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RESEARCH ARTICLE

Adaptive Controller Design and Disturbance Attenuation for Minimum Phase MIMO Linear Systems with Noisy Output Measurements and with Measured Disturbances

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Summary

In this paper, we present a systematic procedure for robust adaptive control design for minimum phase uncertain multiple-input multiple-output linear systems that are right invertible and can be dynamically extended to a linear system with vector relative degree using a dynamic compensator that is known. For this class of systems, it is always possible to dynamically extend them, and/or integrate a select set of output channels, and/or padding dummy state variablea to arrive at a system model that admits uniform vector relative degree and uniform observability indices that is further minimum phase according to [1]. We assume that the uniform vector relative degree is known and an upper bound for the uniform observability indices is known. We also assume that the unknown parameter vector lies in a convex compact set such that the high frequency gain matrix remains invertible for any parameter vector value in the set. These are the assumptions that allow for a successful design of a robust adaptive controller. A numerical example is included to fully illustrate the controller design and the effectiveness of the controller.

KEYWORDS:
nonlinear H^∞ control based robust adaptive control; multiple-input multiple-output linear uncertain systems; minimum phase; extended zero dynamics canonical form; strict observer canonical form.

1 | INTRODUCTION

Robust adaptive control design for uncertain linear systems has attracted a lot of research attention since the 1980s, [2, 3, 4, 5, 6, 7, 8, 9]. A satisfactory solution to the single-input single-output (SISO) linear systems has been obtained in [5] using the game theoretic approach [10]. See [5] for a complete literature review of the robust adaptive control and nonlinear adaptive control methodologies. There, one can further find extensive simulation results comparing our robust adaptive control strategy with those of nonadaptive H^∞-control strategy. The solution to the SISO problem has further been refined in [6], generalized to zero relative degree case [9], generalized to include three degrees of freedom problem [8], and generalized to a class of multiple-input multiple-output (MIMO) linear systems that consists of parallel interconnected SISO linear systems with limited output feedback [11]. The solution in [11] is essentially based on SISO theory as obtained in [8]. The solution methodology has also been successfully generalized to SISO uncertain nonlinear systems in [12]. It is observed that the minimum phase assumption is the key to the success of robust adpative control design for SISO uncertain linear systems. The generalization of the robust adaptive control design to MIMO systems depends critically on the generalization of the minimum phase assumption to MIMO

[0]**Abbreviations:** MIMO, multiple-input and multiple-output; SISO, single-input and single-output.

linear systems. In [13], a generalized minimum phase assumption has been introduced for SISO systems, which is necessary for a successful design of a model reference controller for SISO linear systems. It is proved that, for SISO systems, the generalized minimum phase condition is equivalent to all zeros of the transfer function from control input to the output have negative real parts if the system is controllable from the control input and is observable from the output (Proposition 3 of [13]). More relationships between the generalized minimum phase assumption and its classical counterpart have been obtained in [13]. This generalized minimum phase assumption has been extended to MIMO linear systems in [1]. It is observed that the generalized minimum phase assumption is necessary for a successful design of model reference controller for MIMO linear systems. It is also observed in [1] that the generalized minimum phase assumption is invariant under finite steps of dynamic extensions ([14]). Based on the SISO solution [5], we observe that the key canonical forms of the uncertain linear system are the extended zero dynamics canonical form and the strict observer canonical form. In [15], we established methodologies to extend (dynamically) a given minimum phase uncertain MIMO linear system model to achieve an extended system that admits the extended zero dynamics canonical form and the strict observer canonical form without rendering the system non-minimum phase. This sets the stage for the generalization of the robust adaptive control design to MIMO uncertain linear systems.

In this paper, we present a systematic procedure for robust adaptive control design for uncertain minimum phase MIMO linear systems that are right invertible and can be dynamically extended to a linear system with vector relative degree using a known dynamic compensator. For this class of systems, it is always possible to dynamically extend it [1], and/or integrate a select set of output channels [15], and/or padding dummy state variables [15] to arrive at a system model that admits uniform vector relative degree $r \in \mathbb{Z}_+$ and uniform observability indices $\nu \in \mathbb{N}$ ($r \le \nu$), which is minimum phase according to [1]. We assume that $r \in \mathbb{N}$ is known and an upper bound n for ν is known ($r = 0$ case will be treated in another paper). Thus, the system admits the extended zero dynamics canonical form and the strict observer canonical form. The observable part of the system is then the design model for the system, which is further restricted to be in a block diagonally identical structure for the backbone of the system that is independent of the unknown parameter vector and the control inputs and measurement outputs of the system (this structural assumption does not restrict the class of uncertain systems that is amenable to the robust adaptive control design, but is crucial for the robustness proof to go through for MIMO systems). The design procedure closely resembles that for the SISO case [5]. The general objective of the control design is to attenuate the effect of external disturbance input on the system tracking error. Using a game theoretic approach, we formulate the robust adaptive control problem as a nonlinear H^∞ optimal control problem with a single cost function. By making use of the *cost-to-come* function methodology for nonlinear H^∞ optimal control, we have obtained a closed-form expression for the value function of the identifier for the unknown system, which provides a finite-dimensional estimator structure for the uncertain linear system. Assuming the existence of a known convex compact set for the true values of the system parameters such that the high frequency gain matrix will remain invertible for any parameter values in the set, we introduce a smooth parameter projection scheme for the identifier, which makes it possible to apply the backstepping [16] control design at a later step. With this projection algorithm, the adaptive control system is robust with or without persistently exciting input signals. Using the explicit form of the value function for the identifier, the nonlinear H^∞ adaptive control problem is then transformed into a full-information nonlinear robust control problem, which is subsequently solved using the integrator backstepping methodology. This design procedure has led to a recursive design scheme for two classes of robust adaptive controllers for the minimum phase uncertain MIMO linear system (each one parametrized by the desired disturbance attenuation level γ). The controller actively incorporates the covariance information on the parameter estimates into the control design, and exhibits (in principle) the asymptotic certainty equivalence property, if the worst case covariance matrix converges to zero. However, to guarantee the boundedness of all closed-loop signals, an appropriate cost functional was selected to keep the covariance matrix bounded away from zero. Hence, the asymptotic certainty equivalence structure is in fact never realized. But, when the covariance matrix is close to zero, the controller behaves as a certainty equivalent one. The adaptive controller also achieves the desired disturbance attenuation level for all admissible continuous exogenous disturbance input waveforms and all admissible initial conditions on the infinite horizon. Furthermore, it is proved rigorously that the control law guarantees boundedness of all closed-loop signals under bounded admissible exogenous disturbance inputs, bounded admissible initial conditions, and bounded reference trajectory together with its derivatives up to rth order without the need for any persistency of excitation condition or any stochastic noise assumptions. Asymptotic tracking is achieved when the initial condition is admissible, the reference trajectory together with its derivatives up to rth order are bounded, the admissible disturbance inputs are bounded, and those disturbance inputs with positive attenuation level are of finite energy.

The balance of the paper is organized as follows. In the next section, we list the notations used in the paper. In Section 3, we provide a precise formulation of the problem to be solved, delineate the basic assumptions regarding the underlying system, as well as the input signals, and include a brief discussion of the solution methodology adopted. In Section 4, we present the identification design for the nonlinear H^∞ adaptive control problem, with detailed discussions on the projection algorithm used in the construction. This identifier then becomes the system to be controlled in a worst-case sense, under an equivalent expression for the cost function, transformed to the state space of identifier states. The recursive control design is discussed in Section 5. In Section 6, we present the precise statements and complete proofs of the properties of the closed-loop adaptive systems. The theoretical results are also illustrated on a numerical example in Section 7, which clearly illustrates the effectiveness of the design methodology. The paper ends with the concluding remarks of Section 8, and three appendices presenting some results essential for the derivations in the main body of the paper.

2 | NOTATIONS

Let $\mathbb{R}$ denote the real line; $\mathbb{R}_+ := (0,\infty) \subset \mathbb{R}$; $\mathbb{R}_- := (-\infty, 0) \subset \mathbb{R}$; $\overline{\mathbb{R}_+} := [0,\infty) \subset \mathbb{R}$; $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; $\mathbb{C}$ be the set of complex numbers, where i is the complex unit. For any number $a \in \mathbb{C}$, $\overline{a}$ denotes its complex conjugate and $\mathrm{Re}\,(a)$ denotes its real part. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Unless specified, all signals, constants, and matrices are real. For a continuous function f, we say that it belongs to $\mathcal{C}$; if it is k-times continuously differentiable, we say it belongs to $\mathcal{C}_k$; its lth order derivative is denoted by $\mathrm{D}^l f$ or $f^{(l)}$; its partial derivative with respect to some variable x is denoted by $\frac{\partial f}{\partial x}$. For a $\mathcal{B}_{\mathrm{B}}(\mathbb{R})$-measurable function $f : I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is an interval, we say f is $\bar{\mathrm{L}}_p$, where $p \in [1,\infty) \subset \mathbb{R}$, if $(\int_I |f(\tau)|^p \,\mathrm{d}\tau)^{1/p} < \infty$; the class of all functions g that $g = f$ a.e. in I is denoted by $[f] \in \mathrm{L}_p$; when f is continuous, and we say that f is L_∞ if $\max\{\sup_{t\in I} |f(t)|, 0\} < \infty$. We let $\mathbb{R}^n$ denote the Euclidean space, with norm $|z| := \sqrt{z'z}$, unless specified otherwise. For any matrix A, A' denotes its transpose. We will denote $n \times n$-dimensional real symmetric, positive semidefinite, and positive definite matrices by $\mathcal{S}_n$, $\mathcal{S}_{\mathrm{psd}\,n}$, and $\mathcal{S}_{+\,n}$, and say $Q_1 \le Q_2$, if $Q_2 - Q_1 \in \mathcal{S}_{\mathrm{psd}\,n}$, and $Q_1 < Q_2$, if $Q_2 - Q_1 \in \mathcal{S}_{+\,n}$, $\forall Q_1, Q_2 \in \mathcal{S}_n$; $\mathrm{Tr}\left(Q_1\right)$ denotes the trace of Q_1. For any tensor $A \in \mathrm{B}(\mathbb{R}^{m_1}, \mathrm{B}(\mathbb{R}^{m_2}, \mathcal{Y}))$, $A^{T_{2,1}}$ denotes the transpose of tensor A between the last two indices, and thus $A(x)(y) = A^{T_{2,1}}(y)(x) \in \mathcal{Y}$, $\forall x \in \mathbb{R}^{m_1}$, $\forall y \in \mathbb{R}^{m_2}$. For any $z \in \mathbb{R}^n$ and any $Q \in \mathcal{S}_{\mathrm{psd}\,n}$, $|z|_Q^2$ denotes $z'Qz$. I_n denotes the $n \times n$-dimensional identity matrix. For any matrix A, $A^0 = I$. For any matrix M, $\|M\|_p$ denotes its p-induced norm, $1 \le p \le \infty$; for $p = 2$, we simply write it as $\|M\|$. For any matrices M_1 and M_2, we will write $M_1 \otimes M_2$ to denote the Kronecker product of M_1 and M_2. $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f), \mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$; when this quantity is bounded, we say that $u_{[0,t_f)} \in \mathcal{C}_{\mathrm{b}}([0,t_f), \mathbb{R}^p)$. For an operator $A : \mathcal{X}_1 \to \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are Banach spaces, A' denotes its adjoint operator. For an operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, A^* denotes its Hermitian adjoint. For an $\mathbb{R}^{n\times m\times p}$ tensor A, $A_{:,:,i}$ denotes the $n\times m$-dimensional matrix with the last index fixed at $i = 1, \dots, p$. $\boldsymbol{e}_{m,i}$ denotes the ith unit vector in $\mathbb{R}^m$. For any real (complex) Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, $\mathcal{X}_1^*$ denotes the dual space of $\mathcal{X}_1$, and $\mathcal{X}_1^{**}$ denotes the dual of $\mathcal{X}_1^*$, we will write $\mathrm{B}\left(\mathcal{X}_1, \mathcal{X}_2\right)$ to denote the set of all bounded linear operators from $\mathcal{X}_1$ to $\mathcal{X}_2$. For any Banach space $\mathcal{X}$, $x \in \mathcal{X}$ and $x_* \in \mathcal{X}^*$, we will write $\langle\langle x_*, x \rangle\rangle_{\mathcal{X}}$ to denote the scalar $x_*(x)$; we write $\mathcal{B}_{\mathcal{X}}(x, r)$ to denote the open ball centered at x with radius $r \in \mathbb{R}_+$ in $\mathcal{X}$; and $\mathrm{span}\,(A) \subseteq \mathcal{X}$ denotes the subspace generated by $A \subseteq \mathcal{X}$. For any Hilbert space $\mathcal{H}$, $x, y \in \mathcal{H}$, $\langle x, y \rangle_{\mathcal{H}}$ denotes the inner product of x and y. On $\mathbb{R}$, we will denote $\bar{\mathrm{r}}_{a,b}$ to be the compact interval $[a,b] \subset \mathbb{R}$, where $a \le b$ and $a, b \in \mathbb{R}$; $\mathcal{B}_{\mathrm{B}}(\mathbb{R})$ denotes the Borel measurable subsets of $\mathbb{R}$; and μ_{B} denotes the Borel measure on $\mathbb{R}$. For any sets A, B with $A \subseteq B$, $\chi_{A,B}$ denote the indicator function of the set A on B, i.e., $\chi_{A,B}(x) := \begin{cases} 1 & x \in A \\ 0 & x \in B \setminus A \end{cases}$, $\forall x \in B$; the interior of A is A°, the closure of A is $\overline{A}$, the complement of A is $\widetilde{A}$, all relative to B. For a function $f : X \to \mathcal{Y}$, where X is a set and $\mathcal{Y}$ is a Banach space, we write $\mathcal{P}\circ f : X \to \mathbb{R}$ to be $\mathcal{P}\circ f(x) = \|f(x)\|_{\mathcal{Y}}$, $\forall x \in X$.

