Parameter Identification for Uncertain Plants using H^∞ Methods*†

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Worst-case parameter identifiers for uncertain nonlinear systems with full-state measurements can be obtained using the cost-to-come methodology, where singular perturbation techniques can be used for order reduction and robustification.

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Abstract

In this paper, we demonstrate the effective use of H^∞-filtering and cost-to-come methods for parameter identification in (deterministic) uncertain plants that are linear in the unknown parameters, but nonlinear otherwise. The *cost-to-come* method is an approach that has been used earlier to solve linear and nonlinear H^∞-optimal control and filtering problems. It consists of constructing a *cost-to-come* function, which assists in the design of an "optimal" observer scheme.

The method is used here in the design of a parameter identification scheme for uncertain plants, where measurements on the state of the system are available, but not on its derivative. Two approaches are adopted, in both of which the parameter estimation problem is formulated as an H^∞-filtering problem. One of the approaches uses a more standard prefiltering of the past states, input and disturbance signals. The other approach is a novel design method, which leads to a new class of identification schemes. It involves two subproblems: FSDI (full-state-derivative information) problem, where it is assumed that both the state and its derivative are available to the parameter estimator, and NPFSI (noise perturbed FSI) problem, where the estimator is assumed to measure a noise perturbed measurement of the state. In the latter problem, we use singular perturbation methods to prove asymptotic convergence of the performance of the identifier to that of the unperturbed case, thus providing an asymptotically optimal solution to the FSI (full-state measurement) problem. To illustrate both approaches, several simulation studies on a numerical example are provided.

1 INTRODUCTION

In the control and identification literature, it has been known for quite some time now that a parameter identification problem, with full information on both the state and its derivative (which we will refer to as FSDI information here) can be solved as a state-estimation problem (Åström, 1970). As a result, the Kalman-filter has been used as a design tool for identification in a stochastic setting, leading to the least-squares (LS) algorithm as an optimal identification scheme (Sastry and Bodson, 1989). More recently, H^∞-filtering results (Shaked and Theodor, 1992, Nagpal and Khargonekar, 1991) have been applied to parameter identification in the discrete time (Grimble *et al.*, 1992, Hassibi *et al.*, 1993), with the latter reference also showing that the least-mean-square (LMS) algorithm, which also emerges in this context, is more robust with respect to deterministic disturbances.

In this paper we conduct a more comprehensive study on the use of H^∞ methods in identification in the continuous time, when the plant is perturbed by an unknown driving signal. Our first result is a solution to the problem of continuous-time H^∞-optimal FSI identification of time-invariant parameters in nonlinear plants with linear dependence on the unknown parameters. In this investigation, we use cost-to-come methods which were developed earlier to investigate H^∞-optimal control and filtering problems (Didinsky and Başar, 1992, Didinsky *et al.*., 1993a, 1993b). A byproduct of this analysis is a set of identification schemes, parametrized by their respective disturbance-attenuation levels. In this spectrum, LS represents the largest attenuation level, and for some specific cases LMS represents the smallest attenuation level. We also apply standard convergence analyses (Sastry and Bodson, 1989) to show that under the assumptions of $\mathcal{L}_2$-bounded disturbances and a PE (persistency of excitation) condition, the derived identification schemes exhibit convergence to the true parameter values.

Next, we consider the case where the state is still measured, but not its derivative (which we call FSI information). This, in fact, involves a "singular" optimization problem, and therefore we embed it in a larger class of problems which are well posed. These are noise-perturbed FSI (NPFSI) problems, where the state measurements are perturbed additively by "small" noise (with the $\mathcal{L}_2$ noise multiplied by a small positive parameter ϵ). We show using singular perturbation methods that in the limit as $\epsilon \to 0$, the optimum performance of the NPFSI problem converges to that of the FSDI problem, and hence, for small values of ϵ, the solution of the NPFSI problem provides an excellent approximation to that of the FSI problem. We further show that this property holds not only for the full-order solution of the NPFSI problem, but also for a reduced-order identifier, again obtained using singular perturbation methods and exploiting the "smallness" of ϵ. We also discuss extensions of these results to a larger class of problems where the unknown parameters are time varying.

An alternative approach to avoiding the use of the state derivative involves the more standard prefiltering-based identification. It is a generalization of the idea of filtering the input and the output to produce an auxiliary FSI problem (see, for example, Sastry and Bodson, 1989). We show that the NPFSI identifier has preferred structural properties over those of prefiltering-based identifiers, such as lower dimensions and a simpler selection of characteristic parameters, and furthermore, it performs better in numerical simulations.

The paper is organized as follows: A precise mathematical formulation for the parameter identification problem is given in Section 2. The FSDI problem is solved in Section 3, and an analysis of the convergence of the FSDI identifier is presented in Section 4. The NPFSI problem is formulated and solved in Section 5, which also includes a complete analysis of the limiting behavior of the NPFSI estimator (as ϵ approaches zero), as well as the expression for the reduced-order NPFSI identifier. The prefiltering-based identification problem is solved in Section 6, which also includes a comparison of its solution with the NPFSI identifier. Section 7 discusses extensions to problems with time-varying parameters, and Section 8 presents simulation studies on a numerical example. The paper ends with the concluding remarks of Section 9, and an Appendix which contains proofs of Theorems 7 and 9.

Throughout the paper the following notation and conventions will be adopted, unless otherwise indicated: $\mathcal{L}_2$ denotes the class of Lebesgue measurable, square integrable functions of appropriate dimensions, and $\mathcal{L}_\infty$ denotes Lebesgue measurable functions with finite sup norm, with the interval of interest in both cases being $\mathbf{T} := [0, \infty)$. $\|\cdot\|$ denotes the $\mathcal{L}_2$ norm, and $|\cdot|$ the Euclidean norm, with a subscript in each case denoting an appropriate weighting, such as $\|\cdot\|_R$ or $|\cdot|_P$. If these matrices (i.e., R and P) are singular, then the norms will be interpreted as semi-norms. For a matrix Q, $Q > 0$ (respectively, $Q \geq 0$) means that it is positive definite (respectively, nonnegative definite).

2 PROBLEM FORMULATION

The uncertain plant model is taken to be linear in the parameters to be identified, as well as in the additive unknown noise, but is otherwise allowed to be nonlinear in the state and control input. More precisely, we adopt the parametrized state model:

$$\dot{x} = A(x,u)\theta + b(x,u) + w, \quad x(0) = x_o, \quad t \in \mathbf{T}, \tag{2.1}$$

where θ is an r-dimensional vector of unknown constant parameters (to be identified), x is the n-dimensional state vector, with $x_o \in I\!\!R^n$ being an arbitrary initial state, u is a known input vector of dimension p, possibly governed by a controller policy , and w is an n-dimensional unknown $\mathcal{L}_\infty$ disturbance (written as, $w \in \mathcal{W}$). The $(n \times r)$-dimensional matrix $A(x,u)$ and the n-dimensional vector $b(x,u)$ are jointly continuous in their two arguments, as well as in the time variable t, which we suppress in this formulation. For the special class of linear systems, $A(x,u)$ and $b(x,u)$ are taken as

$$A(x,u) = [A_1 x + B_1 u, A_2 x + B_2 u, \ldots, A_r x + B_r u] \text{ and } b(x,u) = A_0 x + B_0 u, \tag{2.2}$$

where $\{A_i\}_{i=0}^r$ and $\{B_i\}_{i=0}^r$ are known matrices, possibly time varying.

The first goal here is to obtain a "good" estimate of θ at each time t, denoted by $\hat{\theta}(t)$, using the available information up to that time. This estimate should yield a good performance in the presence of the unknown disturbance w, in the sense that it should converge to the true value of the parameter vector, i.e.,

$$\lim_{t\to\infty} \hat{\theta}(t) = \theta, \tag{2.3}$$

and also show good transient behavior. Convergence to the true value may not always be possible due to lack of excitation of the parameters, and hence, a "reasonable" persistency of excitation (PE) condition will have to be prescribed that will guarantee convergence in (2.3).

The estimate $\hat{\theta}(\cdot)$ alluded to above will be determined here by an estimator that uses the past values of the state x and the input u. The estimator will be denoted at time t by $\delta(t, \eta_t)$, where $\eta_t = \{(x(s), u(s)), \forall s \leq t\}$ is the information available for this purpose. Since the state vector x is assumed to be known, it would seem to be reasonable to assume that its derivative $\dot{x}$ is also known to the estimator. This is not a realistic assumption, however, if any consideration is given to robustness of the estimator, as an estimator that depends on the derivative of the state that is obtained via an extrapolation on the past values of the state and the control will be sensitive to measurement disturbances and/or unmodeled dynamics.

Let $\delta \in \Delta$, where Δ is a class of admissible policies satisfying the standard requirements of measurability, adaptability (to the given information fields), etc., and functionally independent of $\dot{x}$. In accordance with the objective delineated above (asymptotic convergence to true parameter value, good transient behavior, insensitivity to unknown disturbances), we introduce a performance index that measures worst-case attenuation from additive disturbance and error in the initial estimate of θ to the estimation error over an interval of interest. Formulating this in an $\mathcal{L}_2$ setting leads to the cost function:

$$L(\delta) = \sup_{x_o, \theta_o, w_{[0,\infty)}} \left\{ \frac{\|\theta - \hat{\theta}\|^2_{Q(t,x,u)}}{\|w\|^2 + |\theta - \overline{\theta}_o|^2_{Q_o}} \;\Big|\; \hat{\theta}(t) = \delta(t, \eta_t) \right\}, \tag{2.4}$$

Assume that $Q(\cdot) \geq 0$, and $Q_o > 0$. $\bar{\theta}_o$ here is a known "bias" term which stands for some initial estimate of θ. Without any loss of generality, Q will be taken to be a matrix-valued function of $(t, x_{[0,t]}, u_{[0,t]})$.

The first objective can now be rephrased as finding an estimator, say $\delta^* \in \Delta$, that minimizes $L(\delta)$:

$$\inf_{\delta \in \Delta} L(\delta) = L(\delta^*) =: \gamma^{*2}. \tag{2.5}$$

This problem is in fact a differential game with δ the minimizer and $(x_0, t, w_{[0,\infty]})$ the maximizer, and with the game kernel being as given inside the braces in (2.4). The square of the disturbance attenuation level γ^* is the upper value of such a game.

A soft-constrained cost function associated with this differential game is

$$J_\gamma(\delta; x_o, \theta_o, w_{[0,\infty)}) = \int_0^\infty \left[|\theta - \delta(t, \eta_t)|^2_{Q(t,x,u)} - \gamma^2 |w(t)|^2 \right] dt - \gamma^2 |\theta - \bar{\theta}_o|^2_{Q_o}, \tag{2.6}$$

parametrized by γ. We know from Başar and Bernhard (1991) that γ^* as defined by (2.5) is the "smallest" value of $\gamma > 0$ for which the differential game just defined, in which δ is the minimizer and the triple $(x_o, \theta_o, w_{[0,\infty)})$ the maximizer, and J_γ is the kernel, has upper value equal to zero.

Toward defining the optimality of an estimator, let us first introduce the following terminology: For a fixed value of $\gamma > 0$, "an estimator δ is *cost-bounding*" if it satisfies the finiteness condition

$$\sup_{x_o,\theta_o,w_{[0,\infty)}} J_\gamma(\delta; x_o, \theta_o, w_{[0,\infty)}) < \infty. \tag{2.7}$$

Assuming that Δ has a nonempty subclass of cost-bounding estimators, then we will say that an estimator $\bar{\delta} \in \Delta$ is *minimax* for a given attenuation level γ, if

$$\inf_{\delta\in\Delta} \sup_{x_o,\theta_o,w_{[0,\infty)}} J_\gamma(\delta; x_o, \theta_o, w_{[0,\infty)}) = \sup_{x_o,\theta_o,w_{[0,\infty)}} J_\gamma(\bar{\delta}; x_o, \theta_o, w_{[0,\infty)}). \tag{2.8}$$

Now, our objective, stated succinctly, is to compute the least value of $\gamma > 0$, denoted again as γ^*, such that J_γ has a cost-bounding solution, and to find a minimax estimator policy, δ_γ, for each $\gamma > \gamma^*$, which has the further property that it generates estimates that converge to the true value of θ.

3 FSDI IDENTIFICATION

In the analysis of this section, we assume that the state vector x is differentiable, and its derivative is also included in the information set. Hence, we have the expanded information set: $\eta_t = \{\dot{x}(s), x(s), u(s),\ s \le t\}$. We will refer to this problem as the FSDI (full-state-derivative information) identification problem.

Since θ is a constant, we associate with it the natural simple dynamics:

$$\dot{\theta} = 0, \qquad \theta(0) = \theta_o, \tag{3.1}$$

where $\theta_o \in I\!\!R^r$ is unknown; this now constitutes an additional state equation for the problem. Next, assuming that γ^* is finite, given any $\gamma > \gamma^*$, we introduce the *cost-to-come function* (Didinsky and Başar, 1992, Didinsky *et al.*., 1993a, 1993b) associated with this imperfect state measurement problem[1]

$$W_\gamma(t,\theta; x_{[0,t]}, \hat{\theta}_{[0,t]}, u_{[0,t]}) := \left\{ \begin{array}{c} \displaystyle\int_0^t \left[|\theta - \hat{\theta}(s)|^2_{Q(s,x,u)} - \gamma^2 |\dot{x} - A(x,u)\theta - b(x,u)|^2 \right] ds \\ -\gamma^2 (|\theta_o - \bar{\theta}_o|^2_{Q_o}) \end{array} \right\}, \tag{3.2}$$

which is a time-truncated version of (2.6), with the dynamic constraint (2.1) used to substitute for w and with $\hat{\theta}(s)$ used to denote the estimate generated by the estimator policy $\delta(s, \eta_s)$.

This cost-to-come function can be rewritten in the simpler quadratic form (by completion of squares)

$$W_\gamma(t,\theta; x_{[0,t]}, \hat{\theta}_{[0,t]}, u_{[0,t]}) = -\gamma^2 |\theta - \bar{\theta}(t)|^2_{\Sigma(t)} - m(t) \tag{3.3}$$

[1] This is an imperfect state measurement problem, since θ is not measured directly, and only through x.

where

$$\dot{\Sigma} = A(x,u)^T A(x,u) - \frac{1}{\gamma^2} Q(x,u), \quad \Sigma(0) = Q_o \tag{3.4}$$

$$\dot{\bar{\theta}} = \Sigma^{-1} A(x,u)^T \left(\dot{x} - A(x,u)\bar{\theta} - b(x,u)\right) + \frac{1}{\gamma^2}\Sigma^{-1} Q(x,u)(\bar{\theta} - \hat{\theta}), \quad \bar{\theta}(0) = \bar{\theta}_o \tag{3.5}$$

$$\dot{m} = -|\bar{\theta} - \hat{\theta}|^2_{Q(x,u)} + \gamma^2 |\dot{x} - A(x,u)\bar{\theta} - b(x,u)|^2, \quad m(0) = 0 \tag{3.6}$$

Using this representation, we immediately arrive at the following result:

Theorem 1 *For the FSDI model, the following hold:*

(i) *γ^* is finite, if, and only if, there exists a $\gamma > 0$ such that the solution of (3.4), denoted by Σ_γ, is positive definite for all $t \in \mathbf{T}$.*

(ii) *Given any $\gamma < \gamma^*$, or any finite γ if γ^* is infinite, the solution of (3.4) is not nonnegative definite.*

(iii) *If γ^* is finite, then for every $\gamma > \gamma^*$, $\{\delta_\gamma(t, \eta_t) = \hat{\theta}(t),\ t \in \mathbf{T}\}$ is a minimax identification policy, where $\hat{\theta}_\gamma(t)$ is generated by (3.5) with $\hat{\theta}(t) = \bar{\theta}(t)$, i.e.,*

$$\dot{\hat{\theta}}_\gamma = \Sigma_\gamma^{-1} A(x,u)^T (\dot{x} - A(x,u)\hat{\theta}_\gamma - b(x,u)), \quad \hat{\theta}_\gamma(0) = \bar{\theta}_o. \tag{3.7}$$

Proof: We first prove the necessity in (i). Assuming that γ^* is finite , choose $\gamma > \gamma^*$. Given any $t \in \mathbf{T}$, let Δ_t be the class of estimator policies, which in addition to the FSDI information have access to the states θ after time t. Obviously, $\Delta \subset \Delta_t$. Also clearly, an optimal strategy in Δ_t will choose its output after time t to be equal to θ. Hence, the following relationship will hold:

$$0 \geq \inf_{\delta \in \Delta_t} \sup_{x_o, \theta_o, w_{[0,\infty)}} J_\gamma(\delta, x_o, \theta_o, w_{[0,\infty)}) = \inf_{\delta_{[0,t]}} \sup_{x_{[0,t]}} \sup_{\theta} \left\{ -\gamma^2 |\theta - \bar{\theta}(t)|^2_{\Sigma_\gamma(t)} + m(t) \right\}.$$

As the supremum over θ in last expression above is finite , it implies that $\Sigma_\gamma(t) > 0$.

For a proof of sufficiency in (i), let $\gamma > 0$ guarantee that $\Sigma_\gamma(t) > 0$ for all $t \in \mathbf{T}$. Choose the identification policy to be the "certainty-equivalence" policy $\delta(t, \eta_t) = \bar{\theta}(t),\ \forall t \in \mathbf{T}$. Then, we have the bound:

$$\int_0^T [|\theta - \delta|^2_Q - \gamma^2 |w|^2] dt - \gamma^2 |\theta_o - \bar{\theta}_o|^2_{Q_o} = W_\gamma(T, \theta; x_{[0,T]}, \bar{\theta}_{[0,T]}, u_{[0,T]}) \leq 0.$$

for all disturbances $(x_o, \theta_o, w_{[0,T]})$ and all $T \in \mathbf{T}$. Taking the limit as $T \uparrow \infty$ leads to the conclusion that the infinite-horizon cost is bounded from above by zero. Note that in addition to sufficiency in (i), this also proves (iii).

