Multi-Time Scale Zero-Sum Differential Games with Perfect State Measurements*

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Abstract

We obtain necessary and sufficient conditions for the existence of "approximate" saddle-point solutions in linear-quadratic zero-sum differential games when the state dynamics are defined on multiple (three) time scales. These different time scales are characterized in terms of two small positive parameters ϵ_1 and ϵ_2, and the terminology "approximate saddle-point solution" is used to refer to saddle-point policies that do not depend on ϵ_1 and ϵ_2, while providing cost levels within $O(\epsilon_1)$ of the full-order game.

It is shown in the paper that, under perfect state measurements, the original game can be decomposed into three subgames – slow, fast and fastest, the composite saddle-point solution of which make up the approximate saddle-point solution of the original game. Specifically, for the minimizing player, it is necessary to use a composite policy that uses the solutions of all three subgames, whereas for the maximizing player, it is sufficient to use a slow policy. In the finite-horizon case this slow policy could be a feedback policy, whereas in the infinite-horizon case it has to be chosen as an open-loop policy that is generated from the solution and dynamics of the slow subgame. These results have direct applications in the H^∞-optimal control of three-time scale singularly perturbed linear systems under perfect state measurements.

Key Words—Zero-sum differential games, singular perturbation, generalized Riccati differential equation, generalized algebraic Riccati equation.

1 Introduction

We consider, in this paper, a class of zero-sum differential games whose open-loop (control-free) linear dynamics exhibit three-time scale separation. This time-scale separation is modeled in terms of a singularly perturbed differential equation (state dynamics) which depends on two small parameters $\epsilon_1 > 0$, $\epsilon_2 > 0$. Given a quadratic cost function, to be minimized by Player 1 and maximized by Player 2, for each fixed $\epsilon_1 > 0$, $\epsilon_2 > 0$, it is wellknown that the feedback saddle-point solution of this differential game is linear in the state vector, and its existence is linked to nonexistence of a conjugate point for a generalized Riccati differential equation (GRDE) in the finite-horizon case, and to nonexistence of a positive definite solution to a related generalized algebraic Riccati equation (GARE) in the infinite-horizon case [1], [2]. Even though the solution is thus known for fixed values of ϵ_1 and ϵ_2, its computation for small values of these parameters is prone to numerical errors and inaccuracies, which makes it necessary to obtain "approximate" solutions that are independent of these parameters, with the additional property that they be in "approximate" saddle-point equilibrium (more precisely known as ϵ-saddle point equilibrium). This is precisely the problem we address in this paper, in both finite and infinite horizons.

In addition to being of independent interest, such problems also arise in H^∞-optimal control when different parts of the system operate on different time scales. The relationship between H^∞-optimal control and zero-sum differential games has been expounded on in the recent book [2], where the reader is referred to for a discussion on this connection. In this context (i.e., the context of H^∞-optimal control), we have in fact studied recently in [3] the two-time scale version of the problem treated here. To put the results of [3] into proper perspective from a pure differential game point of view, let the positive scalar parameter γ^2 denote the weighting on the control energy of Player 2 in the quadratic cost function, and $\epsilon > 0$ denote the singular perturbation parameter. Then, for the full-order system, there is a critical value of γ, dependent on ϵ, say $\gamma^*(\epsilon)$, such that for all $\gamma > \gamma^*(\epsilon)$ the differential game admits a saddle-point solution, and for $\gamma < \gamma^*(\epsilon)$, its upper value is unbounded. For appropriately reduced slow and fast subgames, let γ_s and γ_f stand respectively for the counterpart of $\gamma^*(\epsilon)$, where γ_s and γ_f clearly do not depend on ϵ. We have proven in [3] that an appropriate limit of $\gamma^*(\epsilon)$ as $\epsilon \downarrow 0$ is no larger than the maximum of γ_s and γ_f. Using this result, we construct in [3] an appropriate composite controller for Player 1, from the solution of slow and fast games, that achieves a saddle-point value (SPV) within $O(\epsilon)$ of the SPV of the original game. In the present paper we present a nontrivial extension of this result to the three-time scale case.

The paper is organized as follows. In the next section (Section 2) we formulate the singularly perturbed zero-sum differential game problem, under perfect state measurements. We also provide in that section the solution to the full-order problem. In Section 3, we identify the slow, fast, and fastest subgames. In both the slow and fast subgames, the minimizing player is given an informational advantage to attain separability. In Section 4, a composite strategy for Player 1 is derived. We study the infinite-horizon case in Section 5, where we prove the optimality properties of approximate strategies proposed for both players. In Section 6 we present the counterpart of these results for the finite-horizon case. Section 7 presents a numerical example to illustrate the theory, and Section 8 provides a discussion on further extensions of these results.

2 Problem Formulation

The dynamics of the three-time scale zero-sum differential game considered in this paper are described by:

$$\begin{cases} \dot{x}_1 &=& A_{11}(t)x_1 + A_{12}(t)x_2 + A_{13}(t)x_3 + B_1(t)u + D_1(t)w; \quad x_1(0) = x_{10} \\ \epsilon_1\dot{x}_2 &=& A_{21}(t)x_1 + A_{22}(t)x_2 + A_{23}(t)x_3 + B_2(t)u + D_2(t)w; \quad x_2(0) = x_{20} \\ \epsilon_1\epsilon_2\dot{x}_3 &=& A_{31}(t)x_1 + A_{32}(t)x_2 + A_{33}(t)x_3 + B_3(t)u + D_3(t)w; \quad x_3(0) = x_{30} \end{cases} \tag{1}$$

where $x' := (x_1', x_2', x_3')$ is the n-dimensional state vector, with x_1 of dimension n_1, x_2 of dimension n_2 and x_3 of dimension $n_3 := n - n_1 - n_2$; u and w are the control actions of Player 1 and Player 2, respectively, each belonging to appropriate ($\mathcal{L}^2$) Hilbert spaces $\mathcal{H}_x$, $\mathcal{H}_u$ and $\mathcal{H}_w$, respectively, defined on the time interval $[0, t_f]$. The control inputs u and w are generated by closed-loop control policies μ and ν, respectively, according to

$$u(t) = \mu(t, x_{[0,t]}) \tag{2}$$
$$w(t) = \nu(t, x_{[0,t]}) \tag{3}$$

where $\mu : [0, t_f] \times \mathcal{H}_x \to \mathcal{H}_u$ and $\nu : [0, t_f] \times \mathcal{H}_x \to \mathcal{H}_w$ are piecewise continuous in t and Lipschitz continuous in x, further satisfying the given causality condition. Let us denote the class of all these controllers by $\mathcal{M}_u$ and $\mathcal{M}_w$, respectively. The cost function associated with this system is

$$\begin{aligned} L_\gamma(u, w) &= |x(t_f)|^2_{Q_f} + \int_0^{t_f} (|x(t)|^2_{Q(t)} + |u(t)|^2 - \gamma^2|w(t)|^2)\, dt \\ &\equiv |x(t_f)|^2_{Q_f} + \|x\|^2_Q + \|u\|^2 - \gamma^2\|w\|^2; \quad Q_f \geq 0, \quad Q(.) \geq 0 \end{aligned} \tag{4}$$

where Q_f will show dependence on $\epsilon_i > 0$, $i = 1, 2$, as to be clarified later. Player 1 tries to minimize the cost and Player 2 tries to maximize it. The parameters ϵ_1 and ϵ_2 are small positive scalars. The parameter γ is some positive scalar, in terms of which we are going to parametrize our solutions.

Even though the problem formulated above has been solved completely for every pair $\epsilon := (\epsilon_1, \epsilon_2)$, the computation of saddle-point solution of the game for small values of (ϵ_1, ϵ_2) present serious difficulties, due to numerical stiffness. To remedy this, we pose in this paper the question of whether the saddle-point solution can be determined, for small values of (ϵ_1, ϵ_2), by solving well-behaved ϵ-independent smaller-order problems, as in the case of the singularly perturbed linear-quadratic regulator problem ([4]) and the singularly perturbed linear-quadratic H^∞-optimal control problem. Another question of interest is whether the fast subsystem dynamics can be neglected completely in the design of such controllers, and if so under what conditions.

Before studying these questions, we first present below the solution to the full-order problem, under three basic assumptions:

A1 Q_f and $Q(.)$ in (4) are partitioned as

$$Q_f = \begin{bmatrix} Q_{f11} & \epsilon_1 Q_{f12} & \epsilon_1\epsilon_2 Q_{f13} \\ \epsilon_1 Q_{f21} & \epsilon_1 Q_{f22} & \epsilon_1\epsilon_2 Q_{f23} \\ \epsilon_1\epsilon_2 Q_{f31} & \epsilon_1\epsilon_2 Q_{f32} & \epsilon_1\epsilon_2 Q_{f33} \end{bmatrix}; \qquad Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix}$$

where in each case the ii-block is of dimensions $n_i \times n_i$, $i = 1, 2, 3$.

A2 The matrix functions $A_{ij}(t)$, $Q_{ij}(t)$, $B_i(t)$, $D_i(t)$ $\quad (i = 1, 2, 3 \;\; j = 1, 2, 3)$ are continuously differentiable in $t \geq 0$.

A3 The matrices $A_{33}(t)$, $A_{22}(t) - A_{23}A_{33}^{-1}A_{32}(t)$, $Q_{33}(t)$ and $Q_{22}(t) - Q_{23}Q_{33}^{-1}Q_{32}(t)$ are invertible for all $t \in [0, t_f]$.