Any signal with a hat accent (like $\hat{x}$, $\hat{\theta}$, $\hat{\xi}$) is the worst-case estimate of the corresponding signal without the accent, which is something we design like the control signal. Any signal with a check accent (like $\check{x}$, $\check{\theta}$, $\check{w}$) is some signal we can measure, or the estimate of the corresponding signal without the accent that is produced by the cost-to-come function analysis. Any signal with a grave accent (like $\grave{x}$) is some signal that is unknown in general and is associated with the given unknown MIMO linear system. Any signal without any accent is a signal in the design model. Any signal with tilde accent (like $\tilde{x}$, $\tilde{\theta}$, $\tilde{\xi}$) is the estimation error of the signal without the accent, which equals to the signal without the accent minus the signal with the check accent.

3 | PROBLEM FORMULATION

We consider the adaptive control problem for continuous-time finite-dimensional minimum phase MIMO linear time-invariant systems.

We are given system $\grave{S}$ with state space representation:

$$\dot{\grave{x}} = \grave{A}\grave{x} + \grave{B}\acute{u} + \grave{D}\grave{w}; \quad \grave{x}(0) = \grave{x}_0 \in \grave{\mathcal{D}}_0 \tag{1a}$$

$$\acute{y} = \grave{C}\grave{x} + \grave{F}\acute{u} + \grave{E}\grave{w} \tag{1b}$$

where $\grave{x} \in \mathbb{R}^{\grave{n}}$ is the state vector, $\grave{n} \in \mathbb{Z}_+$; $\grave{x}_0 \in \grave{\mathcal{D}}_0$ is the initial condition, where $\grave{\mathcal{D}}_0 \subseteq \mathbb{R}^{\grave{n}}$ is a subspace ($\grave{\mathcal{D}}_0 = \mathbb{R}^{\grave{n}}$ usually); $\acute{u} \in \mathbb{R}^p$ is the control input, $p \in \mathbb{N}$; $\grave{w} \in \mathbb{R}^{\grave{q}}$ is the disturbance input, $\grave{q} \in \mathbb{Z}_+$; $\acute{y} \in \mathbb{R}^m$ is the measurement output, $m \in \mathbb{N}$; and the matrices $\grave{A}$, $\grave{B}$, $\grave{D}$, $\grave{C}$, $\grave{F}$, and $\grave{E}$ are constant matrices of appropriate dimensions and generally unknown. It is assumed that $m \leq p$; the control inputs are partitioned into $\acute{u} := (\acute{u}_a, u_b)$, where $\acute{u}_a$ is m-dimensional; the disturbance inputs are partitioned into $\grave{w} := (\check{w}, \grave{w}_b)$, where $\check{w} \in \mathbb{R}^{\check{q}}$ are measured disturbance inputs (in addition to the measurements $\acute{y}$), $\check{q} \in \mathbb{Z}_+$; and the waveform of $\grave{w}_{[0,\infty)}$ is assumed to belong to $\grave{\mathcal{W}}_d$ ($= C(\overline{\mathbb{R}_+}, \mathbb{R}^{\grave{q}})$ usually), which is of class $\mathcal{B}_{\grave{q}}$ (see [13]). Thus, we are only considering $\grave{w}_{[0,\infty)}$ that is continuous. In the proof of the main result of the paper, u_b will be treated much like as part of the exogeneous disturbance $\grave{w}_e := (u_b, \grave{w})$, especially like the measured disturbances $\check{w}$, and the set of admissible extended disturbance waveform is $\acute{\mathcal{W}}_d := C(\overline{\mathbb{R}_+}, \mathbb{R}^{p-m}) \times \grave{\mathcal{W}}_d$. We now state a number of assumptions, which are quite natural in this context.

Assumption 1. The system (1) (with control input $\acute{u}_a$, output $\acute{y}$, and extended disturbance input $\grave{w}_e$) is minimum phase with respect to $\grave{\mathcal{D}}_0$ and $\acute{\mathcal{W}}_d$ as defined in [1].

Assumption 2. There exists a known dynamic controller S_{de} with state space representation:

$$\dot{\iota} = A_{\mathrm{de}}\iota + B_{\mathrm{de}}u_a; \quad \iota(0) = \iota_0 \in \mathbb{R}^{n_{\mathrm{de}}} \tag{2a}$$

$$\acute{u}_a = C_{\mathrm{de}}\iota + D_{\mathrm{de}}u_a \tag{2b}$$

where $n_{\mathrm{de}} \in \mathbb{Z}_+$, that is a result of finite number of steps of dynamic extension algorithm [14] such that the composite system of $\grave{S}$ and S_{de} (with control input u_a and output $\acute{y}$) admits well-defined vector relative degree.

By a result of [1], the composite system of $\grave{S}$ and S_{de} (with control input u_a, output $\acute{y}$, and extended disturbance input $\grave{w}_e$) is minimum phase with respect to $\grave{\mathcal{D}}_0 \times \mathbb{R}^{n_{\mathrm{de}}}$ and $\acute{\mathcal{W}}_d$.

In case that the composite system of $\grave{S}$ and S_{de} does not have uniform vector relative degree, by Lemma 2 of [15], we may selectively integrate the components of the output $\acute{y}$ as in the following state space representation S_{oi}:

$$\dot{\varpi} = A_{\mathrm{oi}}\varpi + B_{\mathrm{oi}}\acute{y}; \quad \varpi(0) = \varpi_0 \in \mathbb{R}^{n_{\mathrm{oi}}} \tag{3a}$$

$$y = C_{\mathrm{oi}}\varpi + D_{\mathrm{oi}}\acute{y} \tag{3b}$$

where $n_{\mathrm{oi}} \in \mathbb{Z}_+$ and y is m-dimensional, such that the composite system of S_{oi}, $\grave{S}$, and S_{de} with control input u_a, output y, and extended disturbance input $\grave{w}_e$ is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\acute{\mathcal{W}}_d$, where $\bar{\mathcal{D}}_0 := \mathbb{R}^{n_{\mathrm{oi}}} \times \grave{\mathcal{D}}_0 \times \mathbb{R}^{n_{\mathrm{de}}} \subseteq \mathbb{R}^{n_{\mathrm{oi}}+\grave{n}+n_{\mathrm{de}}}$ and is a subspace, and admits uniform vector relative degree $r \in \mathbb{Z}_+$ from u_a to y. The system S_{oi} and the relative degree r are known.

Denote the composite system of S_{oi}, $\grave{S}$, and S_{de} by $\grave{S}_e$. By Lemma 3 of [15], we can extend the state space of this composite system to arrive at a system $\acute{S}$ with state space representation

$$\dot{\acute{x}} = \acute{A}\acute{x} + \acute{B}u + \acute{D}\grave{w}; \quad \acute{x}(0) = \acute{x}_0 \in \acute{\mathcal{D}}_0 \tag{4a}$$

$$y = \acute{C}\acute{x} + \acute{F}u + \acute{E}\grave{w} \tag{4b}$$

where $\acute{x} \in \mathbb{R}^{n_{\mathrm{oi}}+n_{\mathrm{de}}+\grave{n}+mn-\sum_{i=1}^{m}\nu_i}$ is the state vector; $\nu_1, \ldots, \nu_m$ are the observability indices of $\grave{S}_e$; $n \geq \max_{1\leq i\leq m} \nu_i =: \nu \in \mathbb{Z}_+$, where ν is the observability index [17] of the composite system $\grave{S}_e$, and n is the uniform observability index of the pair $(\acute{A}, \acute{C})$; $\acute{x}_0 \in \acute{\mathcal{D}}_0$ is the initial condition, where $\acute{\mathcal{D}}_0 := \bar{\mathcal{D}}_0 \times \left\{ \mathbf{0}_{mn-\sum_{i=1}^{m}\nu_i} \right\} \subseteq \mathbb{R}^{mn+n_{\mathrm{oi}}+\grave{n}+n_{\mathrm{de}}-\sum_{i=1}^{m}\nu_i}$ is a subspace; $u := (u_a, u_b) \in \mathbb{R}^p$ is the control input; and the matrices $\acute{A}$, $\acute{B}$, $\acute{D}$, $\acute{C}$, $\acute{F}$, and $\acute{E}$ are constant matrices of appropriate dimensions, which are generally unknown. The system (4) (with control input u_a output y and extended disturbance input $\grave{w}_e$) is minimum phase with respect to $\acute{\mathcal{D}}_0$ and $\acute{\mathcal{W}}_d$. The system $\acute{S}$ admits uniform vector relative degree r.

Assumption 3. The upper bound n of the observability index ν of system $\grave{S}_e$ is known. (n is the uniform observability indices of the pair $(\acute{A}, \acute{C})$.)

In this paper, we consider the case $r \in \mathbb{N}$. The case of $r = 0$ requires a separate analysis, and will be addressed in a future paper.

Now, partition the system (4) into observable and unobservable parts as

$$\dot{\acute{x}}_{\bar{o}} = \acute{A}_{\bar{o}}\acute{x}_{\bar{o}} + \acute{A}_{\bar{o}o}\acute{x}_o + \acute{B}_{\bar{o}}u + \acute{D}_{\bar{o}}\grave{w}; \quad \acute{x}_{\bar{o}}(0) = \acute{x}_{\bar{o}0} \tag{5a}$$

$$\dot{\acute{x}}_o = \acute{A}_o\acute{x}_o + \acute{B}_o u + \acute{D}_o\grave{w}; \quad \acute{x}_o(0) = \acute{x}_{o0} \tag{5b}$$

$$y = \acute{C}_o\acute{x}_o + \acute{F}u + \acute{E}\grave{w} \tag{5c}$$

where $\acute{x}_o$ is nm-dimensional, and the pair $(\acute{A}_o, \acute{C}_o)$ is observable.

By Corollary 3 of [15], there exists an invertible matrix $\grave{T}$ such that in $(x_{\bar{o}}, x) := (x_{\bar{o}}, x_1, \dots, x_n) = \grave{T}^{-1}(\acute{x}_{\bar{o}}, \acute{x}_o)$ coordinates, we have that x_i is m-dimensional, $i = 1, \dots, n$, and the system (5) admits the strict observer canonical form representation

$$\dot{x}_{\bar{o}} = A_{\bar{o}}x_{\bar{o}} + A_{\bar{o},1}x_1 + B_{\bar{o},a}u_a + B_{\bar{o},b}u_b + D_{\bar{o}}\grave{w} \tag{6a}$$

$$\dot{x}_i = A_{i,1}x_1 + x_{i+1} + B_{i,a}u_a + B_{i,b}u_b + \acute{D}_i\grave{w}; \quad i = 1, \dots, n-1 \tag{6b}$$

$$\dot{x}_n = A_{n,1}x_1 + B_{n,a}u_a + B_{n,b}u_b + \acute{D}_n\grave{w} \tag{6c}$$

$$y = x_1 + \acute{F}_a u_a + \acute{F}_b u_b + \acute{E}\grave{w} \tag{6d}$$

where all matrices are constant and of appropriate dimensions, $B_{0,a} := \acute{F}_a$, and $B_{i,a} = \mathbf{0}_{m\times m}$, $\forall i = 0, \dots, r-1$, and $B_{r,a}$ is of rank m and is therefore invertible.

For the solvability of the problem, we now make the following natural assumption.