Last, we prove (ii), by contradiction. Let $\gamma < \gamma^*$, or be any positive value if γ^* is infinite, and suppose that $\Sigma_\gamma(t) \geq 0,\ \forall t \in \mathbf{T}$. From (3.4), it follows that

$$\Sigma_\gamma(T) = Q_o + \int_0^T A^T A dt - \frac{1}{\gamma^2} \int_0^T Q dt, \quad \forall T \in \mathbf{T}.$$

As the first two terms on the RHS sum up to a positive definite matrix (since $Q_o > 0$) and last term is nonpositive definite, it follows that $\Sigma_{\gamma+\epsilon}(T) > 0$, $\forall T \in \mathbf{T}$, for any $\epsilon > 0$. If γ^* is finite, then the preceding result contradicts with the hypothesis that γ^* is the least admissible value of γ. If γ^* is infinite, then the above again leads to a contradiction, since $\gamma + \epsilon$ leads to a positive definite Σ, and hence is an admissible attenuation level. $\square$

Remark 1 If we let $\gamma \uparrow \infty$ in (3.4)-(3.5), the limiting filter is precisely the least-squares (LS) estimator for θ (Sastry and Bodson, 1989). $\square$

If we wish to estimate $A(x,u)\theta$, rather than θ itself, then this objective could still be accommodated in the framework of the FSDI identification problem formulated above. In this case, choosing $Q(x,u) = A(x,u)^T A(x,u)$ in Theorem 1 yields the optimal estimator for $A(x,u)\theta$. This result is given below as a corollary to Theorem 1.

Corollary 1 *For the FSDI model and letting $Q(x,u) = A(x,u)^T A(x,u)$, we have:*

1. $\gamma^* = 1$.
2. *For all $\gamma \geq 1$, the policy $\delta_\gamma(t,\eta_t) = \hat{\theta}_\gamma(t)$, where $\hat{\theta}_\gamma$ is generated by*

$$\dot{\hat{\theta}}_\gamma = Q_o^{-1} A(x,u)^T \left(\dot{x} - A(x,u)\hat{\theta}_\gamma - b(x,u)\right), \quad \hat{\theta}_\gamma(0) = \bar{\theta}_o, \tag{3.8}$$

is a minimax policy.

$\square$

Remark 2 The estimator (3.8) is the generalized LMS (least-mean squares) filter. Hence, this corroborates in a more general framework the recent result that the LMS Filter is H^∞-optimal (Hassibi *et al.*, 1993). $\square$

4 PERFORMANCE OF THE FSDI IDENTIFICATION SCHEME

We now analyze the performance of the FSDI identification scheme (3.4)-(3.5). In particular, we will show that with norm-bounded disturbances it is possible to identify uncertain parameters θ, provided that an appropriate persistency of excitation (PE) condition, introduced below, is satisfied.

PE Condition

$$\lim_{T\to\infty} \lambda_{\min}\left(\int_0^T Q(x,u)dt\right) = \infty \tag{4.1}$$

We have the following theorem:

Theorem 2 *Consider the dynamic system (2.1), (3.1), along with the identification scheme (3.4) and (3.7). Let all disturbances belong to $\mathcal{L}_2$, and the PE condition be satisfied. Then, if γ^* is finite, $\hat{\theta}_\gamma$ converges to the true value of θ for all $\gamma > \gamma^*$, i.e.,*

$$\lim_{t\to\infty} \hat{\theta}_\gamma(t) = \theta.$$

Proof: The proof follows the standard line of reasoning used in adaptive control and identification (Sastry and Bodson, 1989). It will consist of two parts: First, it will be shown that

$$\lim_{T\to\infty} \lambda_{\min}(\Sigma_\gamma(T)) = \infty, \quad \forall \gamma > \gamma^*. \tag{4.2}$$

Fixing $\gamma > \gamma^*$, let $\bar{\gamma} = \frac{1}{2}(\gamma + \gamma^*) > \gamma^*$. By definition,

$$\Sigma_{\bar{\gamma}}(T) = Q_o + \int_0^T A^T A dt - \frac{1}{\bar{\gamma}^2}\int_0^T Q dt > 0, \quad \forall T \geq 0.$$

Now, after rewriting $\Sigma_\gamma(T)$ as

$$\begin{aligned} \Sigma_\gamma(T) &= Q_o + \int_0^T A^T A dt - \frac{1}{\bar{\gamma}^2}\int_0^T Q dt - \left(\frac{1}{\gamma^2} - \frac{1}{\bar{\gamma}^2}\right)\int_0^T Q dt \\ &= \Sigma_{\bar{\gamma}}(T) - \left(\frac{1}{\gamma^2} - \frac{1}{\bar{\gamma}^2}\right)\int_0^T Q dt, \end{aligned}$$

and observing that $\left(\frac{1}{\gamma^2} - \frac{1}{\bar{\gamma}^2}\right)$ is negative, (4.2) follows from the PE condition.

Second, let us introduce the positive definite function $V(t, \tilde{\theta}_\gamma) = \tilde{\theta}_\gamma^T \Sigma_\gamma(t) \tilde{\theta}_\gamma$, where the error $\tilde{\theta}_\gamma = \hat{\theta}_\gamma - \theta$ is generated by

$$\dot{\tilde{\theta}}_\gamma = -\Sigma_\gamma^{-1} A^T A \tilde{\theta}_\gamma + \Sigma_\gamma^{-1} A^T w. \tag{4.3}$$

The derivative of $V(t, \tilde{\theta}_\gamma)$ along the trajectory of (4.3) is

$$\dot{V}(t, \tilde{\theta}_\gamma(t)) = -\frac{1}{\gamma^2}\tilde{\theta}_\gamma^T Q \tilde{\theta}_\gamma - |w - A(x,u)\tilde{\theta}_\gamma|^2 + |w|^2 \tag{4.4}$$

which yields the bound (by integration of the RHS of (4.4)):

$$\begin{aligned} V(t, \tilde{\theta}_\gamma(t)) = \tilde{\theta}_\gamma^T(t)\Sigma_\gamma(t)\tilde{\theta}_\gamma(t) &\leq \tilde{\theta}_\gamma(0)^T Q_o \tilde{\theta}_\gamma(0) + \int_0^t [|w|^2 - |w - A(x,u)\tilde{\theta}_\gamma|^2] ds \\ &\leq \tilde{\theta}_\gamma(0)^T Q_o \tilde{\theta}_\gamma(0) + \int_0^t |w|^2 ds. \end{aligned} \tag{4.5}$$

Since the disturbances belong to $\mathcal{L}_2$, it follows in view of conclusion (4.2) that

$$\lim_{t\to\infty} \tilde{\theta}_\gamma(t) = 0. \tag{4.6}$$

□

Remark 3 Although, we have required disturbances to belong to $\mathcal{L}_2$ in Theorem 2 above, the result actually holds for a more general class of disturbances, satisfying the inequality:

$$\int_0^\infty [|w|^2 - |w - A(x,u)\tilde{\theta}|^2]dt < +\infty. \tag{4.7}$$

This can easily be seen from line (4.5) of the proof of Theorem 2. □

5 NPFSI IDENTIFICATION

5.1 NPFSI problem and its solution

In the development above, we used the derivative of the state vector x to prescribe a minimax identification policy. In practice, however, it may not be desirable to differentiate x, as the results could be numerically unstable. Inaccuracies are often due to unmodeled dynamics of the real plant, or due to some measurement error that was not accounted for when the model was built. To cope with such situations, we now introduce a *noise-perturbed* FSI (NPFSI) problem: The state-dynamics (2.1) and (3.1) are replaced by

$$\begin{aligned} \dot{x} &= A(y,u)\theta + b(y,u) + w, \quad x(0) = x_o, \\ \dot{\theta} &= 0, \qquad\qquad\qquad\qquad\quad \theta(0) = \theta_o \end{aligned} \tag{5.1}$$

and the measurement available for identification purposes is

$$y = x + \epsilon v \tag{5.2}$$

where $v \in I\!R^n$ is the measurement disturbance, and $\epsilon > 0$ is a known (generally small) scalar. The value of ϵ would be chosen to reflect our level of confidence in the knowledge of the state x, so that smaller values of ϵ would signify higher levels of confidence. Note that on the RHS of the state equation, we have also replaced x by y; this has been done for reasons of tractability as will be seen shortly, but also such models do indeed arise in practice. Of course, the limiting case $\epsilon \to 0$ would be the same if the RHS of (5.1) had instead x, or similarly if the worst-case measurement disturbance happens to be zero. Hence, here we have an embedding of the FSI problem in a larger class of problems, which enables us not only to solve the original problem, but also to study its robustness to perturbations due to inaccuracies in modeling.

In accordance with the above, the information set (for the identifier) is now taken to be $\eta_t = \{(y(s), u(s)), s \le t\}$, and the soft-constrained cost function (2.6) is accordingly modified to be

$$\begin{aligned} J_\gamma(\delta; x_o, \theta_o, w_{[0,\infty)}) &= \int_0^\infty [|\theta(t) - \delta(t, \eta(t))|^2_{Q(t,y,u)} - \gamma^2(|w(t)|^2 + |v(t)|^2)]dt \\ &\quad -\gamma^2(|\theta_o - \bar{\theta}_o|^2_{Q_o} + |x_o - \bar{x}_o|^2_{P_o}), \end{aligned} \tag{5.3}$$

where $Q(\cdot,\cdot,\cdot) \geq 0$, $P_o = \frac{1}{\epsilon}I$[2], and $\bar{x}_o$ is a known "bias" term. Our objective is still to find the "smallest" value of $\gamma > 0$, to be denoted $\gamma^*(\epsilon)$, which guarantees the existence of a minimax identification policy, and to prescribe a minimax identification policy for all $\gamma > \gamma^*(\epsilon)$.

As in the FSDI case, we will begin with the assumption that $\gamma^*(\epsilon)$ is finite, and introduce the cost-to-come function for all $\gamma > \gamma^*(\epsilon)$, as follows:

$$W_\gamma(t, x, \theta; \hat{\theta}_{[0,t)}, u_{[0,t)}, y_{[0,t]}) :=$$

$$\sup_{(x_o, w_{[0,t]}) \in \Omega(t,x,\theta;\eta_t)} \left\{ \begin{array}{c} \int_0^t [|\theta(s) - \hat{\theta}(s)|^2_{Q(s,y,u)} - \gamma^2(|w(s)|^2 + \frac{1}{\epsilon^2}|y(s) - x(s)|^2)]ds \\ -\gamma^2(|\theta_o - \bar{\theta}_o|^2_{Q_o} + |x_o - \bar{x}_o|^2_{P_o}) \end{array} \right\} \tag{5.4}$$

where

$$\Omega(t, x, \theta; \eta_t) := \left\{ (x_o, w_{[0,t]}) \mid x(t) = x,\ \theta(t) = \theta,\ (y_{[0,t]}, u_{[0,t]}) = \eta_t \right\}.$$

Again, it can be shown that the cost-to-come function admits a quadratic form, which in this case is:

$$W_\gamma(t, x, \theta) = -\gamma^2 \left\| \begin{bmatrix} x \\ \theta \end{bmatrix} - \begin{bmatrix} \bar{x}(t) \\ \bar{\theta}(t) \end{bmatrix} \right\|^2_{\Sigma(t)} - m(t), \tag{5.5}$$

where

$$\dot{\Sigma} = -\Sigma \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ A^T & 0 \end{bmatrix} \Sigma + \begin{bmatrix} \frac{1}{\epsilon^2}I & 0 \\ 0 & -\frac{1}{\gamma^2}Q(y,u) \end{bmatrix} - \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma, \tag{5.6}$$

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{\theta} \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \frac{1}{\epsilon^2}\Sigma^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} (y - \bar{x}) + \frac{1}{\gamma^2}\Sigma^{-1} \begin{bmatrix} 0 \\ Q \end{bmatrix} (\bar{\theta} - \hat{\theta}), \tag{5.7}$$

$$\dot{m} = -|\bar{\theta} - \hat{\theta}|^2_{Q(y,u)} + \frac{\gamma^2}{\epsilon^2}|y - \bar{x}|^2, \quad m(0) = 0 \tag{5.8}$$

and initial conditions for the Riccati equation (5.6) and estimator dynamics (5.9) are

$$\Sigma(0) = \begin{bmatrix} P_o & 0 \\ 0 & Q_o \end{bmatrix} \text{ and } \begin{bmatrix} \bar{x}(0) \\ \bar{\theta}(0) \end{bmatrix} = \begin{bmatrix} \bar{x}_o \\ \bar{\theta}_o \end{bmatrix},$$

respectively.

We now have the following counterpart of Theorem 1.

Theorem 3 *For the NPFSI model with a fixed $\epsilon > 0$, the following are true:*

(i) *$\gamma^*(\epsilon)$ is finite if, and only if, there exists a $\gamma > 0$ such that the solution of (5.6), Σ_γ, is positive definite for all $t \in \mathbf{T}$.*

[2]Inverse of P_o is taken to grow linearly with $\epsilon > 0$ for technical reasons, but it also reflects the intuitive reasoning that the smaller ϵ is the closer should x_o be to the bias term $\bar{x}_o$. Recalling that NPFSI model is used in order to avoid using the derivative of the state, we can readily assume that x_o is a known initial state.

(ii) *Given any $\gamma < \gamma^*(\epsilon)$, or any finite γ if $\gamma^*(\epsilon)$ is infinite, equation (5.6) does not admit a nonnegative-definite solution.*

(iii) *If $\gamma^*(\epsilon)$ is finite, then for every $\gamma > \gamma^*(\epsilon)$, and each $t \in \mathbf{T}$, $\delta(t, \eta_t) = \hat{\theta}_\gamma(t)$ is a minimax policy, where $\hat{\theta}_\gamma$ is generated by (5.7) with $\hat{\theta} = \bar{\theta}$, i. e.,*

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\theta}}_\gamma \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\theta}_\gamma \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \frac{1}{\epsilon^2} \Sigma_\gamma^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} (y - \hat{x}). \tag{5.9}$$

Proof: Let Σ_∞ be the solution to (5.6) with $\gamma = \infty$ (which exists on any finite time interval since it is standard filtering Riccati equation). Then, for any $t \in \mathbf{T}$, and any $\gamma > 0$, the solution to (5.6) is given, whenever it exists, by

$$\Sigma_\gamma(t) = \Sigma_\infty(t) - \frac{1}{\gamma^2} \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} ds.$$

Rewriting Σ_γ in the form above, and using similar arguments as those used in the proof of statement (ii) of Theorem 1, we can readily arrive at statement (ii) of the present theorem. Proofs of (i) and (iii) are also similar to those of appropriate parts of Theorem 1, and hence are omitted. □

Remark 4 The above results on the NPFSI problem readily extend to a more general class of problems where system dynamics (5.1) are described by

$$\begin{aligned} \dot{x} &= Fx + A(y,u)\theta + b(y,u) + Dw, \quad & x(0) = x_o \\ \dot{\theta} &= 0, & \theta(0) = \theta_o \end{aligned} \tag{5.10}$$

and where (5.2) is replaced by

$$y = Cx + \epsilon v. \tag{5.11}$$

Matrices F, D, and C are of appropriate dimensions.

This generalization may represent one of several practical problems. One such class of problems corresponds to the case where the matrix C has a full-row rank which is smaller than n. This describes the situation where the uncertain part of system dynamics are only affected by a subset of the states, which also happen to be the measured states. An alternative situation occurs when the measurement y can be partitioned into two parts: first part affects the uncertain part of state dynamics, and the second part provides additional measurements on the system, without affecting the state dynamics.

Counterparts of Theorems 3 can easily be obtained for the general structure (5.10)-(5.11) by appropriate (obvious) modifications in (5.6) and (5.7) which now will read, respectively:

$$\dot{\Sigma} = -\Sigma \begin{bmatrix} F & A \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} F^T & 0 \\ A^T & 0 \end{bmatrix} \Sigma + \begin{bmatrix} \frac{1}{\epsilon^2} C^T C & 0 \\ 0 & -\frac{1}{\gamma^2} Q(y,u) \end{bmatrix} - \Sigma \begin{bmatrix} DD^T & 0 \\ 0 & 0 \end{bmatrix} \Sigma, \tag{5.12}$$

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} F & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{\theta} \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \frac{1}{\epsilon^2}\Sigma^{-1}C^T(y - C\bar{x}) + \frac{1}{\gamma^2}\Sigma^{-1}\begin{bmatrix} 0 \\ Q \end{bmatrix}(\bar{\theta} - \hat{\theta}), \tag{5.13}$$

A further discussion on this more general structure can be found in (Didinsky, 1994) □

Next, we address the question of whether the existence of a solution to an FSDI identification problem guarantees the existence of a solution to an ϵ-parametrized NPFSI problem, at least for sufficiently small values of ϵ. As it will be shown in the two examples below, the answer to this question may not always be positive.

Before we proceed with the examples, let us first partition Σ

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix},$$

and use this in (5.6) to arrive at the following decomposition:

$$\dot{\Sigma}_1 = \frac{1}{\epsilon^2}I - \Sigma_1\Sigma_1, \quad \Sigma_1(0) = \frac{1}{\epsilon}I \tag{5.14a}$$

$$\dot{\Sigma}_2 = -\Sigma_1(\Sigma_2 + A), \quad \Sigma_2(0) = 0 \tag{5.14b}$$

$$\dot{\Sigma}_3 = -\Sigma_2^T A - A^T\Sigma_2 - \Sigma_2^T\Sigma_2 - \frac{1}{\gamma^2}Q, \quad \Sigma_3(0) = Q_o \tag{5.14c}$$

It can be easily observed that, regardless of the value of γ, the unique solution to (5.14a) is $\Sigma_1 = \frac{1}{\epsilon}I$. In view of the Schur test for positivity, this implies that $\Sigma_\gamma(t) > 0$, $\forall t \in \mathbf{T}$, if, and only if,

$$\Pi_\gamma(t) := \Sigma_3(t) - \epsilon\Sigma_2(t)^T\Sigma_2(t) > 0, \ \forall t \in \mathbf{T}. \tag{5.15}$$

It is useful to observe that Π_γ is the solution of the matrix differential equation

$$\dot{\Pi}(t) = \Sigma_2^T\Sigma_2 - \frac{1}{\gamma^2}Q, \quad \Pi(0) = Q_o \tag{5.16}$$

and Σ_2 is obtained uniquely from the following linear matrix differential equation:

$$\dot{\Sigma}_2 = -\frac{1}{\epsilon}(\Sigma_2 + A), \quad \Sigma_2(0) = 0. \tag{5.17}$$

Now, we present two illuminating examples, in the first of which the NPFSI problem has no solution, whereas in the second one it always has a solution for a sufficiently small $\epsilon > 0$.

