &= -10.1614x_1 - 15.3229x_2 \\
\mu^*_{c\gamma}(x_1, x_2) &= -10.4752x_1 - 15.3229x_2
\end{aligned}$$

Then, we apply $\mu^*_{c\gamma}$, $\mu^*_{s\gamma}$ and $\mu^*_{sf\gamma}$ to the full-order system (139)–(140) and obtain the corresponding disturbance attenuation bounds γ^*_c, γ^*_s and γ^*_{sf}, respectively, which are tabulated in Table 2. Note that only the composite controller $\mu^*_{c\gamma}$ achieves the desired γ bound level.

Table 2: A comparison of H^∞-performance levels under composite, slow and feedforward controllers for Example 1

γ^*_c	3.4818	3.4818	3.4818	3.4818	3.4818
γ^*_s	∞	∞	∞	∞	∞
γ^*_{sf}	∞	∞	∞	∞	∞
ϵ	1	0.1	0.01	0.001	0.0001

Even if one allows a disturbance feedforward design based on the slow subsystem only (which is the controller $\mu^*_{sf\gamma}$), still the fast dynamics cannot be stabilized. $\diamond$

Example 2

Consider the following system and the performance index:

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ 3 \end{bmatrix} w \tag{141}$$

$$L_\gamma = \int_0^\infty (2x_1^2 + 2x_1x_2 + 3x_2^2 + |u|^2 - \gamma^2 |w|^2)\, dt \tag{142}$$

Here, the fast subsystem is open-loop stable. The three basic performance levels in this case are:

$$\begin{aligned} \gamma_{s\infty} &= 1.4240 \\ \gamma_{f\infty} &= 1.2990 \\ \gamma_{of\infty} &= 2.5981 \end{aligned}$$

Note that here the three values satisfy the relationship $\gamma_{f\infty} < \gamma_{s\infty} < \gamma_{of\infty}$. We can also compute the minimax disturbance attenuation level $\gamma^*_\infty(\epsilon)$ of the system (141)–(142) for every fixed ϵ (see Table 3). Note again that as $\epsilon \to 0$, $\gamma^*(\epsilon) \to \max\{\gamma_{s\infty}, \gamma_{f\infty}\}$. Now, we

Table 3: Optimum H^∞-performance level for the full-order system of Example 2

$\gamma^*(\epsilon)$	1.3306	1.3993	1.4212	1.4237	1.4240
ϵ	1	0.1	0.01	0.001	0.0001

choose $\gamma = 1.6 > \max\{\gamma_{s\infty}, \gamma_{f\infty}\}$, and design the optimal controller for the slow and fast subsystems based on this value of γ:

$$\begin{aligned} Z_{s\gamma} &= 3.3788 \\ \mu^*_{s\gamma}(x_1) &= 1.6447x_1 \\ \mu^*_{sf\gamma}(x_1, w) &= -2.4091x_1 - 1.1250w \\ \nu^*_{s\gamma}(x_1) &= -3.6034x_1 \\ Z_{f\gamma} &= 0.69201 \\ \mu^*_{f\gamma}(x_1, x_2) &= -5.88965x_1 - 1.3840x_2 \\ \mu^*_{c\gamma}(x_1, x_2) &= -4.22518x_1 - 1.3840x_2 \end{aligned}$$

Then, we use $\mu^*_{c\gamma}$, $\mu^*_{s\gamma}$ and $\mu^*_{sf\gamma}$ in system (139)–(140) and obtain the corresponding disturbance attenuation bounds γ^*_c, γ^*_s and γ^*_{sf}, respectively, which are tabulated in Table 4. It is clear that in this case both the composite and the feedforward controllers , $\mu^*_{c\gamma}$ and $\mu^*_{sf\gamma}$, can achieve the γ bound level, but the slow controller leads to infinite attenuation. Also, the level achieved by feedforward controller is above the level achieved by $\mu^*_{c\gamma}$. If we want to

Table 4: A comparison of H^∞-performance levels under composite, slow and feedforward controllers for Example 2 with $\gamma = 1.6$

γ_c^*	1.4497	1.4497	1.4497	1.4497	1.4497
γ_s^*	∞	∞	∞	∞	∞
γ_{sf}^*	1.5439	1.5439	1.5439	1.5439	1.5439
ϵ	1	0.1	0.01	0.001	0.0001

totally ignore the fast dynamics in the design of the controller, then we should choose γ to be greater than the $\gamma_{of\infty}$ in the design of the controllers. For instance, if we choose $\gamma = 2.7$, we obtain the values:

$$\begin{aligned}
Z_{s\gamma} &= 2.1713 \\
\mu_{s\gamma}^*(x_1) &= -1.0843x_1 \\
\mu_{sf\gamma}^*(x_1, w) &= -1.5034x_1 - 1.125w \\
\nu_{s\gamma}^*(x_1) &= -0.37258x_1 \\
Z_{f\gamma} &= 0.54480 \\
\mu_{f\gamma}^*(x_1, x_2) &= -2.3352x_1 - 1.0896x_2 \\
\mu_{c\gamma}^*(x_1, x_2) &= -3.4195x_1 - 1.0896x_2
\end{aligned}$$

Then, using $\mu_{c\gamma}^*$, $\mu_{s\gamma}^*$ and $\mu_{sf\gamma}^*$ in system (141)–(142), we obtain the corresponding disturbance attenuation bounds γ_c^*, γ_s^* and γ_{sf}^* respectively, as shown in Table 5. Note that in this case

Table 5: A comparison of H^∞-performance levels under composite, slow and feedforward controllers for Example 2 with $\gamma = 2.7$

γ_c^*	1.3899	1.4152	1.4344	1.4533	1.4533
γ_s^*	3.5891	2.7145	2.5315	2.5990	2.5990
γ_{sf}^*	1.7093	1.6838	1.6838	1.6838	1.6838
ϵ	1	0.1	0.01	0.001	0.0001

all three controllers achieve the 2.7 bound, but obviously, the composite controller $\mu_{c\gamma}^*$ does much better than the other two controllers, $\mu_{s\gamma}^*$ and $\mu_{sf\gamma}^*$, not only for small values of $\epsilon > 0$, but even for $\epsilon = 1$, in which case $\mu_{s\gamma}^*$ does not yield the desired bound (but $\mu_{sf\gamma}^*$ does). $\diamond$

Example 3

Now consider the following system and performance index:

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w \qquad (143)$$

$$L_\gamma = \int_0^\infty (2x_1^2 + 2x_1x_2 + x_2^2 + |u|^2 - \gamma^2 |w|^2)\, dt \qquad (144)$$

We again compute the three basic quantities:

$$\begin{aligned} \gamma_{s\infty} &= 0.63249 \\ \gamma_{f\infty} &= 0.44697 \\ \gamma_{of\infty} &= 0.49941 \end{aligned}$$

In this case, these satisfy the relationship $\gamma_{f\infty} < \gamma_{of\infty} < \gamma_{s\infty}$. We also compute $\gamma^*_\infty(\epsilon)$ for different values of $\epsilon > 0$ (see Table 6). Again, as $\epsilon \to 0$, $\gamma^*(\epsilon) \to \max\{\gamma_{s\infty}, \gamma_{f\infty}\}$. Now, we