Let us further introduce the notation $A(t)$ and $A_\epsilon(t)$, to denote the partitioned matrices

$$A(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ A_{21}(t) & A_{22}(t) & A_{23}(t) \\ A_{31}(t) & A_{32}(t) & A_{33}(t) \end{bmatrix}; \qquad A_\epsilon(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ \frac{1}{\epsilon_1}A_{21}(t) & \frac{1}{\epsilon_1}A_{22}(t) & \frac{1}{\epsilon_1}A_{23}(t) \\ \frac{1}{\epsilon_1\epsilon_2}A_{31}(t) & \frac{1}{\epsilon_1\epsilon_2}A_{32}(t) & \frac{1}{\epsilon_1\epsilon_2}A_{33}(t) \end{bmatrix}$$

Similarly, we introduce the partitioned matrices

$$B(t) := \begin{bmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \end{bmatrix}; \qquad B_\epsilon(t) := \begin{bmatrix} B_1(t) \\ \frac{1}{\epsilon_1}B_2(t) \\ \frac{1}{\epsilon_1\epsilon_2}B_3(t) \end{bmatrix}$$

$$D(t) := \begin{bmatrix} D_1(t) \\ D_2(t) \\ D_3(t) \end{bmatrix}; \qquad D_\epsilon(t) := \begin{bmatrix} D_1(t) \\ \frac{1}{\epsilon_1}D_2(t) \\ \frac{1}{\epsilon_1\epsilon_2}D_3(t) \end{bmatrix}$$

and define

$$S(t;\gamma) := B(t)B'(t) - \frac{1}{\gamma^2}D(t)D'(t); \qquad S_\epsilon(t;\gamma) := B_\epsilon(t)B_\epsilon'(t) - \frac{1}{\gamma^2}D_\epsilon(t)D_\epsilon'(t) \tag{5}$$

with the ij-th block of $S(t;\gamma)$ denoted henceforth by $S_{ij}(t;\gamma)$, $i, j = 1, 2, 3$.

We know from the existing theory on linear-quadratic differential games (see [1], [2]) that for each pair ϵ, there exists a $\gamma^*(\epsilon) \geq 0$ such that for all $\gamma > \gamma^*(\epsilon)$ the zero-sum differential game described by (1) and (4) has a value, $x_0'\tilde{Z}(0,\epsilon)x_0$, where $x_0 := (x'_{10}, x'_{20}, x'_{30})'$, as well as a saddle point, with saddle-point strategies being feedback laws

$$u^*(t) = \mu^*(t, x(t)) = -B_\epsilon'\tilde{Z}(t;\epsilon)x(t), \quad t \geq 0 \tag{6}$$
$$w^*(t) = \nu^*(t, x(t)) = \tfrac{1}{\gamma^2}D_\epsilon'\tilde{Z}(t;\epsilon)x(t), \quad t \geq 0 \tag{7}$$

where $\tilde{Z}(t;\epsilon)$ is the unique bounded nonnegative definite solution of the generalized matrix Riccati differential equation (GRDE)

$$\dot{\tilde{Z}} + A'_\epsilon \tilde{Z} + \tilde{Z} A_\epsilon - \tilde{Z} S_\epsilon \tilde{Z} + Q = 0; \qquad \tilde{Z}(t_f) = Q_f \tag{8}$$

For $\gamma < \gamma^*(\epsilon)$, on the other hand, this GRDE has a conjugate point in the open interval $(0, t_f)$, which says that it has finite escape, and furthermore the associated soft-constrained game has unbounded upper value.

We now turn, in the next two sections, to the original goal of this paper — which is the derivation of the approximate solutions based on a time-scale decomposition. First we identify, in Section 3, the slow and fast subsystems associated with the original problem, and obtain the solutions of three separate ϵ-free zero-sum differential games.

3 A Time-Scale Decomposition

The Slow Sub-Game

To obtain the slow dynamics associated with (1), we let $\epsilon_1 = 0$ (while keeping ϵ_2 fixed) and solve for $z := (x'_2, x'_3)'$ (to be denoted $\bar{z} := (\bar{x}'_2, \bar{x}'_3)'$) in terms of $x_1 =: x_s$, $u =: u_s$, $w =: w_s$, and under the working assumption A3:

$$\bar{z} = -F_{22\epsilon}^{-1}(F_{21\epsilon} x_s + G_{2\epsilon} u_s + H_{2\epsilon} w_s) \tag{9}$$

where

$$F_{22\epsilon} := \begin{bmatrix} A_{22} & A_{23} \\ \frac{1}{\epsilon_2} A_{32} & \frac{1}{\epsilon_2} A_{33} \end{bmatrix}; \quad F_{21\epsilon} := \begin{bmatrix} A_{21} \\ \frac{1}{\epsilon_2} A_{31} \end{bmatrix}$$

$$G_{2\epsilon} := \begin{bmatrix} B_2 \\ \frac{1}{\epsilon_2} B_3 \end{bmatrix}; \quad H_{2\epsilon} := \begin{bmatrix} D_2 \\ \frac{1}{\epsilon_2} D_3 \end{bmatrix}$$

We also introduce some block matrices for notational simplicity:

$$F_{22} := \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}; \quad F_{21} := \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$$

$$G_2 := \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}; \quad F_{12} := \begin{bmatrix} A_{12} & A_{13} \end{bmatrix}$$

$$H_2 := \begin{bmatrix} D_2 \\ D_3 \end{bmatrix}; \quad M_{22} := \begin{bmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{bmatrix}$$

$$M_{21} := \begin{bmatrix} Q_{21} \\ Q_{31} \end{bmatrix}; \quad M_{12} := \begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$$

$$N_{22\epsilon} := G_{2\epsilon} G'_{2\epsilon} - \frac{1}{\gamma^2} H_{2\epsilon} H'_{2\epsilon}$$

$$N_{12\epsilon} := B_1 G'_{2\epsilon} - \frac{1}{\gamma^2} D_1 H'_{2\epsilon}$$

$$N_{21\epsilon} := G_{2\epsilon} B'_1 - \frac{1}{\gamma^2} H_{2\epsilon} D'_1$$

where we define N_{22}, N_{21} and N_{12} likewise with $\epsilon_2 = 1$.

Thus, we can see that expression (9) is in fact ϵ_2-independent and can be written as

$$\bar{z} = -F_{22}^{-1}(F_{21}x_s + G_2 u_s + H_2 w_s) \tag{10}$$

Using this in the first equation of (1), we obtain the reduced-order (slow) dynamics:

$$\dot{x}_s = A_0 x_s + B_0 u_s + D_0 w_s \tag{11}$$

where $A_0 := A_{11} - F_{12}F_{22}^{-1}F_{21}$, $B_0 := B_1 - F_{12}F_{22}^{-1}G_2$, $D_0 := D_1 - F_{12}F_{22}^{-1}H_2$.

Using (10) also in the cost function (4) leads to the reduced (slow) cost (with $x_1 = x_s$):

$$L_s = |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f} (|x_s|^2_{Q_{11}} + \bar{z}' M_{21} x_s + x'_s M_{12} \bar{z} + |\bar{z}|^2_{M_{22}} + |u_s|^2 - \gamma^2 |w_s|^2)\, dt \tag{12}$$

In view of our discussion in [3], we may first endow the minimizer with informational advantage, which will later become superfluous with the additional knowledge of the fast state variable. Thus, at this point, we allow Player 1 to choose strategies which depend on the current action of Player 2. Compatible with the given information structure, we introduce the transformations

$$\begin{aligned}
\tilde{u}_s &= (I + G'_2 F'^{-1}_{22} M_{22} F_{22}^{-1} G_2)^{1/2} [u_s + (I + G'_2 F'^{-1}_{22} M_{22} F_{22}^{-1} G_2)^{-1} \\
&\quad \cdot G'_2 F'^{-1}_{22} M_{22} F_{22}^{-1} ((F_{21} - F_{22} M_{22}^{-1} M_{21}) x_s + H_2 w_s)] \qquad (13) \\
\tilde{w}_s &= (\gamma^2 I - H'_2 (F_{22} M_{22}^{-1} F'_{22} + G_2 G'_2)^{-1} H_2)^{1/2} \\
&\quad \cdot [w_s + (\gamma^2 I - H'_2 (F_{22} M_{22}^{-1} F'_{22} + G_2 G'_2)^{-1} H_2)^{-1} \\
&\quad \cdot H'_2 (F_{22} M_{22}^{-1} F'_{22} + G_2 G'_2)^{-1} (F_{22} M_{22}^{-1} M_{21} - F_{21}) x_s] \qquad (14)
\end{aligned}$$

where the latter one is valid under the condition

$$\gamma^2 I - H'_2 (F_{22} M_{22}^{-1} F'_{22} + G_2 G'_2)^{-1} H_2 > 0 \tag{15}$$

which can equivalently be written as

$$N_{22}(\gamma) + F_{22} M_{22}^{-1} F'_{22} > 0 \tag{16}$$

If we use (13) and (14) in (11) and (12), we arrive at the following standard LQ differential game, which has no cross terms between state, control and disturbance in the cost (the derivation follows the one described in [3]):

$$\dot{x}_s = \tilde{A}_0 x_s + \tilde{B}_0 \tilde{u}_s + \tilde{D}_0 \tilde{w}_s; \quad x_s(0) = 0 \tag{17}$$

$$L_{\gamma s} = |x_s(t_f)|^2_{Q_{f11}} + \int_0^{t_f} (|x_s|^2_{\tilde{Q}} + |\tilde{u}_s|^2 - |\tilde{w}_s|^2)\, dt \tag{18}$$

The coefficient matrices above are given by:

$$\begin{aligned}
\tilde{A}_0(\gamma) &= A_{11} - F_{12}M_{22}^{-1}M_{21} - (N_{12} + F_{12}M_{22}^{-1}F_{22}')(N_{22} + F_{22}M_{22}^{-1}F_{22}')^{-1} \\
&\quad \cdot (F_{21} - F_{22}M_{22}^{-1}M_{21}) && (19)\\
\tilde{B}_0 &= B_0(I + G_2'F_{22}'^{-1}M_{22}F_{22}^{-1}G_2)^{-1/2} && (20)\\
\tilde{D}_0(\gamma) &= (D_0 - B_0G_2'(F_{22}M_{22}^{-1}F_{22}' + G_2G_2')^{-1}H_2) \\
&\quad \cdot (\gamma^2 I - H_2'(F_{22}M_{22}^{-1}F_{22}' + G_2G_2')^{-1}H_2)^{-1/2} && (21)\\
\tilde{Q}(\gamma) &= Q_{11} - M_{12}M_{22}^{-1}M_{21} + (F_{21}' - M_{12}M_{22}^{-1}F_{22}')(N_{22} + F_{22}M_{22}^{-1}F_{22}')^{-1} \\
&\quad \cdot (F_{21} - F_{22}M_{22}^{-1}M_{21}) && (22)
\end{aligned}$$