Example 1: Consider the scalar dynamical system

$$\dot{x} = x\theta + w, \quad x(0) = x_o$$

and the associated cost function

$$J_\gamma(\delta) = \int_0^\infty [x^2(\theta(t) - \delta(t, \eta_t))^2 - \gamma^2 w^2]dt - \gamma^2 Q_o(\theta_o - \bar{\theta}_o)^2.$$

Formulated as an FSDI problem, we have $\gamma^* = 1$, by Corollary 1. If, however, $\dot{x}$ is not available, and x is measured with a small additive noise, we have the NPFSI problem with dynamics

$$\dot{x} = y\theta + w, \quad x(0) = x_o$$

measurement

$$y = x + \epsilon v,$$

and cost function

$$J_\gamma(\delta) = \int_0^\infty [y^2(\theta(t) - \delta(t))^2 - \gamma^2(w^2 + v^2)]dt - \gamma^2(Q_o(\theta_o - \bar{\theta}_o)^2 + \frac{1}{\epsilon}(x_o - \bar{x}_o)^2).$$

For this system, (5.16) and (5.17) become, respectively

$$\begin{aligned} \dot{\Pi} &= \Sigma_2^2 - \frac{1}{\gamma^2}y^2, \quad \Pi(0) = Q_o \\ \dot{\Sigma}_2 &= -\frac{1}{\epsilon}(\Sigma_2 + y), \quad \Sigma_2(0) = 0 \end{aligned}$$

Consider the output measurement $y(t) = a \sin \omega t$, $t \geq 0$. Note that this is a legitimate output of the FSDI system, which can be seen by taking $\theta = 0$, $x_o = 0$, and $w(t) = a\omega \cos(\omega t)$. For the same output in the NPFSI case, Σ_2 admits the analytic solution:

$$\Sigma_2(t) = -a\left[\frac{1}{1+\epsilon^2\omega^2}\sin \omega t - \frac{\epsilon\omega}{1+\epsilon^2\omega^2}\cos \omega t + \frac{\epsilon\omega}{1+\epsilon^2\omega^2}e^{-\frac{1}{\epsilon}t}\right].$$

By choosing $\omega \gg \frac{1}{\epsilon}$, Σ_2 can be made arbitrarily small (positive) for all $t \in \boldsymbol{T}$, whereas if at the same time a is sufficiently large, then $\Pi(t)$ can be made negative. Therefore, given any $\epsilon > 0$ and $\gamma > 1$, one can construct an output signal for which the NPFSI problem will not have a solution.

$\square$

Example 2: This second example pertains to a system for which the existence of a minimax FSDI identifier implies the existence of a minimax NPFSI identifier. Consider the case of A and Q being constant matrices. In this case, $\Sigma_2(t)$ and $\Pi(t)$ can be solved for the NPFSI problem explicitly:

$$\begin{aligned} \Sigma_2(t) &= -(1 - e^{-\frac{1}{\epsilon}t})A \\ \Pi(t) &= \Sigma^{FSI}(t) - \frac{\epsilon}{2}[(2 - e^{-\frac{1}{\epsilon}t})^2 - 1]A^T A \\ &\geq \Sigma^{FSI}(t) - \frac{3\epsilon}{2}A^T A, \end{aligned}$$

where Σ^{FSI} is the solution of (3.4). Now, if γ^* is finite, it implies that given any $\gamma > \gamma^*$, there exists a $\delta(\gamma) > 0$ such that $\Sigma^{FSI}(t) \geq \delta(\gamma)I$, $\forall t \in \mathbf{T}$. Then, for all

$$\epsilon < \frac{2\delta(\gamma)}{3\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}(\Lambda)$ stands for the largest eigenvalue of the nonnegative definite matrix Λ, the NPFSI problem has a solution. □

5.2 Modification of the NPFSI problem

The main reason why an NPFSI problem may fail to have a solution was best illustrated by Example 1 above. If the matrix A changes faster than Σ_2 can track it, eventually the matrix Π may have one of its eigenvalues turn negative. Hence, it becomes necessary to place some restrictions on the output measurement waveforms in the NPFSI problem. Toward this end, we first provide here (to set the stage for further analysis) a somewhat modified NPFSI problem, with restricted measurement waveforms.

A Modified NPFSI problem: Given the state dynamics (5.1), fixed input $\{u(t),\ t \in \mathbf{T}\}$, a fixed parameter $\epsilon > 0$, and a prespecified set $\mathbf{Y}$ of time functions $y_{[0,\infty)}$, introduce the set $\mathcal{W}(\mathbf{Y})$ as:

$$\begin{aligned}\mathcal{W}(\mathbf{Y}) \ := \ & \{(w_{[0,\infty)}, v_{[0,\infty)}, x_o, \theta_o) \mid \text{The output } y_{[0,\infty)} \text{ generated by this} \\ & \text{quadruple belongs to the set } \mathbf{Y}.\ \}\end{aligned} \tag{5.18}$$

Consider the following two formulations:

1. **(Finite-horizon problem)** Given some $T \in \mathbf{T}$, find the "smallest" $\gamma > 0$, denoted by $\gamma_Y^*(\epsilon, T)$, such that for each $\gamma > \gamma_Y^*(\epsilon, T)$, there exists a $\delta_\gamma \in \Delta$ such that

$$\sup_{(x_o,\theta_o,w_{[0,T]},v_{[0,T]})\in\mathcal{W}(\mathbf{Y})} J_\gamma^T(\delta_\gamma; x_o, \theta_o, w_{[0,T]}, v_{[0,T]}) \le 0,$$

   where

$$\begin{aligned}J_\gamma^T(\delta; x_o, \theta_o, w_{[0,T]}, v_{[0,T]}) \ := \ & \int_0^T [|\theta(t) - \delta(t, \eta(t))|^2_{Q(t,y,u)} - \gamma^2(|w(t)|^2 + |v(t)|^2)]dt \\ & -\gamma^2(|\theta_o - \bar\theta_o|^2_{Q_o} + |x_o - \bar x_o|^2_{P_o}).\end{aligned}$$

2. **(Infinite-horizon problem)** Find the "smallest" $\gamma > 0$, denoted by $\gamma_Y^*(\epsilon)$, such that for each $\gamma > \gamma_Y^*(\epsilon)$, there exists a $\delta_\gamma \in \Delta$ such that

$$\sup_{(x_o,\theta_o,w_{[0,\infty)},v_{[0,\infty)})\in\mathcal{W}(\mathbf{Y})} J_\gamma(\delta_\gamma; x_o, \theta_o, w_{[0,\infty)}, v_{[0,\infty)}) \le 0.$$

□

Theorem 4 *Given the modified NPFSI problem as above, on finite- or infinite-horizons, the following hold for each fixed $\epsilon > 0$:*

(i) $\gamma_Y^*(\epsilon, T)$ *(respectively, $\gamma_Y^*(\epsilon)$) is finite if, and only if, there exists a $\gamma > 0$ such that $\Pi_\gamma(t) > 0$ for all $t \in [0, T]$ (respectively, $t \in \mathbf{T}$), where Π_γ is the solution of (5.16).*

(ii) *Given any $\gamma < \gamma_Y^*(\epsilon, T)$ (respectively, $\gamma_Y^*(\epsilon)$), or any finite γ if $\gamma^*(\epsilon)$ (respectively, $\gamma_Y^*(\epsilon)$) is infinite, the solution of (5.16) is not nonnegative definite on the time interval $[0, T]$ (respectively, $\mathbf{T}$).*

(iii) *If $\gamma_Y^*(\epsilon, T)$ (respectively, $\gamma_Y^*(\epsilon)$) is finite, then for every $\gamma > \gamma_Y^*(\epsilon, T)$ (respectively, $\gamma_Y^*(\epsilon)$), and each $t \in \mathbf{T}$, $\delta_\gamma(t, \eta_t) = \hat{\theta}_\gamma(t)$, where $\hat{\theta}_\gamma$ is generated by (5.9), is a minimax policy.*

Proof: Since the proof of Theorem 3 did not depend explicitly on the set of all possible measurements, it can be replicated here to prove the desired result. □

We will be particularly interested in two classes of parametrized sets:

$$\mathbf{Y}_{\gamma,\epsilon,T} := \left\{ y_{[0,\infty)} \mid \Pi_\gamma(t) > 0, \ \forall t \in [0, T] \right\}; \tag{5.19}$$

$$\mathbf{Y}_{\gamma,\epsilon} := \left\{ y_{[0,\infty)} \mid \Pi_\gamma(t) > 0, \ \forall t \in \mathbf{T} \right\}, \tag{5.20}$$

where Π_γ is the solution of (5.16). In terms of these two sets, we immediately have the following corollary to Theorem 4:

Corollary 2 *For the modified NPFSI problem formulated in this subsection, the following two statements are true:*

(i) *If the set $\mathbf{Y}_{\gamma,\epsilon,T}$ is nonempty, then $\gamma^*_{\mathbf{Y}_{\gamma,\epsilon,T}}(\epsilon, T)$ is finite.*

(ii) *If the set $\mathbf{Y}_{\gamma,\epsilon}$ is nonempty, then $\gamma^*_{\mathbf{Y}_{\gamma,\epsilon}}(\epsilon)$ is finite.*

□

We are now left with the question of whether the sets $\mathbf{Y}_{\gamma,\epsilon,T}$ and $\mathbf{Y}_{\gamma,\epsilon}$ contain sufficiently rich classes of output functions to be of practical value. Toward this end, we introduce the following two classes of outputs:

$\mathcal{LB}^A(L, C)$: This is the set of all time functions $y_{[0,\infty)}$ that satisfy the following two conditions for some positive constants L and C:

1. $\|A(y(t_1)) - A(y(t_2))\|_2 \leq L|t_1 - t_2|, \quad \forall t_1, t_2 \in \mathbf{T}$, where $\|\cdot\|_2$ stands for the induced $\mathcal{L}_2$-norm of a matrix.
2. $\|A(y(t))\|_2 \leq C$ for all $t \in \mathbf{T}$.

$\mathcal{LBP}^A(L, C, \rho)$: This is the set of all time functions $y_{[0,\infty)}$ such that there corresponds to each element of this set a $\bar{y}_{[0,\infty)} \in \mathcal{LB}^A(L, C)$, with the property

$$\|A(y(t)) - A(\bar{y}(t))\|_2 \leq \rho, \quad \forall t \in \mathbf{T}$$

Remark 5 In the case of linear-quadratic time-invariant problems, the above sets of outputs correspond to outputs which are themselves Lipschitz continuous and bounded, perturbed by small (possibly high-frequency) disturbances. In the frequency domain, each such output has a spectrum that is similar to that of a low-pass filter, i.e., the spectrum can be split into two regions: higher magnitudes in lower frequencies, and diminishing magnitudes in higher frequencies.

Alternatively, an assumption of bounded disturbances in (5.1)-(5.2) will guarantee that $y_{[0,T]}$ belongs to the above two classes of outputs. $\square$

Before we present a result that provides an upper bound on the maximum value of ϵ for which $\mathcal{LBP}^A(L,C,\rho) \subset \mathbf{Y}_{\gamma,\epsilon,T}$, we will state the following three lemmas, the first of which will require no proof.

Lemma 1 *Given any two matrices A and B of equal dimensions, if $\|A \pm B\|_2 \leq C$, for some constant $C > 0$, then*

$$-C(\|A\|_2 + \|B\|_2)I \leq A^T A - B^T B \leq C(\|A\|_2 + \|B\|_2)I.$$

$\square$

Lemma 2 *Given $y_{[0,\infty)}$ that satisfies $\|A(y,u,t)\|_2 \leq C$, for some constant $C > 0$, for all $t \in \mathbf{T}$, then $\|\Sigma_2(t)\|_2 \leq C$, for all $t \in \mathbf{T}$, where Σ_2 is the solution of (5.17).*

Proof: Since the solution of (5.17) is

$$\Sigma_2(t) = -\frac{1}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-\tau)} A(\tau) d\tau,$$

using properties of matrix norms, we can easily obtain the bound

$$\|\Sigma_2(t)\|_2 \leq \frac{C}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-\tau)} d\tau = C(1 - e^{-\frac{1}{\epsilon}t}) \leq C.$$

$\square$

Lemma 3 *If the set $\mathcal{LBP}^A(L,C,\rho)$ is nonempty, given any $y_{[0,\infty)} \in \mathcal{LBP}^A(L,C,\rho)$, the following inequality holds:*

$$\|\Sigma_2(t) + A(y(t))\|_2 \leq \|A(y(0))\|_2 e^{-\frac{1}{\epsilon}t} + L\epsilon(1 - e^{-\frac{1}{\epsilon}t}) + 2\rho(1 - e^{-\frac{1}{\epsilon}t}).$$

Proof: Given $y_{[0,\infty)} \in \mathcal{LBP}^A(L,C,\rho)$, let $\bar{y}_{[0,\infty)} \in \mathcal{LB}^A(L,C)$ be an output waveform that satisfies $\|A(y(t)) - A(\bar{y}(t)\|_2 \leq \rho$, for all $t \in \mathbf{T}$. The existence of such a function is guaranteed by the definition of the set $\mathcal{LBP}^A(L,C,\rho)$. We start by rewriting (5.17) in integral form

$$\begin{aligned}
\Sigma_2(t) &= -\tfrac{1}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-\tau)} A(y(\tau)) d\tau \\
&\leq -A(y(t)) + A(y(0))e^{-\frac{1}{\epsilon}t} + \rho(1 - e^{-\frac{1}{\epsilon}t})\, I + \int_0^t e^{-\frac{1}{\epsilon}(t-\tau)} dA(\bar{y}(\tau)) \\
&\quad -\tfrac{1}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-\tau)} [A(y(\tau)) - A(\bar{y}(\tau))] d\tau.
\end{aligned} \tag{5.21}$$

Then, it follows that

$$\|\Sigma_2(t) + A(y(t))\|_2 \le \|A(y(0))\|_2 e^{-\frac{1}{\epsilon}t} + \rho(1 - e^{-\frac{1}{\epsilon}t}) + \int_0^t e^{-\frac{1}{\epsilon}(t-\tau)} L d\tau + \frac{\rho}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-\tau)} d\tau.$$

Completing the integration on the RHS leads to the inequality of the lemma. $\square$

These three lemmas now lead to the following result:

Theorem 5 *Given a $\gamma > \gamma^*$, and positive scalars L, C, T, ϵ, and ρ, assume that the set $\mathcal{LBP}^A(L, C, \rho)$ is nonempty, and $2\epsilon C(C + Lt) + 4Ct\rho < \bar{\rho}(t)$, for all $t \in [0, T]$, where $\bar{\rho}(t)$ is any positive function such that*

$$\Sigma_\gamma^F(t) := Q_o + \int_0^t (A(y,u)^T A(y,u) - \frac{1}{\gamma^2} Q(y,u)) \, d\tau \ge \bar{\rho}(t) I,$$

$\forall t \in [0, T]$, $\forall y_{[0,\infty)} \in \mathcal{LBP}^A(L, C, \rho)$. Then, $\mathcal{LBP}^A(L, C, \rho) \subset \mathbf{Y}_{\gamma,\epsilon,T}$. Moreover, the filter (5.6),(5.9) attains the disturbance attenuation level γ for all disturbances in the set $\mathcal{W}(\mathcal{LBP}^A(L, C, \rho))$.

Proof: For any $y_{[0,\infty)} \in \mathcal{LBP}^A(L, C, \rho)$, we have the following sequence of inequalities:

$$\begin{aligned} \Pi_\gamma(t) &= \Sigma_\gamma^F(t) + \int_0^t (\Sigma_2^T \Sigma_2 - A^T A) d\tau \\ &\ge \bar{\rho}(t) I - \int_0^t [2C(Ce^{-\frac{1}{\epsilon}\tau} + L\epsilon(1 - e^{-\frac{1}{\epsilon}\tau}) + 2\rho(1 - e^{-\frac{1}{\epsilon}\tau})] I d\tau \\ &\ge (\bar{\rho}(t) - 2C\epsilon(C + Lt) - 2Ct\rho) I > 0, \end{aligned} \tag{5.22}$$

where the first inequality is a consequence of Lemmas 1, 2, and 3, and the second inequality is a consequence of the hypotheses of the theorem. $\square$

The existence of a positive function $\bar{\rho}(t)$ used in Theorem 5 above is guaranteed by the fact that given any $\gamma > \gamma^*$, there exists a scalar $\bar{\rho} > 0$, possibly dependent on the value of γ and T, such that $\Sigma_\gamma^F(t) \ge \bar{\rho} I$, for all $t \in [0, T]$. However, as $T \to \infty$, the existence of $\bar{\rho} > 0$ may no longer be true, i.e., given some L, C, ρ, and $\gamma > \gamma^*$, there might not be any ϵ such that $\mathcal{LBP}^A(L, C, \rho) \subset \mathbf{Y}_{\gamma,\epsilon}$. To deal with such (infinite-horizon) problems, let us introduce a new class of outputs:

$\mathcal{LBPP}_E^A(L, C, \rho, \bar{\rho}; \gamma)$: This is the set of all waveforms $y_{[0,\infty)} \in \mathcal{LBP}^A(L, C, \rho)$ that satisfy the condition $\Sigma_\gamma^F(t) \ge \bar{\rho} t I$, for all $t \in \mathbf{T}$, and for some constant $\bar{\rho} > 0$.[3]

Theorem 6 *Given any $\gamma > \gamma^*$, and positive scalars L, C, ϵ, $\bar{\rho}$, and ρ, with $\bar{\rho} > 4C\rho$, assume that $\mathcal{LBPP}_E^A(L, C, \rho, \bar{\rho}; \gamma)$ is nonempty. Then, for any $\epsilon < \dfrac{\bar{\rho} - 4C\rho}{2CL}$, the following hold:*

[3] The additional symbol $\mathcal{P}_E$ in the denotation of this set stands for persistency of excitation. The condition $\Sigma_\gamma^F(t) \ge \bar{\rho} t I$ is a persistency of excitation condition, which, in fact, is more conservative than PE of Section 4.

(i) *$\mathcal{LBPP}_E^A(L, C, \rho, \bar{\rho}; \gamma) \subset \mathbf{Y}_{\gamma,\epsilon}$, and the filter (5.6),(5.9) attains the disturbance attenuation level γ for the modified NPFSI problem.*

(ii) *For all disturbances in the set $\mathcal{W}(\mathcal{LBPP}_E^A(L, C, \rho, \bar{\rho}; \gamma)) \bigcap \mathcal{L}_2$, we have $\lim_{t\to\infty} \hat{\theta}_\gamma(t) = \theta$.*

Proof: Statement (i) follows from Theorem 5. To prove the second statement, we have to show that Σ_3 satisfying (5.14c) has all its eigenvalues increasing to ∞. Since, by (5.15), $\Sigma_3 = \Pi + \epsilon \Sigma_2^T \Sigma_2$, and Π has all its eigenvalues increasing to ∞, the result follows. $\square$

Since the above Theorem holds for both finite- and infinite-horizon cases, then it holds also for the set of all $y_{[0,\infty)}$ that are piecewise continuous in the set $\mathcal{LBP}_E^A(L, C, \bar{\rho})$ but have finitely many discontinuities. To put this observation into precise mathematical terms, we introduce a new set:

$\mathcal{LBP}_{EPW}^A(L, C, \bar{\rho}, T_m, K)$: This is the set of all waveforms $y_{[0,\infty)}$ that satisfy the following four conditions for some positive constants L, C, $\bar{\rho}$, T_m and integer K:

1. There exist $K < \infty$ time instances $0 < t_1 < \cdots < t_K < \infty$, ($t_K < T$ in the finite-horizon case) such that
$$\min_{k \in \{1,\ldots,K\}} t_k - t_{k-1} \geq T_m.$$
2. $\|A(y(t')) - A(y(t))\|_2 \leq L|t' - t|, \quad \forall t', t \in [0, t_1]$, or $\forall t', t \in [t_k, t_{k+1}], k = 1, \ldots, K-1$, or $\forall t', t \in [t_K, \infty)$.
3. $\|A(y(t))\|_2 \leq C$ for all $t \in \mathbf{T}$.
4. $\Sigma_\gamma^F(t) \geq \bar{\rho}(t+1)I$, for all $t \in \boldsymbol{T}$ (or $t \in [0, T]$ in the finite horizon case).