Table 6: Optimum H^∞-performance level for the full-order system of Example 3

$\gamma^*(\epsilon)$	0.55946	0.61434	0.63023	0.63223	0.63243
ϵ	1	0.1	0.01	0.001	0.0001

choose $\gamma = 0.7 > \max\{\gamma_{s\infty}, \gamma_{f\infty}\}$, and design the optimal controller for the slow and fast subsystems based on this value of γ:

$$\begin{aligned} Z_{s\gamma} &= 5.5839 \\ \mu^*_{s\gamma}(x_1) &= -16.370x_1 \\ \mu^*_{sf\gamma}(x_1, w) &= -11.968x_1 - 0.20000w \\ \nu^*_{s\gamma}(x_1) &= 22.013x_1 \\ Z_{f\gamma} &= 0.26880 \\ \mu^*_{f\gamma}(x_1, x_2) &= 1.0272x_1 - 0.26880x_2 \\ \mu^*_{c\gamma}(x_1, x_2) &= -15.343x_1 - 0.26880x_2 \end{aligned}$$

Using $\mu^*_{c\gamma}$, $\mu^*_{s\gamma}$ and $\mu^*_{sf\gamma}$ in system (143)–(144), we obtain the values shown in Table 7 for the corresponding disturbance attenuation bounds γ^*_c, γ^*_s and γ^*_{sf}, respectively. We can see that all three controllers achieve the 0.7 bound, but $\mu^*_{sf\gamma}$ does a little better than the other two. This is mainly due to the fact that in this case $\gamma_{of\infty} < \gamma_{s\infty}$. ◇

We observe from the preceding analysis the following essential features of approximate designs based on time scale separation:

Table 7: A comparison of H^{∞}-performance levels under composite, slow and feedforward controllers for Example 3

γ_c^*	0.69507	0.69507	0.69507	0.69507	0.69507
γ_s^*	0.69713	0.69713	0.69713	0.69713	0.69713
γ_{sf}^*	0.69309	0.69309	0.69309	0.69309	0.69309
ϵ	1	0.1	0.01	0.001	0.0001

1. When the fast subsystem is open-loop unstable, only the composite controller $\mu_{c\gamma}^*$ can achieve a $\gamma > \bar{\gamma}_{\infty}$ bound.

2. Even if the fast subsystem is open-loop stable, the composite controller may achieve a better disturbance attenuation bound than the $\mu_{sf\gamma}^*$ when $\gamma_{s_\infty} < \gamma_{of_\infty}$.

3. In the case of $\gamma_{s_\infty} \geq \gamma_{of_\infty} \geq \gamma_{f_\infty}$, all three controllers (i.e., $\mu_{s\gamma}^*$, $\mu_{sf\gamma}^*$ and $\mu_{c\gamma}^*$) can achieve a $\gamma > \bar{\gamma_\infty}$. The composite controller does not necessarily lead to a much better performance than the other two.

6 CONCLUSION

In this paper, we have provided a complete analysis of the singularly perturbed linear-quadratic H^{∞}-optimal control problem with perfect state measurements in both finite and infinite horizons, by relating it to a class of singularly perturbed differential games. One of the main results of the paper is the existence and construction of a composite controller, independent of the singular perturbation parameter, under which the associated differential game has a bounded upper value. This composite controller leads to a specific (pre-computable) H^{∞}-performance bound for the full-order control problem for sufficiently small values of the singular perturbation parameter. For the infinite-horizon case, we have actually proved recently, in Pan and Başar (1992), that this performance bound is tight;[13] we have not included this result here since the arguments are quite technical and lengthy, and its counterpart in the finite-horizon case is still an open problem.

In the present paper, we have also obtained the conditions under which a controller designed based on the slow subsystem only can achieve a desired performance level. Yet another controller whose performance is studied is the one that uses slow state feedback and a feedforward term from the disturbance, and it has been shown that the performance of this controller is generally inferior to that of the composite controller. One of the important messages of the paper has been that the "best" composite controller is not simply the sum of

[13]In the recent work, Pan and Başar (1992), we prove a weak converse of Theorem 2, which says that $\lim_{\epsilon \to 0+} \gamma^*(\epsilon) = \bar{\gamma}$ provided $\gamma_s > \gamma_f$ and the matrix $Q_f > 0$ for small enough, but positive, ϵ.

the "best" slow and the "best" fast controllers (contrary to what is observed in the singularly perturbed LQR problem), but involves a more intricate construction.

One immediate, but not trivial, extension of these results would be to the sampled-data measurement case, so as to obtain the counterparts of the results of Başar (1991) in the singularly perturbed case. This work has already been completed and the results will be presented elsewhere Pan and Başar (1991b). Another extension is to the imperfect measurement case (the four-block H^∞-optimal control problem) which has also recently been completed Pan and Başar (1991a). One other extension would be to the multiple-time scale problems, which is a topic currently under study.

APPENDICES

A SIMPLIFICATION OF SUBSYSTEM PARAMETER MATRICES

First, we introduce some matrices to simplify our proof:

$$\pi := A_{22}Q_{22}^{-1}A'_{22} \tag{145}$$
$$\rho := I + B'_2A'^{-1}_{22}Q_{22}A_{22}^{-1}B_2 \tag{146}$$
$$\sigma := \gamma^2 I - D'_2(A_{22}Q_{22}^{-1}A'_{22} + B_2B'_2)^{-1}D_2 \tag{147}$$
$$\phi := \gamma^2 I - D'_2A'^{-1}_{22}Q_{22}A_{22}^{-1}D_2 \tag{148}$$
$$\chi := I + B'_2(A_{22}Q_{22}^{-1}A'_{22} - \frac{1}{\gamma^2}D_2D'_2)^{-1}B_2 \tag{149}$$

In terms of these matrices, we can obtain the following expressions using some simple matrix operations and matrix inversion identities:

$$\rho = I + B'_2\pi^{-1}B_2 \tag{150}$$
$$\rho^{-1} = I - B'_2(\pi + B_2B'_2)^{-1}B_2 \tag{151}$$
$$\rho^{-1}B'_2\pi^{-1} = B'_2(\pi + B_2B'_2)^{-1} \tag{152}$$
$$\sigma = \gamma^2 I - D'_2(\pi + B_2B'_2)^{-1}D_2 \tag{153}$$
$$\sigma^{-1} = \frac{1}{\gamma^2}I + \frac{1}{\gamma^4}D'_2(\pi + S_{22})^{-1}D_2 \tag{154}$$
$$\sigma^{-1}D'_2(\pi + B_2B'_2)^{-1} = \gamma^2 D'_2(\pi + S_{22})^{-1} \tag{155}$$
$$\phi = \gamma^2 I - D'_2\pi^{-1}D_2 \tag{156}$$
$$\phi^{-1} = \frac{1}{\gamma^2}(I + \frac{1}{\gamma^2}D'_2(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}D_2) \tag{157}$$
$$\phi^{-1}D'_2\pi^{-1} = \frac{1}{\gamma^2}D'_2(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1} \tag{158}$$
$$\chi^{-1} = I - B'_2(\pi + S_{22})^{-1}B_2 \tag{159}$$

$$\chi^{-1}B_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1} \quad = \quad B_2'(\pi + S_{22})^{-1} \tag{160}$$