For each $\gamma \geq 0$, we associate the following generalized RDE with this standard LQ differential game:

$$\dot{\tilde{Z}}_s + \tilde{A}_0'\tilde{Z}_s + \tilde{Z}_s\tilde{A}_0 - \tilde{Z}_s(\tilde{B}_0\tilde{B}_0' - \tilde{D}_0\tilde{D}_0')\tilde{Z}_s + \tilde{Q} = 0; \qquad \tilde{Z}_s(t_f) = Q_{f11}. \tag{23}$$

Let us introduce

$$S_0 := \tilde{B}_0\tilde{B}_0' - \tilde{D}_0\tilde{D}_0'$$

which can be rewritten in terms of the original system matrices (see [3] for details):

$$\begin{aligned}
S_0 &= S_{11} + F_{12}M_{22}^{-1}F_{12}' - (N_{12} + F_{12}M_{22}^{-1}F_{22}')(N_{22} + F_{22}M_{22}^{-1}F_{22}')^{-1} \\
&\quad \cdot (N_{21} + F_{22}M_{22}^{-1}F_{12}') && (24)
\end{aligned}$$

In view of this , let us introduce the set:

$$\begin{aligned}
\tilde{\Gamma}_s &:= \{\gamma' > 0 : \forall\gamma \geq \gamma', N_{22}(\gamma) + F_{22}M_{22}^{-1}F_{22}' > 0 \text{ and (23) has a bounded} && (25)\\
&\qquad \text{nonnegative definite solution over } [0, t_f].\} && (26)
\end{aligned}$$

and further define

$$\gamma_s := \inf\{\gamma \in \tilde{\Gamma}_s\} \tag{27}$$

Then, it follows from the analysis of [2], chapter 8, that the transformed game with cost function $L_{\gamma s}$ has a bounded value, $x_{10}'Z_{s\gamma}x_{10}$[1], if $\gamma > \gamma_s$, and only if $\gamma \geq \gamma_s$. For $\gamma > \gamma_s$, let $Z_{s\gamma}$ be the unique nonnegative definite solution of (23). Then, there exist strongly time consistent (feedback) saddle-point policies for the transformed game, given by

$$\begin{aligned}
\tilde{u}_{s\gamma}^* &= \tilde{\mu}_{s\gamma}^*(t, x_s(t)) = -\tilde{B}_0'Z_{s\gamma}x_s(t) && (28)\\
\tilde{w}_{s\gamma}^* &= \tilde{\nu}_{s\gamma}^*(t, x_s(t)) = \tilde{D}_0'Z_{s\gamma}x_s(t) && (29)
\end{aligned}$$

[1] Note that under condition (16), $\tilde{Q}(\gamma) \geq 0$.

Applying the inverse transformation of (14) to (29), we arrive (following the derivation given in [3]) at the following expression:

$$\begin{aligned} w^*_{s\gamma} &= \nu^*_{s\gamma}(t, x_s(t)) = \frac{1}{\gamma^2}(D_1' Z_{s\gamma} - H_2'(N_{22} + F_{22}M_{22}^{-1}F_{22}')^{-1}((N_{21} + F_{22}M_{22}^{-1}F_{12}')Z_{s\gamma} \\ &\quad -(F_{21} - F_{22}M_{22}^{-1}M_{21})))x_s(t) \end{aligned} \tag{30}$$

The inverse transformation of (13) applied to (28) yields:

$$\begin{aligned} u^*_{sf\gamma} &= \mu^*_{sf\gamma}(t, x_s(t), w_s(t)) = -(I + G_2' F_{22}'^{-1} M_{22} F_{22}^{-1} G_2)^{-1}(B_0' Z_{s\gamma} \\ &\quad - G_2' F_{22}'^{-1} M_{22} F_{22}^{-1}(F_{22}M_{22}{-}1M_{21} - F_{21}))x_s(t) \\ &\quad - G_2'(G_2 G_2' + F_{22}M_{22}^{-1}F_{22}')^{-1} H_2 w_s \end{aligned} \tag{31}$$

which is a controller that achieves the finite value in the slow game. Note that this controller involves a feedforward term in w_s, as expected. If, however, we wish to obtain one which depends only on the state, we simply substitute (30) into (31), to arrive at the following expression:

$$\begin{aligned} u^*_{s\gamma} &= \mu^*_{s\gamma}(t, x_s(t)) = (-B_1' Z_{s\gamma} + G_2'(N_{22} + F_{22}M_{22}^{-1}F_{22}')^{-1}((N_{21} + F_{22}M_{22}^{-1}F_{12}')Z_{s\gamma} \\ &\quad -(F_{21} - F_{22}M_{22}^{-1}M_{21})))x_s(t) \end{aligned} \tag{32}$$

By the "ordered interchangeability" property of multiple saddle point equilibria (see [1]), $\mu^*_{s\gamma}$ is also in equilibrium with $\nu^*_{s\gamma}$, but not for all $\gamma > \gamma_s$. In other words, there will exist some $\gamma \in \tilde{\Gamma}_s$ for which under $\mu^*_{s\gamma}$ the upper value of the game will be infinite, even though it is finite if $\mu^*_{sf\gamma}$ given by (31) is used. It is shown in [3] that the pair $(\mu^*_{s\gamma}, \nu^*_{s\gamma})$ in fact provides a saddle-point solution to the slow game, but under a more restrictive condition (than (15)).

The Fast Sub-Games

Let $z_f := z - \bar{z}$, $u_f := u - u_s$, $w_f := w - w_s$ and $\tau = \frac{t'-t}{\epsilon_1}$ where we take t to be frozen, and t' to vary on the same scale as t. We define the fast subsystem and the associated cost (as in the standard regulator problem; see [4]) by:

$$\frac{d}{d\tau} z_f^t = F_{22\epsilon}(t) z_f^t + G_{2\epsilon}(t) u_f^t + H_{2\epsilon}(t) w_f^t; \quad z_f^t(0) = z_f(t) \tag{33}$$

$$L_{\gamma f}^t = \int_0^\infty (|z_f^t|^2_{M_{22}(t)} + |u_f^t|^2 - \gamma^2 |w_f^t|^2)\, d\tau \tag{34}$$

The GARE associated with this infinite-horizon game, for each t, is:

$$F_{22\epsilon}'(t) Z_f + Z_f F_{22\epsilon}(t) + M_{22}(t) - Z_f N_{22\epsilon}(t) Z_f = 0 \tag{35}$$

This is now a two-time scale zero-sum differential game, which is of the type that has been studied extensively in [3]. Following the derivations there, one can further separate the zero-sum game (33) and (35) into two subgames: one defined on the τ time scale and the other one defined on an even faster time scale, ξ time scale, which we will define shortly.

Thus, the subgame defined on the τ time scale (which we will call fast subgame) can be obtained by setting $\epsilon_2 = 0$ in (33) and (35) and solving for x^t_{3f} (to be denoted $\bar{x}^t_{3f}$) in terms of $x^t_{2f} =: x^t_{fs}$, $u^t_f =: u^t_{fs}$, $w^t_f =: w^t_{fs}$, and under the working assumption A3:

$$\bar{x}^t_{3f} = -A_{33}^{-1}(A_{32}x^t_{fs} + B_3 u^t_{fs} + D_3 w^t_{fs}) \tag{36}$$

Using this in the first equation of (33), we obtain the reduced-order dynamics:

$$\dot{x}^t_{fs} = F_0 x^t_{fs} + G_0 u^t_{fs} + H_0 w^t_{fs} \tag{37}$$

where

$$\begin{aligned} F_0 &:= A_{22} - A_{23}A_{33}^{-1}A_{32} && (38)\\ G_0 &:= B_2 - A_{23}A_{33}^{-1}B_3 && (39)\\ H_0 &:= D_2 - A_{23}A_{33}^{-1}D_3 && (40) \end{aligned}$$

Using (10) also in the cost function (4) leads to the reduced (slow) cost:

$$L^t_{fs} = \int_0^\infty (|x^t_{fs}|^2_{Q_{22}} + x^t_{fs}{}'Q_{23}\bar{x}^t_{3f} + \bar{x}^t_{3f}{}'Q_{32}x^t_{fs} + |\bar{x}^t_{3f}|^2_{Q_{33}} + |u^t_{fs}|^2 - \gamma^2|w^t_{fs}|^2)\, dt \tag{41}$$

As in the slow subgame case, we again endow Player 1 with informational advantage, which will be void when the fastest state is available. Now, we introduce the transformations:

$$\begin{aligned} \tilde{u}^t_{fs} &= (I + B_3'A_{33}'^{-1}Q_{33}A_{33}^{-1}B_3)^{1/2}[u^t_{fs} + (I + B_3'A_{33}'^{-1}Q_{33}A_{33}^{-1}B_3)^{-1} \\ &\quad \cdot B_3'A_{33}'^{-1}Q_{33}A_{33}^{-1}((A_{32} - A_{33}Q_{33}^{-1}Q_{32})x^t_{fs} + D_3 w^t_{fs})] && (42)\\ \tilde{w}^t_{fs} &= (\gamma^2 I - D_3'(A_{33}Q_{33}^{-1}A_{33}' + B_3B_3')^{-1}D_3)^{1/2} \\ &\quad \cdot[w^t_{fs} + (\gamma^2 I - D_3'(A_{33}Q_{33}^{-1}A_{33}' + B_3B_3')^{-1}D_3)^{-1} \\ &\quad \cdot D_3'(A_{33}Q_{33}^{-1}A_{33}' + B_3B_3')^{-1}(A_{33}Q_{33}^{-1}Q_{32} - A_{32})x^t_{fs}] && (43) \end{aligned}$$

where the latter one is valid under the condition

$$\gamma^2 I - D_3'(A_{33}Q_{33}^{-1}A_{33}' + B_3B_3')^{-1}D_3 > 0 \tag{44}$$

which can equivalently be written as

$$S_{33}(\gamma) + A_{33}Q_{33}^{-1}A_{33}' > 0 \tag{45}$$

If we use (42) and (43) in (37) and (41), we arrive at the following standard LQ differential game:

$$\frac{d}{d\tau}x^t_{fs} = \tilde{F}_0 x^t_{fs} + \tilde{G}_0\tilde{u}^t_{fs} + \tilde{H}_0\tilde{w}^t_{fs}; \quad x_{fs}(0) = x_2^f(t) \tag{46}$$

$$L_{\gamma fs} = \int_0^\infty (|x^t_{fs}|^2_{\tilde{M}} + |\tilde{u}^t_{fs}|^2 - |\tilde{w}^t_{fs}|^2)\, dt \tag{47}$$