Then, we have the following extension of Theorems 5 and 6:

Corollary 3 *Consider the finite- and infinite-horizon modified NPFSI problems, and let $\gamma > \gamma^*$. Then:*

1. *There exists an $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, the set $\mathcal{LBP}_{EPW}^A(L, C, \bar{\rho}, T_m, K)$ is a subset of $\bar{\boldsymbol{Y}}_{\gamma,\epsilon}$ ($\bar{\boldsymbol{Y}}_{\gamma,\epsilon,T}$ in the finite-horizon case). Furthermore, the filter (5.6),(5.9) achieves the disturbance attenuation level γ for the modified NPFSI problem.*
2. *In the infinite-horizon case, if the disturbances are in the set $\mathcal{W}(\mathcal{LBP}_{EPW}^A(L, C, \bar{\rho}, T_m, K)) \bigcap \mathcal{L}_2$, then the parameter estimate $\hat{\theta}_\gamma(t)$ converges to the true parameter θ as $t \to \infty$.* $\square$

5.3 A reduced-order identifier

Theorem 6 of the previous subsection has established the optimality of the identifier described by (5.6) and (5.9). But the computation of this identifier is much more complex compared to that of the filter given by (3.4) and (3.7) in the FSDI case. To remedy this, we undertake in this subsection a further analysis, that will lead to a structural simplification by exploiting the smallness of ϵ. We first note that for $\gamma > \gamma^*$, the NPFSI filter (5.6) and (5.9) can be written approximately as the following identification policy $\bar{\delta}(t, u_{[0,t]}, y_{[0,t]})$:

$$\bar{\delta}(t, u_{[0,t]}, y_{[0,t]}) = \hat{\theta}_\gamma(t) \tag{5.23}$$

$$\dot{\hat{x}} = A(y,u)\hat{\theta}_\gamma + b(y,u) + \frac{1}{\epsilon}(y - \hat{x}); \quad \hat{x}(0) = \bar{x}_o \tag{5.24}$$

$$\dot{\hat{\theta}}_\gamma = \frac{1}{\epsilon}\Sigma_\gamma^{F^{-1}} A(y,u)^T (y - \hat{x}); \quad \hat{\theta}_\gamma(0) = \bar{\theta}_o \tag{5.25}$$

where we have replaced Σ_2 by A and have ignored higher order terms of ϵ in the expression for Σ^{-1}, and Σ_γ^F is generated by the differential equation (3.4) with x replaced by y. The inversion can further be avoided since $\Sigma_\gamma^{F^{-1}}$ satisfies the following Riccati differential equation:

$$\frac{d}{dt}(\Sigma_\gamma^{F^{-1}}) = -\Sigma_\gamma^{F^{-1}}\left(A(y,u)^T A(y,u) - \frac{1}{\gamma^2}Q(y,u)\right)\Sigma_\gamma^{F^{-1}}; \qquad \Sigma_\gamma^{F^{-1}}(0) = Q_o^{-1} \tag{5.26}$$

In this subsection, we will study robustness properties of the approximate identification policy $\bar{\delta}$ defined by (5.23)–(5.25). Toward this end, we first introduce two sets $\bar{\boldsymbol{Y}}_{\gamma,\epsilon,T}$ and $\bar{\boldsymbol{Y}}_{\gamma,\epsilon}$ as follows:

$\bar{\boldsymbol{Y}}_{\gamma,\epsilon,T}$: The set of measurement waveforms $y_{[0,T]}$ for which the policy $\bar{\delta}$ attains the performance level γ for the modified NPFSI problem on the time interval $[0,T]$.

$\bar{\boldsymbol{Y}}_{\gamma,\epsilon}$: The set of measurement waveforms $y_{[0,\infty)}$ for which the policy $\bar{\delta}$ attains the performance level γ for the modified NPFSI problem on $\boldsymbol{T}$.

Obviously, by their definitions, the identification policy $\bar{\delta}$ is robust and attains the disturbance attenuation level γ on $\bar{\boldsymbol{Y}}_{\gamma,\epsilon,T}$ and $\bar{\boldsymbol{Y}}_{\gamma,\epsilon}$ for the finite- and infinite-horizon cases, respectively. The interesting question that remains is how large these sets are. To answer this question, let us consider the following set of measurements:

$\mathcal{LBP}_E^A(L,C,\bar{\rho})$: This is the set of all waveforms $y_{[0,\infty)} \in \mathcal{LB}^A(L,C)$ that satisfy the further condition that for some positive scalar $\bar{\rho}$, $\Sigma_\gamma^F(t) \geq \bar{\rho}(t+1)I$, for all $t \in \boldsymbol{T}$ (or $t \in [0,T]$ in the finite horizon case). [4]

Now we have the following result on the optimality of the reduced-order identifier:

[4] This is a "persistency of excitation" condition in this context, and the set introduced here is equivalent to $\mathcal{LBPP}_E^A(L,C,0,\bar{\rho})$.

Theorem 7 *Consider the finite- and infinite-horizon modified NPFSI problems, and let $\gamma > \gamma^*$. Then:*

(i) *There exists an $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, the set $\mathcal{LBP}_E^A(L, C, \bar{\rho})$ is a subset of $\bar{\boldsymbol{Y}}_{\gamma,\epsilon}$ ($\bar{\boldsymbol{Y}}_{\gamma,\epsilon,T}$ in the finite-horizon case). Furthermore, the identification policy $\bar{\delta}$ defined by (5.23)–(5.25) achieves the disturbance attenuation level γ for the modified NPFSI problem.*

(ii) *In the infinite-horizon case, if in addition the disturbances $w_{[0,\infty)}$ and $v_{[0,\infty)}$ belong to $\mathcal{L}_2$, then the parameter estimate $\hat{\theta}_\gamma(t)$ converges to the true parameter value θ as $t \to \infty$.*

Proof: See Appendix A. □

Note that Theorem 7 holds for both finite- and infinite-horizon cases. It can further be extended to yield the following result:

Corollary 4 *Consider the finite- and infinite-horizon modified NPFSI problems, and let $\gamma > \gamma^*$. Then:*

(i) *There exists an $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, the set $\mathcal{LBP}_{EPW}^A(L, C, \bar{\rho}, T_m, K)$ is a subset of $\bar{\boldsymbol{Y}}_{\gamma,\epsilon}$ ($\bar{\boldsymbol{Y}}_{\gamma,\epsilon,T}$ in the finite-horizon case). Furthermore, the identification policy $\bar{\delta}$ defined by (5.23)–(5.25) achieves the disturbance attenuation level γ for the modified NPFSI problem.*

(ii) *In the infinite-horizon case, if in addition the disturbances $w_{[0,\infty)}$ and $v_{[0,\infty)}$ belong to $\mathcal{L}_2$, then the parameter estimate $\hat{\theta}_\gamma(t)$ converges to the true parameter value θ as $t \to \infty$.*

□

Remark 6 In Remark 4, it was mentioned that the cost-to-come method can be used to address NPFSI problems of the more general form (5.10)-(5.11). However, the result on the reduced-order identifier may not always be extended to such generalization. One of the setups for which reduction follows the same lines as the case discussed in this subsection is when $F = 0$ and $D = I$ in (5.10), and where (5.11) has the form

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I \\ \bar{C} \end{bmatrix} x + \epsilon v,$$

with $\bar{C}$ being a full row-rank matrix. In this case only y_1 is used in A, b, and Q to replace x, and y_2 represents additional measurement on x for identification purposes. Counterpart of Theorem 7 can easily be obtained for this structure by appropriate modifications in (5.24) and (5.25), which now will read, respectively:

$$\dot{\hat{x}} = A(y,u)\hat{\theta}_\gamma + b(y,u) + \frac{1}{\epsilon}\left[(y_1 - \hat{x}) + \bar{C}^T(y_2 - \bar{C}\hat{x})\right]; \quad \hat{x}(0) = \bar{x}_o \tag{5.27}$$

$$\dot{\hat{\theta}}_\gamma = \frac{1}{\epsilon}\Sigma_\gamma^{F^{-1}} A(y,u)^T \left[(y_1 - \hat{x}) + \bar{C}^T(y_2 - \bar{C}\hat{x})\right]; \quad \hat{\theta}_\gamma(0) = \bar{\theta}_o \tag{5.28}$$

□

6 PREFILTERING-BASED IDENTIFICATION

In this section, we will take a more classical approach toward designing an estimator that does not depend on $\dot{x}$. In particular, we will use the idea of prefiltered input and output (Sastry and Bodson, 1989), adapted to the present context. This will give us an opportunity to compare the performances of the full- and reduced-order identifiers derived in the previous section with that of a competing design. To introduce the prefiltering-based identification scheme, first we multiply both sides of (2.1) by $e^{\lambda t}$, and then, integrate both sides with respect to time from 0 to t:

$$\int_0^t e^{\lambda\tau}\dot{x}d\tau = \int_0^t e^{\lambda\tau}[A(x,u)\theta + b(x,u) + w]d\tau.$$

Next, using integration by parts, we obtain the following equality:

$$x_t = \tilde{A}_t\theta + \tilde{b}_t + \omega_t, \tag{6.1}$$

where

$$\begin{aligned}
\dot{\tilde{A}} &= -\lambda\tilde{A} + A(x,u), \quad \tilde{A}_{t|t=0} = 0,\\
\dot{\tilde{b}} &= -\lambda\tilde{b} + b(x,u) + \lambda x, \quad \tilde{b}_{t|t=0} = x_o,\\
\dot{\omega} &= -\lambda\omega + w, \quad \omega_{t|t=0} = 0.
\end{aligned}$$

Now, the algebraic relationship (6.1) can be viewed as providing a noisy measurement of θ, in a way similar to (2.1) that was used in the FSDI case. In this setup, an estimator will be designed to attenuate the new measurement disturbance $\omega_{[0,\infty)}$ and, hence, in (2.6) w will be replaced by ω. To guarantee that $\omega \in \mathcal{L}_2$, the filtering parameter λ must be chosen to be strictly positive.

As in the FSDI case, let us introduce the cost-to-come function

$$W_\gamma(t,\theta; x_{[0,t]}, \hat{\theta}_{[0,t]}, u_{[0,t]}) := \int_0^t \left[|\theta - \hat{\theta}|_Q^2 - \gamma^2|x - \tilde{A}\theta - \tilde{b}|^2\right] ds - \gamma^2|\theta - \bar{\theta}_o|_{Q_o}^2,$$

which can be rewritten in the simpler quadratic form

$$W_\gamma(t,\theta; x_{[0,t]}, \hat{\theta}_{[0,t]}, u_{[0,t]}) = -\gamma^2|\theta - \bar{\theta}_t|_{\Sigma_t}^2 - m_t,$$

where

$$\dot{\Sigma} = \tilde{A}^T\tilde{A} - \frac{1}{\gamma^2}Q, \quad \Sigma_{t|t=0} = Q_o, \tag{6.2}$$

$$\dot{\bar{\theta}} = \Sigma^{-1}\tilde{A}^T\left(x - \tilde{A}\theta - \tilde{b}\right) + \frac{1}{\gamma^2}\Sigma^{-1}Q_t(\bar{\theta} - \hat{\theta}), \quad \bar{\theta}_{t|t=0} = \bar{\theta}_o, \tag{6.3}$$

$$\dot{m} = -|\bar{\theta} - \hat{\theta}|_Q^2 + \gamma^2|x - \tilde{A}\theta - \tilde{b}|^2, \quad m_{t|t=0} = 0.$$

Using this representation leads to the following counterpart of Theorems 1 and 2:

Theorem 8 *For the identification problem with the prefiltering, the following are true:*

(i) γ^* *is finite if, and only if, there exists a* $\gamma > 0$ *such that* $\Sigma_\gamma(t) > 0$ *for all* $t \in \mathbf{T}$*, where* Σ_γ *is the solution to (6.2).*

(ii) *Given any* $\gamma < \gamma^*$*, or any finite* γ *if* γ^* *is not finite, the unique solution of (6.2) is not nonnegative definite.*

(iii) *If* γ^* *is finite, then for every* $\gamma > \gamma^*$*, the policy* $\{\delta(t, \eta_t) = \hat{\theta}_\gamma(t),\ t \in \mathbf{T}\}$*, where* $\hat{\theta}_\gamma$ *is generated by (6.3) with* $\hat{\theta} = \bar{\theta}$*, i.e.,*

$$\dot{\hat{\theta}}_\gamma = \Sigma_\gamma^{-1} A(x,u)^T (x - \tilde{A}\hat{\theta}_\gamma - \tilde{b}), \quad \hat{\theta}_\gamma(0) = \bar{\theta}_o, \tag{6.4}$$

is minimax.

(iv) *Let all disturbances belong to* $\mathcal{L}_2$*, and the PE condition (4.1) be satisfied. If* γ^* *is finite, then for all* $\gamma > \gamma^*$*,* $\hat{\theta}_\gamma$ *generated by (6.4) converges to the true value of* θ*, i.e.,*

$$\lim_{t \to \infty} \hat{\theta}_\gamma(t) = \theta.$$

□

Remark 7 As a natural counterpart of Remark 3, the proof of Theorem 8 can be extended to guarantee parameter convergence for the larger class of disturbances ω, which satisfy the following finiteness condition:

$$\int_0^\infty [|\omega|^2 - |\omega - \tilde{A}\tilde{\theta}_\gamma|^2] dt < +\infty, \tag{6.5}$$

□

Remark 8 One of the stumbling blocks associated with using prefiltering estimation is the choice of the parameter λ. In the NPFSI problem, it is obvious that the smaller $\epsilon > 0$ is, the better is the performance (at least theoretically) of the NPFSI estimator. A similar straightforward recommendation may not be obvious in prefiltering estimation.

For example, a smaller value of λ would imply a larger $\tilde{A}$, which would be better for excitation of parameter estimates. On the other hand, a smaller λ also implies a larger disturbance vector ω, which would be detrimental to disturbance attenuation. The situation is reversed when λ is chosen to be large. In this case, ω would typically become small, which implies the possibility of better attenuation. But then, $\tilde{A}$ also becomes small, which would imply less excitation for identification.

In this situation, the choice of λ will probably require a more extensive search, and may depend on the specific problem at hand. For example, for different adaptive control schemes, the parameter λ may have to be chosen differently. □

Remark 9 One important factor in a realistic comparison of any two filter structures is the number of internal states. Everything else being comparable, it would definitely be preferred to have an estimator of smaller dimensions (i.e., fewer internal states). From this point of view, the approximate NPFSI filter (5.23)-(5.25) fares much better than the prefiltering estimator.[5] The former has internal states $\hat{x}$, $\hat{\theta}$, and Σ^F, which amounts to $n + r + \frac{r(r+1)}{2}$ states, while the prefiltering estimator (6.2)-(6.4) has states $\tilde{A}$, $\tilde{b}$, $\hat{\theta}$, and Σ, which amounts to $(n \cdot r)^2 + n + r + \frac{r(r+1)}{2}$ states. The number of states for the prefiltering estimator can be brought down to $p + n + r + r^2$ in the case of a linear system (where p is the dimension of input vector u), by first separately filtering x and u, and then using the outputs of these filters to construct $\tilde{A}$ and $\tilde{b}$. Nonetheless, even in this special case, the dimension of the prefiltering estimator remains larger than that of the NPFSI estimator. In Section 8, it will be shown (in the context of a numerical example) that the NPFSI filter also exhibits better robustness in performance. □

7 AN EXTENSION TO PROBLEMS WITH TIME-VARYING PARAMETERS

An additional advantage of the FSDI and NPFSI methods over prefiltering is that they both can be applied to a wider class of identification problems which involve time-varying parameters. Consider for example the model:

$$\dot{\theta} = F\theta + Gd; \quad \theta(0) = \theta_o, \tag{7.1}$$

where F and G are matrices of appropriate dimensions, possibly time-varying, and d is a parametric disturbance. In this case, the overall system, state dynamics (2.1) and parameter dynamics (7.1), become

$$\left\{ \begin{array}{rcll} \dot{x} & = & A(x,u)\theta + b(x,u) + w, & x(0) = x_o \\ \dot{\theta} & = & F\theta + Gd, & \theta(0) = \theta_o \end{array} \right. . \tag{7.2}$$

For the FSDI case, the cost-to-come function is now defined as

$$W_\gamma(t, \theta; x_{[0,t]}, \hat{\theta}_{[0,t]}, u_{[0,t]}) := \sup_{(\theta_o, d_{[0,t]}) \in \Omega^{FSI}(t,\theta;x_{[0,t]},u_{[0,t]})} \left\{ \begin{array}{l} \int_0^t \left[|\theta(s) - \hat{\theta}(s)|_Q^2 - \gamma^2(|\dot{x} - A\theta - b|^2 + |d|^2 \right] ds \\ -\gamma^2 |\theta_o - \bar{\theta}_o|_{Q_o}^2 \end{array} \right\} \tag{7.3}$$

[5]The approximate NPFSI filter is of lower dimension than the NPFSI filter (5.6)-(5.9); its performance, however, is comparable to that of the NPFSI filter when ϵ is small, as shown in Theorem 7.

where

$$\Omega^{FSI}(t,\theta;x_{[0,t]},u_{[0,t]}) := \left\{(\theta_o, d_{[0,t]}) \mid \theta(t)=\theta\right\}.$$

As in the earlier case, provided that this cost-to-come function exists (i.e., it is finite), it can be rewritten in the simpler quadratic form

$$W_\gamma(t,\theta;x_{[0,t]},\hat{\theta}_{[0,t]},u_{[0,t]}) = -\gamma^2|\theta-\bar{\theta}(t)|^2_{\Sigma(t)} - m(t) \tag{7.4}$$

where

$$\begin{aligned}
\dot{\Sigma} &= -\Sigma F - F^T\Sigma - \Sigma G G^T \Sigma + A(x,u)^T A(x,u) - \frac{1}{\gamma^2}Q(x,u), \quad \Sigma(0)=Q_o && (7.5)\\
\dot{\bar{\theta}} &= F\bar{\theta} + \Sigma^{-1}A(x,u)^T\left(\dot{x} - A(x,u)\bar{\theta} - b(x,u)\right) + \frac{1}{\gamma^2}\Sigma^{-1}Q(\bar{\theta}-\hat{\theta}), \quad \bar{\theta}(0)=\bar{\theta}_o && (7.6)\\
\dot{m} &= -|\bar{\theta}-\hat{\theta}|^2_{Q(x,u)} + \gamma^2|\dot{x}-A(x,u)\bar{\theta}-b(x,u)|^2, \quad m(0)=0 && (7.7)
\end{aligned}$$

Note that, in contrast with (3.4), Σ here satisfies a matrix Riccati equation, instead of a Lyapunov equation.