Transformations of the slow subsystem under the FI information pattern

We will deal with the transformations described in (40) and (41) one by one. Substitution of (40) into (34) and (39) yields the following system:

$$\begin{aligned} \dot{x}_s &= A^\# x_s + B^\# \tilde{u}_s + D^\# w_s; \quad x_s(0) = x_1(0) && (161) \\ L_{\gamma s} &= |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f} (|x_s|^2_{Q\#} + 2x_s' P^\# w_s + |w_s|^2_{R\#} + |\tilde{u}_s|^2)\, dt && (162) \end{aligned}$$

where

$$\begin{aligned} A^\# &= A_0 + B_0\rho^{-1}B_2'\pi^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\ &= A_0 + B_0B_2'(\pi + B_2B_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\ B^\# &= B_0\rho^{-1/2} \\ D^\# &= D_0 - B_0\rho^{-1}B_2'\pi^{-1}D_2 \\ &= D_0 - B_0B_2'(\pi + B_2B_2')^{-1}D_2 \\ Q^\# &= Q_{11} - A_{21}'A_{22}'^{-1}Q_{21} - Q_{12}A_{22}^{-1}A_{21} + A_{21}'\pi^{-1}A_{21} \\ &\quad -(Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')\pi^{-1}B_2\rho^{-1}B_2'\pi^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\ &= Q_{11} - A_{21}'A_{22}'^{-1}Q_{21} - Q_{12}A_{22}^{-1}A_{21} + A_{21}'\pi^{-1}A_{21} \\ &\quad -(Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')\pi^{-1}B_2B_2'(\pi + B_2B_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\ &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')(\pi + B_2B_2')^{-1} \\ &\quad \cdot(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\ P^\# &= -Q_{12}A_{22}^{-1}D_2 + A_{21}'\pi^{-1}D_2 + (Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')\pi^{-1}B_2\rho^{-1}B_2'\pi^{-1}D_2 \\ &= -Q_{12}A_{22}^{-1}D_2 + A_{21}'\pi^{-1}D_2 + (Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')(\pi + B_2B_2')^{-1}B_2B_2'\pi^{-1}D_2 \\ &= -(Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')(\pi + B_2B_2')^{-1}D_2 \\ R^\# &= -\gamma^2 I + D_2'\pi^{-1}D_2 - D_2'\pi^{-1}B_2\rho^{-1}B_2'\pi^{-1}D_2 \\ &= -\gamma^2 I + D_2'\pi^{-1}D_2 - D_2'\pi^{-1}B_2B_2'(\pi + B_2B_2')^{-1}D_2 \\ &= -\gamma^2 I + D_2'(\pi + B_2B_2')^{-1}D_2 \\ &= -\sigma \end{aligned}$$

Now, substitute (41) into (161) and (162) to obtain:

$$\begin{aligned} \tilde{A}_0 &= A_0 + B_0B_2'(\pi + B_2B_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) - (D_0 - B_0B_2'(\pi + B_2B_2')^{-1}D_2) \\ &\quad \cdot\sigma^{-1}D_2'(\pi + B_2B_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\ &= A_0 + B_0B_2'(\pi + B_2B_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) - (D_0 - B_0B_2'(\pi + B_2B_2')^{-1}D_2) \\ &\quad \cdot\frac{1}{\gamma^2}D_2'(\pi + S_{22})^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \end{aligned}$$

$$
\begin{aligned}
&= A_0 + B_0 B_2'(\pi + B_2 B_2')^{-1}(I + \frac{1}{\gamma^2} D_2 D_2'(\pi + S_{22})^{-1}) \\
&\quad \cdot (A_{22} Q_{22}^{-1} Q_{21} - A_{21}) - \frac{1}{\gamma^2} D_0 D_2'(\pi + S_{22})^{-1}(A_{22} Q_{22}^{-1} Q_{21} - A_{21}) \\
&= A_0 + (S_{12} - A_{12} A_{22}^{-1} S_{22})(\pi + S_{22})^{-1}(A_{22} Q_{22}^{-1} Q_{21} - A_{21}) \\
&= A_{11} - A_{12} Q_{22}^{-1} Q_{21} + (S_{12} + A_{12} Q_{22}^{-1} A_{22}')(\pi + S_{22})^{-1} \\
&\quad \cdot (A_{22} Q_{22}^{-1} Q_{21} - A_{21}) \\
\tilde{B}_0 &= B^{\#} \\
\tilde{D}_0 &= D^{\#} \sigma^{-1/2} \\
\tilde{Q} &= Q^{\#} + (Q_{12} Q_{22}^{-1} A_{22}' - A_{21}')(\pi + B_2 B_2')^{-1} D_2 \sigma^{-1} D_2' \\
&\quad \cdot (\pi + B_2 B_2')^{-1}(A_{22} Q_{22}^{-1} Q_{21} - A_{21}) \\
&= Q^{\#} + (Q_{12} Q_{22}^{-1} A_{22}' - A_{21}')(\pi + B_2 B_2')^{-1} \frac{1}{\gamma^2} D_2 D_2' \\
&\quad \cdot (\pi + S_{22})^{-1}(A_{22} Q_{22}^{-1} Q_{21} - A_{21}) \\
&= Q_{11} - Q_{12} Q_{22}^{-1} Q_{21} + (Q_{12} Q_{22}^{-1} A_{22}' - A_{21}')(\pi + S_{22})^{-1} \\
&\quad \cdot (A_{22} Q_{22}^{-1} Q_{21} - A_{21})
\end{aligned}
$$

Now, we can compute S_0 as follows:

$$
\begin{aligned}
S_0 &= \tilde{B}_0 \tilde{B}_0' - \tilde{D}_0 \tilde{D}_0' \\
&= B_0 B_0' - B_0 B_2'(\pi + B_2 B_2')^{-1} B_2 B_0' - D_0 \sigma^{-1} D_0' \\
&\quad + B_0 B_2'(\pi + S_{22})^{-1} \frac{1}{\gamma^2} D_2 D_0' + \frac{1}{\gamma^2} D_0 D_2'(\pi + S_{22})^{-1} B_2 B_0' \\
&\quad - B_0 B_2'(\pi + B_2 B_2')^{-1} \frac{1}{\gamma^2} D_2 D_2'(\pi + S_{22})^{-1} B_2 B_0' \\
&= B_0 B_0' - B_0 B_2'(\pi + S_{22})^{-1} B_2 B_0' - \frac{1}{\gamma^2} D_0 D_0' - \frac{1}{\gamma^2} D_0 D_2'(\pi + S_{22})^{-1} \frac{1}{\gamma^2} D_2 D_0' \\
&\quad + B_0 B_2'(\pi + S_{22})^{-1} \frac{1}{\gamma^2} D_2 D_0' + \frac{1}{\gamma^2} D_0 D_2'(\pi + S_{22})^{-1} B_2 B_0' \\
&= S_{11} - A_{12} A_{22}^{-1} S_{21} - S_{12} A_{22}'^{-1} A_{12}' + A_{12} A_{22}^{-1} S_{22} A_{22}'^{-1} A_{12}' \\
&\quad - (S_{12} - A_{12} A_{22}^{-1} S_{22})(\pi + S_{22})^{-1}(S_{21} - S_{22} A_{22}'^{-1} A_{12}') \\
&= S_{11} + A_{12} Q_{22}^{-1} A_{12}' - (S_{12} + A_{12} Q_{22}^{-1} A_{22}')(\pi + S_{22})^{-1}(S_{21} + A_{22} Q_{22}^{-1} A_{12}')
\end{aligned}
$$