The coefficient matrices above are given by:

$$\begin{aligned}
\tilde{F}_0(\gamma) &= A_{22} - A_{23}Q_{33}^{-1}Q_{32} - (S_{23} + A_{23}Q_{33}^{-1}A'_{33})(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1} \\
&\quad \cdot (A_{32} - A_{33}Q_{33}^{-1}Q_{32}) && (48) \\
\tilde{G}_0 &= G_0(I + B'_3 A'_{33}{}^{-1} Q_{33} A_{33}^{-1} B_3)^{-1/2} && (49) \\
\tilde{H}_0(\gamma) &= (H_0 - B_0 B'_3 (A_{33}Q_{33}^{-1}A'_{33} + B_3 B'_3)^{-1} D_3) \\
&\quad \cdot (\gamma^2 I - D'_3 (A_{33}Q_{33}^{-1}A'_{33} + B_3 B'_3)^{-1} D_3)^{-1/2} && (50) \\
\tilde{M}(\gamma) &= Q_{22} - Q_{23}Q_{33}^{-1}Q_{32} + (A'_{32} - Q_{23}Q_{33}^{-1}A'_{33})(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1} \\
&\quad \cdot (A_{32} - A_{33}Q_{33}^{-1}Q_{32}) && (51)
\end{aligned}$$

For each $\gamma \geq 0$, we associate the following generalized GARE with this standard LQ differential game:

$$\tilde{F}'_0 \tilde{Z}_{fs} + \tilde{Z}_{fs}\tilde{F}_0 - \tilde{Z}_{fs}(\tilde{G}_0\tilde{G}'_0 - \tilde{H}_0\tilde{H}'_0)\tilde{Z}_{fs} + \tilde{M} = 0 \tag{52}$$

Let us introduce

$$N_0 := \tilde{G}_0\tilde{G}'_0 - \tilde{H}_0\tilde{H}'_0$$

which can be rewritten in terms of the original system matrices as follows:

$$\begin{aligned}
N_0 &= S_{22} + A_{23}Q_{33}^{-1}A'_{23} - (S_{23} + A_{23}Q_{33}^{-1}A'_{33})(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1} \\
&\quad \cdot (S_{32} + A_{33}Q_{33}^{-1}A'_{23}) && (53)
\end{aligned}$$

In view of this , let us introduce the set:

$$\begin{aligned}
\tilde{\Gamma}^t_{fs} &:= \{\gamma' > 0 : \forall \gamma \geq \gamma', S_{33}(\gamma) + A_{33}Q_{33}^{-1}A'_{33} > 0 \text{ and (23) has a bounded} \\
&\qquad \text{positive definite solution.}\} && (54)
\end{aligned}$$

and further define

$$\gamma^t_{fs} := \inf\{\gamma \in \tilde{\Gamma}^t_{fs}\} \tag{55}$$

$$\gamma_{fs} := \sup\nolimits_{t \in [0,t_f]} \gamma^t_{fs} \tag{56}$$

Then, it follows from the analysis of [2], chapter 8, that the transformed game with cost function $L^t_{\gamma fs}$ has a bounded value for all $t \in [0, t_f]^2$, if $\gamma > \gamma_{fs}$, and only if $\gamma \geq \gamma_{fs}$. For $\gamma > \gamma_{fs}$, let $Z_{fs\gamma}(t)$ be the unique positive definite solution of (52). Then, there exist strongly time consistent (feedback) saddle-point policies for the transformed game, for each $t \in [0, t_f]$, given by

$$\tilde{u}^{t*}_{fs\gamma} = \tilde{\mu}^{t*}_{fs\gamma}(t, x^t_{fs}(\tau)) = -\tilde{G}'_0 Z_{fs\gamma} x^t_{fs}(\tau) \tag{57}$$

$$\tilde{w}^{t*}_{fs\gamma} = \tilde{\nu}^{t*}_{fs\gamma}(t, x^t_{fs}(\tau)) = \tilde{H}'_0 Z_{fs\gamma} x^t_{fs}(\tau) \tag{58}$$

---

[2] Note that under condition (45), $\tilde{M}(\gamma) \geq 0$.

Applying the inverse transformation of (43) to (58) and setting $\tau = 0$, we arrive at the following expression in the t time scale:

$$\begin{aligned} w^*_{fs\gamma} &= \nu^*_{fs\gamma}(t, x_{2f}(t)) = \frac{1}{\gamma^2}(D'_2 Z_{fs\gamma} - D'_3(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1}((S_{32} + A_{33}Q_{33}^{-1}A'_{23}) \\ &\cdot Z_{fs\gamma} - (A_{32} - A_{33}Q_{33}^{-1}Q_{32})))x_{2f}(t) \end{aligned} \tag{59}$$

The inverse transformation of (42) applied to (28) yields:

$$\begin{aligned} u^*_{fsf\gamma} &= \mu^*_{sf\gamma}(t, x_{2f}(t), w_f(t)) = -(I + B'_3 {A'_{33}}^{-1} Q_{33} A_{33}^{-1} B_3)^{-1}(G'_0 Z_{fs\gamma} \\ &- B'_3 {A'_{33}}^{-1} Q_{33} A_{33}^{-1}(A_{33}Q_{33}{-}1Q_{32} - A_{32}))x_{2f}(t) \\ &- B'_3(B_3 B'_3 + A_{33}Q_{33}^{-1}A'_{33})^{-1} D_3 w_f \end{aligned} \tag{60}$$

which is a controller that achieves the finite value in the slow game. We then substitute (59) into (60), to arrive at the following expression:

$$\begin{aligned} u^*_{fs\gamma} &= \mu^*_{fs\gamma}(t, x_{2f}(t)) = (-B'_2 Z_{fs\gamma} + B'_3(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1}((S_{32} + A_{33}Q_{33}^{-1}A'_{23}) \\ &\cdot Z_{fs\gamma} - (A_{32} - A_{33}Q_{33}^{-1}Q_{32})))x_{2f}(t) \end{aligned} \tag{61}$$

To complete the study of the optimality of the slow strategy (32) for Player 1, we need to consider also the open-loop version of this game, which is the case when μ_{fs} is chosen to depend only on the initial state. As discussed in [1], this game admits a finite value if the GARE (52) admits a positive definite solution with $B_2 = 0$ and $B_3 = 0$. We let γ_{ofs} be the infimum of all γ's such that, $\forall \gamma' > \gamma$, the GARE (52) admits a positive definite solution with $B_2 = 0$ and $B_3 = 0$, and for all $t \in [0, t_f]$.

The fastest subgame is defined as follows. Let $x_{ff} := x_{3f} - \bar{x}_{3f}$, $u_{ff} := u_f - u_{fs}$, $w_{ff} := w_f - w_{fs}$ and $\xi = \frac{\tau' - \tau}{\epsilon_2}$ where we take τ to be frozen, and τ' to vary on the same time scale as τ. We define the fastest subgame by:

$$\frac{d}{d\xi} x^\tau_{ff} = A_{33}(t) x^\tau_{ff} + B_3(t) u^\tau_{ff} + D_3(t) w^\tau_{ff}; \quad x^\tau_{ff}(0) = x_{ff}(\tau) \tag{62}$$

$$L^\tau_{\gamma ff} = \int_0^\infty (|x^\tau_{ff}|^2_{Q_{33}(t)} + |u^\tau_{ff}|^2 - \gamma^2 |w^\tau_{ff}|^2)\, d\xi \tag{63}$$

The GARE associated with this infinite-horizon game, for each τ, is:

$$A'_{33}(t) Z_{ff} + Z_{ff} A_{33}(t) + Q_{33}(t) - Z_{ff} S_{33}(t) Z_{ff} = 0 \tag{64}$$

We now let γ^t_{ff} denote the infimum of the set of γ's such that GARE (64) admits a positive definite solution, and let $Z_{ff\gamma}(t)$ be the minimal positive definite solution for (64). Then, for any $\gamma > \gamma^t_{ff}$, the feedback controller that attains the finite upper value of the fastest game is

$$\begin{aligned} u^{\tau *}_{ff\gamma}(\xi) &= \mu^{\tau *}_{ff\gamma}(x^\tau_{ff}(\xi)) = -B'_3(t) Z_{ff\gamma}(t) x^\tau_{ff}(\xi) \\ \Rightarrow \quad u^*_{ff\gamma}(t) &= -B'_3(t) Z_{ff\gamma}(t) x_{ff}(t) \end{aligned} \tag{65}$$

where we have substituted $\tau = 0$ and $\xi = 0$ into the above equation to arrive at (65). Substitute (36), (61) and (59) into (65), to obtain

$$\begin{aligned}\mu^*_{ff\gamma}(t, z_f(t)) &= -B_3' Z_{ff\gamma} x_{3f} - B_3' Z_{ff\gamma} Q_{33}^{-1} (A_{23}' Z_{fs\gamma} + Q_{32} - A_{33}'(S_{33} + A_{33} Q_{33}^{-1} \\ &\quad \cdot A_{33}')^{-1}((S_{32} + A_{33} Q_{33}^{-1} A_{23}') Z_{s\gamma} - (A_{32} - A_{33} Q_{33}^{-1} Q_{32}))) x_{2f}(t) \quad (66)\end{aligned}$$

We now define

$$\gamma_{ff} := \sup_{t \in [0, t_f]} \gamma_{ff}^t \quad (67)$$

Then, for every $\gamma > \gamma_{ff}$, the GARE (64) admits a positive definite solution for all $t \in [0, t_f]$.