Assumption 4. The output equation (6d) is independent of u_b and $\grave{w}$ (if it depends on u_b but not $\grave{w}$, we just need to set $u_b \equiv \mathbf{0}_{p-m}$ in the final control design).

By further introducing a disturbance transformation $w_b = \grave{M}\grave{w}_b$, where w_b is mq_b-dimensional, $q_b \in \mathbb{N}$, and $\grave{M}$ is an unknown constant matrix, we may obtain the following design model for the dynamics of $x = (x_1, \dots, x_n)$ in (6)

$$\dot{x} = Ax + \check{A}y + Bu_a + \check{B}_b u_b + \check{D}\check{w} + Dw_b + (A_{211,1}y + A_{211,2}u_b + A_{211,3}\check{w} + A_{212}u_a)\theta \tag{7a}$$

$$y = Cx + Ew_b \tag{7b}$$

where the matrices A, $\check{A}$, B, $\check{B}_b$, $\check{D}$, D, C, and E are known matrices of appropriate dimensions; $\theta \in \Theta \subseteq \mathbb{R}^\sigma$ is the unknown parameter vector of the system; $A_{211,1}$, $A_{211,2}$, $A_{211,3}$, and A_{212}, are known second-order $\mathbb{R}^{nm}$-valued tensors of appropriate dimensions; and further we have the following structures.

$$A = \begin{bmatrix} a_{1,1}I_m & a_{1,2}I_m & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} \\ a_{2,1}I_m & a_{2,2}I_m & a_{2,3}I_m & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \mathbf{0}_{m\times m} \\ a_{n-1,1}I_m & a_{n-1,2}I_m & \cdots & a_{n-1,n-1}I_m & a_{n-1,n}I_m \\ a_{n,1}I_m & a_{n,2}I_m & \cdots & a_{n,n-1}I_m & a_{n,n}I_m \end{bmatrix} =: A_1 \otimes I_m; \quad B = \begin{bmatrix} \mathbf{0}_{(r-1)m\times m} \\ B_r \\ \vdots \\ B_n \end{bmatrix}; \quad A_{212} = (a_{212,i,j,k})_{mn\times\sigma\times m}$$

$$C = \begin{bmatrix} I_m & \mathbf{0}_{m\times(n-1)m} \end{bmatrix} =: C_1 \otimes I_m; \quad E = \begin{bmatrix} e_{1,1}I_m & \cdots & e_{1,q_b}I_m \end{bmatrix} =: E_1 \otimes I_m; \quad D = \begin{bmatrix} d_{1,1}I_m & \cdots & d_{1,q_b}I_m \\ \vdots & & \vdots \\ d_{n,1}I_m & \cdots & d_{n,q_b}I_m \end{bmatrix} =: D_1 \otimes I_m$$

$a_{i,j} \in \mathbb{R}$, $\forall i,j = 1, \dots, n$ with $j \le i+1$; $a_{i,i+1}$ is nonzero, $i = 1, \dots, n-1$; and $a_{212,i,j,k} = 0$, $\forall i = 1, \dots, (r-1)m$, and B_i are $m \times m$-dimensional matrices, $i = r, \dots, n$, $e_{1,j} \in \mathbb{R}$ and $d_{i,j} \in \mathbb{R}$, $i = 1, \dots, n$, $j = 1, \dots, q_b$. The structures of B and A_{212} are the result of our knowledge of the uniform vector relative degree r of the system. We will further partition $\check{w}$ into $(\check{w}_1, \check{w}_2)$, where $\check{w}_i$ is of dimension $\check{q}_i$, $i = 1, 2$. Then, partition $A_{211,3}$ accordingly as $A_{211,3}\check{w} = A_{211,3,1}\check{w}_1 + A_{211,3,2}\check{w}_2$ with $A_{211,3,1} := (a_{211,3,1,i,j,k})_{mn\times\sigma\times\check{q}_1}$ and $a_{211,3,1,i,j,k} = 0$, $\forall 1 \le i \le (r-1)m$; and partition $\check{D}$ accordingly as $\check{D}\check{w} = \check{D}_1\check{w}_1 + \check{D}_2\check{w}_2$ with $\check{D}_1$ having the first $(r-1)m$ rows equal to $\mathbf{0}_{(r-1)m\times\check{q}_1}$. This follows from the fact that y has relative degree at least r with respect to $\check{w}_1$.

Assumption 5. There exists a known smooth nonnegative proper convex function $P(\theta)$, such that the true value of θ lies in convex compact set $\Theta := \left\{ \bar{\theta} \in \mathbb{R}^\sigma \;\middle|\; P(\bar{\theta}) \le 1 \right\}$. Furthermore, $\forall \bar{\theta} \in \Theta$, the matrix $B_r + A_{212,r}^{T_{2,1}}\bar{\theta} =: B_{p0}(\bar{\theta})$ is invertible, where $A_{212,r}$ is the 2nd order $\mathbb{R}^m$-valued sub-tensor of A_{212} consisting of $((r-1)m+1)$st to (rm)th indices in the output dimension, all indices in the first dimension and all indices in the second dimension..

$B_{p0}(\theta)$ being invertible follows from the fact that system (6) admits uniform vector relative degree r from u_a to y. We define a class of parametrized convex compact sets $\Theta_\rho := \{ \bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) \le \rho \}, \forall \rho > 1$.

Assumption 6. Associated with system (6), we are given an m-dimensional reference trajectory $y_d(t)$ that y is to track. The reference trajectory y_d is r times continuously differentiable. The signal y_d and the first r derivatives of y_d are available for control design, that is the vector $Y_d := (y_d, y_d^{(1)}, \ldots, y_d^{(r)})$.

The objective of the control design is to achieve asymptotic tracking of the reference trajectory while rejecting the uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y^{(r)}_{d[0,\infty)}) \in \acute{\mathcal{D}}_0 \times \Theta \times \grave{\mathcal{W}}_d \times \mathcal{C}(\overline{\mathbb{R}_+}, \mathbb{R}^m) =: \grave{\mathcal{W}}$, which comprises the initial state of the system $\acute{\mathcal{S}}$, the true values of the unknown parameters, and the disturbance input waveforms, and the rth order derivative of the reference waveform. We will obtain a class of causal robust adaptive controllers,

$$u(t) = (u_a(t), u_b(t)) = \mu(t, y_{[0,t]}, \check{w}_{[0,t]}, Y_{d[0,t]}) \tag{8}$$

$\forall t \in \overline{\mathbb{R}_+}$ to achieve the desired the tracking and disturbance attenuation objectives (to be delineated shortly). Let us denote the class of these causal admissible controllers by $\mathcal{M}$. Thus, after the design of the controller μ, the actual controller is the composition of $\mathcal{S}_{\mathrm{de}}$, μ, and $\mathcal{S}_{\mathrm{oi}}$, to be denoted by $\bar{\mu}$.

The control design objective is now made precise in the following.

Definition 1. A controller μ is said to be achieve *disturbance attenuation level* 0 *with respect to* $\check{w}_1$ *and disturbance attenuation level* $\gamma \in \mathbb{R}_+$ *with respect to* $\check{w}_2$ *and* w_b, if there exist nonnegative functions $l(t, \theta, x_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, Y_{d[0,t]})$ and $l_0(\check{x}_0, \check{\theta}_0)$ such that for all $t_f \ge 0$ the following dissipation inequality holds :

$$\sup_{(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}} J_{\gamma t_f} \le 0 \tag{9}$$

where

$$\begin{aligned} J_{\gamma t_f} := \int_0^{t_f} &\left(\left| C x(\tau) - y_d(\tau) \right|^2 + l(\tau, \theta, x_{[0,\tau]}, y_{[0,\tau]}, \check{w}_{[0,\tau]}, Y_{d[0,\tau]}) - \gamma^2 \left| \check{w}_2(\tau) \right|^2 \right. \\ &\left. - \gamma^2 \left| w_b(\tau) \right|^2 \right) \mathrm{d}\tau - \gamma^2 \left| (\theta - \check{\theta}_0, x(0) - \check{x}_0) \right|^2_{\bar{Q}_0} - l_0(\check{x}_0, \check{\theta}_0) \end{aligned} \tag{10}$$

Here, $\check{\theta}_0$ is the initial guess of the unknown parameters; $\check{x}_0$ is the initial guess of the unknown initial state $x(0)$; and $(\sigma + mn) \times (\sigma + mn)$-dimensional matrix $\bar{Q}_0 \in \mathcal{S}_{+(\sigma+mn)}$ is the quadratic weighting on the initial estimation error, quantifying our level of confidence in the a priori estimates of θ and $x(0)$; and $\bar{Q}_0^{-1}$ admits the structure $\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1} \Phi_0' \end{bmatrix}$, where $Q_0 \in \mathcal{S}_{+\sigma}$ and $\Pi_0 \in \mathcal{S}_{+mn}$, respectively.[1]

Note that, in the above definition, the negative weighting on the disturbance input $\grave{w}$ is through the negative weightings on the transformed disturbance inputs w_b and $\check{w}_2$. The motivation behind the above definition is to guarantee that, for each time instant $t_f \ge 0$, the squared L_2 norm of the output tracking error $x_1 - y_d$ on $[0, t_f]$ is bounded by γ^2 times the squared L_2 norm of the transformed disturbance input $w_{b[0,t_f]}$ plus γ^2 times the squared L_2 norm of the measured disturbance $\check{w}_{2[0,t_f]}$ plus some constant that depends only on the initial condition of the system. When the disturbance inputs $\grave{w}_b$ and $\check{w}_2$ have finite L_2 norms on $[0, \infty)$, then the L_2 norm of the tracking error $x_1 - y_d$ is also finite, which further implies that $\lim_{t\to\infty}(x_1(t) - y_d(t)) = \mathbf{0}_m$, under additional stability conditions of the closed-loop system. On the other hand, for nonvanishing disturbance inputs $\grave{w}_b$ and $\check{w}_2$, whose truncated squared L_2 norms increase linearly with t_f, the rate of increase for an upper bound of the truncated squared L_2 norm of the tracking error $x_1 - y_d$ is also linear, and is bounded by γ^2 times the rate for the disturbance $(\check{w}_2, w_b)$. Clearly, when such an objective is achieved, the closed-loop system will be robust with respect to the disturbance $\grave{w}$, but the exact attenuation level with respect to $\grave{w}$ will in general depend on the unknown transformation matrix $\grave{M}$. Under Assumption 5, $\grave{M}$ can be selected to have a known bound for its norm, which then guarantees a known bound for the attenuation level from $\grave{w}$ to the tracking error.

The problem formulated above can be brought into the framework of H^∞ optimal control for affine-quadratic nonlinear systems with imperfect state measurements. Toward this end, we expand the system dynamics (7) by adjoining the simple

[1] At this point, Π_0 is quite arbitrary. Later, to simplify the structure of the adaptive controller to be derived, we will choose it to be the solution of an algebraic Riccati equation.

dynamics of θ: $\dot{\theta} = \mathbf{0}_\sigma$. Let ξ denote the expanded state $\xi = (\theta, x)$, which satisfies the following dynamics:

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times mn} \\ A_{211,1}y + A_{211,2}u_b + A_{211,3}\check{w} + A_{212}u_a & A \end{bmatrix}\xi + \begin{bmatrix} \mathbf{0}_{\sigma\times m} \\ \check{A} \end{bmatrix} y + \begin{bmatrix} \mathbf{0}_{\sigma\times m} \\ B \end{bmatrix} u_a \\
&\quad + \begin{bmatrix} \mathbf{0}_{\sigma\times(p-m)} \\ \check{B}_b \end{bmatrix} u_b + \begin{bmatrix} \mathbf{0}_{\sigma\times\check{q}} \\ \check{D} \end{bmatrix} \check{w} + \begin{bmatrix} \mathbf{0}_{\sigma\times mq_b} \\ D \end{bmatrix} w_b \\
&=: \bar{A}\xi + \bar{\check{A}}y + \bar{B}u_a + \bar{\check{B}}_b u_b + \bar{\check{D}}\check{w} + \bar{D}w_b && \text{(11a)} \\
y &= \begin{bmatrix} \mathbf{0}_{m\times\sigma} & C \end{bmatrix}\xi + Ew_b =: \bar{C}\xi + Ew_b && \text{(11b)}
\end{aligned}$$