Below, we now provide the details of the steps that led from (56)– (57) to (58)–(60):

$$
\begin{aligned}
w_{s\gamma}^* &= \sigma^{-1}[D^{\#\prime} Z_{s\gamma} - D_2'(\pi + B_2 B_2')^{-1}(A_{22} Q_{22}^{-1} Q_{21} - A_{21})] x_s \\
&= [\frac{1}{\gamma^2} D_0' Z_{s\gamma} + \frac{1}{\gamma^2} D_2'(\pi + S_{22})^{-1} D_2 D_0' Z_{s\gamma} - \frac{1}{\gamma^2} D_2'(\pi + S_{22})^{-1} B_2 B_0' Z_{s\gamma} \\
&\quad - \frac{1}{\gamma^2} D_2'(\pi + S_{22})^{-1}(A_{22} Q_{22}^{-1} Q_{21} - A_{21})] x_s
\end{aligned}
$$

$$
\begin{aligned}
&= [\frac{1}{\gamma^2}D_1'Z_{s\gamma} - \frac{1}{\gamma^2}D_2'A_{22}'^{-1}A_{12}'Z_{s\gamma} - \frac{1}{\gamma^2}D_2'(\pi + S_{22})^{-1}(S_{21} - A_{12}A_{22}^{-1}S_{22})Z_{s\gamma} \\
&\quad -\frac{1}{\gamma^2}D_2'(\pi + S_{22})^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21})]x_s \\
&= [\frac{1}{\gamma^2}D_1'Z_{s\gamma} - \frac{1}{\gamma^2}D_2'(\pi + S_{22})^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\
&\quad -(A_{21} - A_{22}Q_{22}^{-1}Q_{21}))]x_s \\
u_{sf\gamma}^* &= -\rho^{-1}[B_0'Z_{s\gamma}x_s + B_2'\pi^{-1}(D_2w_s - (A_{22}Q_{22}^{-1}Q_{21} - A_{21})x_s)] \\
&= -(B_1'Z_{s\gamma}x_s - B_2'(\pi + B_2B_2')^{-1}B_2B_1'Z_{s\gamma}x_s + B_2'(\pi + B_2B_2')^{-1} \\
&\quad \cdot(D_2w_s - (A_{22}Q_{22}^{-1}Q_{21} - A_{21})x_s - A_{22}Q_{22}^{-1}A_{12}'Z_{s\gamma}x_s)) \\
&= -B_1'Z_{s\gamma}x_s + B_2'(\pi + B_2B_2')^{-1}((B_2B_1' + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\
&\quad +(A_{22}Q_{22}^{-1}Q_{21} - A_{21}))x_s - B_2'(\pi + B_2B_2')^{-1}D_2w_s \\
u_{s\gamma}^* &= -B_1'Z_{s\gamma}x_s + B_2'(\pi + B_2B_2')^{-1}((B_2B_1' + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\
&\quad +(A_{22}Q_{22}^{-1}Q_{21} - A_{21}))x_s - B_2'(\pi + B_2B_2')^{-1}D_2[\frac{1}{\gamma^2}D_1'Z_{s\gamma} \\
&\quad -\frac{1}{\gamma^2}D_2'(\pi + S_{22})^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\
&\quad -(A_{21} - A_{22}Q_{22}^{-1}Q_{21}))]x_s \\
&= -B_1'Z_{s\gamma}x_s + B_2'(\pi + B_2B_2')^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\
&\quad +(A_{22}Q_{22}^{-1}Q_{21} - A_{21}))x_s + B_2'(\pi + B_2B_2')^{-1}\frac{1}{\gamma^2}D_2D_2'(\pi + S_{22})^{-1} \\
&\quad \cdot((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21}))x_s \\
&= -B_1'Z_{s\gamma}x_s + B_2'(\pi + S_{22})^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\
&\quad -(A_{21} - A_{22}Q_{22}^{-1}Q_{21}))x_s
\end{aligned}
$$

Transformations of the slow subsystem under the SF information pattern

Similar to that in the FI information case, we consider the transformations introduced in (61) and (62) one at a time. Substitution of (61) into (34) and (39) leads to the following system:

$$\dot{x}_s = A^{\square}x_s + B^{\square}u_s + D^{\square}\tilde{w}_s; \quad x_s(0) = x_1(0) \tag{163}$$

$$L_{\gamma s} = |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f}(|x_s|^2_{Q^{\square}} + 2x_s'P^{\square}u_s + |u_s|^2_{R^{\square}} - |\tilde{w}_s|^2)\,dt \tag{164}$$

where

$$
\begin{aligned}
A^{\square} &= A_0 + D_0\phi^{-1}D_2'\pi^{-1}(A_{21} - A_{22}Q_{22}^{-1}Q_{21}) \\
&= A_0 - \frac{1}{\gamma^2}D_0D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21})
\end{aligned}
$$

$$
\begin{aligned}
B^{\square} &= B_0 + D_0\phi^{-1}D_2'\pi^{-1}B_2 \\
&= B_0 + \frac{1}{\gamma^2}D_0D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}B_2 \\
D^{\square} &= D_0\phi^{-1/2} \\
Q^{\square} &= Q_{11} - A_{21}'A_{22}'^{-1}Q_{21} - Q_{12}A_{22}^{-1}A_{21} + A_{21}'\pi^{-1}A_{21} \\
&\quad +(Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')\pi^{-1}D_2\phi^{-1}D_2'\pi^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&= Q_{11} - A_{21}'A_{22}'^{-1}Q_{21} - Q_{12}A_{22}^{-1}A_{21} + A_{21}'\pi^{-1}A_{21} \\
&\quad +(Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')\pi^{-1}\frac{1}{\gamma^2}D_2D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1} \\
&\quad \cdot(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
P^{\square} &= -Q_{12}A_{22}^{-1}B_2 + A_{21}'\pi^{-1}B_2 - (Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')\pi^{-1}D_2\phi^{-1}D_2'\pi^{-1}B_2 \\
&= -Q_{12}A_{22}^{-1}B_2 + A_{21}'\pi^{-1}B_2 - (Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}\frac{1}{\gamma^2}D_2D_2'\pi^{-1}B_2 \\
&= -(Q_{12}Q_{22}^{-1}A_{22}' - A_{21}')(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}B_2 \\
R^{\square} &= I + B_2'\pi^{-1}B_2 + B_2'\pi^{-1}D_2\phi^{-1}D_2'\pi^{-1}B_2 \\
&= I + B_2'\pi^{-1}B_2 - B_2'\pi^{-1}\frac{1}{\gamma^2}D_2D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}B_2 \\
&= I + B_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}B_2 \\
&= \chi
\end{aligned}
$$