We need to consider the open-loop version of this fastest game also. Let γ_{off} be the infimum of the set of all γ's such that, $\forall \gamma' > \gamma$, the following GARE admits a positive definite solution for all $t \in [0, t_f]$:

$$A_{33}'(t) Z_{off} + Z_{off} A_{33}(t) + Q_{33}(t) + Z_{off} \frac{1}{\gamma^2} D_3 D_3'(t) Z_{off} = 0 \quad (68)$$

Then, $\forall \gamma > \gamma_{off}$, the fastest game has a finite value even if the first player has access only to open-loop information.

Let

$$\bar{\gamma} := \max\{\gamma_{ff}, \gamma_{fs}, \gamma_s\} \quad (69)$$

We will shortly see that this value will play an important role in the characterization of an approximate saddle-point solution for our problem.

4 Composite Strategies

We now form our composite strategies for Players 1 and 2. Notice the difference between the roles of Players 1 and 2: the lower value of the game is bounded below by zero for all values of γ, whereas the upper value is possibly infinite for small enough γ. Thus, we must use a composite strategy (which we will define shortly) for Player 1 as discussed in [3], whereas only a slow open-loop strategy, the open-loop version of (30) (which will be discussed shortly), will suffice for Player 2. The appropriateness of such a choice will be established in the next two sections for both finite and infinite horizons.

First, we introduce the fast part of the composite strategy for Player 1:

$$\mu^*_{cf\gamma}(t, z_f) = \mu^*_{fs\gamma}(t, z_f) + \mu^*_{ff\gamma}(t, z_f) \quad (70)$$

where $\mu^*_{fs\gamma}$ and $\mu^*_{ff\gamma}$ were defined by (61) and (66), respectively, for $\gamma > \bar{\gamma}$. After some manipulations (parallel to those that can be found in [3]), this composite controller can be rewritten as

$$\mu^*_{cf\gamma}(t, z_f) = -G_2' \begin{bmatrix} Z_{fs\gamma} & 0 \\ Z_{fc} & Z_{ff\gamma} \end{bmatrix} \begin{bmatrix} x_{2f} \\ x_{3f} \end{bmatrix} := -G_2' \check{Z}_f z_f \quad (71)$$

where

$$
\begin{aligned}
Z_{fc} &:= Z_{ff\gamma}Q_{33}^{-1}(A'_{23}Z_{fs\gamma}+Q_{32})-(I+Z_{ff\gamma}Q_{33}^{-1}A'_{33})(S_{33}+A_{33}Q_{33}^{-1}A'_{33})^{-1} \\
&\quad \cdot((S_{32}+A_{33}Q_{33}^{-1}A'_{23})Z_{fs\gamma}-(A_{32}-A_{33}Q_{33}^{-1}Q_{32})) && (72)\\
&\equiv Z_{ff\gamma}U+V && (73)\\
&\equiv Z_{ff\gamma}(U_1Z_{fs\gamma}+U_2)+V_1Z_{fs\gamma}+V_2 && (74)
\end{aligned}
$$

$$
\begin{aligned}
U_1 &:= Q_{33}^{-1}A'_{23}-Q_{33}^{-1}A'_{33}(S_{33}+A_{33}Q_{33}^{-1}A'_{33})^{-1}(S_{32}+A_{33}Q_{33}^{-1}A'_{23}) && (75)\\
U_2 &:= Q_{33}^{-1}Q_{32}+Q_{33}^{-1}A'_{33}(S_{33}+A_{33}Q_{33}^{-1}A'_{33})^{-1}(A_{32}-A_{33}Q_{33}^{-1}Q_{32}) && (76)\\
V_1 &:= -(S_{33}+A_{33}Q_{33}^{-1}A'_{33})^{-1}(S_{32}+A_{33}Q_{33}^{-1}A'_{23}) && (77)\\
V_2 &:= (S_{33}+A_{33}Q_{33}^{-1}A'_{33})^{-1}(A_{32}-A_{33}Q_{33}^{-1}Q_{32}) && (78)
\end{aligned}
$$

To form the composite strategy for Player 1, we need to define the following matrices to simplify our notation.

$$
\begin{aligned}
\bar{U}_1 &:= M_{22}^{-1}F'_{12}-M_{22}^{-1}F'_{22}(N_{22}+F_{22}M_{22}^{-1}F'_{22})^{-1}(N_{21}+F_{22}M_{22}^{-1}F'_{12}) && (79)\\
\bar{U}_2 &:= M_{22}^{-1}M_{21}+M_{22}^{-1}F'_{22}(N_{22}+F_{22}M_{22}^{-1}F'_{22})^{-1}(F_{21}-F_{22}M_{22}^{-1}M_{21}) && (80)\\
\bar{V}_1 &:= -(N_{22}+F_{22}M_{22}^{-1}F'_{22})^{-1}(N_{21}+F_{22}M_{22}^{-1}F'_{12}) && (81)\\
\bar{V}_2 &:= (N_{22}+F_{22}M_{22}^{-1}F'_{22})^{-1}(F_{21}-F_{22}M_{22}^{-1}M_{21}) && (82)\\
\bar{U} &:= \bar{U}_1Z_{s\gamma}+\bar{U}_2 && (83)\\
\bar{V} &:= \bar{V}_1Z_{s\gamma}+\bar{V}_2 && (84)
\end{aligned}
$$

Substitute (30) and (32) into (10), and notice that

$$
\begin{aligned}
w_s^* &= \frac{1}{\gamma^2}(D'_1Z_{s\gamma}+H'_2\bar{V})x_s && (85)\\
u_s^* &= -(B'_1Z_{s\gamma}+G'_2\bar{V})x_s && (86)
\end{aligned}
$$

in view of which we arrive at the following expression for $\bar{z}$:

$$
\bar{z}=\bar{U}x_s \qquad (87)
$$

where $\bar{U}$ was defined by (83).

We now introduce the composite strategy for Player 1:

$$
\mu_{c\gamma}^*(t,x)=\mu_{s\gamma}^*(t,x_1)+\mu_{cf\gamma}^*(t,z_f) \qquad (88)
$$

By substituting (87), (71) and the definition for z_f into the above expression, we obtain the control law:

$$
\mu_{c\gamma}^*(t,x)=-B'\begin{bmatrix} Z_{s\gamma} & 0 & 0 \\ \check{Z}_f\bar{U}+\bar{V} & Z_{fs\gamma} & 0 \\ & Z_{fc} & Z_{ff\gamma}\end{bmatrix}x(t) \qquad (89)
$$

Let us define $\check{Z}_c := \check{Z}_f\bar{U}+\bar{V}$ for future use.

We now turn to show the optimality of the composite strategy (89), slow strategy (32) for Player 1 and the slow strategy (30) for Player 2, in the next two sections.

5 The Infinite-Horizon Case

We first discuss the infinite horizon case. Let A, B, D and Q be time invariant and Q_f be zero. Furthermore, suppose that (A_ϵ, B_ϵ) is controllable, and (A_ϵ, Q) is observable for every pair ϵ. Then, for each pair ϵ, there exists a $\gamma^*_\infty(\epsilon)$ such that, for all $\gamma > \gamma^*_\infty(\epsilon)$, the infinite-horizon soft-constrained game has a finite value, with the minimizing strategy for Player 1 and the corresponding maximizing strategy for Player 2 being

$$u^*(t) = \mu^\infty(x) = -B'_\epsilon \tilde{Z}_\infty(\epsilon)x(t), \quad t \geq 0 \tag{90}$$

$$w^*(t) = \nu^\infty(x) = \frac{1}{\gamma^2} D'_\epsilon \tilde{Z}_\infty(\epsilon)x(t), \quad t \geq 0 \tag{91}$$

where $\tilde{Z}_\infty > 0$ is the minimal positive definite solution of the generalized algebraic Riccati equation (GARE)

$$A'_\epsilon \tilde{Z}_\infty + \tilde{Z}_\infty A_\epsilon - \tilde{Z}_\infty S_\epsilon \tilde{Z}_\infty + Q = 0 \tag{92}$$

Toward the end of finding ϵ-free solutions, we follow steps similar to those used in the finite-horizon case. We first decompose the system into slow, fast and fastest subgames. After an appropriate transformation, the slow game with FI controller is described by

$$\dot{x}_s = \tilde{A}_0 x_s + \tilde{B}_0 \tilde{u}_s + \tilde{D}_0 \tilde{w}_s; \quad x_s(0) = x_{10} \tag{93}$$

$$L_{\gamma s} = \int_0^\infty (|x_s|^2_{\tilde{Q}} + |\tilde{u}_s|^2 - |\tilde{w}_s|^2)\, dt \tag{94}$$

where $\tilde{A}_0$, $\tilde{B}_0$, $\tilde{D}_0$ and $\tilde{Q}$ are as defined before, with the only difference being that they are now time invariant. The associated GARE is

$$\tilde{A}'_0 \tilde{Z}_s + \tilde{Z}_s \tilde{A}_0 - \tilde{Z}_s S_0 \tilde{Z}_s + \tilde{Q} = 0 \tag{95}$$

Let us define the following set as the counterpart of (26):

$$\begin{aligned} \tilde{\Gamma}_{s\infty} \;:=\; & \{\gamma' > 0 : \forall \gamma \geq \gamma', N_{22} + F_{22} M_{22}^{-1} F'_{22} > 0, \text{(95) has a positive} \\ & \quad \text{definite solution } Z_{s\gamma}, \text{ and } \tilde{A}'_0(\gamma) - S_0(\gamma) Z_{s\gamma} \text{ is Hurwitz.}\} \end{aligned}$$

and further define

$$\gamma_{s\infty} := \inf\{\gamma \in \tilde{\Gamma}_{s\infty}\}. \tag{96}$$

For every $\gamma > \gamma_{s\infty}$[3], let $Z_{s\gamma}$ be the minimal positive definite solution to (95). Then, for each $\gamma > \gamma_{s\infty}$, the minimizing strategy for Player 1 that attains the finite value is the time-invariant version of (28), and the maximizing strategy for Player 2 is the time-invariant

[3] Again, for $\gamma_{s\infty}$ to be finite, it will be sufficient to have the pair $(\tilde{A}_0, \tilde{B}_0)$ controllable at $\gamma = \infty$, which is equivalent to having the pair (A_0, B_0) controllable.

version of (29). Then we can solve for $\mu^*_{s\gamma}$ and $\nu^*_{s\gamma}$, which are the time-invariant versions of (30) and (31), respectively.[4]

The fast part of the system is the same as in the finite-horizon case, where the coefficient matrices are now constants. The fast game is described by (36) and (41), and the GARE is the same as (52). We will use $\gamma_{fs\infty}$ to denote the counterpart of γ_{fs}, and let $Z_{fs\gamma}$ denote the minimal positive definite solution to (52). Then, the minimizing strategy for Player 1 (after the inverse transformation) is given by (61) and the corresponding maximizing strategy for Player 2 is the same as (61). We also define $\gamma_{ofs\infty}$ as the counterpart of γ_{ofs}.