The worst-case optimization of the cost function (10) can be carried out in two steps: first a maximization over $\acute{x}_0$, θ, and w_b, given the the measurements available to the controller, and then maximization over $\check{w}$, y, and Y_d. The idea is that the controller can observe the underlying system only through the measurements, and hence once the measurement waveform is fixed, the control input is an open-loop time function with respect to the underlying dynamics. This is precisely the idea that underpins the *cost-to-come* function methodology, leading to the following identity for each fixed $t_f > 0$:

$$\begin{aligned}
&\sup_{(\acute{x}_0,\theta,\acute{w}_{[0,\infty)},y^{(r)}_{d\,[0,\infty)})\in\acute{\mathcal{W}}} J_{\gamma t_f} \\
&= \sup_{y_{[0,\infty)}\in\mathcal{C},Y_{d\,[0,\infty)}\in\mathcal{C},\check{w}_{[0,\infty)}\in\mathcal{C}} \; \sup_{(\acute{x}_0,\theta,\acute{w}_{[0,\infty)},y^{(r)}_{d\,[0,\infty)})\in\acute{\mathcal{W}}|y_{[0,\infty)},Y_{d\,[0,\infty)},\check{w}_{[0,\infty)}} J_{\gamma t_f} \\
&\le \sup_{y_{[0,\infty)}\in\mathcal{C},Y_{d\,[0,\infty)}\in\mathcal{C},\check{w}_{[0,\infty)}\in\mathcal{C}} \; \sup_{(\acute{x}_0,\theta,w_{b[0,\infty)},y^{(r)}_{d\,[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},Y_{d\,[0,\infty)},\check{w}_{[0,\infty)}} J_{\gamma t_f} \qquad (12)
\end{aligned}$$

where the right-hand sup operator

$$\sup_{(\acute{x}_0,\theta,w_{b[0,\infty)},y^{(r)}_{d\,[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},Y_{d\,[0,\infty)},\check{w}_{[0,\infty)}}$$

is over all initial conditions $\acute{x}_0 \in \mathbb{R}^{n_{\text{oi}}+n_{\text{de}}+\check{n}+mn-\sum_{i=1}^m \nu_i}$, parameter value $\theta \in \Theta$, and disturbance waveforms $w_{b[0,\infty)} \in \mathcal{C}$ that generate the output waveform $y_{[0,\infty)}$ with $\check{w}_{[0,\infty)}$ and $Y_{d\,[0,\infty)}$ fixed and known. In the above, we have elected to be conservative that we supremize with respect to $w_{b[0,\infty)}$, instead of $\acute{w}_{b[0,\infty)}$. This is done solely for the consideration of the existence of a finite-dimensional solution for the problem.

The right-hand supremization, which will be carried out first, corresponds to the evaluation of the worst-case performance for any set of known measurement waveforms, which renders the control input waveform independent of the actual disturbance input waveform, since the control input is generated as a function of the output waveform and the reference trajectory. This is the identification design step, discussed next in Section 4. Because of the special structure of the problem under consideration, an upper bound of the value function for this step of the optimization, which is related to the *cost-to-come* function for this problem, can be obtained explicitly by utilizing the results of Appendix B of [5].

The left-hand supremization, which will be carried out second, corresponds to the computation of the worst-case measurement waveform against a given control law. Since the control law is restricted to be a causal function of the measurements and the reference trajectory, it plays a critical role in the determination of the achievability of the objective (9). This is the control design step, which is discussed in Section 5.

The design function $l(t, \theta, x_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, Y_{d[0,t]})$ is selected based on two considerations: the existence of a solution to the problem; and the ease of analysis of stability and robustness of the resulting closed-loop system. It is built up in the identifier and the controller design steps. In the identifier design step, the weighting functions are selected to provide necessary stability properties, and to yield a desirable structure for the identifier that is amenable to the later backstepping design procedure. In particular, they are selected to maintain a predetermined positive definite lower bound for the worst-case covariance matrix of the parameter estimates, which is necessary for the robustness of the closed-loop system.

In the controller design step, we employ a backstepping procedure for the design of the input u_a, which also yields an upper bound of the value function for the closed-loop system. Based on this upper bound function, the choice of u_b can be determined to further decrease the negative drift of the value function. But the choice for u_b must be bounded, since u_a is the only control input that is allowed to have infinite control authority. Therefore, the choice for u_b will be passed through a saturation function to allow for the stability analysis to go through. We prove later that all signals in the closed-loop system are uniformly bounded in time for any uniformly bounded admissible disturbance input waveforms, any uniformly bounded reference trajectories together with its derivatives up to rth order, and any bounded admissible initial condition.

This completes the formulation of the robust adaptive control problem and the general solution method to be adopted. We now turn to the identification and control designs in the next two sections.

4 | DESIGN OF A WORST-CASE IDENTIFIER

In this section, we present the identification design for the adaptive control problem formulated. For this step, the measurement waveforms $y_{[0,\infty)}$, $\check{w}_{[0,\infty)}$, $Y_{d[0,\infty)}$, and therefore the control waveforms $u_{a[0,\infty)}$ and $u_{b[0,\infty)}$, are assumed to be fixed and known. We consider the cost function:

$$J^t_{i\gamma} = \int_0^t (\left|Cx(\tau) - y_d(\tau)\right|^2 + \left|\xi(\tau) - \hat{\xi}(\tau)\right|^2_{\bar{Q}} - \gamma^2 \left|w_b(\tau)\right|^2) \,\mathrm{d}\tau - \gamma^2 \left|(\theta - \check{\theta}_0, x(0) - \check{x}_0)\right|^2_{\bar{Q}_0} \tag{13}$$

where the first positive definite term is required by the objective of the adaptive control design (10); the second nonnegative definite term is introduced for robustness considerations of the complete adaptive system, where $\hat{\xi}$ is the worst-case estimate for the expanded state ξ, which is like a control signal yet to be determined; the two negative-definite weighting terms involving the disturbance w_b and the initial conditions are again required by the objective of the adaptive control design (9). The nonnegative-definite weighting function $\bar{Q}$ will exhibit a special structure to be delineated shortly. Compared with the cost function (9), we have neglected here some terms which are constant for this step of optimization.

To avoid singularity in estimation, we assume that

Assumption 7. The matrix E is of full row rank, or equivalently, $E_1 E_1' =: \zeta^{-2} \in \mathbb{R}_+$.

Note that $N := EE' = \zeta^{-2} I_m \in \mathcal{S}_{+m}$. By expressing the above cost function completely in the ξ state variables, we can apply Lemma 10 of [5] to obtain an equivalent, more transparent, expression for $J^t_{i\gamma}$.

Let $\bar{\Sigma}$ and $\check{\xi}$ be defined by

$$\begin{aligned}
\dot{\bar{\Sigma}} &= (\bar{A} - \bar{L}N^{-1}\bar{C})\bar{\Sigma} + \bar{\Sigma}(\bar{A} - \bar{L}N^{-1}\bar{C})' + \frac{1}{\gamma^2}\bar{D}\bar{D}' - \frac{1}{\gamma^2}\bar{L}N^{-1}\bar{L}' - \bar{\Sigma}(\gamma^2\bar{C}'N^{-1}\bar{C} - \bar{C}'\bar{C} - \bar{Q})\bar{\Sigma}; \\
\bar{\Sigma}(0) &= \frac{1}{\gamma^2}\bar{Q}_0^{-1} = \frac{1}{\gamma^2}\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi_0' \end{bmatrix}
\end{aligned} \tag{14a}$$

$$\begin{aligned}
\dot{\check{\xi}} &= (\bar{A} + \bar{\Sigma}(\bar{C}'\bar{C} + \bar{Q}))\check{\xi} - \bar{\Sigma}(\bar{C}'y_d + \bar{Q}\hat{\xi}) + \bar{\check{A}}y + \bar{\check{B}}_b u_b + \bar{B}u_a + \bar{\check{D}}\check{w} + (\gamma^2\bar{\Sigma}\bar{C}' + \bar{L})N^{-1}(y - \bar{C}\check{\xi}); \\
\check{\xi}(0) &= \begin{bmatrix} \check{\theta}_0 \\ \check{x}_0 \end{bmatrix}
\end{aligned} \tag{14b}$$

where $\bar{L} := \bar{D}E'$ is given by $\bar{L} = \begin{bmatrix} \mathbf{0}_{\sigma\times m} \\ L \end{bmatrix}$ with $L := DE' = (D_1E_1') \otimes I_m =: L_1 \otimes I_m$.

Then, the cost function (13) can equivalently be written as (from Lemma 10 of [5])

$$\begin{aligned}
J^t_{i\gamma} = &- \left|\xi(t) - \check{\xi}(t)\right|^2_{(\bar{\Sigma}(t))^{-1}} + \int_0^t (\left|\bar{C}\check{\xi}(\tau) - y_d(\tau)\right|^2 + \left|\check{\xi}(\tau) - \hat{\xi}(\tau)\right|^2_{\bar{Q}} - \gamma^2\left|y(\tau) - \bar{C}\check{\xi}(\tau)\right|^2_{N^{-1}} \\
&- \gamma^2\left|w_b(\tau) - w_*(\tau, \xi_{[0,\tau]}, y_{[0,\tau]}, u_{[0,\tau]}, \check{w}_{[0,\tau]}, y_{d[0,\tau]}, \hat{\xi}_{[0,\tau]})\right|^2) \,\mathrm{d}\tau
\end{aligned} \tag{15}$$

where

$$w_*(\tau, \xi_{[0,\tau]}, y_{[0,\tau]}, u_{[0,\tau]}, \check{w}_{[0,\tau]}, y_{d[0,\tau]}, \hat{\xi}_{[0,\tau]}) = E'N^{-1}(y(\tau) - \bar{C}\xi(\tau)) + \frac{1}{\gamma^2}(I_{mq_b} - E'N^{-1}E)\bar{D}'(\bar{\Sigma}(\tau))^{-1}(\xi(\tau) - \check{\xi}(\tau)) \tag{16}$$

Furthermore, an upper bound of the value function for this estimation step is W:

$$W(t, \xi, \check{\xi}) := |\xi - \check{\xi}|^2_{\bar{\Sigma}^{-1}} \tag{17}$$

whose time derivative is given by

$$\begin{aligned}
\dot{W} = &- \left|\bar{C}\xi - y_d\right|^2 + \left|y_d - \bar{C}\check{\xi}\right|^2 - \left|\xi - \hat{\xi}\right|^2_{\bar{Q}} + \left|\check{\xi} - \hat{\xi}\right|^2_{\bar{Q}} + \gamma^2|w_b|^2 \\
&- \gamma^2\left|y - \bar{C}\check{\xi}\right|^2_{N^{-1}} - \gamma^2\left|w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}, \hat{\xi}_{[0,t]})\right|^2
\end{aligned} \tag{18}$$

Partition $\check{\xi} := (\check{\theta}, \check{x})$ and $\hat{\xi} := (\hat{\theta}, \hat{x})$ compatible with the partition of $\xi = (\theta, x)$. Our intention is to keep $\check{\theta}$ within a vicinity of Θ such that the matrix $B_r + A_{212,r}^{T_{2,1}}\check{\theta} = B_{p0}(\check{\theta})$ is always invertible, by using a smooth projection algorithm for the backstepping design procedure to be presented in the next section to work.

Define

$$\rho_M := \inf_{\det(B_r + A_{212,r}^{T_{2,1}}\theta)=0} P(\theta) \tag{19}$$

By Assumption 5, we have $\rho_M \in (1, \infty] \subset \mathbb{R}_e$. Choose $\rho_o \in (1, \rho_M) \subset \mathbb{R}$. We will design the smooth projection algorithm such that the estimate $\check{\theta}$ lies in the open set

$$\Theta_o := \{\, \theta \in \mathbb{R}^\sigma \mid P(\theta) < \rho_o \,\} \subset \Theta_{\rho_o}$$

It is immediate that this implies that $B_r + A_{212,r}^{T_{2,1}}\check{\theta}$ is invertible, $\forall \check{\theta} \in \Theta_{\rho_o}$, and there exists $c_0 \in \overline{\mathbb{R}_+}$, such that $\left\| (B_r + A_{212,r}^{T_{2,1}}\check{\theta})^{-1} \right\| \le c_0$, $\forall \check{\theta} \in \Theta_{\rho_o}$.