For the NPFSI case, the cost-to-come function becomes

$$W_\gamma(t,x,\theta;\hat{\theta}_{[0,t)},u_{[0,t)},y_{[0,t]}) :=$$

$$\sup_{x_o,w_{[0,t]},d_{[0,t]}\in\Omega^{NPFSI}(t,x,\theta;u_{[0,t)},y_{[0,t]})}\left\{\begin{array}{l}\int_0^t [|\theta(s)-\hat{\theta}(s)|^2_Q - \gamma^2(|w(s)|^2+|d(s)|^2\\ \quad +\frac{1}{\epsilon^2}|y(s)-x(s)|^2)]ds - \gamma^2(|\theta_o-\bar{\theta}_o|^2_{Q_o}\\ \quad +|x_o-\bar{x}_o|^2_{P_o})\end{array}\right\} \tag{7.8}$$

where

$$\Omega^{NPFSI}(t,x,\theta;u_{[0,t)},y_{[0,t]}) := \left\{(x_o,w_{[0,t]},d_{[0,t]}) \mid x(t)=x,\ \theta(t)=\theta\right\}.$$

If finite, this cost-to-come function admits a solution in quadratic form:

$$W_\gamma(t,x,\theta) = -\gamma^2\left|\begin{bmatrix} x\\ \theta\end{bmatrix} - \begin{bmatrix}\bar{x}(t)\\ \bar{\theta}(t)\end{bmatrix}\right|^2_{\Sigma(t)} - m(t), \tag{7.9}$$

where

$$\dot{\Sigma} = -\Sigma\begin{bmatrix}0 & A\\ 0 & F\end{bmatrix} - \begin{bmatrix}0 & 0\\ A^T & F^T\end{bmatrix}\Sigma + \begin{bmatrix}\frac{1}{\epsilon^2}I & 0\\ 0 & -\frac{1}{\gamma^2}Q(y,u)\end{bmatrix} - \Sigma\begin{bmatrix}I & 0\\ 0 & GG^T\end{bmatrix}\Sigma, \tag{7.10}$$

$$\begin{aligned}
\begin{bmatrix}\dot{\bar{x}}\\ \dot{\bar{\theta}}\end{bmatrix} &= \begin{bmatrix}0 & A\\ 0 & F\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{\theta}\end{bmatrix} + \begin{bmatrix}b\\ 0\end{bmatrix} + \frac{1}{\epsilon^2}\Sigma^{-1}\begin{bmatrix}I\\ 0\end{bmatrix}(y-\bar{x}) + \frac{1}{\gamma^2}\Sigma^{-1}\begin{bmatrix}0\\ Q\end{bmatrix}(\bar{\theta}-\hat{\theta}), && (7.11)\\
\dot{m} &= -|\bar{\theta}-\hat{\theta}|^2_{Q(y,u)} + \frac{\gamma^2}{\epsilon^2}|y-\bar{x}|^2, \quad m(0)=0. && (7.12)
\end{aligned}$$

The initial conditions for the Riccati equation (7.10) and estimator dynamics (7.11) are

$$\Sigma(0) = \begin{bmatrix} \frac{1}{\epsilon}I & 0 \\ 0 & Q_o \end{bmatrix} \text{ and } \begin{bmatrix} \bar{x}(0) \\ \bar{\theta}(0) \end{bmatrix} = \begin{bmatrix} \bar{x}_o \\ \bar{\theta}_o \end{bmatrix},$$

respectively.

In both cases, the dynamics of Σ are too complicated to permit us to make statements as strong and conclusive as those in Theorems 1, 2 and 3. However, still the following useful result can be obtained, which parallels those of Theorems 1 and 3:

Proposition 1 *For both the FSDI and NPFSI problems defined in this section, the following statements are true:*

(i) *If there exists a $\gamma > 0$ such that $\Sigma_\gamma(t) > 0$ for all $t \in \mathbf{T}$, then γ^* is finite.*

(ii) *If $\Sigma_\gamma(t)$ is not nonnegative definite for all $t \in \mathbf{T}$, then either γ^* is infinite or $\gamma^* > \gamma$.*

(iii) *If γ^* is finite, then for every $\gamma > \gamma^*$, and each $t \in \mathbf{T}$, $\delta(t, \eta_t) = \hat{\theta}_\gamma(t)$ is a minimax policy, where $\hat{\theta}_\gamma$ is generated by (7.6) or (7.11), by setting $\hat{\theta} = \bar{\theta}$.*

□

Remark 10 The reason why we cannot make a stronger statement than that of Proposition 1 above is because the condition $\Sigma_\gamma \geq 0$ could be sufficient for the maximum of W_γ over all θ to exist in FSDI case, and a similar maximum to exist over all (x, θ) in the NPFSI case. In the case of constant unknown parameters (cf. Theorem 1), we had shown that $\Sigma_\gamma \geq 0$, for all$\gamma > \gamma^*$, and has at least one zero eigenvalue, only at $\gamma = \gamma^*$. Here, however, such a property may not hold. □

Performance Analysis

In the general case of time-varying parameters, it is harder to carry out the analysis on the relationship between the existence of a minimax FSDI identification scheme and NPFSI identification scheme. This can be done, however, at least partially, for a subset of the class (7.2), as discussed below.

In Section 5, we had shown that in the NPFSI case, Σ_γ, solution of equation (5.6), is positive definite if, and only if, Π_γ is positive definite. The latter was shown to be positive definite whenever $\gamma > \gamma^*$ for a sufficiently rich class of measurement waveforms $y_{[0,\infty)}$. A similar result can obtained in the time-varying parameters case when $G = 0$ in (7.1).

In this case, the solution of the Riccati differential equation (7.10), Σ, can be decomposed as

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix}$$

where

$$\dot{\Sigma}_1 = \frac{1}{\epsilon^2}I - \Sigma_1\Sigma_1, \quad \Sigma_1(0) = \frac{1}{\epsilon}I \tag{7.13}$$

$$\dot{\Sigma}_2 = -\Sigma_1 A - \Sigma_2 F - \Sigma_1\Sigma_2, \quad \Sigma_2(0) = 0 \tag{7.14}$$

$$\dot{\Sigma}_3 = -\Sigma_2^T A - \Sigma_3 F - A^T\Sigma_2 - F^T\Sigma_3 - \Sigma_2^T\Sigma_2 - \frac{1}{\gamma^2}Q, \quad \Sigma_3(0) = Q_o \tag{7.15}$$

By observation, $\Sigma_1 \equiv \frac{1}{\epsilon}I$, which implies that $\Sigma > 0$, if, and only if, $\Pi := \Sigma_3 - \epsilon\Sigma_2^T\Sigma_2 > 0$. Similar to the constant parameters case, Π is the solution of the following linear matrix equation:

$$\dot{\Pi} = -\Pi F - F^T\Pi + \Sigma_2^T\Sigma_2 - \frac{1}{\gamma^2}Q, \quad \Pi(0) = Q_o, \tag{7.16}$$

which carries an obvious similarity to the Σ equation in the FSDI case. Therefore, as in the constant parameter case, we can show that whenever there exists a solution to the FSDI problem, there will exist a solution to a modified NPFSI problem for a sufficiently large class of possible output waveforms.

Asymptotic Convergence

Studying the convergence properties of the identification schemes (7.5)-(7.6) or (7.10)-(7.11) in the time-varying setting is now generally more complicated. This is due to the fact that it does not readily follow from the PE condition that $\lambda_{\min}(\Sigma_\gamma(T)) \uparrow \infty$ as $T \to \infty$, unless we restrict our attention to certain classes of time-varying parameters.

One such class, which permits us to make conclusive statements, is the case of time-varying parameters that asymptotically converge to constant values. This can be captured as a special case of (7.1), where now the parameter dynamics are:

$$\dot{\theta} = \begin{bmatrix} \bar{F} & I \\ 0 & 0 \end{bmatrix}\theta, \qquad \theta(0) = \theta_o, \tag{7.17}$$

where $\bar{F}$ is a constant Hurwitz matrix. We assume here a natural partitioning of the parameter vector $\theta = (\theta_1, \theta_2)$, where only θ_1 affects the x-dynamics (2.1), i.e., $A(x,u) = [\bar{A}(x,u)\ 0]$.

For this case, the dynamics of Σ in the FSDI case (7.5) become

$$\dot{\Sigma} = -\Sigma F - F^T\Sigma + A(x,u)^T A(x,u) - \frac{1}{\gamma^2}Q(x,u) \tag{7.18}$$

Letting $\Phi(t)$ denote the state transition matrix associated with the linear matrix equation (7.18), i.e.,

$$\dot{\Phi} = -F^T\Phi, \quad \Phi(0) = I,$$

we have

$$\Sigma_\gamma(T) = \Phi(T)Q_o\Phi(T)^T + \int_0^T \Phi(T-t)(A^T A - \frac{1}{\gamma^2}Q)\Phi(T-t)^T dt. \tag{7.19}$$

From the above equation, it can again be concluded that Σ_γ will be positive semi-definite, with at least one zero eigenvalue, only at $\gamma = \gamma^*$. Hence, for this case of time-varying parameters, a statement similar to that of Theorem 1 will still hold. Similarly, Corollary 1 holds, if we again choose $Q = A^T A$.

To analyze the convergence properties of $\hat{\theta}_1$ (since only $\hat{\theta}_1$ affects the state dynamics), we observe that the $(1,1)$-block of Σ_γ, denoted $\Sigma_\gamma^{(1,1)}$, satisfies the Lyapunov equation:

$$\dot{\Sigma}_\gamma^{(1,1)} = -\Sigma_\gamma^{(1,1)}\bar{F} - \bar{F}^T\Sigma_\gamma^{(1,1)} + \bar{A}^T\bar{A} - \frac{1}{\gamma^2}Q^{(1,1)} \tag{7.20}$$

where $Q^{(1,1)}$ is the $(1,1)$-block of matrix Q. Denoting by $\Phi_\gamma^{(1,1)}(t)$ the associated state transition matrix, it follows that

$$\begin{aligned}\Sigma_\gamma^{(1,1)}(T) = \Phi_\gamma^{(1,1)}(T)Q_o^{(1,1)}\Phi_\gamma^{(1,1)}(T)^T + \int_0^T \Phi_\gamma^{(1,1)}(T-t)\bar{A}^T\bar{A}\Phi_\gamma^{(1,1)}(T-t)^T dt \\ -\frac{1}{\gamma^2}\int_0^T \Phi_\gamma^{(1,1)}(T-t)Q^{(1,1)}\Phi_\gamma^{(1,1)}(T-t)^T dt.\end{aligned} \tag{7.21}$$

Hence, using the same line of proof as in the case of Theorem 2, it follows that if all disturbances belong to $\mathcal{L}_2$, and the following counterpart of the PE condition (4.1):

$$\lim_{T\to\infty} \lambda_{\min}\left(\int_0^T \Phi_\gamma^{(1,1)}(T-t)Q^{(1,1)}\Phi_\gamma^{(1,1)}(T-t)^T dt\right) = \infty, \tag{7.22}$$

holds, then $\bar{\theta}_1 \to \theta_1$.

Remark 11 The new PE condition (7.22) is less restrictive than the old one (4.1). This is due to the fact that as $\bar{F}$ is Hurwitz, $-\bar{F}$ is anti-stable, thus making $\Phi_\gamma^{(1,1)}$ grow exponentially at a rate which is no smaller than $\lambda_{\min}(-\bar{F})$. For the case $\bar{F} = \lambda I$, $\lambda < 0$, for example, condition (7.22) becomes

$$\lim_{T\to\infty} \lambda_{\min}\left(\int_0^T e^{-2\lambda(T-t)}Q^{(1,1)}dt\right) = \infty,$$

which is clearly strictly less restrictive than condition (4.1). □

Reduced-order identifier

Another special case in the NPFSI problem that is amenable to analysis is when $F = 0$ and $G = I$. First, we obtain a reduced-order identifier $\hat{\delta}(t, u_{[0,t]}, y_{[0,t]})$ corresponding to the full-order identifier (7.10)–(7.11):

$$\hat{\delta}(t, u_{[0,t]}, y_{[0,t]}) = \hat{\theta}_\gamma(t) \tag{7.23}$$

$$\dot{\hat{x}} = A(y,u)\hat{\theta}_\gamma + b(y,u) + \frac{1}{\epsilon}(y - \hat{x}); \quad \hat{x}(0) = \bar{x}_o \tag{7.24}$$

$$\dot{\hat{\theta}}_\gamma = \frac{1}{\epsilon}\Sigma_\gamma^{F^{-1}}A(y,u)^T(y - \hat{x}); \quad \hat{\theta}_\gamma(0) = \bar{\theta}_o \tag{7.25}$$

where $\Sigma_\gamma^{F^{-1}}$ satisfies the following Riccati differential equation:

$$\frac{d}{dt}(\Sigma_\gamma^{F^{-1}}) = -\Sigma_\gamma^{F^{-1}}(A(y,u)^T A(y,u) - \frac{1}{\gamma^2}Q(y,u))\Sigma_\gamma^{F^{-1}} + I; \qquad \Sigma_\gamma^{F^{-1}}(0) = Q_o^{-1}. \quad (7.26)$$

This approximate filter can be shown to attain the disturbance attenuation level γ for the modified NPFSI problem for sufficiently small $\epsilon > 0$ for the following set of output waveforms:

$\mathcal{LBP}_{EPW}^{ATV}(L, C, \bar{\rho}, T_m, K)$: This is the set of all measurement waveforms $y_{[0,\infty)}$ that satisfy the following four conditions for some constants L, C, T_m and integer K:

1. There exist K $(< \infty)$ time instances $0 < t_1 < \cdots < t_K < \infty$, ($t_K < T$ in the finite-horizon case) such that
$$\min_{k\in\{1,\ldots,K\}} t_k - t_{k-1} \geq T_m.$$
2. $\|A(y(t')) - A(y(t))\|_2 \leq L|t' - t|, \quad \forall t', t \in [0, t_1]$, or $\forall t', t \in [t_k, t_{k+1}], k = 1, \ldots, K-1$, or $\forall t', t \in [t_K, \infty)$.
3. $\|A(y(t))\|_2 \leq C$ for all $t \in \mathbf{T}$.
4. $\Sigma_\gamma^F(t) \geq \bar{\rho}(t+1)I$, for all $t \in \boldsymbol{T}$ (or $t \in [0, T]$ in the finite horizon case).

This is summarized in the following theorem:

Theorem 9 *Consider the finite- and infinite-horizon modified NPFSI problems, and let $\gamma > \gamma^*$. Then, there exists an $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, the filter $\hat{\delta}$ defined by (7.23)–(7.26) achieves the disturbance attenuation level γ for the modified NPFSI problem with the measurement set $\mathcal{LBP}_{EPW}^{ATV}(L, C, \bar{\rho}, T_m, K)$.*

Proof. See Appendix B. □

8 AN EXAMPLE

To illustrate the results obtained in the previous sections, we will now consider a second-order dynamic system with two unknown parameters θ_1 and θ_2:

$$\dot{x}_1 = x_2 + w_1 \quad (8.1)$$
$$\dot{x}_2 = -\theta_1 x_1 - (\theta_1 + 1)x_2 + \theta_2 u + w_2 \quad (8.2)$$

We will first consider the FSDI problem, i.e., we will assume that both $x := (x_1, x_2)^T$ and its derivative $\dot{x}$ are available to the estimator. Thus, the problem can be rewritten in the standard form as (2.1) with

$$A(x,u) = \begin{bmatrix} 0 & 0 \\ -x_1 - x_2 & u \end{bmatrix}; \qquad b(x,u) = \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix}.$$

The performance index is taken to be

$$L(\delta) = \sup_{\theta_o, w_{[0,\infty)}} \frac{\|\theta - \delta\|^2_{A^TA}}{\|w\|^2 + |\theta_o - \bar{\theta}_o|^2} \tag{8.3}$$

By Corollary 1, the optimal disturbance attenuation level for this filtering problem is $\gamma^* = 1$. To compare the robustness properties of various minimax identifiers, we chose four different values of γ: 1, 1.1, 1.2 and ∞. Note that when $\gamma = \infty$, the corresponding filter is the well-known LS filter, and when $\gamma = 1$, the filter is the LMS filter. We chose the true values for the two unknown parameters to be

$$\theta_1^* = 5; \qquad \theta_2^* = 3.$$

The initial conditions for the system and the identifier are arbitrarily chosen to be

$$\hat{\theta}_1(0) = 3; \qquad \hat{\theta}_2(0) = 2; \qquad x_1(0) = 2; \qquad x_2(0) = 1.$$

Three simulations are presented here, involving the maximum eigenvalue of Σ^{-1} and the estimates of the two unknown parameters. The three cases are:

1. No disturbance, but a sinusoidal control input:

$$u(t) = 2\sin(10t + 1); \quad w_1(t) = 0; \quad w_2(t) = 0$$

The results are shown in Figure 1.