The fastest subgame is described in the same way as (62) and (63). The associated GARE is the same as (64). Thus, we may find the minimizing strategy for Player 1 to be the same as (66). We define $\gamma_{ff\infty}$ as the counterpart of γ_{ff}. For each $\gamma > \gamma_{ff\infty}$, we let $Z_{ff\gamma}$ denote the minimal positive definite solution to the GARE (64). We also define $\gamma_{off\infty}$ as the counterpart of γ_{off} in the infinite-horizon case.

As the counterpart of (69), we define

$$\bar{\gamma}_\infty := \max\{\gamma_{ff\infty}, \gamma_{fs\infty}, \gamma_{s\infty}\} \tag{97}$$

This quantity will also play an important role in our subsequent analysis in this section.

Now, we can form the composite strategy for Player 1, which is in exactly the same form as (88). For the Player 2, on the other hand, we form the open-loop representation of the strategy (30) as follows:

$$\begin{aligned} w^*_{os\gamma}(t) &= \nu^*_{os\gamma}(t) = \frac{1}{\gamma^2}(D_1' Z_{s\gamma} - H_2'(N_{22} + F_{22}M_{22}^{-1}F_{22}')^{-1}((N_{21} + F_{22}M_{22}^{-1}F_{12}')Z_{s\gamma} \\ &\quad -(F_{21} - F_{22}M_{22}^{-1}M_{21})))x^*_s(t) \end{aligned} \tag{98}$$

where $x^*_s(t)$ is generated by the following dynamic system:

$$\dot{x}^*_s(t) = (\tilde{A}_0 - S_0 Z_{s\gamma})x^*_s(t); \quad x^*_s(0) = x_{10} \tag{99}$$

where the dynamic equation for x^*_s is obtained by substituting (30) and (32) into (11).

This brings us to the point where we can state the main results concerning the infinite-horizon zero-sum differential game.

Theorem 1 *Consider the singularly perturbed zero-sum differential game (1) and (4), with $t_f = \infty$, $Q_f = 0$ and A, B, D, Q time-invariant. If assumptions A1-A3 hold and further the pair $(A_0, Q_{11} - M_{12}M_{22}^{-1}M_{21})$ is observable and the pairs (A_0, B_0), (F_0, G_0) and (A_{33}, B_3) are controllable, then*

1. *$\gamma^*_\infty(\epsilon) \le \bar{\gamma}_\infty$, asymptotically as $\epsilon \to 0$, where $\bar{\gamma}_\infty$, as defined in (97), is finite.*
2. *$\lim_{\epsilon_1 \to 0^+} \lim_{\epsilon_2 \to 0^+} \gamma^*_\infty(\epsilon) = \bar{\gamma}_\infty = \lim_{\epsilon_2 \to 0^+} \lim_{\epsilon_1 \to 0^+} \gamma^*_\infty(\epsilon)$.*

[4]Note that here $\nu^*_{s\gamma}$ is maximizing under $\mu^*_{s\gamma}$, but $\mu^*_{s\gamma}$ is not necessarily minimizing under $\nu^*_{s\gamma}$; see [2] pp. 94–96. $\mu^*_{s\gamma}$ is minimizing, however, under the open-loop representation of $\nu^*_{s\gamma}$.

3. $\forall \gamma > \bar{\gamma}_\infty$, $\exists \epsilon_1^* > 0, \epsilon_2^* > 0$ *such that* $\forall \epsilon_1 \in [0, \epsilon_1^*)$ *and* $\epsilon_2 \in [0, \epsilon_2^*)$, *the GARE (92) admits a positive definite solution, and consequently, the game has a finite value,* $J^*(\epsilon)$, *which can be approximated by*

$$J^*(\epsilon) = x_{10}' Z_{s\gamma} x_{10} + O(\epsilon_1) \tag{100}$$

4. $\forall \gamma > \bar{\gamma}_\infty$, *if Player 1 uses the composite strategy* $\mu_{c\gamma}^*$ *defined in (88), then* $\exists \tilde{\epsilon}_1^* > 0, \tilde{\epsilon}_2^* > 0$ *such that* $\forall \epsilon_1 \in [0, \tilde{\epsilon}_1^*)$ *and* $\epsilon_2 \in [0, \tilde{\epsilon}_2^*)$, *we have*

$$\sup_{\nu \in \mathcal{M}_w} L_\gamma(\mu_{c\gamma}, \nu) \leq J^*(\epsilon) + O(\epsilon_1) \tag{101}$$

5. $\forall \gamma > \max\{\gamma_{s\infty}, \gamma_{ofs\infty}, \gamma_{off\infty}\}$, *if Player 1 uses the slow strategy* $\mu_{s\gamma}^*$ *defined in (32), then* $\exists \bar{\epsilon}_1^* > 0, \bar{\epsilon}_2^* > 0$ *such that* $\forall \epsilon_1 \in [0, \bar{\epsilon}_1^*)$ *and* $\epsilon_2 \in [0, \bar{\epsilon}_2^*)$, *we have*

$$\sup_{\nu \in \mathcal{M}_w} L_\gamma(\mu_{s\gamma}, \nu) \leq J^*(\epsilon) + O(\epsilon_1) \tag{102}$$

6. $\forall \gamma > \bar{\gamma}_\infty$, *if Player 2 uses the slow open-loop strategy* $\nu_{os\gamma}^*$ *defined in (98) and (99), then* $\exists \check{\epsilon}_1^* > 0, \check{\epsilon}_2^* > 0$ *such that* $\forall \epsilon_1 \in [0, \check{\epsilon}_1^*)$ *and* $\epsilon_2 \in [0, \check{\epsilon}_2^*)$, *we have*

$$\inf_{\mu \in \mathcal{M}_u} L_\gamma(\mu, \nu_{os\gamma}) \geq J^*(\epsilon) - O(\epsilon_1) \tag{103}$$

Proof Statements 1), 3), 4) and 5) can be proved in a way analogous to those of Theorem 1 of [3], with the details being somewhat different. We just give a brief description of the line of reasoning here; the proof mainly relies on the application of the implicit function theorem.

For 1) and 3), we consider the GARE (92). We assume that the solution to (92) admits the structure

$$\tilde{Z} := \begin{bmatrix} \tilde{Z}_{11} & \epsilon_1 \tilde{Z}_{12} & \epsilon_1\epsilon_2 \tilde{Z}_{13} \\ \epsilon_1 \tilde{Z}_{21} & \epsilon_1 \tilde{Z}_{22} & \epsilon_1\epsilon_2 \tilde{Z}_{23} \\ \epsilon_1\epsilon_2 \tilde{Z}_{31} & \epsilon_1\epsilon_2 \tilde{Z}_{32} & \epsilon_1\epsilon_2 \tilde{Z}_{33} \end{bmatrix} \tag{104}$$

where the partitioning is compatible with that of Q_f. By substituting this structure into (92), we arrive at a system of coupled algebraic equations for $\tilde{Z}_{11}$, $\tilde{Z}_{12}$, $\tilde{Z}_{13}$, $\tilde{Z}_{22}$, $\tilde{Z}_{23}$ and $\tilde{Z}_{33}$. By a bruteforce method as in the proof of Theorem 1 in [3], one can show that $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\tilde{Z}_{11} = Z_{s\gamma}$, $[\tilde{Z}_{12}, \tilde{Z}_{13}] = \check{Z}_c'$, $\tilde{Z}_{22} = Z_{fs\gamma}$, $\tilde{Z}_{23} = Z_{fc}'$ and $\tilde{Z}_{33} = Z_{ff\gamma}$ satisfies the system of coupled algebraic equations. Then checking the nonsingularity of the Jacobian matrix of the system of coupled algebraic equations in a similar manner as in the proof of Theorem 1 in [3], prepares the ground for the application of the implicit function theorem, to establish 1) and 3).

Statements 4) and 5), can be established in a manner similar to the proof of 3) and 4) of Theorem 1 in [3], which also invoke the implicit function theorem.

For 6), the proof follows the line of reasoning used in the proof of Theorem 1 [5], the only difference being that we now have an infinite-horizon one-stage minimization problem. This allows the use of the implicit function theorem to obtain an approximation for the solution to multi-time scale GARE's, instead of approximating GRDE's. It appears that this is the first such result in the literature, with the only related result being in [6], which however deals with state approximation for multi-time scale linear systems.