Now, substitute (62) into (163) and (164) to obtain:

$$
\begin{aligned}
\hat{A}_0 &= A_0 - \frac{1}{\gamma^2}D_0D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&\quad +(B_0 + \frac{1}{\gamma^2}D_0D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}B_2)\chi^{-1}B_2' \\
&\quad \cdot(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&= A_0 - \frac{1}{\gamma^2}D_0D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&\quad +(B_0 + \frac{1}{\gamma^2}D_0D_2'(\pi - \frac{1}{\gamma^2}D_2D_2')^{-1}B_2)B_2'(\pi + S_{22})^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&= A_0 + B_0B_2'(\pi + S_{22})^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&\quad -\frac{1}{\gamma^2}D_0D_2'(\pi + S_{22})^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&= A_0 + (S_{12} - A_{12}A_{22}^{-1}S_{22})(\pi + S_{22})^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21})
\end{aligned}
$$

$$
\begin{aligned}
&= A_{11} - A_{12}Q_{22}^{-1}Q_{21} + (S_{12} + A_{12}Q_{22}^{-1}A'_{22})(\pi + S_{22})^{-1} \\
&\quad \cdot(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
\hat{B}_0 &= B^{\square}\chi^{-1/2} \\
\hat{D}_0 &= D^{\square} \\
\hat{Q} &= Q^{\square} - (Q_{12}Q_{22}^{-1}A'_{22} - A'_{21})(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}B_2\chi^{-1}B'_2 \\
&\quad \cdot(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&= Q^{\square} - (Q_{12}Q_{22}^{-1}A'_{22} - A'_{21})(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}B_2B'_2(\pi + S_{22})^{-1} \\
&\quad \cdot(A_{22}Q_{22}^{-1}Q_{21} - A_{21}) \\
&= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (Q_{12}Q_{22}^{-1}A'_{22} - A'_{21})(\pi + S_{22})^{-1} \\
&\quad \cdot(A_{22}Q_{22}^{-1}Q_{21} - A_{21})
\end{aligned}
$$

Now, we can compute $\hat{S}_0$ as follows:

$$
\begin{aligned}
\hat{S}_0 &= \hat{B}_0\hat{B}'_0 - \hat{D}_0\hat{D}'_0 \\
&= B_0\chi^{-1}B'_0 + \frac{1}{\gamma^2}D_0D'_2(\pi + S_{22})^{-1}B_2B'_0 + B_0B'_2(\pi + S_{22})^{-1}D_2D'_0\frac{1}{\gamma^2} \\
&\quad + \frac{1}{\gamma^2}D_0D'_2(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}B_2B'_2(\pi + S_{22})^{-1}D_2D'_0\frac{1}{\gamma^2} \\
&\quad - \frac{1}{\gamma^2}D_0D'_0 - \frac{1}{\gamma^2}D_0D'_2(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}D_2D'_0\frac{1}{\gamma^2} \\
&= B_0B'_0 - B_0B'_2(\pi + S_{22})^{-1}B_2B'_0 + \frac{1}{\gamma^2}D_0D'_2(\pi + S_{22})^{-1}B_2B'_0 \\
&\quad + B_0B'_2(\pi + S_{22})^{-1}D_2D'_0\frac{1}{\gamma^2} - \frac{1}{\gamma^2}D_0D'_0 - \frac{1}{\gamma^2}D_0D'_2(\pi + S_{22})^{-1}D_2D'_0\frac{1}{\gamma^2} \\
&= S_{11} + A_{12}Q_{22}^{-1}A'_{12} - (S_{12} + A_{12}Q_{22}^{-1}A'_{22})(\pi + S_{22})^{-1}(S_{21} + A_{22}Q_{22}^{-1}A'_{12})
\end{aligned}
$$

We now give the different steps of the derivation that led from (74) to (60):

$$
\begin{aligned}
u^*_{s\gamma} &= \chi^{-1}[-(B'_0 + B'_2(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}D_2D'_0\frac{1}{\gamma^2}Z_{s\gamma} \\
&\quad + B'_2(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21})x_s] \\
&= [-B'_0Z_{s\gamma} + B'_2(\pi + S_{22})^{-1}B_2B'_0Z_{s\gamma} + B'_2(\pi + S_{22})^{-1} \\
&\quad \cdot(-D_2D'_0\frac{1}{\gamma^2})Z_{s\gamma} + (A_{22}Q_{22}^{-1}Q_{21} - A_{21})]x_s \\
&= [-B'_0Z_{s\gamma} + B'_2(\pi + S_{22})^{-1}(S_{21} - S_{22}A'_{22}{}^{-1}A'_{12})Z_{s\gamma} \\
&\quad + B'_2(\pi + S_{22})^{-1}(A_{22}Q_{22}^{-1}Q_{21} - A_{21}]x_s \\
&= -B'_1Z_{s\gamma}x_s + B'_2(\pi + S_{22})^{-1}((S_{21} + A_{22}Q_{22}^{-1}A'_{12})Z_{s\gamma}
\end{aligned}
$$

$$-(A_{21} - A_{22}Q_{22}^{-1}Q_{21}))x_s$$

Finally, we give the steps that lead from (79) to (80) for the fast subsystem:

$$\begin{aligned}
u_{f\gamma}^*(t) &= -B_2'(t)Z_{f\gamma}(t)(x_2 - \bar{x}_2) \\
&= -B_2'Z_{f\gamma}(x_2 + A_{22}^{-1}(A_{21}x_1 + B_2u_{sf\gamma}^* + D_2w_{s\gamma}^*)) \\
&= -B_2'Z_{f\gamma}x_2 - B_2'Z_{f\gamma}A_{22}^{-1}(A_{21} - B_2B_1'Z_{s\gamma} + B_2B_2'(S_{22} + \pi)^{-1} \\
&\quad \cdot((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})) + \frac{1}{\gamma^2}D_2D_1'Z_{s\gamma} \\
&\quad -\frac{1}{\gamma^2}D_2D_2'(\pi + S_{22})^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_1 \\
&= -B_2'Z_{f\gamma}x_2 - B_2'Z_{f\gamma}A_{22}^{-1}(A_{21} - S_{21}Z_{s\gamma} + S_{22}(S_{22} + \pi)^{-1} \\
&\quad \cdot((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_1 \\
&= -B_2'Z_{f\gamma}x_2 - B_2'Z_{f\gamma}A_{22}^{-1}(A_{21} - S_{21}Z_{s\gamma} - \pi(S_{22} + \pi)^{-1} \\
&\quad \cdot((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})) \\
&\quad +(S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21}))x_1 \\
&= -B_2'Z_{f\gamma}x_2 - B_2'Z_{f\gamma}Q_{22}^{-1}(-A_{22}'(S_{22} + \pi)^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\gamma} \\
&\quad -(A_{21} - A_{22}Q_{22}^{-1}Q_{21})) + A_{12}'Z_{s\gamma} + Q_{21}))x_1
\end{aligned}$$

B A USEFUL LEMMA

The following lemma will be used thoughout in the derivations of Appendix C.