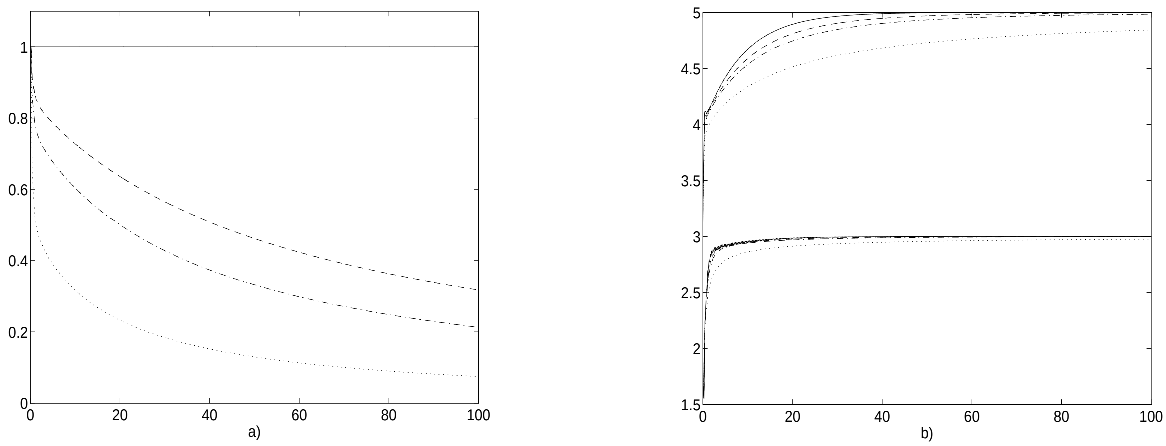

Figure 1: FSDI identifiers without disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\gamma = \infty$ case; dashdot lines for the $\gamma = 1.2$ case; dash lines for the $\gamma = 1.1$ case; solid lines for the $\gamma = 1$ case.

2. Both disturbances (w_1 and w_2) are band-limited white noises with sample rate 1 and covariance 0.1. The simulation results are illustrated in Figure 2.

3. All the inputs to the system are sine waves:

$$u(t) = 2\sin(t); \quad w_1(t) = 0.1\sin(3t+1); \quad w_2(t) = 0.2\sin(5t+2)$$

   The simulation results are plotted in Figure 3.

From these three simulations, it can be observed that the filter with a smaller γ appears to have a faster convergence rate. Note, however, that the estimates generated by the LMS filter, which is the one with $\gamma = 1$, do not converge as $t \to \infty$ when there is noise present in the system, since the observer gain does not vanish for this filter. Although LMS filter is the best in terms of disturbance attenuation, it is not the best choice when the objective is to obtain convergent parameter estimates. It appears that the filter with a γ value close to γ^* (but not equal to it) performs the best overall.

Next, we consider the NPFSI case. The system is approximated by the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -y_1 - y_2 & u \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_2 \\ -y_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{8.4}$$

where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Hence, the regressor in this case is

$$A = \begin{bmatrix} 0 & 0 \\ -y_1 - y_2 & u \end{bmatrix}$$

The performance index is taken to be

$$L(\delta) = \sup_{x_o, \theta_o, w_{[0,\infty)}} \frac{\|\theta - \delta\|^2_{A^T A}}{\|w\|^2 + \|v\|^2 + |\theta_o - \bar{\theta}_o|^2_I + |x_o - \bar{x}_o|^2_{\frac{1}{\epsilon}I}} \tag{8.5}$$

The optimal disturbance attenuation level is again $\gamma^* = 1$, which can be achieved as $\epsilon \to 0$, according to Theorem 3. Hence, we fix the γ level to be 1.2. The true values of the parameters and the initial conditions for x and $\hat{\theta}$ are taken to be the same as in the FSI case. We take the initial conditions for $\hat{x}_1$ and $\hat{x}_2$ to be 2 and 1, respectively, which are exactly the same as the initial conditions for the plant, and simulate the filter response for two values of ϵ: 0.05 and 0.01.

For the full-order NPFSI filter (5.6),(5.9), we have the following three simulations:

1. First, we simulate the system response without any disturbances but with a sinusoidal control input:

$$u(t) = 2\sin(10t+1)$$

   The results are shown in Figure 4.

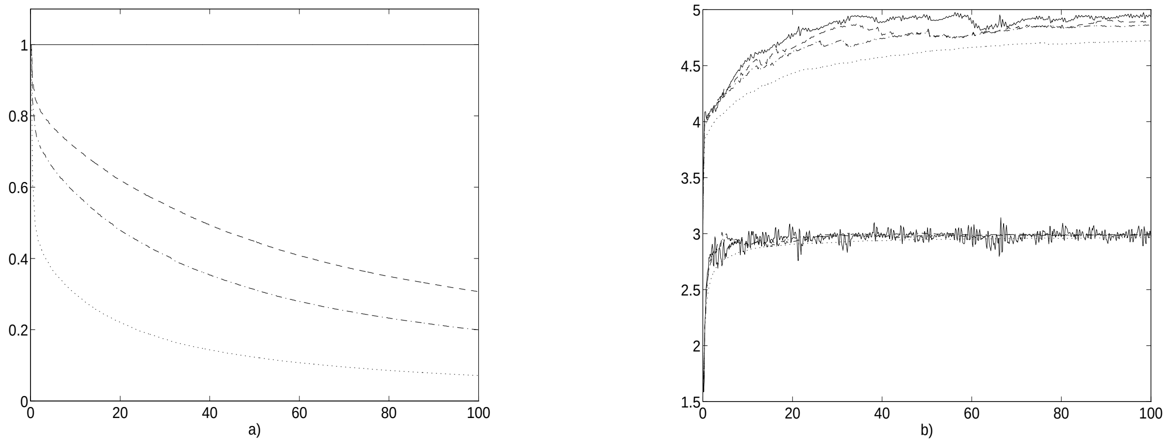

Figure 2: FSDI identifiers with white noise disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\gamma = \infty$ case; dashdot lines for the $\gamma = 1.2$ case; dash lines for the $\gamma = 1.1$ case; solid lines for the $\gamma = 1$ case.

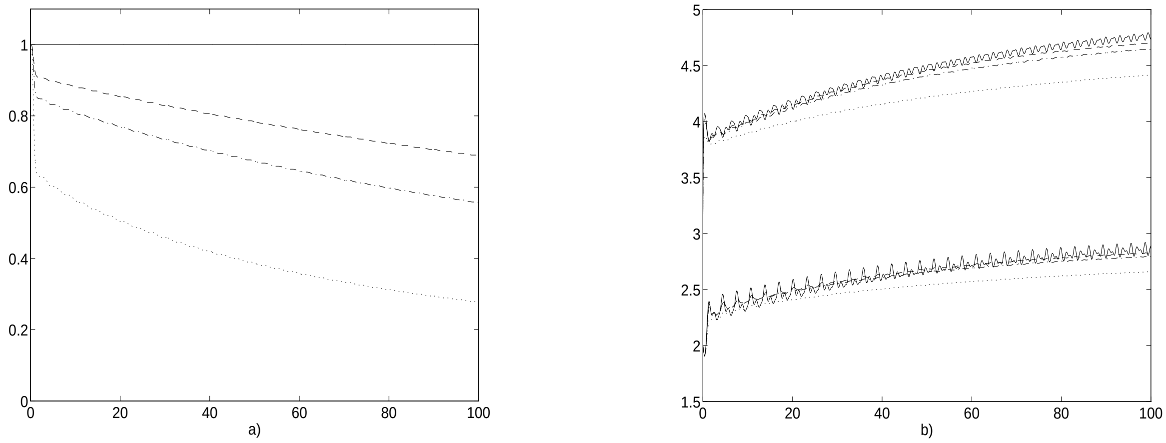

Figure 3: FSDI identifiers with sine wave inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\gamma = \infty$ case; dashdot lines for the $\gamma = 1.2$ case; dash lines for the $\gamma = 1.1$ case; solid lines for the $\gamma = 1$ case.

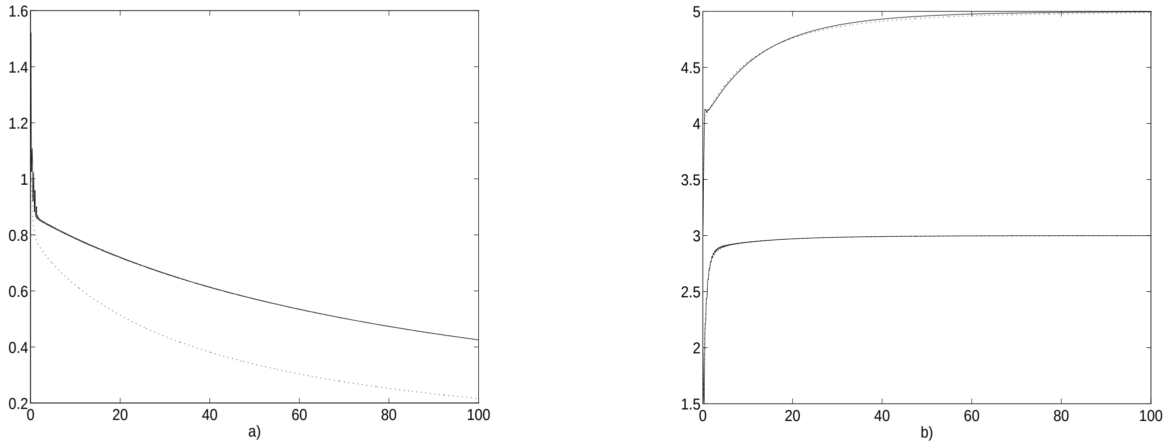

Figure 4: NPFSI identifiers without disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\epsilon = 0.01$ case; solid lines for the $\epsilon = 0.05$ case.

2. Next, all disturbance inputs (w_1, w_2, v_1 and v_2) are taken as band-limited white noises with sample rate 1 and covariance 0.1. The simulation results are illustrated in Figure 5.

3. Finally, we take all the inputs to the system to be sine waves of different frequencies:

$$u(t) = 2\sin t; \qquad w_1(t) = 0.1\sin(3t+1); \qquad w_2(t) = 0.2\sin(5t+2);$$

$$v_1(t) = 0.1\sin(6t+1); \qquad v_2(t) = 0.1\sin(0.5t+4)$$

   The simulation results are plotted in Figure 6.

All three figures indicate that the case $\epsilon = 0.01$ results in a performance that is closer to that of the corresponding FSDI identifiers.

Next, we simulated the reduced-order identifier $\bar{\delta}$ defined by (5.23)–(5.25) for the system under the same three sets of different control and disturbance inputs:

1. No disturbance, but a sinusoidal control input:

$$u(t) = 2\sin(10t+1)$$

   The results are shown in Figure 7.

2. The disturbances are band limited white noises with sample rate 1 and covariance 0.1. The simulation results are depicted in Figure 8.

3. All the inputs to the system are taken to be sine waves as in Figure 6. The simulation results are plotted in Figure 9.

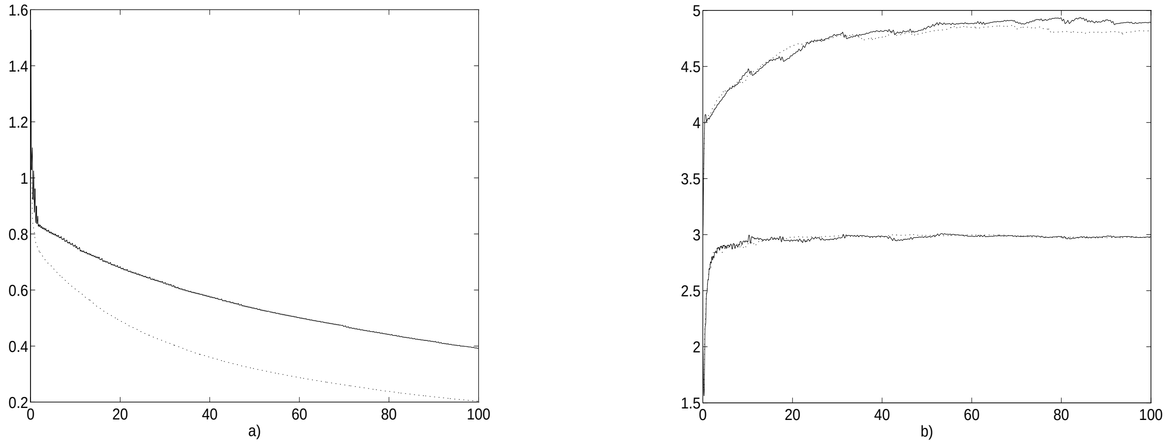

Figure 5: NPFSI identifiers with white noise disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\epsilon = 0.01$ case; solid lines for the $\epsilon = 0.05$ case.

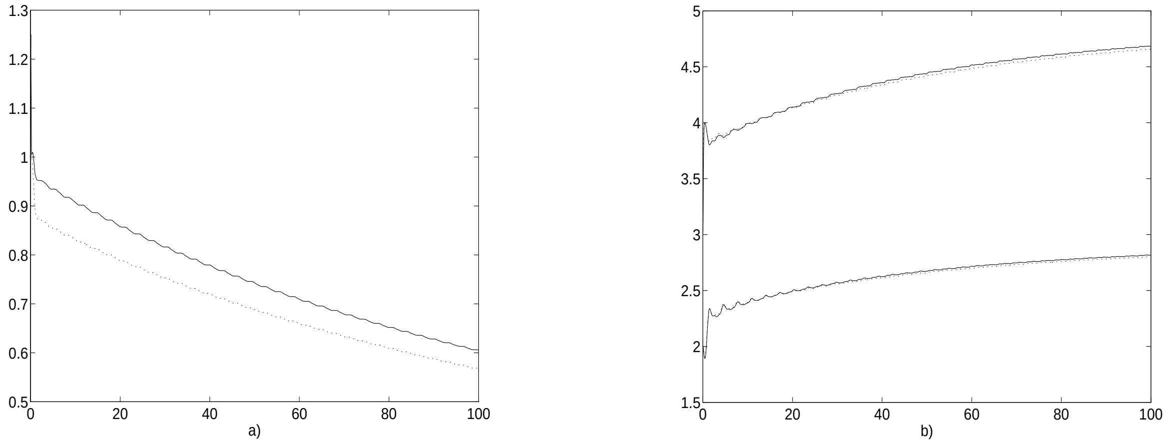

Figure 6: NPFSI identifiers with sinusoidal disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\epsilon = 0.01$ case; solid lines for the $\epsilon = 0.05$ case.

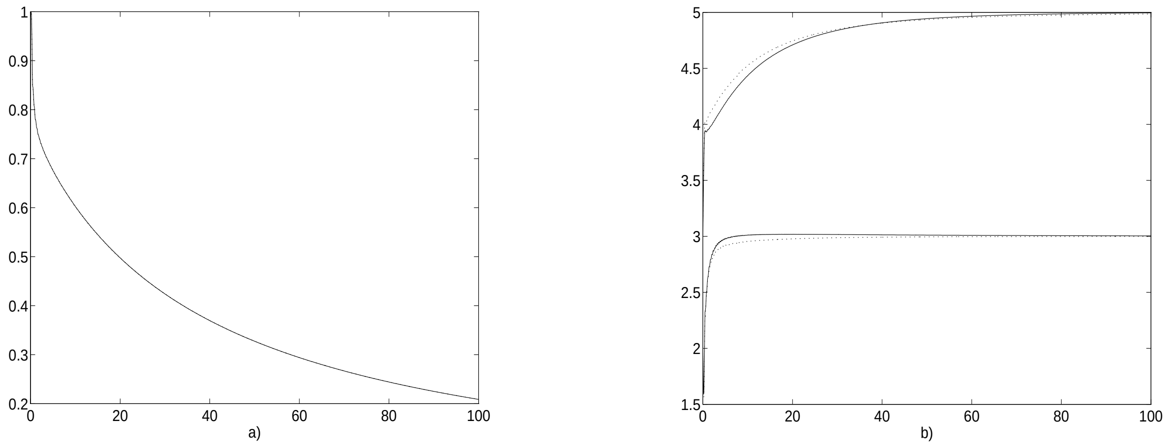

Figure 7: NPFSI reduced-order identifiers without disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\epsilon = 0.01$ case; solid lines for the $\epsilon = 0.05$ case.

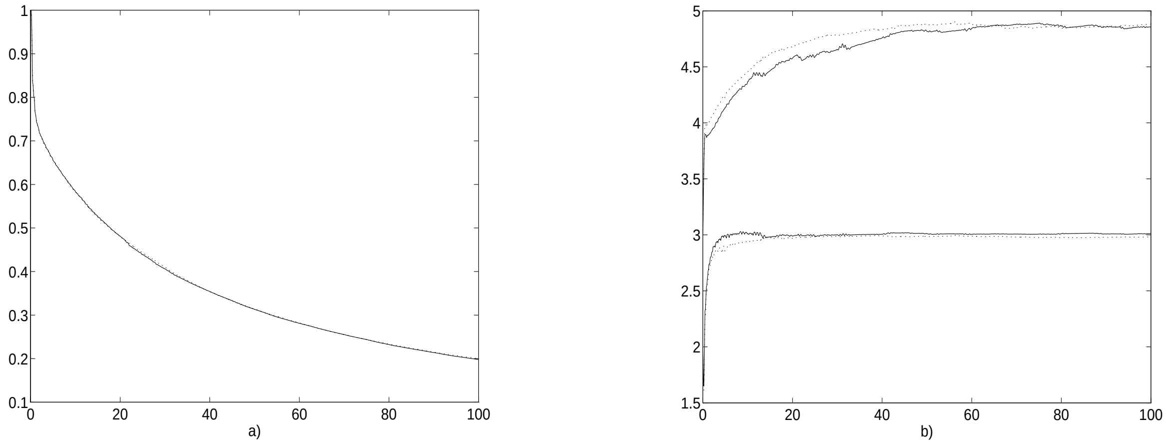

Figure 8: NPFSI reduced-order identifiers with white noise disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\epsilon = 0.01$ case; solid lines for the $\epsilon = 0.05$ case.

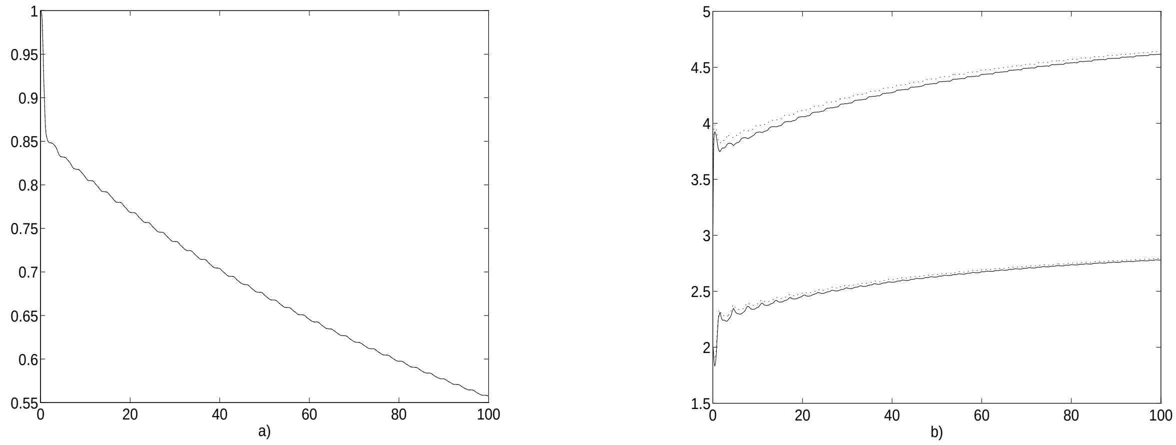

Figure 9: NPFSI reduced-order identifiers with sinusoidal disturbance inputs
a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\epsilon = 0.01$ case; solid lines for the $\epsilon = 0.05$ case.