By Proposition 4 on Page 178 of [18], we have

$$\frac{\partial P}{\partial \theta}(\check{\theta})(\theta - \check{\theta}) \le P(\theta) - P(\check{\theta}) \le 1 - P(\check{\theta}); \qquad \forall \check{\theta} \in \mathbb{R}^\sigma \tag{20}$$

We now add to the right-hand side of the dynamics (14b) for $\check{\xi}$ the following term when $P(\check{\theta}) > 1$:

$$-\frac{\exp(-\frac{1}{P(\check{\theta})-1})}{(\rho_o - P(\check{\theta}))^3}\bar{\Sigma}\left[\, \frac{\partial P}{\partial \theta}(\check{\theta}) \;\; \mathbf{0}_{1\times nm} \,\right]'$$

Hence, we have

$$\begin{aligned}\dot{\check{\xi}} \;=\; & -(1 - \chi_{\Theta,\mathrm{R}^\sigma}(\check{\theta}))\frac{\exp(-\frac{1}{P(\check{\theta})-1})}{(\rho_o - P(\check{\theta}))^3}\bar{\Sigma}\left[\, \frac{\partial P}{\partial \theta}(\check{\theta}) \;\; \mathbf{0}_{1\times nm} \,\right]' + \bar{A}\check{\xi} - \bar{\Sigma}\bar{C}'(y_d - \bar{C}\check{\xi}) + \bar{\bar{A}}y \\ & - \bar{\Sigma}\bar{Q}(\hat{\xi} - \check{\xi}) + \bar{\bar{B}}_b u_b + \bar{B}u_a + \bar{\bar{D}}\check{w} + (\gamma^2\bar{\Sigma}\bar{C}' + \bar{L})N^{-1}(y - \bar{C}\check{\xi}); \quad \check{\xi}(0) = \begin{bmatrix} \check{\theta}_0 \\ \check{x}_0 \end{bmatrix}\end{aligned} \tag{21}$$

It is easy to verify that the following nonlinear functions P_r and p_r

$$P_r(\check{\theta}) := (1 - \chi_{\Theta,\mathrm{R}^\sigma}(\check{\theta}))\frac{\exp(-\frac{1}{P(\check{\theta})-1})}{(\rho_o - P(\check{\theta}))^3}(\frac{\partial P}{\partial \theta}(\check{\theta}))' =: p_r(\check{\theta})(\frac{\partial P}{\partial \theta}(\check{\theta}))' = \frac{\kappa_1(P(\check{\theta}) - 1)}{(\rho_o - P(\check{\theta}))^3}(\frac{\partial P}{\partial \theta}(\check{\theta}))' \tag{22}$$

are C_∞ on the set Θ_o, where κ_1 is as defined in Definition 2. In view of this, the derivative of the value function W given by (17) is equal to

$$\begin{aligned}\dot{W} \;=\; & -\left|\bar{C}\xi - y_d\right|^2 + \left|y_d - \bar{C}\check{\xi}\right|^2 - \left|\xi - \hat{\xi}\right|^2_{\bar{Q}} + \left|\check{\xi} - \hat{\xi}\right|^2_{\bar{Q}} + \gamma^2|w_b|^2 \\ & -\gamma^2\left|y - \bar{C}\check{\xi}\right|^2_{N^{-1}} - \gamma^2\left|w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}, \hat{\xi}_{[0,t]})\right|^2 + 2(\theta - \check{\theta})'P_r(\check{\theta})\end{aligned}$$

The last term above appears because of the modification in the dynamics of $\check{\xi}$. We now have the following inequality:

$$2(\theta - \check{\theta})'P_r(\check{\theta}) \;=\; 2\frac{\kappa_1(P(\check{\theta}) - 1)}{(\rho_o - P(\check{\theta}))^3}\frac{\partial P}{\partial \theta}(\check{\theta})(\theta - \check{\theta}) \le 2\frac{\kappa_1(P(\check{\theta}) - 1)}{(\rho_o - P(\check{\theta}))^3}(1 - P(\check{\theta})) = 2p_r(\check{\theta})(1 - P(\check{\theta})) \le 0; \quad \forall \check{\theta} \in \Theta_o$$

which shows that the last term in the expression for $\dot{W}$ is nonpositive, is zero on the set Θ, and approaches $-\infty$ as $\check{\theta}$ approaches the boundary of the set Θ_o (i. e., $P(\check{\theta})$ approaches ρ_o).

To further deduce the existence of the covariance matrix $\bar{\Sigma}$ and the structure of the identifier, we pursue the following line of detailed analysis. First, partition the worst-case covariance matrix $\bar{\Sigma}$ (compatible with the partition of ξ) as

$$\bar{\Sigma} = \begin{bmatrix} \Sigma & \bar{\Sigma}_{12} \\ \bar{\Sigma}_{21} & \bar{\Sigma}_{22} \end{bmatrix} \tag{23}$$

and introduce the quantities:

$$\Phi := \bar{\Sigma}_{21}\Sigma^{-1} \tag{24a}$$

$$\Pi := \gamma^2(\bar{\Sigma}_{22} - \bar{\Sigma}_{21}\Sigma^{-1}\bar{\Sigma}_{12}) \tag{24b}$$

Next, choose the following structure for the weighting matrix $\bar{Q}$:

$$\bar{Q} = \bar{\Sigma}^{-1} \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times mn} \\ \mathbf{0}_{mn\times\sigma} & \Delta \end{bmatrix} \bar{\Sigma}^{-1} + \begin{bmatrix} \epsilon\Phi' C'(\gamma^2 N^{-1} - I_m) C\Phi & \mathbf{0}_{\sigma\times mn} \\ \mathbf{0}_{mn\times\sigma} & \mathbf{0}_{mn\times mn} \end{bmatrix} \tag{25}$$

where $\Delta := \gamma^{-2}\beta_\Delta \Pi + \gamma^{-2}(\Delta_1 \otimes I_m)$ with $\beta_\Delta \in \overline{\mathbb{R}_+}$ being a constant and $\Delta_1 \in \mathcal{S}_{+n}$ being an $n\times n$-dimensional positive-definite matrix; and ϵ is a scalar function defined by

$$\epsilon(t) := \frac{\operatorname{Tr}\left((\Sigma(t))^{-1}\right)}{K_c} \tag{26}$$

with $K_c \in [\gamma^2 \operatorname{Tr}\left(Q_0\right), \infty) \subset \mathbb{R}$ being a constant corresponding to the preselected maximum level for the quantity $\operatorname{Tr}\left((\Sigma(t))^{-1}\right)$. The Riccati differential equation (RDE) for $\bar{\Sigma}$ is expressed as

$$\begin{aligned} \dot{\bar{\Sigma}} &= (\bar{A} - \bar{L}N^{-1}\bar{C})\bar{\Sigma} + \bar{\Sigma}(\bar{A} - \bar{L}N^{-1}\bar{C})' + \frac{1}{\gamma^2}\bar{D}\bar{D}' - \frac{1}{\gamma^2}\bar{L}N^{-1}\bar{L}' - \bar{\Sigma}(\gamma^2\bar{C}'N^{-1}\bar{C} - \bar{C}'\bar{C} \\ &\quad - \begin{bmatrix} \epsilon\Phi' C'(\gamma^2 N^{-1} - I_m) C\Phi & \mathbf{0}_{\sigma\times mn} \\ \mathbf{0}_{mn\times\sigma} & \mathbf{0}_{mn\times mn} \end{bmatrix})\bar{\Sigma} + \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \Delta \end{bmatrix}; \\ \bar{\Sigma}(0) &= \gamma^{-2} \begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi_0' \end{bmatrix} \end{aligned}$$

By Lemma 6 of [5], we obtain the following differential equations for Σ, Φ, and Π:

$$\dot{\Sigma} = -(1-\epsilon(t))\Sigma\Phi' C'(\gamma^2 N^{-1} - I_m) C\Phi\Sigma; \quad \Sigma(0) = \gamma^{-2} Q_0^{-1} \tag{27a}$$

$$\dot{\Phi} = (A - LN^{-1}C - \Pi C'(N^{-1} - \frac{1}{\gamma^2} I_m)C)\Phi + A_{211,1}y + A_{211,2}u_b + A_{211,3}\check{w} + A_{212}u_a; \quad \Phi(0) = \Phi_0 \tag{27b}$$

$$\dot{\Pi} = (A - LN^{-1}C)\Pi + \Pi(A - LN^{-1}C)' - \Pi C'(N^{-1} - \frac{1}{\gamma^2} I_m)C\Pi + DD' - LN^{-1}L' + \gamma^2\Delta; \quad \Pi(0) = \Pi_0 \tag{27c}$$

We note that in order to guarantee the boundedness of the matrix Σ, we can pick γ such that $\gamma^2 N^{-1} \geq I_m$, i. e., $\gamma^2\zeta^2 \geq 1$ or $\gamma \geq \zeta^{-1}$. For the RDE (27c), we note that the pairs (A, C) and $(A, DD' - LN^{-1}L' + \gamma^2\Delta)$ are both observable. Then, the RDE (27c) admits a unique positive-definite solution on $[0, \infty)$, and the solution converges, as $t \to \infty$, to the unique positive-definite solution of the corresponding algebraic Riccati equation (28) below, if (28) admits a stabilizing positive-definite solution.

$$\begin{aligned} (A - LN^{-1}C + \frac{\beta_\Delta}{2} I_{mn})\Pi + \Pi(A - LN^{-1}C + \frac{\beta_\Delta}{2} I_{mn})' - \Pi C'(\zeta^2 I_m - \frac{1}{\gamma^2} I_m)C\Pi + DD' \\ -LN^{-1}L' + \Delta_1 \otimes I_m = \mathbf{0}_{mn\times mn} = \mathbf{0}_{n\times n} \otimes I_m \end{aligned} \tag{28}$$

Clearly, if $\gamma > \zeta^{-1}$, then (28) admits a unique positive-definite stabilizing solution. Because of the structure for A, D, C, and E, the above algebraic Riccati equation (28) admits a solution $\Pi = \Pi_1 \otimes I_m$, where Π_1 satisfies the algebraic Riccati equation (29) below in Assumption 8 —an assumption we make to clarify the possible choices of γ.

Assumption 8. The desired disturbance attenuation level γ satisfies $\gamma \geq \zeta^{-1}$ and is such that the following algebraic Riccati equation admits a positive-definite stabilizing solution Π_1:

$$(A_1 - \zeta^2 L_1 C_1 + \frac{\beta_\Delta}{2} I_n)\Pi_1 + \Pi_1(A_1 - \zeta^2 L_1 C_1 + \frac{\beta_\Delta}{2} I_n)' - \Pi_1 C_1'(\zeta^2 - \frac{1}{\gamma^2})C_1\Pi_1 + D_1 D_1' - \zeta^2 L_1 L_1' + \Delta_1 = \mathbf{0}_{n\times n} \tag{29}$$

that is, the matrix $A_1 - L_1\zeta^2 C_1 + \frac{\beta_\Delta}{2} I_n - \Pi_1 C_1'(\zeta^2 - \frac{1}{\gamma^2})C_1$ is Hurwitz.

Under Assumption 8, the RDE (27c) admits a positive-definite solution on the infinite horizon $[0, \infty)$. To further simplify the controller structure and allow a proof of the closed-loop robustness, we assume that $\Pi_0 = \Pi = \Pi_1 \otimes I_m$, where Π_1 is the positive-definite solution to (29). This implies that $\Pi(t) \equiv \Pi = \Pi_0 = \Pi_1 \otimes I_m$, where $\Pi(t)$ is the solution to (27c). Then, the matrix $A_f := A - LN^{-1}C - \Pi C'(N^{-1} - \frac{1}{\gamma^2} I_m)C = (A_1 - L_1\zeta^2 C_1 - \Pi_1 C_1'(\zeta^2 - \gamma^{-2})C_1) \otimes I_m =: A_{f1} \otimes I_m$ is Hurwitz.

From its definition, the function $\epsilon(t)$ can be shown to be less than or equal to 1 for any $t \geq 0$. Therefore, the covariance matrix Σ is nonincreasing. This result is summarized in the following lemma.

Lemma 1. Consider the matrix differential equation (27a) for the covariance matrix Σ. Let Assumption 8 hold. Then, the matrix Σ is uniformly upper and lower bounded as follows:

$$K_c^{-1} I_{\sigma\times\sigma} \leq \Sigma(t) \leq \Sigma(0) = \gamma^{-2} Q_0^{-1}$$

Proof. Let $[0, t_f]$ denote the maximum-length interval on which $\mathrm{Tr}(\Sigma^{-1}(t)) \le K_c$. Then, on this interval we have: $\dot{\Sigma} \le \mathbf{0}_{\sigma\times\sigma}$. If t_f is finite, then, we have $\mathrm{Tr}\left((\Sigma(t_f))^{-1}\right) = K_c$, and $\dot{\Sigma} = \mathbf{0}_{\sigma\times\sigma}$ on the interval $[t_f, \infty)$. This proves that t_f cannot be finite. Hence, the matrix Σ is nonincreasing on $[0, \infty)$, and this verifies the upper bound.