Lemma 3 *Let U_1, U_2, V_1 and V_2 be defined as in (96)–(99). Then the following relationships hold.*

1. $$A_{22}U_1 + S_{22}V_1 = -S_{21} \tag{165}$$
2. $$A_{22}U_2 + S_{22}V_2 = A_{21} \tag{166}$$
3. $$Q_{22}U_2 - A_{22}'V_2 = Q_{21} \tag{167}$$
4. $$Q_{22}U_1 - A_{22}'V_1 = A_{12}' \tag{168}$$
5. $$A_{22}U + S_{22}V = A_{21} - S_{21}Z_{s\gamma} \tag{169}$$
6. $$Q_{22}U - A_{22}'V = Q_{21} + A_{12}'Z_{s\gamma} \tag{170}$$

Proof

$$\begin{aligned}
1)\ \ & A_{22}U_1 + S_{21} \\
&= S_{21} + A_{22}Q_{22}^{-1}A_{12}' - A_{22}Q_{22}^{-1}A_{22}'(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1}(S_{21} + A_{22}Q_{22}^{-1}A_{12}')
\end{aligned}$$

$$
\begin{aligned}
&= -S_{22}(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(S_{21}+A_{22}Q_{22}^{-1}A_{12}')\\
&= -S_{22}V_1
\end{aligned}
$$

2) $A_{22}U_2 - A_{21}$

$$
\begin{aligned}
&= -A_{21}+A_{22}Q_{22}^{-1}Q_{21}+A_{22}Q_{22}^{-1}A_{22}'(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(A_{21}-A_{22}Q_{22}^{-1}Q_{21})\\
&= -S_{22}(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(A_{21}-A_{22}Q_{22}^{-1}Q_{21})\\
&= -S_{22}V_2
\end{aligned}
$$

3) $Q_{22}U_2 - Q_{21}$

$$
\begin{aligned}
&= Q_{21}-Q_{21}+A_{22}'(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(A_{21}-A_{22}Q_{22}^{-1}Q_{21})\\
&= A_{22}'V_2
\end{aligned}
$$

4) $Q_{22}U_1 - A_{12}'$

$$
\begin{aligned}
&= A_{12}'-A_{12}'-A_{22}'(S_{22}+A_{22}Q_{22}^{-1}A_{22}')^{-1}(S_{21}+A_{22}Q_{22}^{-1}A_{12}')\\
&= A_{22}'V_1
\end{aligned}
$$

5) and 6) are direct extensions of 1), 2), 3) and 4). □

C VERIFICATION OF SOME USEFUL RESULTS

1. Proof of the fact that $Z_{11}=Z_{s\gamma}$, $Z_{12}=Z_c'$, $Z_{22}=Z_{f\gamma}$ satisfy (30)–(32)

It is immediate to see that (32) is satisfied. For (31),

$$
\begin{aligned}
\text{LHS} &= A_{21}'Z_{f\gamma}+Z_{s\gamma}A_{12}+(V'+U'Z_{f\gamma})A_{22}+Q_{12}-Z_{s\gamma}S_{12}Z_{f\gamma}\\
&\quad -(V'+U'Z_{f\gamma})S_{22}Z_{f\gamma}\\
&= (A_{21}'-Z_{s\gamma}S_{12}-V'S_{22})Z_{f\gamma}+Z_{s\gamma}A_{12}+V'A_{22}+Q_{12}\\
&\quad +U'Z_{f\gamma}A_{22}-U'Z_{f\gamma}S_{22}Z_{f\gamma}\\
&= U'A_{22}'Z_{f\gamma}+U'Q_{22}+U'Z_{f\gamma}A_{22}-U'Z_{f\gamma}S_{22}Z_{f\gamma}\\
&= U'(A_{22}'Z_{f\gamma}+Z_{f\gamma}A_{22}+Q_{22}-Z_{f\gamma}S_{22}Z_{f\gamma})\\
&= 0
\end{aligned}
$$

where in going from the second to the third step Lemma 3 has been used.

For (30),

$$
\begin{aligned}
\text{LHS} &= A_{11}'Z_{s\gamma}+A_{21}'(Z_{f\gamma}U+V)+Z_{s\gamma}A_{11}+(V'+U'Z_{f\gamma})A_{21}+Q_{11}-Z_{s\gamma}S_{11}Z_{s\gamma}\\
&\quad -(V'+U'Z_{f\gamma})S_{21}Z_{s\gamma}-Z_{s\gamma}S_{12}(Z_{f\gamma}U+V)-(V'+U'Z_{f\gamma})S_{22}(Z_{f\gamma}U+V)\\
&= A_{11}'Z_{s\gamma}+Z_{s\gamma}A_{11}+Q_{11}-Z_{s\gamma}S_{11}Z_{s\gamma}+(A_{21}'-Z_{s\gamma}S_{12})(Z_{f\gamma}U+V)\\
&\quad +(V'+U'Z_{f\gamma})(A_{21}-S_{21}Z_{s\gamma}-S_{22}Z_{f\gamma}U-S_{22}V)\\
&= A_{11}'Z_{s\gamma}+Z_{s\gamma}A_{11}+Q_{11}-Z_{s\gamma}S_{11}Z_{s\gamma}+(U'A_{22}'+V'S_{22})Z_{f\gamma}U
\end{aligned}
$$

$$
\begin{aligned}
& \quad +(A_{21}' - Z_{s\gamma}S_{12})V + (V' + U'Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})U \\
&= A_{11}'Z_{s\gamma} + Z_{s\gamma}A_{11} + Q_{11} - Z_{s\gamma}S_{11}Z_{s\gamma} + V'(S_{22}Z_{f\gamma} + A_{22} - S_{22}Z_{f\gamma})U \\
& \quad +(A_{21}' - Z_{s\gamma}S_{12})V + U'(A_{22}'Z_{f\gamma} + Z_{f\gamma}A_{22} - Z_{f\gamma}S_{22}Z_{f\gamma})U \\
&= A_{11}'Z_{s\gamma} + Z_{s\gamma}A_{11} + Q_{11} - Z_{s\gamma}S_{11}Z_{s\gamma} + (V'A_{22} - U'Q_{22})U \\
& \quad +(A_{21}' - Z_{s\gamma}S_{12})V \\
&= A_{11}'Z_{s\gamma} + Z_{s\gamma}A_{11} + Q_{11} - Z_{s\gamma}S_{11}Z_{s\gamma} - (Q_{12} + Z_{s\gamma}A_{12})U \\
& \quad +(A_{21}' - Z_{s\gamma}S_{12})V \\
&= \tilde{A}_0'Z_{s\gamma} + Z_{s\gamma}\tilde{A}_0 - Z_{s\gamma}S_0Z_{s\gamma} + \tilde{Q} \\
&= 0
\end{aligned}
$$

where we have used Lemma 3 throughout the simplification.