For 2), we first prove the second equality. Fix a $\gamma > \bar{\gamma}_\infty$; by Theorem 1 in [3], the GARE (35) admits a positive definite solution for sufficiently small $\epsilon_2 > 0$. Fix an $\epsilon_2 > 0$ such that GARE (35) admits a positive definite solution; then, by Theorem 1 in [3], the GARE (92) admits a positive definite solution for sufficiently small $\epsilon_1 > 0$. This shows that

$$\bar{\gamma}_\infty \geq \lim_{\epsilon_2 \to 0^+} \lim_{\epsilon_1 \to 0^+} \gamma^*(\epsilon)$$

Now, fix a $\gamma < \bar{\gamma}_\infty$; then either $\max\{\gamma_{fs\infty}, \gamma_{ff\infty}\} < \gamma < \gamma_{s\infty}$ or $\gamma < \max\{\gamma_{fs\infty}, \gamma_{ff\infty}\}$. In the case $\max\{\gamma_{fs\infty}, \gamma_{ff\infty}\} < \gamma < \gamma_{s\infty}$, fix a sufficiently small $\epsilon_2 > 0$ such that GARE (35) admits a positive definite solution (by Theorem 1 in [3], such an ϵ_2 exists); then, by Theorem 2 in [7], the GARE (92) does not admit any positive definite solution for sufficiently small $\epsilon_1 > 0$. In the case $\gamma < \max\{\gamma_{fs\infty}, \gamma_{ff\infty}\}$, by Theorem 2 in [7], the GARE (35) does not admit any positive definite solution for sufficiently small $\epsilon_2 > 0$. Fix such an $\epsilon_2 > 0$; then, by Theorem 2 in [7], the GARE (92) does not admit any positive definite solution for sufficiently small $\epsilon_1 > 0$. Hence, we have

$$\bar{\gamma}_\infty \leq \lim_{\epsilon_2 \to 0^+} \lim_{\epsilon_1 \to 0^+} \gamma^*(\epsilon)$$

Consequently,

$$\bar{\gamma}_\infty = \lim_{\epsilon_2 \to 0^+} \lim_{\epsilon_1 \to 0^+} \gamma^*(\epsilon)$$

The first equality in 2) can be proved in a similar way, since the following GARE

$$\bar{A}'_{0\epsilon}\bar{Z}_s + \bar{Z}_s\bar{A}_{0\epsilon} - \bar{Z}_s\bar{S}_{0\epsilon}\bar{Z}_s + \bar{Q} = 0 \tag{105}$$

has its slow GARE (as $\epsilon_1 \to 0^+$) exactly the GARE (95), where in (105) we have used the matrices:

$$\begin{aligned}
\bar{A}_{0\epsilon}(\gamma) &= \bar{F}_{11\epsilon} - \bar{F}_{12\epsilon}Q_{33}^{-1}\bar{M}_{21} - (\bar{N}_{12\epsilon} + \bar{F}_{12\epsilon}Q_{33}^{-1}A'_{33})(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1} \\
&\quad \cdot(\bar{F}_{21} - A_{33}Q_{33}^{-1}\bar{M}_{21}) \qquad (106) \\
\bar{Q}(\gamma) &= \bar{M}_{11} - \bar{M}_{12}Q_{33}^{-1}\bar{M}_{21} + (\bar{F}'_{21} - \bar{M}_{12}Q_{33}^{-1}A'_{33})(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1} \\
&\quad \cdot(\bar{F}_{21} - A_{33}Q_{33}^{-1}\bar{M}_{21}) \qquad (107) \\
\bar{S}_{0\epsilon} &= \bar{N}_{11\epsilon} + \bar{F}_{12\epsilon}Q_{33}^{-1}\bar{F}'_{12\epsilon} - (\bar{N}_{12\epsilon} + \bar{F}_{12\epsilon}Q_{33}^{-1}A'_{33})(S_{33} + A_{33}Q_{33}^{-1}A'_{33})^{-1} \\
&\quad \cdot(\bar{N}_{21\epsilon} + A_{33}Q_{33}^{-1}\bar{F}'_{12\epsilon}) \qquad (108)
\end{aligned}$$

$$
\begin{aligned}
\bar{F}_{11\epsilon} &:= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \frac{1}{\epsilon_1}A_{21}(t) & \frac{1}{\epsilon_1}A_{22}(t) \end{bmatrix}; \quad \bar{F}_{12\epsilon} := \begin{bmatrix} A_{13}(t) \\ \frac{1}{\epsilon_1}A_{23}(t) \end{bmatrix} \\
\bar{F}_{21} &:= \begin{bmatrix} A_{31}(t) & A_{32}(t) \end{bmatrix}; \quad \bar{M}_{21} := \begin{bmatrix} Q_{31}(t) & Q_{32}(t) \end{bmatrix} \\
\bar{M}_{11} &:= \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}; \quad \bar{M}_{12} := \begin{bmatrix} Q_{13}(t) \\ Q_{23}(t) \end{bmatrix} \\
\bar{G}_{1\epsilon} &:= \begin{bmatrix} B_1(t) \\ \frac{1}{\epsilon_1}B_2(t) \end{bmatrix}; \quad \bar{H}_{1\epsilon} := \begin{bmatrix} D_1(t) \\ \frac{1}{\epsilon_1}D_2(t) \end{bmatrix} \\
\bar{N}_{11\epsilon} &:= \bar{G}_{1\epsilon}\bar{G}'_{1\epsilon} - \frac{1}{\gamma^2}\bar{H}_{1\epsilon}\bar{H}'_{1\epsilon}; \quad \bar{N}_{12\epsilon} := \bar{G}_{1\epsilon}B'_3 - \frac{1}{\gamma^2}\bar{H}_{1\epsilon}D'_3 \\
\bar{N}_{21\epsilon} &:= \bar{N}'_{12\epsilon}
\end{aligned}
$$

The proof of the above statement is straightforward but lengthy, and hence is not included in the paper.

Thus, the theorem is proved. □

6 The Finite-Horizon Case

Now, we return to the finite-horizon version of the problem. We have the following theorem.

Theorem 2 *For the singularly perturbed zero-sum differential game (1) and (4), let assumptions A1–A3 be satisfied, the pairs $(F_0(t), G_0(t))$ and $(A_{33}(t), B_3(t))$ be controllable for each $t \in [0, t_f]$, and the following conditions hold:*

$Q_{f22} \le Z_{fs\gamma}(t_f)$ *and* $Q_{f33} \le Z_{ff\gamma}(t_f)$, *with* γ *fixed.*

Then,

1. $\lim_{\epsilon_1 \to 0^+} \lim_{\epsilon_2 \to 0^+} \gamma^*(\epsilon) \le \bar{\gamma} \ge \lim_{\epsilon_2 \to 0^+} \lim_{\epsilon_1 \to 0^+} \gamma^*(\epsilon)$, *where* $\bar{\gamma}$ *as defined in (69), is finite.*

2. $\forall \gamma > \bar{\gamma}$,

$$\lim_{\epsilon_1 \to 0^+} \lim_{\epsilon_2 \to 0^+} J^*(\epsilon) = \lim_{\epsilon_2 \to 0^+} \lim_{\epsilon_1 \to 0^+} J^*(\epsilon) = x'_{1\,0} Z_{s\gamma} x_{1\,0} \tag{109}$$

3. $\forall \gamma > \bar{\gamma}$, *if Player 1 uses the composite strategy* $\mu^*_{c\gamma}$ *defined in (88), then*

$$\lim_{\epsilon_1 \to 0^+} \lim_{\epsilon_2 \to 0^+} \sup_{\nu \in \mathcal{M}_w} L_\gamma(\mu_{c\gamma}, \nu) = x'_{1\,0} Z_{s\gamma} x_{1\,0} \tag{110}$$

and

$$\lim_{\epsilon_2 \to 0^+} \lim_{\epsilon_1 \to 0^+} \sup_{\nu \in \mathcal{M}_w} L_\gamma(\mu_{c\gamma}, \nu) = x'_{1\,0} Z_{s\gamma} x_{1\,0} \tag{111}$$

4. $\forall\gamma > \max\{\gamma_s, \gamma_{ofs}, \gamma_{off}\}$, *if Player 1 uses the slow strategy* $\mu^*_{s\gamma}$ *defined in (32), then*

$$\lim_{\epsilon_1\to 0^+}\lim_{\epsilon_2\to 0^+}\sup_{\nu\in\mathcal{M}_w} L_\gamma(\mu_{s\gamma},\nu) = x'_{10}Z_{s\gamma}x_{10} \tag{112}$$

and

$$\lim_{\epsilon_2\to 0^+}\lim_{\epsilon_1\to 0^+}\sup_{\nu\in\mathcal{M}_w} L_\gamma(\mu_{s\gamma},\nu) = x'_{10}Z_{s\gamma}x_{10} \tag{113}$$

5. $\forall\gamma > \bar{\gamma}$, *if Player 2 uses the slow strategy* $\nu^*_{s\gamma}$ *defined in (30), then*

$$\lim_{\epsilon_1\to 0^+}\lim_{\epsilon_2\to 0^+}\inf_{\mu\in\mathcal{M}_u} L_\gamma(\mu,\nu_{s\gamma}) = x'_{10}Z_{s\gamma}x_{10} \tag{114}$$

and

$$\lim_{\epsilon_2\to 0^+}\lim_{\epsilon_1\to 0^+}\inf_{\mu\in\mathcal{M}_u} L_\gamma(\mu,\nu_{s\gamma}) = x'_{10}Z_{s\gamma}x_{10} \tag{115}$$

Proof To prove 1) and 2), we first fix a $\gamma > \bar{\gamma}$, and note that by Theorem 1 in [3], the GARE (35) admits a positive definite solution for each $t \in [0, t_f]$ and sufficiently small $\epsilon_2 > 0$. Now, fix such an $\epsilon_2 > 0$; by Theorem 1 in [3], the GRDE (8) admits a nonnegative definite solution on the time interval $[0, t_f]$, which admits a approximation of the form (104) with the 11-block being $Z_{s\gamma}(t) + O(\epsilon_1)$. Then,

$$\bar{\gamma}_\infty \geq \lim_{\epsilon_2\to 0^+}\lim_{\epsilon_1\to 0^+}\gamma^*(\epsilon)$$

and

$$\lim_{\epsilon_2\to 0^+}\lim_{\epsilon_1\to 0^+} J^*(\epsilon) = x'_{10}Z_{s\gamma}x_{10}$$

We note that the following GRDE

$$\dot{\bar{Z}}_s + \bar{A}'_{0\epsilon}\bar{Z}_s + \bar{Z}_s\bar{A}_{0\epsilon} - \bar{Z}_s\bar{S}_{0\epsilon}\bar{Z}_s + \bar{Q} = 0 \tag{116}$$

has its slow GRDE exactly the GRDE (23) (the derivation of this is lengthy but straightforward). A similar reasoning as above also leads to

$$\lim_{\epsilon_1\to 0^+}\lim_{\epsilon_2\to 0^+}\gamma^*(\epsilon) \leq \bar{\gamma}_\infty$$

and

$$\lim_{\epsilon_1\to 0^+}\lim_{\epsilon_2\to 0^+} J^*(\epsilon) = x'_{10}Z_{s\gamma}x_{10}$$

Hence, 1) and 2) are established.