Again, all figures indicate that the case $\epsilon = 0.01$ results in a performance that is closer to that of the corresponding FSI filter. Compared with the NPFSI full-order filter, there is some difference for the parameter estimates in the case $\epsilon = 0.05$. But, in the case $\epsilon = 0.01$, the differences are negligible.

Finally, we consider the prefiltering identification scheme. The best performance of the identifier can be achieved by using very small prefilter parameter λ when there is no disturbance inputs. But, with band-limited white noise inputs, better performance is observed with larger values of λ. Hence, the parameter λ was chosen to be 0.1. The performance index is taken to be

$$L(\delta) = \sup_{\theta_o, \omega_{[0,\infty)}} \frac{\|\theta - \delta\|^2_{\tilde{A}^T\tilde{A}}}{\|\omega\|^2 + |\theta_o - \bar{\theta}_o|^2} \tag{8.6}$$

The optimal disturbance attenuation level is again $\gamma^* = 1$. The true values of the parameters and the initial conditions for x and $\hat{\theta}$ are taken to be the same as in the FSDI case. We simulate the system response for two different values of γ: 1.1 and ∞ with three different sets of inputs as in the FSDI identifier case.

1. Simulation results for the system response without any disturbances are shown in Figure 10.

2. Next, we take all disturbance inputs to be band-limited white noises with sample rate 1 and covariance 0.1. The simulation results are illustrated in Figure 11.

3. Finally, we take all the inputs to the system to be sine waves of different frequencies as for the FSDI identifier. The simulation results are plotted in Figure 12.

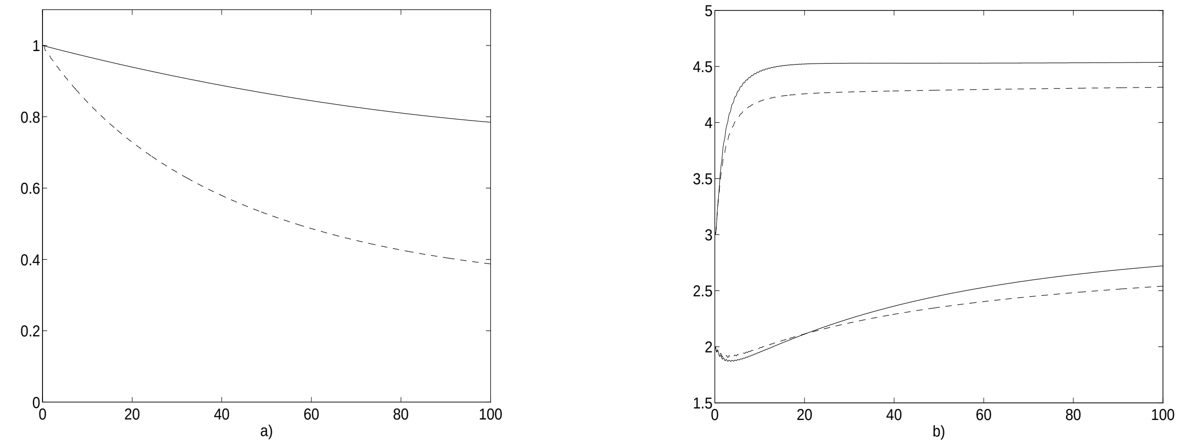

Figure 10: Prefiltering identifier without disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\gamma = \infty$ case; solid lines for the $\gamma = 1.1$ case.

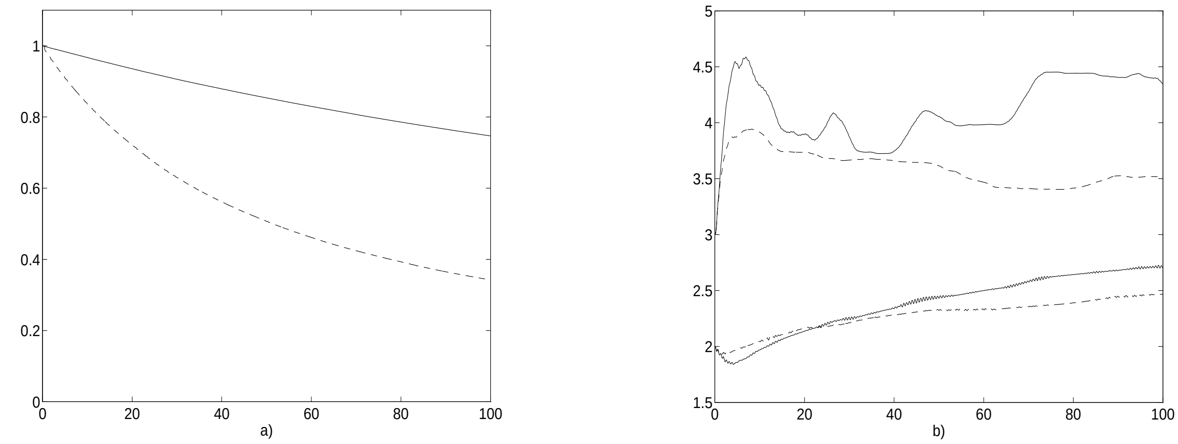

Figure 11: Prefiltering identifier with white noise disturbance inputs

a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\gamma = \infty$ case; solid lines for the $\gamma = 1.1$ case.

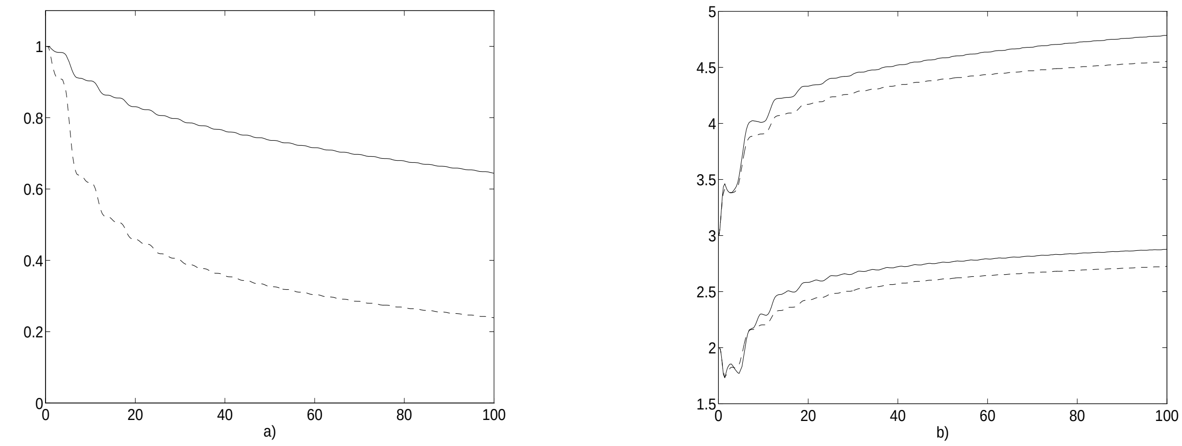

Figure 12: Prefiltering identifier with sinusoidal disturbance inputs
a) The maximum eigenvalue of $\Sigma^{-1}(t)$. b) The estimates of the unknown parameters. Dotted lines for the $\gamma = \infty$ case; solid lines for the $\gamma = 1.1$ case.

We observe that the identification is much slower than those of the FSDI, NPFSI and NPFSI reduced-order identifiers when the system does not have any disturbance inputs. When the system is subject to additional disturbance inputs, the identifier may perform worse (for white noise disturbances) or marginally better (for sinusoidal disturbances) than those identifiers. Overall, the performance of the prefilter based identifier for a fixed value of λ depends very much on the specific disturbance inputs. Hence, we conclude that the NPFSI identifier and the NPFSI reduced-order identifier are clearly better than the prefilter based identifier, since they can be designed to yield excellent performance under a much larger classes of disturbances.

9 CONCLUDING REMARKS

We have shown the effective use of H^∞-based techniques in optimal/robust parameter identification, for both constant and time-varying parameters, and have applied them to three classes of problems: FSDI, NPFSI and prefilter-based identification. As the performance criterion, we have chosen the least worst-case $\mathcal{L}_2$ attenuation of the disturbance. We have obtained a set of identification schemes which are parametrized by accessible attenuation levels $\gamma > \gamma^*$. In the FSDI case with constant parameters, where the derivatives of the state variables are available for identification, these estimators range from the well-known LS estimator ($\gamma = \infty$) to the generalized LMS estimator ($\gamma = \gamma^*$) for certain choices of weighting functions, where under the optimality criterion selected, LMS is more robust. This conclusion is also corroborated by the simulations performed on specific examples.

The NPFSI problem is formulated as a noise-perturbed state-measurement problem. The effect of the fictitious noise perturbation is captured by the (positive small) parameter ϵ. For the case of constant parameters, we have shown that as ϵ takes smaller values approaching zero, then the best achievable attenuation level of the NPFSI estimator, as well as the reduced-order NPFSI estimator, approaches that of the corresponding FSDI estimator. This limiting behavior was also corroborated by our simulations. The approximate NPFSI filter actually showed robustness and better convergence properties against a larger class of disturbances. It has the potential to provide a fairly accurate approximation to measurement schemes that contain the derivative of the state in other contexts also, such as adaptive control.

Using a conventional approach to design estimators which are independent of the derivatives of the state variables, prefiltering based minimax identification led to a class of estimators parametrized by the desired performance level $\gamma > 0$. In this case, the identification problem was formulated so as to attenuate the filtered disturbance signals. Obviously, when the original disturbances are $\mathcal{L}_2$, so will be the filtered signals. But the performance of these identification schemes turn out to be generally worse than those of the NPFSI identifier, as well as the reduced-order NPFSI identifier.

Last, we have made some investigation of problems with time-varying parameters. We have shown that even though they are generally harder to analyze, still there exist certain subclasses of time-varying problems for which we can make conclusive statements, regarding both robustness and convergence. One such class of problems is one with convergent parameters, and yet another one is the class of problems with noise driven parameters; both have been discussed in the paper.

It is important to note that the full-order and reduced-order identifiers obtained in this paper provide conservative, but nevertheless robust, designs for all problems which can be viewed as a subclasses of the NPFSI problem formulated in Section 5. One such class is one where in the systems dynamics (5.1) w is replaced by Dw where $D'D = I$. This would characterize a situation where it is known with certainty that some components of the system state are not corrupted by noise. Since choosing either all or some components of w to be zero is an admissible choice for the disturbance, the minimax identifiers obtained in Section 5 (as well as in Section 6 and 7) and their properties are equally valid for this partial case also. Likewise, by choosing $v = 0$ in the measurement equation, the NPFSI problem can be turned into an FSI problem, and since again $v = 0$ is an admissible choice for the measurement disturbance, the identifiers obtained provide robust designs for the FSI problem as well, as already shown through the singular perturbation analysis of Section 5.

An important application of the identification schemes developed in this paper would be in the (adaptive) control of uncertain systems of the type described by (2.1) where, for example, the control input enters linearly. The NPFSI identifier could then be used alongside a certainty-equivalent controller obtained, for example, using an optimal control or H^∞-control approach with full knowledge of the state and the unknown parameters. The preliminary results presented in Başar, Didinsky and Pan (1994) in the context of specific control examples indicate that with such an identifier in the control loop, the system shows superior performance in the presence of additive disturbances, as compared with other competing adaptive-control based certainty-equivalent controllers. Further research would be needed to investigate the performance of the NPFSI identifier in the control loop, and in particular whether the control signals thus generated would lead to satisfaction of the PE condition, as well as the other various conditions introduced in this paper.

Acknowledgment

Authors would like to thank Prof. Petar Kokotović, Prof. Sean Meyn, and R. Ravikanth for useful discussions on the topic of this paper.

A Proof of Theorem 7

We will prove the theorem in the infinite-horizon case; the extension to the finite-horizon case will then be straightforward.

Let $\xi := (x^T, \theta^T)^T$ and $\hat{\xi} := (\hat{x}^T, \hat{\theta}^T)^T$. Define $\tilde{\xi} := \xi - \hat{\xi}$. Then, $\tilde{\xi}$ satisfies the following differential equation:

$$\dot{\tilde{\xi}} = \begin{bmatrix} 0 & A(y,u) \\ 0 & 0 \end{bmatrix} \tilde{\xi} - \frac{1}{\epsilon} \begin{bmatrix} I \\ {\Sigma_\gamma^F}^{-1} A(y,u)^T \end{bmatrix} (y - \hat{x}) + \begin{bmatrix} I \\ 0 \end{bmatrix} w; \quad \tilde{\xi}(0) = \begin{bmatrix} x_o - \bar{x}_o \\ \theta_o - \bar{\theta}_o \end{bmatrix}.$$

Using the approximate filter $\bar{\delta}$ in the cost function (5.3) results in the following expression:

$$\begin{aligned} \hat{J}(x_o, \theta_o, w_{[0,\infty)}, v_{[0,\infty)}) &:= J(\bar{\delta}; x_o, \theta_o, w_{[0,\infty)}, v_{[0,\infty)}) \\ &= -\gamma^2 |\tilde{\xi}_0|^2_{\begin{bmatrix} P_0 & 0 \\ 0 & Q_o \end{bmatrix}} + \int_0^\infty (|\tilde{\xi}|^2_{\begin{bmatrix} -\frac{\gamma^2}{\epsilon^2} I & 0 \\ 0 & Q(y,u) \end{bmatrix}} + \frac{2\gamma^2}{\epsilon^2} \tilde{\xi}^T \begin{bmatrix} y - \hat{x} \\ 0 \end{bmatrix} \\ &\quad - \gamma^2 |w|^2 - \frac{\gamma^2}{\epsilon^2} |y - \hat{x}|^2)\, dt \end{aligned} \tag{A.1}$$

Fix a $\gamma > \gamma^*$. To prove part (i) of the theorem, we need only show that

$$\sup_{(w_{[0,\infty)}, v_{[0,\infty)}, x_o, \theta_o) \in \mathcal{W}(\mathcal{LBP}_E^A(L,C,\bar{\rho}))} \hat{J} \le 0$$

For this, it will be sufficient to prove that

$$\sup_{(w_{[0,t]}, v_{[0,t]}, x_o, \theta_o) \in \mathcal{W}(\mathcal{LBP}_E^A(L,C,\bar{\rho}))} \hat{J}^t \le 0; \qquad \forall t \in \boldsymbol{T}$$

where $\hat{J}^t$ is defined as

$$\begin{aligned} \hat{J}^t(x_o, \theta_o, w_{[0,t]}, v_{[0,t]}) &:= -\gamma^2 |\tilde{\xi}_0|^2_{\begin{bmatrix} P_0 & 0 \\ 0 & Q_o \end{bmatrix}} + \int_0^t (|\tilde{\xi}|^2_{\begin{bmatrix} -\frac{\gamma^2}{\epsilon^2} I & 0 \\ 0 & Q(y,u) \end{bmatrix}} + \frac{2\gamma^2}{\epsilon^2} \tilde{\xi}^T \begin{bmatrix} y - \hat{x} \\ 0 \end{bmatrix} \\ &\quad - \gamma^2 |w|^2 - \frac{\gamma^2}{\epsilon^2} |y - \hat{x}|^2)\, dt. \end{aligned}$$

Define the cost-to-come function for the above nonlinear maximization problem as follows:

$$\hat{W}_\gamma(t,x,\theta;y_{[0,t]},u_{[0,t]}) := \sup_{(\theta_o,x_o,w_{[0,t]})\in\Omega(t,x,\theta)} \hat{J}^t$$

where

$$\Omega(t,x,\theta;\eta(t)) := \{(\theta_o,x_o,w_{[0,t]}) | x(t)=x, \theta(t)=\theta, (y_{[0,t]},u_{[0,t]})=\eta(t)\ \}.$$

For each $t\in\boldsymbol{T}$, we have the following identity:

$$\sup_{(w_{[0,\infty)},v_{[0,\infty)},x_o,\theta_o)\in\mathcal{W}(\mathcal{LBP}^A_E(L,C,\bar{\rho}))} \hat{J}^t = \sup_{y_{[0,\infty)}\in\mathcal{LBP}^A_E(L,C,\bar{\rho})} \sup_{x,\theta} \hat{W}_\gamma(t,x,\theta;y_{[0,t]},u_{[0,t]})$$

Introduce a function $V(t,x,\theta;y_{[0,t]},u_{[0,t]}) = \tilde{\xi}(t)^T\hat{\Sigma}_\gamma(t;y_{[0,t]},u_{[0,t]})\tilde{\xi}(t)$, where $\hat{\Sigma}_\gamma(t;y_{[0,t]},u_{[0,t]})$ is generated by the following Riccati differential equation:

$$\begin{aligned}\dot{\hat{\Sigma}}_\gamma + \hat{\Sigma}_\gamma \begin{bmatrix} -\frac{1}{\epsilon}I & A \\ -\frac{1}{\epsilon}\Sigma_\gamma^{F^{-1}}A^T & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{\epsilon}I & -\frac{1}{\epsilon}A\Sigma_\gamma^{F^{-1}} \\ A^T & 0 \end{bmatrix}\hat{\Sigma}_\gamma + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma^2}Q \end{bmatrix} \\ +\hat{\Sigma}_\gamma \begin{bmatrix} 2I & A\Sigma_\gamma^{F^{-1}} \\ -\Sigma_\gamma^{F^{-1}}A^T & \Sigma_\gamma^{F^{-1}}A^TA\Sigma_\gamma^{F^{-1}} \end{bmatrix}\hat{\Sigma}_\gamma = 0; \qquad \hat{\Sigma}_\gamma(0) = \begin{bmatrix} P_0 & 0 \\ 0 & Q_o \end{bmatrix},\end{aligned} \tag{A.2}$$

and note that using (A.2) and a "completion of squares" argument, $\hat{J}^t$ can be re-written as follows:

$$\begin{aligned}\hat{J}^t &= -\gamma^2\tilde{\xi}(t)^T\hat{\Sigma}_\gamma(t)\tilde{\xi}(t) - \gamma^2\int_0^t \Big(\frac{1}{\epsilon^2}\Big|y-x+\epsilon\begin{bmatrix} I & A\Sigma_\gamma^{F^{-1}}\end{bmatrix}\hat{\Sigma}_\gamma\tilde{\xi}\Big|^2 \\ &\quad + \Big|w-\begin{bmatrix} I & 0\end{bmatrix}\hat{\Sigma}_\gamma\tilde{\xi}\Big|^2\Big)\,dt.\end{aligned} \tag{A.3}$$

Then, we have the bound

$$\hat{W}_\gamma(t,x,\theta;y_{[0,t]},u_{[0,t]}) \le -\gamma^2 V(t,x,\theta;y_{[0,t]},u_{[0,t]})$$

Claim. *For $\gamma>\gamma^*$, there exists an $\epsilon_0>0$ such that the Riccati differential equation (A.2) admits a solution $\hat{\Sigma}_\gamma(t)>0$ on $\boldsymbol{T}$ for any $y_{[0,\infty)}\in\mathcal{LBP}^A_E(L,C,\bar{\rho})$, for $\epsilon\in(0,\epsilon_0]$. Furthermore, $\hat{\Sigma}_\gamma(t)$ can be approximated by :*

$$\hat{\Sigma}_\gamma(t) = \begin{bmatrix} \frac{1}{\epsilon}I + \hat{\Sigma}_{1b}(t/\epsilon) + O(\epsilon) & -A + O(\epsilon) + \hat{\Sigma}_{2b}(t/\epsilon) \\ -A^T + O(\epsilon) + \hat{\Sigma}_{2b}(t/\epsilon)^T & \Sigma_\gamma^F(t) + \bar{\rho}(t+1)O(\epsilon) \end{bmatrix}$$

uniformly on $t\in\boldsymbol{T}$ and $y_{[0,\infty)}\in\mathcal{LBP}^A_E(L,C,\bar{\rho})$, where $\hat{\Sigma}_{1b}$ and $\hat{\Sigma}_{2b}$ are some exponentially decaying (boundary layer) terms. □

Given the validity of the above claim (which is yet to be proved), for sufficiently small $\epsilon>0$, we have the inequality

$$\sup_{(w_{[0,\infty)},v_{[0,\infty)},x_o,\theta_o)\in\mathcal{W}(\mathcal{LBP}^A_E(L,C,\bar{\rho}))} \hat{J}^t \le -\inf_{y_{[0,\infty)}\in\mathcal{LBP}^A_E(L,C,\bar{\rho})} \inf_{x,\theta} \gamma^2 V(t,x,\theta;y_{[0,t]},u_{[0,t]}) \le 0,$$

which completes the proof of part (i).