2. Verification of the fact that $\epsilon = 0$, $\tilde{Y}_{11} = Z_{s\gamma}$, $\tilde{Y}_{21} = Z_{of\gamma}U + V$, $\tilde{Y}_{22} = Z_{of\gamma}$ satisfy (128)–(130)

The verification of (130) is straightforward. For (129), using Lemma 3:

$$
\begin{aligned}
\text{LHS} &= A_{21}'Z_{of\gamma} + Z_{s\gamma}A_{12} + V'A_{22} + U'Z_{of\gamma}A_{22} + Q_{12} - Z_{s\gamma}B_1B_2'Z_{of\gamma} \\
& \quad -V'B_2B_2'Z_{of\gamma} + \frac{1}{\gamma^2}Z_{s\gamma}D_1D_2'Z_{of\gamma} + \frac{1}{\gamma^2}(V' + U'Z_{of\gamma})D_2D_2'Z_{of\gamma} \\
&= A_{21}'Z_{of\gamma} + (Z_{s\gamma}A_{12} + V'A_{22} + Q_{12}) - Z_{s\gamma}S_{12}Z_{of\gamma} - V'S_{22}Z_{of\gamma} \\
& \quad +U'(Z_{of\gamma}A_{22} + \frac{1}{\gamma^2}Z_{of\gamma}D_2D_2'Z_{of\gamma}) \\
&= (A_{21}' - Z_{s\gamma}S_{12} - V'S_{22} - U'A_{22}')Z_{of\gamma} + (Z_{s\gamma}A_{12} + V'A_{22} + Q_{12} - U'Q_{22}) \\
&= 0
\end{aligned}
$$

For (128), on the other hand, using again Lemma 3,

$$
\begin{aligned}
\text{LHS} &= A_{11}'Z_{s\gamma} - Z_{s\gamma}B_1B_1'Z_{s\gamma} - V'B_2B_1'Z_{s\gamma} + Z_{s\gamma}A_{11} - Z_{s\gamma}B_1B_1'Z_{s\gamma} - Z_{s\gamma}B_1B_2'V \\
& \quad +(A_{21}' - Z_{s\gamma}B_1B_2' - V'B_2B_2' + \frac{1}{\gamma^2}D_1D_2')(Z_{of\gamma}U + V) \\
& \quad +(V' + U'Z_{of\gamma})(A_{21} - B_2B_1'Z_{s\gamma} - B_2B_2'V + \frac{1}{\gamma^2}D_2D_1''Z_{s\gamma}) \\
& \quad +\frac{1}{\gamma^2}Z_{s\gamma}D_1D_1'Z_{s\gamma} + Q_{11} + Z_{s\gamma}B_1B_1'Z_{s\gamma} + V'B_2B_1'Z_{s\gamma} + Z_{s\gamma}B_1B_2'V \\
& \quad +V'B_2B_2'V + \frac{1}{\gamma^2}(V' + U'Z_{of\gamma})D_2D_2'(Z_{of\gamma}U + V) \\
&= A_{11}'Z_{s\gamma} + Z_{s\gamma}A_{11} - Z_{s\gamma}S_{11}Z_{s\gamma} + Q_{11} + (A_{21}' - Z_{s\gamma}S_{12} - V'S_{22} \\
& \quad +\frac{1}{\gamma^2}U'Z_{of\gamma}D_2D_2')(Z_{of\gamma}U + V) + V'(A_{21} - S_{21}Z_{s\gamma}) \\
& \quad +U'Z_{of\gamma}(A_{21} - S_{21}Z_{s\gamma} - B_2B_2'V)
\end{aligned}
$$

$$
\begin{aligned}
&= A_{11}'Z_{s\gamma} + Z_{s\gamma}A_{11} - Z_{s\gamma}S_{11}Z_{s\gamma} + Q_{11} + U'A_{22}'(Z_{of\gamma}U + V) \\
&\quad + \frac{1}{\gamma^2}U'Z_{of\gamma}D_2D_2'Z_{of\gamma}U + V'(A_{21} - S_{21}Z_{s\gamma}) \\
&\quad + U'Z_{of\gamma}(A_{22}U + S_{22}V - S_{22}V) \\
&= A_{11}'Z_{s\gamma} + Z_{s\gamma}A_{11} - Z_{s\gamma}S_{11}Z_{s\gamma} + Q_{11} + U'(A_{22}'Z_{of\gamma} + Z_{of\gamma}A_{22} \\
&\quad + \frac{1}{\gamma^2}Z_{of\gamma}D_2D_2'Z_{of\gamma})U + U'A_{22}'V + V'(A_{21} - S_{21}Z_{s\gamma}) \\
&= A_{11}'Z_{s\gamma} + Z_{s\gamma}A_{11} - Z_{s\gamma}S_{11}Z_{s\gamma} + Q_{11} + U'(-Q_{21} - A_{12}'Z_{s\gamma}) \\
&\quad + V'(A_{21} - S_{21}Z_{s\gamma}) \\
&= \tilde{A}_0'Z_{s\gamma} + Z_{s\gamma}\tilde{A}_0 - Z_{s\gamma}S_0Z_{s\gamma} + \tilde{Q} \\
&= 0
\end{aligned}
$$

3. Verification of the nonsingularity of Ξ_{11}

In Theorem 1, in order to be able to use the Implicit Function Theorem, we needed the condition that Ξ_{11} is nonsingular. We now verify this result here. First, it is easy to see that Ξ_{11} has to be in the form (102), where R is given by :

$$
\begin{aligned}
R &= A_{11} - (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1}A_{21} - S_{11}Z_{s\gamma} \\
&\quad - S_{12}(Z_{f\gamma}U + V) + (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1}S_{21}Z_{s\gamma} \\
&\quad + (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1}S_{22}(Z_{f\gamma}U + V)
\end{aligned}
$$

This can further be simplified to (by using Lemma 3 of Appendix B):

$$
\begin{aligned}
R &= A_{11} - S_{11}Z_{s\gamma} - (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1}(A_{21} - S_{21}Z_{s\gamma} \\
&\quad - S_{22}(Z_{f\gamma}U + V)) - S_{12}(Z_{f\gamma}U + V) \\
&= A_{11} - S_{11}Z_{s\gamma} - (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1}(A_{22} - S_{22}Z_{f\gamma})U \\
&\quad - S_{12}(Z_{f\gamma}U + V) \\
&= A_{11} - S_{11}Z_{s\gamma} - A_{12}U - S_{12}V \\
&= \tilde{A}_0 - S_0Z_{s\gamma}
\end{aligned}
$$

Since $\tilde{A}_0 - S_0Z_{s\gamma}$ is Hurwitz, for $\gamma > \gamma_{s\infty}$, the eigenvalues Ξ_{11} are in the left half plane. From this it follows that Ξ_{11} is nonsingular.