Since $t_f = \infty$, we have $\mathrm{Tr}\left((\Sigma(t))^{-1}\right) \le K_c$ on the interval $[0, \infty)$. Next, we observe the following inequality:

$$\mathrm{Tr}\left((\Sigma(t))^{-1}\right) \ge \lambda_{\max}((\Sigma(t))^{-1}) = \frac{1}{\lambda_{\min}(\Sigma(t))}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote, respectively, the minimum and maximum eigenvalues of a symmetric matrix. Therefore, we have $\lambda_{\min}(\Sigma(t)) \ge K_c^{-1}$, which yields the desired lower bound. □

In actual implementation, it is preferred not to invert the matrix Σ on line. Computation of such an inverse for the purpose of evaluating ϵ can in fact be avoided (see [5]). Let $s_\Sigma(t) := \mathrm{Tr}\left((\Sigma(t))^{-1}\right)$; thus, we have

$$\dot{s}_\Sigma = (1-\epsilon)(\gamma^2\zeta^2 - 1)\,\mathrm{Tr}\left(C\Phi\Phi' C'\right); \quad s_\Sigma(0) = \gamma^2\,\mathrm{Tr}\left(Q_0\right); \quad \epsilon(t) = K_c^{-1} s_\Sigma(t) \tag{30}$$

For ease of reference, we now summarize collectively the equations describing the identifier derived heretofore.

$$
\begin{aligned}
&(A_1 - L_1\zeta^2 C_1 + \frac{\beta_\Delta}{2} I_n)\Pi_1 + \Pi_1(A_1 - L_1\zeta^2 C_1 + \frac{\beta_\Delta}{2} I_n)' - \Pi_1 C_1'(\zeta^2 - \gamma^{-2})C_1\Pi_1 + D_1 D_1' - L_1\zeta^2 L_1' + \Delta_1 = \mathbf{0}_{n\times n} && \text{(31a)}\\
&\dot{\Sigma} = -(1-\epsilon)(\gamma^2\zeta^2 - 1)\Sigma\Phi' C' C\Phi\Sigma; \quad \Sigma(0) = \gamma^{-2} Q_0^{-1} && \text{(31b)}\\
&\dot{s}_\Sigma = (1-\epsilon)(\gamma^2\zeta^2 - 1)\,\mathrm{Tr}\left(C\Phi\Phi' C'\right); \qquad s_\Sigma(0) = \gamma^2\,\mathrm{Tr}\left(Q_0\right) && \text{(31c)}\\
&\epsilon = K_c^{-1} s_\Sigma && \text{(31d)}\\
&A_f = A_{f1} \otimes I_m; \quad A_{f1} = A_1 - L_1\zeta^2 C_1 - \Pi_1 C_1'(\zeta^2 - \gamma^{-2})C_1 && \text{(31e)}\\
&\dot{\Phi} = A_f\Phi + A_{211,1}y + A_{211,2}u_b + A_{211,3}\check{w} + A_{212}u_a; \quad \Phi(0) = \Phi_0 && \text{(31f)}\\
&\dot{\check{\theta}} = -\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'(y_d - C\check{x}) - \left[\Sigma\ \Sigma\Phi'\right]\bar{Q}\xi_c + \gamma^2\zeta^2\Sigma\Phi' C'(y - C\check{x}); \quad \check{\theta}(0) = \check{\theta}_0 && \text{(31g)}\\
&\dot{\check{x}} = -\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) + \check{A}y + (A_{211,1}y + A_{211,2}u_b + A_{211,3}\check{w} + A_{212}u_a)\check{\theta}\\
&\qquad + Bu_a + \check{B}_b u_b + \check{D}\check{w} - \left[\Phi\Sigma\ \ \gamma^{-2}\Pi + \Phi\Sigma\Phi'\right]\bar{Q}\xi_c + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi' C' + L)(y - C\check{x}); \quad \check{x}(0) = \check{x}_0 && \text{(31h)}
\end{aligned}
$$

where $\xi_c := \hat{\check{\xi}} - \check{\xi}$. Associated with this identifier, we have the upper bound of the value function:

$$W = |\xi - \check{\xi}|^2_{\bar{\Sigma}^{-1}} = |\theta - \check{\theta}|^2_{\Sigma^{-1}} + \gamma^2|x - \check{x} - \Phi(\theta - \check{\theta})|^2_{\Pi^{-1}} \tag{32}$$

whose time derivative is given by

$$
\begin{aligned}
\dot{W} = &-\left|Cx - y_d\right|^2 + \left|C\check{x} - y_d\right|^2 - \left|\xi - \hat{\check{\xi}}\right|^2_{\bar{Q}} + \left|\xi_c\right|^2_{\bar{Q}} + \gamma^2\left|w_b\right|^2 + 2(\theta - \check{\theta})' P_r(\check{\theta})\\
&-\gamma^2\zeta^2\left|y - C\check{x}\right|^2 - \gamma^2\left|w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}, \hat{\check{\xi}}_{[0,t]})\right|^2
\end{aligned} \tag{33}
$$

Also, the cost function (13) can equivalently be written as:

$$
\begin{aligned}
J_{i\gamma}^t = &-\left|\xi(t) - \check{\xi}(t)\right|^2_{(\bar{\Sigma}(t))^{-1}} + \int_0^t \Big(\left|C\check{x}(\tau) - y_d(\tau)\right|^2 + \left|\xi_c(\tau)\right|^2_{\bar{Q}(\tau, y_{[0,\tau]}, \check{w}_{[0,\tau]}, Y_{d[0,\tau]}, u_{[0,\tau]}, \hat{\check{\xi}}_{[0,\tau]})} + 2(\theta - \check{\theta}(\tau))' P_r(\check{\theta}(\tau))\\
&-\gamma^2\zeta^2\left|y(\tau) - C\check{x}(\tau)\right|^2 - \gamma^2\left|w_b(\tau) - w_*(\tau, \xi_{[0,\tau]}, y_{[0,\tau]}, u_{[0,\tau]}, \check{w}_{[0,\tau]}, y_{d[0,\tau]}, \hat{\check{\xi}}_{[0,\tau]})\right|^2\Big)\,\mathrm{d}\tau
\end{aligned} \tag{34}
$$

Note that the matrix Φ may be suitably generated by prefilters of signals of y, u_b, $\check{w}$, and u_a, to replace dynamics (31f) as follows to further simplify the identifier structure. The pairs $(A_1, \boldsymbol{e}_{n,n})$ and $(A_{f1}, \boldsymbol{e}_{n,n})$ are controllable. This implies that

$$M_f := \left[\, A_{f1}^{n-1}\boldsymbol{e}_{n,n}\ \cdots\ A_{f1}\boldsymbol{e}_{n,n}\ \boldsymbol{e}_{n,n} \,\right] \tag{35a}$$

is invertible.

It is then straightforward to verify that the following prefiltering system for y, u_b, $\check{w}$, and u_a generates the matrix Φ on line.

$$
\begin{aligned}
&\dot{\eta}_i = A_{f1}\eta_i + \boldsymbol{e}_{n,n}y_i; \qquad \eta_i(0) = \mathbf{0}_n;\ i = 1, \ldots, m && \text{(35b)}\\
&\dot{\lambda}_{ai} = A_{f1}\lambda_{ai} + \boldsymbol{e}_{n,n}u_{ai}; \qquad \lambda_{ai}(0) = \mathbf{0}_n;\ i = 1, \ldots, m && \text{(35c)}\\
&\dot{\lambda}_{bi} = A_{f1}\lambda_{bi} + \boldsymbol{e}_{n,n}u_{bi}; \qquad \lambda_{bi}(0) = \mathbf{0}_n;\ i = 1, \ldots, p - m && \text{(35d)}\\
&\dot{\check{\eta}}_{1i} = A_{f1}\check{\eta}_{1i} + \boldsymbol{e}_{n,n}\check{w}_{1i}; \qquad \check{\eta}_{1i}(0) = \mathbf{0}_n;\ i = 1, \ldots, \check{q}_1 && \text{(35e)}\\
&\dot{\check{\eta}}_{2i} = A_{f1}\check{\eta}_{2i} + \boldsymbol{e}_{n,n}\check{w}_{2i}; \qquad \check{\eta}_{2i}(0) = \mathbf{0}_n;\ i = 1, \ldots, \check{q}_2 && \text{(35f)}
\end{aligned}
$$

$$\dot{\lambda}_o = A_{f1}\lambda_o; \qquad \lambda_o(0) = e_{n,n} \tag{35g}$$

$$\begin{aligned}\Phi = &\sum_{i=1}^{m}\left((\left[A_{f1}^{n-1}\eta_i \ \cdots \ A_{f1}\eta_i \ \eta_i \right] M_f^{-1}) \otimes I_m \right) A_{211,1,:,:,i} + \sum_{i=1}^{p-m}\left((\left[A_{f1}^{n-1}\lambda_{bi} \ \cdots \ A_{f1}\lambda_{bi} \ \lambda_{bi} \right] M_f^{-1}) \otimes I_m \right) A_{211,2,:,:,i} \\ &+ \sum_{i=1}^{\check{q}_1}\left((\left[A_{f1}^{n-1}\check{\eta}_{1i} \ \cdots \ A_{f1}\check{\eta}_{1i} \ \check{\eta}_{1i} \right] M_f^{-1}) \otimes I_m \right) A_{211,3,1,:,:,i} + \sum_{i=1}^{m}\left((\left[A_{f1}^{n-1}\lambda_{ai} \ \cdots \ A_{f1}\lambda_{ai} \ \lambda_{ai} \right] M_f^{-1}) \otimes I_m \right) A_{212,:,:,i} \\ &+ \sum_{i=1}^{\check{q}_2}\left((\left[A_{f1}^{n-1}\check{\eta}_{2i} \ \cdots \ A_{f1}\check{\eta}_{2i} \ \check{\eta}_{2i} \right] M_f^{-1}) \otimes I_m \right) A_{211,3,2,:,:,i} + \left((\left[A_{f1}^{n-1}\lambda_o \ \cdots \ A_{f1}\lambda_o \ \lambda_o \right] M_f^{-1}) \otimes I_m \right) \Phi_0\end{aligned} \tag{35h}$$

where $y = (y_1, \dots, y_m)$, $u_a = (u_{a1}, \dots, u_{am})$, $u_b = (u_{b1}, \dots, u_{b\,p-m})$, $\check{w}_1 = (\check{w}_{11}, \dots, \check{w}_{1\check{q}_1})$, and $\check{w}_2 = (\check{w}_{21}, \dots, \check{w}_{2\check{q}_2})$.

This completes the identification design step. We now turn, in the next section, to the control design for the uncertain system, with the identifier above in place.

5 | CONTROLLER DESIGN

In this section, we describe the controller design for the uncertain system under consideration. The *key* identity obtained from the previous section is the equivalent form (34) of the cost function (or the expression (33) for the total derivative of W). Based on the equivalence (12), we now need to supremize $J_{i\gamma}^{t}$ over all measurement waveforms. In (34) and (33), we see that the cost function is given in terms of the estimated state variable $\check{x}$, $\check{\theta}$, and $\bar{\Sigma}$, whose dynamics are driven by the measurements y, $\check{w}$, y_d, and inputs u and $\hat{\xi}$, which are signals we either measure or can construct. This is then a full-information control design problem, which is truly nonlinear in nature. Instead of considering y as the maximizing variable, we can equivalently deal with the transformed variable:

$$v := y - C\check{x} =: y - \check{x}_1 \tag{36}$$

In terms of v, we have

$$\dot{\Phi} = A_f\Phi + A_{211,1}\check{x}_1 + A_{211,2}u_b + A_{211,3}\check{w} + A_{212}u_a + A_{211,1}v \tag{37a}$$

$$\dot{\check{\theta}} = -\Sigma P_r(\check{\theta}) - \Sigma\Phi' C'(y_d - \check{x}_1) - \left[\Sigma \ \Sigma\Phi'\right]\bar{Q}\xi_c + \gamma^2\zeta^2\Sigma\Phi' C' v \tag{37b}$$

$$\begin{aligned}\dot{\check{x}} = &-\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) + \check{A}\check{x}_1 + (A_{211,1}\check{x}_1 + A_{211,2}u_b + A_{211,3}\check{w} + A_{212}u_a)\check{\theta} \\ &+ Bu_a + \check{B}_b u_b + \check{D}\check{w} - \left[\Phi\Sigma \ \gamma^{-2}\Pi + \Phi\Sigma\Phi'\right]\bar{Q}\xi_c + (\check{A} + (A_{211,1}^{T_{2,1}}\check{\theta}) + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi' C' + L))v\end{aligned} \tag{37c}$$

$$\begin{aligned}\dot{W} = &-\left|x_1 - y_d\right|^2 + \left|\check{x}_1 - y_d\right|^2 - \left|\xi - \hat{\xi}\right|_{\bar{Q}}^2 + \left|\xi_c\right|_{\bar{Q}}^2 + \gamma^2\left|w_b\right|^2 \\ &+2(\theta - \check{\theta})' P_r(\check{\theta}) - \gamma^2\zeta^2\left|v\right|^2 - \gamma^2\left|w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}, \hat{\xi}_{[0,t]})\right|^2\end{aligned} \tag{37d}$$

The control design will make use of the integrator backstepping methodology [16]. We will further reveal the structure in the estimator dynamics that allows for the application of $(r+1)$-steps of integrator backstepping.