The proofs of 3), 4) and 5) basically follow the same lines: First, substitute the chosen strategy into the state equation (1) as well as the cost function (4) to reduce the problem to a one person maximization or minimization problem, whose solution depends on a solution of a GRDE. Then, by applying Theorems 1 and 2 of [3], the desired results follow. □

7 An Example

We present in this section a numerical example for the infinite horizon case. As stressed earlier, the quantities $\gamma_{s\infty}$, $\gamma_{fs\infty}$, $\gamma_{ofs\infty}$, $\gamma_{ff\infty}$ and $\gamma_{off\infty}$ play important roles in the computation of an approximate value for $\gamma^*_\infty(\epsilon)$, as well as the optimality of the strategies (32), (89) and (98), (99)[5].

Consider the following three-dimensional system and the performance index:

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon_1 \dot{x}_2 \\ \epsilon_1\epsilon_2\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} w; \quad \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \tag{117}$$

$$L_\gamma = \int_0^\infty \left(x' \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + |u|^2 - \gamma^2 |w|^2 \right) dt \tag{118}$$

where $x = (x_1', x_2', x_3')'$. By using a particular search algorithm, we readily compute the five basic quantities:

$$\gamma_{s\infty} = 1.1074 \quad \gamma_{fs\infty} = 1.1032 \quad \gamma_{ff\infty} = 0.55416$$
$$\gamma_{ofs\infty} = 1.8294 \quad \gamma_{off\infty} = 0.66596$$

We now compute the threshold level $\gamma^*_\infty(\epsilon)$ of the full-order system (117)–(118) for different values of ϵ as shown in Table 1. Note that as $\epsilon \to 0$, $\gamma^*(\epsilon) \to \max\{\gamma_{s\infty}, \gamma_{fs\infty}, \gamma_{ff\infty}\}$. Now,

Table 1: $\gamma^*(\epsilon)$ for the full-order system

ϵ_1	0.1			0.01			0.001		
ϵ_2	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001
$\gamma^*(\epsilon)$	1.0463	1.0458	1.0458	1.0993	1.0993	1.0993	1.1066	1.1066	1.1066

we choose $\gamma = 1.3 > \max\{\gamma_{s\infty}, \gamma_{fs\infty}, \gamma_{ff\infty}\}$ (but $\gamma < \max\{\gamma_{s\infty}, \gamma_{ofs\infty}, \gamma_{off\infty}\}$), and design approximate strategies for both players based on this value of γ:

$$Z_{s\gamma} = 1.0188 \quad Z_{fs\gamma} = 0.32215 \quad Z_{ff\gamma} = 0.15972$$
$$\mu^*_{s\gamma}(x_1) = -5.2534 x_1 \quad \nu^*_{os\gamma}(x^*_s) = 3.2285 x^*_s$$
$$\mu^*_{c\gamma}(x) = -4.8590 x_1 - 1.0467 x_2 - 0.31945 x_3$$

where

$$\dot{x}^*_s = -5.8755 x^*_s$$

[5]In these examples, the relative accuracy is 0.002.

Then, we apply $\mu^*_{c\gamma}$, $\mu^*_{s\gamma}$ and $\nu^*_{os\gamma}$ to the full-order system (117)–(118) and obtain the corresponding costs $L^*_{c\gamma}(\epsilon) := \sup_{\nu\in\mathcal{M}_w} L_\gamma(\mu_{c\gamma},\nu)$, $L^*_{s\gamma}(\epsilon) := \sup_{\nu\in\mathcal{M}_w} L_\gamma(\mu_{s\gamma},\nu)$ and $L^*_{o\gamma}(\epsilon) := \inf_{\mu\in\mathcal{M}_u} L_\gamma(\mu,\nu_{os\gamma})$, respectively, as well as the value of the game, $J^*(\epsilon)$, which are tabulated in Table 2. We see that, as $\epsilon\to 0$, the quantities $J^*(\epsilon)$, $L^*_{c\gamma}(\epsilon)$ and $L^*_{o\gamma}(\epsilon)$ all converge to

Table 2: $J^*(\epsilon)$, $L^*_{s\gamma}(\epsilon)$, $L^*_{c\gamma}(\epsilon)$ and $L^*_{o\gamma}(\epsilon)$ for the full-order system with $\gamma = 1.3$

ϵ_1	0.1			0.01			0.001		
ϵ_2	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001
$J^*(\epsilon)$	4.3193	4.2510	4.2442	4.1006	4.0932	4.0925	4.0776	4.0769	4.0768
$L^*_{s\gamma}(\epsilon)$	∞	∞	∞	∞	∞	∞	∞	∞	∞
$L^*_{c\gamma}(\epsilon)$	4.3821	4.3060	4.2984	4.1016	4.0941	4.0933	4.0776	4.0769	4.0768
$L^*_{o\gamma}(\epsilon)$	4.2991	4.2269	4.2196	4.1003	4.0928	4.0920	4.0776	4.0768	4.0768

$x'_{10}Z_{s\gamma}x_{10} = 4.0750$, consistent with the statement of Theorem 1. If Player 1 uses only a slow policy, however, Player 2 can drive the value to $+\infty$.

Now, let us choose a larger value of γ: $\gamma = 1.9 > \max\{\gamma_{s\infty}, \gamma_{ofs\infty}, \gamma_{off\infty}\}$. Based on this value of γ, we design the approximate strategies for both players:

$$Z_{s\gamma} = 0.65885 \quad Z_{fs\gamma} = 0.24662 \quad Z_{ff\gamma} = 0.15508$$
$$\mu^*_{s\gamma}(x_1) = -2.4792x_1 \quad \nu^*_{os\gamma}(x^*_s) = 0.60502x^*_s$$
$$\mu^*_{c\gamma}(x) = -2.9361x_1 - 0.85989x_2 - 0.31015x_3$$

where

$$\dot{x}^*_s = -4.5445x^*_s$$

Then, we apply $\mu^*_{c\gamma}$, $\mu^*_{s\gamma}$ and $\nu^*_{os\gamma}$ to the full-order system (117)–(118) and obtain the corresponding costs $L^*_{c\gamma}(\epsilon)$, $L^*_{s\gamma}(\epsilon)$ and $L^*_{o\gamma}(\epsilon)$, respectively, as well as the value of the game, $J^*(\epsilon)$, which are tabulated in Table 3. This time, the four quantities $J^*(\epsilon)$, $L^*_{s\gamma}(\epsilon)$, $L^*_{c\gamma}(\epsilon)$ and $L^*_{o\gamma}(\epsilon)$ all converge to the value $x'_{10}Z_{s\gamma}x_{10} = 2.6354$, as $\epsilon\to 0$. This again corroborates the statement of Theorem 1.

8 Conclusion

In this paper, we have provided a complete analysis of the three-time scale linear-quadratic zero-sum differential game with perfect state measurements in both finite and infinite horizons. One of the main results of the paper is the existence and construction of a composite strategy for the minimizing player, independent of the singular perturbation parameters, which is parametrized by γ. This composite policy leads to a bounded value for the game,

Table 3: $J^*(\epsilon)$, $L^*_{s\gamma}(\epsilon)$, $L^*_{c\gamma}(\epsilon)$ and $L^*_{o\gamma}(\epsilon)$ for the full-order system with $\gamma = 1.9$

ϵ_1	0.1			0.01			0.001		
ϵ_2	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001
$J^*(\epsilon)$	2.8744	2.8794	2.9272	2.6676	2.6628	2.6623	2.6386	2.6382	2.6381
$L^*_{s\gamma}(\epsilon)$	2.9865	2.9938	3.0700	2.6790	2.6731	2.6725	2.6397	2.6391	2.6391
$L^*_{c\gamma}(\epsilon)$	2.8775	2.8825	2.9316	2.6677	2.6628	2.6623	2.6386	2.6382	2.6381
$L^*_{o\gamma}(\epsilon)$	2.9196	2.8734	2.8686	2.6665	2.6618	2.6614	2.6385	2.6381	2.6380

which approximates the optimal one to the order of $O(\epsilon_1)$ whenever the game has a finite value, for sufficiently small ϵ_1 and ϵ_2, in the infinite-horizon case. In the finite-horizon case, a somewhat weaker but similar result is obtained. In the paper, we have also studied the optimality of the slow policy for the minimizing player. It has been shown that the slow policy is not as robust as the composite policy, in the sense that it attains a value close to the optimal one only when γ exceeds a specific bound. For the maximizing player, the slow policy is sufficient to guarantee a value close to the optimal one in the finite-horizon case. In infinite-horizon case, the open-loop version of the slow policy must be used to achieve the ϵ-optimal value. The paper has also included a numerical example to illustrate the theory.

The results obtained here admit immediate extensions to the general n-time scale problem. Another interesting extension would be to the case when Player 1 has partial state information, so as to obtain the counterpart of the results of [8] for the n ($n \geq 3$) time scale game; this extension, however, is not that immediate.

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