To prove part (ii), we first rewrite the expression for V as follows, by making use of (A.2) and the definition of $\hat{J}^t$:

$$V(t,x,\theta;y_{[0,t]},u_{[0,t]}) = \tilde{\xi}(0)^T\hat{\Sigma}_\gamma(0)\tilde{\xi}(0) + \int_0^t (|w|^2+|v|^2 - \left|w - \begin{bmatrix} I & 0 \end{bmatrix}\hat{\Sigma}_\gamma\tilde{\xi}\right|^2 - \left|v + \begin{bmatrix} I & A\Sigma_\gamma^{F^{-1}} \end{bmatrix}\hat{\Sigma}_\gamma\tilde{\xi}\right|^2 - |\tilde{\xi}|^2_{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma^2}Q \end{bmatrix}})\,dt.$$

If the disturbances $w_{[0,\infty)}$ and $v_{[0,\infty)}$ are $\mathcal{L}_2$ functions, then,

$$V(t,x,\theta;y_{[0,t]},u_{[0,t]}) \le C \quad \forall t\in \boldsymbol{T} \text{ and for some constant } C>0.$$

Since $\Sigma_\gamma^F(t) \ge \bar{\rho}(t+1)I$ for all $t\in\boldsymbol{T}$ by the definition of the set $\mathcal{LBP}_E^A(L,C,\bar{\rho})$, then, $\lim_{t\to\infty}\hat{\theta}_\gamma(t)=\theta$, which completes the proof of (ii).

Now, we proceed with the proof of the Claim. Suppose that $\hat{\Sigma}_\gamma$ admits the following partitioning:

$$\hat{\Sigma}_\gamma = \begin{bmatrix} \epsilon^{-1}\hat{\Sigma}_1 & \hat{\Sigma}_2 \\ \hat{\Sigma}_2^T & \hat{\Sigma}_3 \end{bmatrix}$$

Let $\tilde{\Sigma}_1 := \hat{\Sigma}_1 - I$, $\tilde{\Sigma}_2 := \hat{\Sigma}_2 + A$ and $\tilde{\Sigma}_3 := \frac{1}{\bar{\rho}(t+1)}(\hat{\Sigma}_3 - \Sigma_\gamma^F)$. If we use these in the Riccati differential equation (A.2), we obtain the following set of coupled matrix differential equations for $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ and $\tilde{\Sigma}_3$ [6]:

$$\begin{aligned}
&\epsilon\dot{\tilde{\Sigma}}_1 + 2\tilde{\Sigma}_1 + 2\tilde{\Sigma}_1\tilde{\Sigma}_1 + \epsilon(\tilde{\Sigma}_1 A\Sigma_\gamma^{F^{-1}}\tilde{\Sigma}_2^T - \tilde{\Sigma}_1 A\Sigma_\gamma^{F^{-1}}A^T + \tilde{\Sigma}_2\Sigma_\gamma^{F^{-1}}A^T\tilde{\Sigma}_1 - A\Sigma_\gamma^{F^{-1}}A^T\tilde{\Sigma}_1 \\
&\quad + \epsilon\tilde{\Sigma}_2\Sigma_\gamma^{F^{-1}}A^TA\Sigma_\gamma^{F^{-1}}\tilde{\Sigma}_2^T - \epsilon A\Sigma_\gamma^{F^{-1}}A^TA\Sigma_\gamma^{F^{-1}}\tilde{\Sigma}_2^T - \epsilon\tilde{\Sigma}_2\Sigma_\gamma^{F^{-1}}A^TA\Sigma_\gamma^{F^{-1}}A^T \\
&\quad + \epsilon A\Sigma_\gamma^{F^{-1}}A^TA\Sigma_\gamma^{F^{-1}}A^T) = 0; \quad \tilde{\Sigma}_1(0)=0 \\
&\epsilon\dot{\tilde{\Sigma}}_2 + \tilde{\Sigma}_2 + 2\tilde{\Sigma}_1\tilde{\Sigma}_2 + \tilde{\Sigma}_1 A\Sigma_\gamma^{F^{-1}}\bar{\rho}(t+1)\tilde{\Sigma}_3 + \epsilon(\dot{A} + \tilde{\Sigma}_2\Sigma_\gamma^{F^{-1}}A^T\tilde{\Sigma}_2 - A\Sigma_\gamma^{F^{-1}}A^T\tilde{\Sigma}_2 \\
&\quad + \tilde{\Sigma}_2\bar{\rho}(t+1)\Sigma_\gamma^{F^{-1}}A^TA\tilde{\Sigma}_3 - A\bar{\rho}(t+1)\Sigma_\gamma^{F^{-1}}A^TA\tilde{\Sigma}_3) = 0; \quad \tilde{\Sigma}_2(0) = A(0,y(0),u(0)) \\
&\dot{\tilde{\Sigma}}_3 + \frac{1}{t+1}\tilde{\Sigma}_3 + \frac{1}{t+1}(2\tilde{\Sigma}_2^T\tilde{\Sigma}_2 + \tilde{\Sigma}_2^TA\bar{\rho}(t+1)\Sigma_\gamma^{F^{-1}}\tilde{\Sigma}_3 + \tilde{\Sigma}_3\bar{\rho}(t+1)\Sigma_\gamma^{F^{-1}}A^T\tilde{\Sigma}_2 \\
&\quad + \tilde{\Sigma}_3\bar{\rho}(t+1)\Sigma_\gamma^{F^{-1}}A^TA\bar{\rho}(t+1)\Sigma_\gamma^{F^{-1}}\tilde{\Sigma}_3) = 0; \quad \tilde{\Sigma}_3(0)=0
\end{aligned}$$

Introduce the notation $\overrightarrow{B}$ to denote the stacked-up vector of all column vectors of a matrix B. Further introduce two candidate "Lyapunov" functions:

$$V_1(t,\tilde{\Sigma}_1,\tilde{\Sigma}_2) = \frac{1}{2}(\frac{1}{\epsilon^2}|\overrightarrow{\tilde{\Sigma}_1}|^2 + |\overrightarrow{\tilde{\Sigma}_2}|^2); \qquad V_2(t,\tilde{\Sigma}_3) = \frac{1}{2}|\overrightarrow{\tilde{\Sigma}_3}|^2,$$

[6] Observe that the time function A is Lipschitz continuous with Lipschitz constant L for any $y_{[0,\infty)} \in \mathcal{LBP}_E^A(L,C,\bar{\rho})$, and hence $\dot{A}$ exists and $\|\dot{A}\|_2 \le L$ almost everywhere (Royden, 1988). We will let $\dot{A}(t)=0$ on the set where $\dot{A}$ is not defined; then $A(t) = \int_0^t \dot{A}\,dt + A(0)$. Hence, the set of differential equations to follow is properly defined.

whose initial conditions are, respectively,

$$V_1(0) = \frac{1}{2}|\overrightarrow{A(0)}|^2 \le \bar{C}, \quad \text{for some constant } \bar{C} > 0 \; ; \qquad V_2(0) = 0.$$

To obtain useful bounds on the derivatives of V_1 and V_2, the following lemma will be repeatedly used.

Lemma 4 *For any two matrices $B_{l\times m}$ and $D_{m\times n}$ such that $\|B\|_2 \le M_b$ and $\|D\|_2 \le M_d$, for some positive constants M_b and M_c, the following inequality holds,*

$$|\overrightarrow{B}| \le MM_b; \qquad |\overrightarrow{BD}| \le \bar{M}|\overrightarrow{B}||\overrightarrow{D}|$$

where M and $\bar{M}$ are some positive constants, which depend only on the dimensions of the matrices B and D. □

Now, since $\bar{\rho}(t+1)\Sigma_\gamma^{F^{-1}} \le I$ for all $t \in \boldsymbol{T}$, we have the following bounds on the derivatives of V_1 and V_2:

$$\begin{aligned} \epsilon\dot{V}_1 &\le -2V_1 + \epsilon M_1 V_1^{3/2} + \epsilon M_2 V_1\sqrt{V_2} + \epsilon M_3\sqrt{V_1 V_2} + \epsilon M_4\sqrt{V_1} + \epsilon M_5 V_1 \\ \dot{V}_2 &\le -\frac{2}{t+1}V_2 + \frac{1}{t+1}(M_6 V_2\sqrt{V_1} + M_7 V_1\sqrt{V_2} + M_8 V_2^{3/2}) \end{aligned}$$

where M_i, $i = 1, \ldots, 7$ are some positive constants that depend on the dimension of the problem and $\bar{\rho}$, L and C.

Let

$$M := \max\{M_6\sqrt{\bar{C}} + M_8, M_7, 2\bar{C}\}.$$

Consider the set of $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ and $\tilde{\Sigma}_3$, such that $V_1 \le \bar{C}$ and $V_2 \le \epsilon^2 M^4$. Then, on this set, we have

$$\dot{V}_2 \le MV_2 + \epsilon M^3 V_1; \qquad \epsilon \in (0, 1/M^2]$$

Let

$$N := 2(M_3 + M_4).$$

If, in addition, $V_1 \ge N^2\epsilon^2$, then

$$\epsilon\dot{V}_1 \le -V_1; \qquad \epsilon \in (0, \epsilon_1], \; \epsilon_1 := \min\{1/(2M_1\sqrt{\bar{C}} + 2M_2 + 2M_5), 1/M^2, \sqrt{\bar{C}}/(\sqrt{2}N), 1/4\}.$$

Also, for some constant $\lambda \in [-4, 0)$, we have

$$V_1(\lambda\epsilon\ln\epsilon) \le 2N^2\epsilon^2; \quad V_1(t) \le \bar{C}\exp(-t/\epsilon); \quad t \in [0, \lambda\epsilon\ln\epsilon].$$

Hence, on the time interval $[0, \lambda\epsilon\ln\epsilon]$, V_2 is bounded above by

$$V_2(t) \le \int_0^t \exp(M(t-s))\epsilon M^3 V_2(s)\, ds \le \frac{1}{2}M^4\epsilon^2 \qquad \forall\epsilon \in (0, \epsilon_1].$$

Thus, at $t = \lambda\epsilon\ln\epsilon$, $V_1 \le 2N^2\epsilon^2$ and $V_2 \le \frac{1}{2}M^4\epsilon^2$ for $\epsilon \in (0, \epsilon_1]$.

Consider the set of $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ and $\tilde{\Sigma}_3$, such that $V_1 \le 2N^2\epsilon^2$ and $V_2 \le M^4\epsilon^2$. On this set, if in addition $V_1 \ge N^2\epsilon^2$, we have

$$\epsilon\dot{V}_1 \le -V_1 \le 0 \qquad \forall\epsilon \in (0, \epsilon_1],$$

since $\bar{C} > 2N^2\epsilon^2 \ge V_1$.

Again, on this set, if in addition $V_2 \ge \frac{1}{2}M^4\epsilon^2$, then

$$\begin{aligned}\dot{V}_2 &\le \frac{1}{t+1}(-2 + \epsilon\sqrt{2}M_6N + \epsilon 2^{3/2}\frac{M_7N^2}{M^2} + \epsilon M_8M^2)V_2 \\ &\le 0; \qquad \epsilon \in (0, \epsilon_0], \quad \epsilon_0 := \min\{\epsilon_1, 2/(\sqrt{2}M_6N + 2^{3/2}\frac{M_7N^2}{M^2} + M_8M^2)\}.\end{aligned}$$

Hence, for $\epsilon \in (0, \epsilon_0]$, $V_1 \le N^2\epsilon^2$ and $V_2 \le \frac{1}{2}M^4\epsilon^2$ for $t \ge \lambda\epsilon\ln\epsilon$.

Thus, we have shown that for a fixed $\gamma > \gamma^*$, there exists an $\epsilon_0 > 0$ such that for any $y_{[0,\infty)} \in \mathcal{LBP}_E^A(L, C, \bar{\rho})$,

$$\begin{aligned}&V_1 \le \bar{C}; \quad V_2 \le \epsilon^2M^4; \quad t \in [0, \lambda\epsilon\ln\epsilon] \\ &V_1 \le 2N^2\epsilon^2; \quad V_2 \le \epsilon^2M^4; \quad t \in [\lambda\epsilon\ln\epsilon, \infty)\end{aligned}$$

This completes the proof of the Claim, and hence the proof of the theorem.

B Proof of Theorem 9

We first note that Σ_γ^F is bounded from above by the solution to the following Riccati differential equation:

$$\dot{\Sigma} = -\Sigma\Sigma + A^TA; \quad \Sigma(0) = Q_o.$$

Since $\|A\|_2 \le C$, we have $\Sigma_\gamma^F \le CI$ for all $t \in \boldsymbol{T}$ ($t \in [0, T]$ in the finite-horizon case).

Without loss of generality, we will prove the theorem only in the infinite-horizon case with $K = 0$. By a line of reasoning similar to that used in the proof of Theorem 7, in Appendix A, it will be sufficient to show that the following Riccati differential equation admits a positive definite solution $\hat{\Sigma}$ on $\boldsymbol{T}$:

$$\begin{aligned}&\dot{\hat{\Sigma}}_\gamma + \hat{\Sigma}_\gamma \begin{bmatrix} -\frac{1}{\epsilon}I & A \\ -\frac{1}{\epsilon}\Sigma_\gamma^{F^{-1}}A^T & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{\epsilon}I & -\frac{1}{\epsilon}A\Sigma_\gamma^{F^{-1}} \\ A^T & 0 \end{bmatrix}\hat{\Sigma}_\gamma + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma^2}Q \end{bmatrix} \\ &+\hat{\Sigma}_\gamma \begin{bmatrix} 2I & A\Sigma_\gamma^{F^{-1}} \\ -\Sigma_\gamma^{F^{-1}}A^T & I + \Sigma_\gamma^{F^{-1}}A^TA\Sigma_\gamma^{F^{-1}} \end{bmatrix}\hat{\Sigma}_\gamma = 0; \qquad \hat{\Sigma}_\gamma(0) = \begin{bmatrix} P_0 & 0 \\ 0 & Q_o \end{bmatrix}. \qquad \text{(B.1)}\end{aligned}$$

Suppose that $\hat{\Sigma}_\gamma$ admits the following partitioning:

$$\hat{\Sigma}_\gamma = \begin{bmatrix} \epsilon^{-1}\hat{\Sigma}_1 & \hat{\Sigma}_2 \\ \hat{\Sigma}_2^T & \hat{\Sigma}_3 \end{bmatrix}$$

Let $\tilde{\Sigma}_1 := \hat{\Sigma}_1 - I$, $\tilde{\Sigma}_2 := \hat{\Sigma}_2 + A$ and $\tilde{\Sigma}_3 := (\hat{\Sigma}_3 - \Sigma_\gamma^F)$. Define two candidate "Lyapunov" functions:

$$V_1(t, \tilde{\Sigma}_1, \tilde{\Sigma}_2) = \frac{1}{2}(\frac{1}{\epsilon^2}|\overrightarrow{\tilde{\Sigma}_1}|^2 + |\overrightarrow{\tilde{\Sigma}_2}|^2); \qquad V_2(t, \tilde{\Sigma}_3) = \frac{1}{2}|\overrightarrow{\tilde{\Sigma}_3}|^2$$

Observe the fact that the matrix

$$I \otimes \Sigma_\gamma^F + \Sigma_\gamma^F \otimes I$$

is a symmetric matrix with eigenvalues larger than or equal to $2\bar{\rho} > 0$. Hence, a "Lyapunov" analysis similar to that used in the proof of Theorem 7 shows that $\hat{\Sigma}$ can be approximated by :

$$\hat{\Sigma}_\gamma(t) = \left[\begin{array}{cc} \frac{1}{\epsilon}I + \hat{\Sigma}_{1b}(t/\epsilon) + O(\epsilon) & -A + O(\epsilon) + \hat{\Sigma}_{2b}(t/\epsilon) \\ -A^T + O(\epsilon) + \hat{\Sigma}_{2b}(t/\epsilon)^T & \Sigma_\gamma^F(t) + O(\epsilon) \end{array} \right]$$

uniformly on $t \in \boldsymbol{T}$ and $y_{[0,\infty)} \in \mathcal{LBP}_{EPW}^{ATV}(L, C, \bar{\rho}, T_m, K)$, where $\hat{\Sigma}_{1b}$ and $\hat{\Sigma}_{2b}$ are some exponentially decaying (boundary layer) terms. Hence, the structural assumption made was accurate, and this completes the proof of the theorem.

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