Note that A_f admits the same structure as the matrix A, with the first m columns changed by feedback. Then, the Φ dynamics can be rewritten as

$$\dot{\Phi}_1 = \hat{a}_{1,1}\Phi_1 + a_{1,2}\Phi_2 + A_{211,1,1}\check{x}_1 + A_{211,2,1}u_b + A_{211,3,2,1}\check{w}_2 + A_{211,1,1}v \tag{38a}$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{\Phi}_{r-1} = \hat{a}_{r-1,1}\Phi_1 + \cdots + a_{r-1,r}\Phi_r + A_{211,1,r-1}\check{x}_1 + A_{211,2,r-1}u_b + A_{211,3,2,r-1}\check{w}_2 + A_{211,1,r-1}v \tag{38b}$$

where $\Phi := \left[\Phi_1' \ \cdots \ \Phi_n'\right]'$ and Φ_i are $m \times \sigma$-dimensional matrices, $i = 1, \dots, n$; $A_{211,1,i}$ is the 2nd order $\mathbb{R}^m$-valued sub-tensor of $A_{211,1}$ that consists of the $((i-1)m+1)$st to (im)th indices in the output dimension, all indices in the first dimension, and all indices in the second dimension, $i = 1, \dots, n$; $A_{211,2,i}$ is the 2nd order $\mathbb{R}^m$-valued sub-tensor of $A_{211,2}$ that consists of the $((i-1)m+1)$st to (im)th indices in the output dimension, all indices in the first dimension, and all indices in the second dimension, $i = 1, \dots, n$; $A_{211,3,2,i}$ is the 2nd order $\mathbb{R}^m$-valued sub-tensor of $A_{211,3,2}$ that consists of the $((i-1)m+1)$st to (im)th indices in the output dimension, all indices in the first dimension, and all indices in the second dimension, $i = 1, \dots, n$. In the above, u_a and $\check{w}_1$ do not appear due to our assumption on their relative degrees.

We partition $\check{x} = (\check{x}_1, \dots, \check{x}_n)$, with $\check{x}_i$ being m-dimensional, $i = 1, \dots, n$. The rest of the relevant dynamics for the integrator backstepping control design are summarized in the following:

$$\dot{s}_\Sigma = (1 - K_c^{-1} s_\Sigma)(\gamma^2\zeta^2 - 1)\,\mathrm{Tr}\left(\Phi_1\Phi_1'\right) \tag{38c}$$

$$\dot{\Sigma} = -(1 - K_c^{-1} s_\Sigma)(\gamma^2\zeta^2 - 1)\Sigma\Phi_1'\Phi_1\Sigma \tag{38d}$$

$$\dot{\check{\theta}} = \delta(y_d, \check{x}_1, \Phi_1, \check{\theta}, \Sigma) + \varphi(\Phi, \Sigma)\bar{Q}\xi_c + h_\theta(\Phi_1, \Sigma)v \tag{38e}$$

$$\begin{aligned}\dot{\check{x}}_i =\;& f_i(y_d, \check{x}_1, \dots, \check{x}_i, \check{\theta}, \Phi_1, \Phi_i, \Sigma) + a_{i,i+1}\check{x}_{i+1} + \varrho_i(\Phi, \Sigma)\bar{Q}\xi_c + (A_{211,2,i}^{T_{2,1}}\check{\theta} + \check{B}_{b,i})u_b \\ & +(A_{211,3,2,i}^{T_{2,1}}\check{\theta} + \check{D}_{2,i})\check{w}_2 + h_i(\check{\theta}, \Phi_1, \Phi_i, \Sigma)v; \quad i = 1, \dots, r-1\end{aligned} \tag{38f}$$

$$\begin{aligned}\dot{\check{x}}_r =\;& f_r(y_d, \check{x}_1, \dots, \check{x}_r, \check{\theta}, \Phi_1, \Phi_r, \Sigma) + a_{r,r+1}\check{x}_{r+1} + \varrho_r(\Phi, \Sigma)\bar{Q}\xi_c + (A_{211,2,r}^{T_{2,1}}\check{\theta} + \check{B}_{b,r})u_b \\ & +(A_{211,3,2,r}^{T_{2,1}}\check{\theta} + \check{D}_{2,r})\check{w}_2 + (A_{212,r}^{T_{2,1}}\check{\theta} + B_r)u_a + (A_{211,3,1,r}^{T_{2,1}}\check{\theta} + \check{D}_{1,r})\check{w}_1 + h_r(\check{\theta}, \Phi_1, \Phi_r, \Sigma)v\end{aligned} \tag{38g}$$

where $A_{211,3,1,i}$ is the 2nd order $\mathbb{R}^m$-valued sub-tensor of $A_{211,3,1}$ that consists of the $((i-1)m+1)$st to (im)th indices in the output dimension, all indices in the first dimension, and all indices in the second dimension, $i = 1, \dots, n$; $\check{B}_{b,i}$ is the $m \times (p-m)$ dimensional submatrix of $\check{B}_b$ that consists of the $(m(i-1)+1)$st to (mi)th rows, $i = 1, \dots, n$; $\check{D}_{2,i}$ is the $m \times \check{q}_2$ dimensional submatrix of $\check{D}_2$ that consists of the $(m(i-1)+1)$st to (mi)th rows, $i = 1, \dots, n$; $\check{D}_{1,i}$ is the $m \times \check{q}_1$ dimensional submatrix of $\check{D}_1$ that consists of the $(m(i-1)+1)$st to (mi)th rows, $i = 1, \dots, n$; the nonlinear functions φ, h_θ, f_i, ϱ_i, h_i and δ are C_∞ as long as $\check{\theta} \in \Theta_o$, $s_\Sigma \in \mathbb{R}_+$, and $\Sigma \in \mathcal{S}_{+\sigma}$.

Toward applying the integrator backstepping procedure to the above system, we first observe that s_Σ, Σ, and $\check{\theta}$ are always bounded by the particular choice of the identifier. The system structure allows the integrator backstepping from the output $\check{x}_1 - y_d$ to step back to $\check{x}_2$, …, $\check{x}_r$, and then step back to the control input u_a. The inputs u_b are unstructured, and cannot be used in this process, and will be set to $\mathbf{0}_{p-m}$. The input $\hat{\xi}$ (or $\hat{\xi}_c$ to be precise) has nonnegative weighting in the cost function, which can not be used in the backstepping process, and $\hat{\xi}_c$ will be set to $\mathbf{0}_{\sigma+mn}$. The choices of u_b and $\hat{\xi}_c$ will be determined after the upper bound of the value function for the closed-loop system has been obtained to further assist the stabilization and disturbance attenuation objective. Thus, the backstepping procedure can only stabilize rm states. We will carry out the control design as if they were bounded, and prove later that they are indeed so under the derived control law.

Based on the equivalent form (34) of the cost function, or the expression (33) for the total derivative of W, we need only achieve 0 level of disturbance attenuation with respect to $\check{w}_1$, γ level of disturbance attenuation with respect to $\check{w}_2$, and $\gamma\zeta$ level of disturbance attenuation with respect to the equivalent disturbance v. Note that $\check{w}_1$ does not appear in (38) except in $\check{x}_r$ dynamics (38g). Then, the effect of $\check{w}_1$ on $\check{x}_1$ can be cancelled out entirely by the control input u_a. The measured disturbance input $\check{w}_2$ enters (38) before u_a enters the dynamics. This means that $\check{w}_2$ must be attenuated like v in the control design. The main backstepping lemma we will apply at each of the $r+1$ steps is Lemma 6 or Lemma 7, both of Appendix B.

Step 0: Due to robustness concerns, not related to the objectives of this paper, we will include this step in the backstepping design. Introduce the dynamics

$$\dot{\tilde{\eta}} = \lambda_m\tilde{\eta} + y - y_d; \qquad \tilde{\eta}(0) = \mathbf{0}_m \tag{39}$$

where $\tilde{\eta}$ is an m-dimensional additional state variable, $\lambda_m \in \mathbb{R}_-$ is a design parameter chosen as $\lambda_m \approx \max(\mathrm{Re}\left(\lambda(A_{f1})\right)) \in \mathbb{R}_-$, where $\lambda(A_{f1})$ denotes the eigenvalues of A_{f1}. Then, we have $\dot{\tilde{\eta}} = \lambda_m\tilde{\eta} + \check{x}_1 - y_d + v$. There exist positive-definite matrices $Z, Y \in \mathcal{S}_{+m}$ (which may or may not be chosen as diagonal matrices) such that

$$2\lambda_m Z + \frac{1}{\gamma^2\zeta^2}ZZ + Y = \mathbf{0}_{m\times m} \tag{40}$$

Then, if we choose the value function $V_0 = |\tilde{\eta}|_Z^2$, we have

$$\dot{V}_0 = \gamma^2\zeta^2\,|v| - \gamma^2\zeta^2\left|v - \frac{1}{\gamma^2\zeta^2}Z\tilde{\eta}\right|^2 - |\tilde{\eta}|_Y^2 + 2\tilde{\eta}'Z(\check{x}_1 - y_d) \tag{41}$$

Then, the desired virtual control law for $\check{x}_1$ is y_d.

We will now distinguish two exhaustive and mutually exclusive cases: $r = 1$ and $r > 1$. First, consider the case $r > 1$.

Step 1: Define $z_1 := \check{x}_1 - y_d$. To apply Lemma 6 (or Lemma 7 for a much simplified controller), we identify

$$\begin{gathered} X_{1o} := (y_d, \check{\theta}, \Sigma, s_\Sigma, \tilde{\eta}) \to x_o; \quad X_{1a} := \check{x}_1 \to x_a; \quad X_{1d} := (y_d^{(1)}, \Phi_1) \to x_d; \quad \check{x}_2 \to u; \quad (\check{w}_2, v) \to w \\ \infty \to k; \quad \mathcal{D}_{1o} := \mathbb{R}^m \times \Theta_o \times \mathcal{S}_{+\sigma} \times \mathbb{R}_+ \times \mathbb{R}^m \to D_o; \quad \mathcal{D}_{1a} := \mathbb{R}^m \to D_a; \quad \mathcal{D}_{1d} := \mathbb{R}^m \times \mathbb{R}^{m\times\sigma} \to D_d \\ V_0 \to V_o; \quad \mathbb{R}^m \to \mathcal{U}; \quad \mathcal{D}_w := \mathbb{R}^{\check{q}_2+m} \to D_w; \quad \left|\check{w}_2\right|^2 + \zeta^2\,|v|^2 \to \left\|(\check{w}_2, v)\right\|_{\mathcal{W}}^2; \quad \mathcal{D}_{1o} \times \mathcal{D}_{1d} \to D_1 \end{gathered}$$

$$\mathcal{X}_{1d} := \mathcal{D}_{1d} \to \mathcal{X}_d; \quad (\mathbf{0}_{\check{q}_2}, \tfrac{1}{\gamma^2\zeta^2} Z\tilde{\eta}) \to \sigma_o; \quad y_d \to \alpha_o; \quad |\tilde{\eta}|_Y^2 \to l_o; \quad \mathcal{W} := (\mathcal{D}_w, \mathbb{R}, \|\cdot\|_{\mathcal{W}}) \to \mathcal{W}$$

and

$$\begin{aligned}
f_o &\leftarrow \begin{bmatrix} y_d^{(1)} \\ \delta(y_d, \check{x}_1, \Phi_1, \check{\theta}, \Sigma) \\ -(1 - K_c^{-1} s_\Sigma)(\gamma^2\zeta^2 - 1)\Sigma\Phi_1'\Phi_1\Sigma \\ (1 - K_c^{-1} s_\Sigma)(\gamma^2\zeta^2 - 1)\operatorname{Tr}(\Phi_1\Phi_1') \\ \lambda_m\tilde{\eta} + \check{x}_1 - y_d \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^\sigma \times \mathcal{S}_\sigma \times \mathbb{R} \times \mathbb{R}^m =: \mathcal{X}_{1o} \to \mathcal{X}_o \\
h_o &\leftarrow \begin{bmatrix} \mathbf{0}_{m\times(\check{q}_2+m)} \\ \mathbf{0}_{\sigma\times\check{q}_2} \;\; h_\theta(\Phi_1, \Sigma) \\ \mathbf{0}_{\sigma\times\sigma\times(\check{q}_2+m)} \\ \mathbf{0}_{1\times(\check{q}_2+m)} \\ \mathbf{0}_{m\times\check{q}_2} \;\; I_m \end{bmatrix} \in \mathrm{B}\left(\mathcal{W}, \mathcal{X}_{1o}\right) \\
f_a &\leftarrow f_1(y_d, \check{x}_1, \check{\theta}, \Phi_1, \Phi_1, \Sigma) \in \mathbb{R}^m =: \mathcal{X}_{1a} \to \mathcal{X}_a \\
g_a &\leftarrow a_{1,2} I_m \in \mathrm{B}\left(\mathbb{R}^m, \mathcal{X}_{1a}\right) \\
h_a &\leftarrow \left[\, A_{211,3,2,1}^{T_{2,1}}\check{\theta} + \check{D}_{2,1} \;\; h_1(\check{\theta}, \Phi_1, \Phi_1, \Sigma) \,\right] \in \mathrm{B}\left(\mathcal{W}, \mathcal{X}_{1a}\right)
\end{aligned}$$