For (117), on the other hand, R is given by:

$$
\begin{aligned}
R &= A_{11} - B_1B_1'Z_{s\gamma} - B_1B_2'Z_c - (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1} \\
&\quad \cdot(A_{21} - B_2B_1'Z_{s\gamma} - B_2B_2'Z_c) + \frac{1}{\gamma^2}D_1D_1'Z_{s\gamma} + \frac{1}{\gamma^2}D_1D_2'Z_c \\
&\quad - (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1}\frac{1}{\gamma^2}D_2D_1'Z_{s\gamma} \\
&\quad - (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1}\frac{1}{\gamma^2}D_2D_2'Z_c
\end{aligned}
$$

which simplifies to:

$$\begin{aligned}
R &= A_{11} - S_{11}Z_{s\gamma} - S_{12}(Z_{f\gamma}U + V) - (A_{12} - S_{12}Z_{f\gamma})(A_{22} - S_{22}Z_{f\gamma})^{-1} \\
&\quad \cdot (A_{21} - S_{21}Z_{s\gamma} - S_{22}(Z_{f\gamma}U + V)) \\
&= \tilde{A}_0 - S_0 Z_{s\gamma}
\end{aligned}$$

from which it again follows that Ξ_{11} is nonsingular.

Now, for (132), R can be rewritten as follows:

$$\begin{aligned}
R &= A_{11} - B_1B_1'Z_{s\gamma} - B_1B_2'V - (A_{12} + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma})(A_{22} + \frac{1}{\gamma^2}D_2D_2'Z_{of\gamma})^{-1}\hat{A}_{21} \\
&\quad + \frac{1}{\gamma^2}D_1D_1'Z_{s\gamma} + \frac{1}{\gamma^2}D_1D_2'(Z_{of\gamma}U + V) \\
&\quad - (A_{12} + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma})(A_{22} + \frac{1}{\gamma^2}D_2D_2'Z_{of\gamma})^{-1}\frac{1}{\gamma^2}D_2D_1'Z_{s\gamma} \\
&\quad - (A_{12} + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma})(A_{22} + \frac{1}{\gamma^2}D_2D_2'Z_{of\gamma})^{-1}\frac{1}{\gamma^2}D_2D_2'(Z_{of\gamma}U + V) \\
&= A_{11} - S_{11}Z_{s\gamma} - S_{12}V + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma}U - (A_{12} + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma}) \\
&\quad \cdot (A_{22} + \frac{1}{\gamma^2}D_2D_2'Z_{of\gamma})^{-1}(A_{12} - S_{21}Z_{s\gamma} - S_{22}V + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma}U) \\
&= A_{11} - S_{11}Z_{s\gamma} - S_{12}V + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma}U - (A_{12} + \frac{1}{\gamma^2}D_1D_2'Z_{of\gamma}) \\
&\quad \cdot (A_{22} + \frac{1}{\gamma^2}D_2D_2'Z_{of\gamma})^{-1}(A_{22} + \frac{1}{\gamma^2}D_2D_2'Z_{of\gamma})U \\
&= A_{11} - S_{11}Z_{s\gamma} - S_{12}V - A_{12}U \\
&= A_{11} - S_{12}V_2 - A_{12}U_2 - (S_{11} + S_{12}V_1 + A_{12}U_1)Z_{s\gamma} \\
&= \tilde{A}_0 - S_0 Z_{s\gamma},
\end{aligned}$$

again, the nonsingularity of Ξ_{11} follows.

D THEOREM L OF KOKOTOVIĆ AND YACKEL (1972)

Consider the initial value problem

$$\frac{d\xi}{dt} = f(t, \xi, \eta, \epsilon) \qquad \xi(t_0) = \xi_0 \tag{171}$$

$$\epsilon\frac{d\eta}{dt} = g(t, \xi, \eta, \epsilon) \qquad \eta(t_0) = \eta_0 \tag{172}$$

where ϵ is a small positive parameter, and ξ and η are n and m dimensional vectors, respectively. Formally setting $\epsilon = 0$ in the system (171)–(172) yields the degenerate system

$$\frac{d\bar{\xi}}{dt} = f(t, \bar{\xi}, \bar{\eta}, 0) \qquad \bar{\xi}(t_0) = \xi_0 \tag{173}$$

$$0 = g(t, \bar{\xi}, \bar{\eta}, 0) \tag{174}$$

Since (174) may have several roots, suppose that a particular root $\bar{\eta} = \phi(t, \bar{\xi})$ is of interest and substitute it in (173). The n-dimensional system

$$\frac{d\bar{\xi}}{dt} = f(t, \bar{\xi}, \phi(t, \bar{\xi}), 0), \qquad \bar{\xi}(t_0) = \xi_0 \tag{175}$$

is a reduced system of (171)–(172).

Introducing a new time variable τ, (172) can be rewritten in the form of a boundary layer system,

$$\frac{d\omega}{d\tau} = g(\alpha, \beta, \omega, 0) \qquad \omega(t_0) = \eta_0 \tag{176}$$

where $\alpha = t_0$ and $\beta = \bar{\xi}(t_0)$ are fixed parameters. In the space of variables ξ, η, ϵ, t we define a region $\Re : \|\xi - \bar{\xi}(t)\| < r, \ \|\eta - \bar{\eta}(t)\| < r, \quad 0 \leq \epsilon \leq \epsilon^0, \quad t_0 \geq t \geq t_f$ where $r > |\eta_0 - \bar{\eta}(t_0)| \geq 0$.

Theorem 3 *Let the following conditions be satisfied.*

L1: *f, $\partial f/\partial \xi$, $\partial f/\partial \eta$, g, $\partial g/\partial \xi$, $\partial g/\partial \eta$ are of class C^0 in $(\xi, \eta, t, \epsilon) \in \Re$.*

L2: *The solution $\omega(\tau)$ of (176) exists on $\tau \in [0, \infty)$, is unique, and is asymptotically stable with respect to the root $\phi(t_0, \xi_0)$ of (174).*

L3: *The solution $\bar{\xi}(t)$ of the reduced system (175) exists and is unique on $t \in [t_0, t_f]$.*

L4: *The real parts of the eigenvalues of the Jacobian matrix*

$$\frac{\partial g}{\partial \eta}(t, \bar{\xi}, \bar{\eta}, 0)$$

are negative on $[t_0, t_f]$, for $\bar{\eta} = \phi(t, \bar{\xi})$.

Then for sufficiently small ϵ, the system (171)–(172) has a unique solution $\xi(t, \epsilon)$, $\eta(t, \epsilon)$ on $t \in [t_0, t_f]$ satisfying the initial conditions $\xi(t_0, \epsilon) = \xi_0$, $\eta(t_0, \epsilon) = \eta_0$. Furthermore,

$$\lim_{\epsilon \to 0} \xi(t, \epsilon) = \bar{\xi}(t) \quad \text{on} \quad [t_0, t_f] \tag{177}$$

$$\lim_{\epsilon \to 0} \eta(t, \epsilon) = \bar{\eta}(t) \quad \text{on} \quad (t_0, t_f] \tag{178}$$

where the limit (177) is uniform in t on $[t_0, t_f]$ and the limit (178) is uniform in t on any interval $(t_1, t_f]$, where $t_0 < t_1 < t_f$.

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