Choose two $\mathcal{C}_\infty$ mappings $\gamma_1 : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$ and $\beta_1 : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$, where $\hat{\Theta}_{\rho_o}$ is an arbitrary open set in $\mathbb{R}^\sigma$ that $\hat{\Theta}_{\rho_o} \supset \Theta_{\rho_o} \supset \Theta_o$. Next, in the application of Lemma 6, we make the following substitutions.

$$\gamma_1 \to Z; \quad (I_m + \beta_1)z_1 \to \phi; \quad V_1 \to V; \quad \alpha_1 \to \alpha$$

Then, $V_1 = V_0 + \left| z_1 \right|^2_{\gamma_1(\check{\theta})}$, $V_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \to \overline{\mathbb{R}_+}$, and $\alpha_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}^m$ are smooth and such that

$$\dot{V}_1\Big|_{\check{x}_2=\alpha_1(X_{1o},X_{1a},X_{1d})} = -l_1(X_{1o}, X_{1a}, X_{1d}) + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\zeta^2|v|^2 - \gamma^2\left|(\check{w}_2, v) - \bar{v}_1\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & N^{-1} \end{bmatrix}};$$

$$\forall (X_{1o}, X_{1d}) \in \mathcal{D}_{1o} \times \mathcal{D}_{1d}, \ \forall X_{1a} \in \mathcal{D}_{1a}, \ \forall (\check{w}_2, v) \in \mathcal{D}_w$$

where $l_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \overline{\mathbb{R}_+}$ and $\bar{v}_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathcal{W}$ are smooth and appropriately defined; $l_1(X_{1o}, X_{1a}, X_{1d}) \geq |\tilde{\eta}|_Y^2 + \left|z_1\right|^2 + \left|z_1\right|^2_{\beta_1(\check{\theta})} \geq 0$, $\forall (X_{1o}, X_{1a}, X_{1d}) \in \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d}$.

If $\check{x}_2$ had been the actual control input, then we would have used the following virtual control law: $\check{x}_2 = \alpha_1(X_{1o}, X_{1a}, X_{1d})$ to guarantee the dissipation inequality with supply rate:

$$-\left|\check{x}_1 - y_d\right|^2 - |\tilde{\eta}|_Y^2 - \left|z_1\right|^2_{\beta_1(\check{\theta})} + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\zeta^2|v|^2$$

This completes this step of the backstepping design.

Step i, $1 < i < r$: We inductively assume that we have completed $i - 1$ steps of the backstepping procedure, and obtained

$$X_{jo} := (y_d, \check{\theta}, \Sigma, s_\Sigma, \tilde{\eta}, \check{x}_1, y_d^{(1)}, \Phi_1, \ldots, \check{x}_{j-1}, y_d^{(j-1)}, \Phi_{j-1}); \quad j = 1, \ldots, i-1 \tag{42a}$$

$$\mathcal{D}_{jo} := \mathbb{R}^m \times \Theta_o \times \mathcal{S}_{+\sigma} \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times\sigma} \times \cdots \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times\sigma}; \quad j = 1, \ldots, i-1 \tag{42b}$$

$$X_{ja} := \check{x}_j; \quad \mathcal{D}_{ja} := \mathbb{R}^m; \quad j = 1, \ldots, i-1 \tag{42c}$$

$$X_{jd} := (y_d^{(j)}, \Phi_j); \quad \mathcal{D}_{jd} := \mathbb{R}^m \times \mathbb{R}^{m\times\sigma}; \quad j = 1, \ldots, i-1 \tag{42d}$$

$$\alpha_j : \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}^m; \quad j = 1, \ldots, i-1 \tag{42e}$$

$$\beta_j : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}; \quad j = 1, \ldots, i-1 \tag{42f}$$

$$\bar{v}_{i-1} : \mathcal{D}_{i-1\,o} \times \mathcal{D}_{i-1\,a} \times \mathcal{D}_{i-1\,d} \to \mathcal{W} \tag{42g}$$

$$z_j = \check{x}_j - \alpha_{j-1}(X_{j-1\,o}, X_{j-1\,a}, X_{j-1\,d}); \quad j = 1, \ldots, i-1 \tag{42h}$$

$$\gamma_j : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}; \quad j = 1, \ldots, i-1 \tag{42i}$$

$$V_{i-1} = |\tilde{\eta}|_Z^2 + \sum_{j=1}^{i-1}\left|z_j\right|^2_{\gamma_j(\check{\theta})}; \quad V_{i-1} : \mathcal{D}_{i-1\,o} \times \mathcal{D}_{i-1\,a} \to \overline{\mathbb{R}_+}; \tag{42j}$$

$$\dot{V}_{i-1}\big|_{\check{x}_i=\alpha_{i-1}(X_{i-1\,o},X_{i-1\,a},X_{i-1\,d})} = -l_{i-1}(X_{i-1\,o},X_{i-1\,a},X_{i-1\,d}) + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\zeta^2\left|v\right|^2 - \gamma^2\left|(\check{w}_2,v)-\bar{v}_{i-1}\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & N^{-1}\end{bmatrix}};$$

$$\forall(X_{i-1\,o},X_{i-1\,d})\in\mathcal{D}_{i-1\,o}\times\mathcal{D}_{i-1\,d},\ \forall X_{i-1\,a}\in\mathcal{D}_{i-1\,a},\ \forall(\check{w}_2,v)\in\mathcal{D}_w \tag{42k}$$

where the nonlinear functions α_j, β_j, and γ_j, $j=1,\ldots,i-1$, $\bar{v}_{i-1}$, l_{i-1}, and V_{i-1} are smooth on their domains of definition; and $l_{i-1}(X_{i-1\,o},X_{i-1\,a},X_{i-1\,d}) \ge \left|z_1\right|^2 + |\tilde{\eta}|^2_Y + \sum_{j=1}^{i-1}\left|z_j\right|^2_{\beta_j(\check{\theta})} \ge 0$, $\forall(X_{i-1\,o},X_{i-1\,a},X_{i-1\,d})\in\mathcal{D}_{i-1\,o}\times\mathcal{D}_{i-1\,a}\times\mathcal{D}_{i-1\,d}$.

At the current step i, we again apply Lemma 6 (or Lemma 7 for a much simplified controller). Toward that end, introduce

$$z_i = \check{x}_i - \alpha_{i-1}(X_{i-1\,o},X_{i-1\,a},X_{i-1\,d}) \tag{43}$$

and make the following substitution to apply Lemma 6.

$$\begin{gathered} X_{io} := (X_{i-1\,o},X_{i-1\,a},X_{i-1\,d}) \to x_o; \quad X_{ia} := \check{x}_i \to x_a; \quad X_{id} := (y_d^{(i)},\Phi_i) \to x_d; \quad \check{x}_{i+1} \to u \\ \mathcal{D}_{io} := \mathcal{D}_{i-1\,o}\times\mathcal{D}_{i-1\,a}\times\mathcal{D}_{i-1\,d} \to D_o; \quad \mathcal{D}_{ia} := \mathbb{R}^m \to D_a; \quad (\check{w}_2,v)\to w; \quad \mathcal{D}_{id} := \mathbb{R}^m\times\mathbb{R}^{m\times\sigma} \to D_d \\ \mathcal{D}_w \to D_w; \quad \left|\check{w}_2\right|^2 + \zeta^2\left|v\right|^2 \to \left\|(\check{w}_2,v)\right\|^2_{\mathcal{W}}; \quad \mathcal{D}_{io}\times\mathcal{D}_{id} \to D_1; \quad \mathcal{W}\to\mathcal{W}; \quad \mathcal{X}_{id} := \mathcal{D}_{id} \to \mathcal{X}_d \\ \mathbb{R}^m \to \mathcal{U}; \quad V_{i-1}\to V_o; \quad \bar{v}_{i-1}\to\sigma_o; \quad \alpha_{i-1}\to\alpha_o; \quad l_{i-1}\to l_o; \quad \infty \to k \end{gathered}$$

and

$$\begin{aligned} f_o &\leftarrow \begin{bmatrix} f_{o\,\text{Step}\,i-1} \\ f_{a\,\text{Step}\,i-1} + g_{a\,\text{Step}\,i-1}\check{x}_i \\ y_d^{(i)} \\ \hat{a}_{i-1,1}\Phi_1 + a_{i-1,2}\Phi_2 + \cdots + a_{i-1,i}\Phi_i + A_{211,1,i-1}\check{x}_1 \end{bmatrix} \in \mathcal{X}_{i-1\,o}\times\mathcal{X}_{i-1\,a}\times\mathcal{X}_{i-1\,d} =: \mathcal{X}_{io} \to \mathcal{X}_o \\ h_o &\leftarrow \begin{bmatrix} h_{o\,\text{Step}\,i-1} \\ h_{a\,\text{Step}\,i-1} \\ \mathbf{0}_{m\times(\check{q}_2+m)} \\ A_{211,3,2,i-1} \;\; A_{211,1,i-1} \end{bmatrix} \in \mathrm{B}\left(\mathcal{W},\mathcal{X}_{io}\right) \\ f_a &\leftarrow f_i(y_d,\check{x}_1,\ldots,\check{x}_i,\check{\theta},\Phi_1,\Phi_i,\Sigma) \in \mathbb{R}^m =: \mathcal{X}_{ia} \to \mathcal{X}_a \\ g_a &\leftarrow a_{i,i+1}I_m \in \mathrm{B}\left(\mathbb{R}^m,\mathcal{X}_{ia}\right) \\ h_a &\leftarrow \left[\, A^{T_{2,1}}_{211,3,2,i}\check{\theta} + \check{D}_{2,i} \;\; h_i(\check{\theta},\Phi_1,\Phi_i,\Sigma) \,\right] \in \mathrm{B}\left(\mathcal{W},\mathcal{X}_{ia}\right) \end{aligned}$$

Note that

$$X_{io} := (y_d,\check{\theta},\Sigma,s_\Sigma,\tilde{\eta},\check{x}_1,y_d^{(1)},\Phi_1,\ldots,\check{x}_{i-1},y_d^{(i-1)},\Phi_{i-1})$$

Choose two C_∞ mappings $\gamma_i : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$ and $\beta_i : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$. Next, in the application of Lemma 6 (or Lemma 7), we make the following substitutions:

$$\gamma_i \to Z; \quad \beta_i z_i \to \phi; \quad V_i \to V; \quad \alpha_i \to \alpha$$

Then, $V_i = V_{i-1} + \left|z_i\right|^2_{\gamma_i(\check{\theta})}$, $V_i : \mathcal{D}_{io}\times\mathcal{D}_{ia} \to \overline{\mathbb{R}_+}$, and $\alpha_i : \mathcal{D}_{io}\times\mathcal{D}_{ia}\times\mathcal{D}_{id} \to \mathbb{R}^m$ are smooth and such that

$$\dot{V}_i\big|_{\check{x}_{i+1}=\alpha_i(X_{io},X_{ia},X_{id})} = -l_i(X_{io},X_{ia},X_{id}) + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\zeta^2\left|v\right|^2 - \gamma^2\left|(\check{w}_2,v)-\bar{v}_i\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & N^{-1}\end{bmatrix}};$$

$$\forall(X_{io},X_{id})\in\mathcal{D}_{io}\times\mathcal{D}_{id},\ \forall X_{ia}\in\mathcal{D}_{ia},\ \forall(\check{w}_2,v)\in\mathcal{D}_w$$

where $l_i : \mathcal{D}_{io}\times\mathcal{D}_{ia}\times\mathcal{D}_{id} \to \overline{\mathbb{R}_+}$ and $\bar{v}_i : \mathcal{D}_{io}\times\mathcal{D}_{ia}\times\mathcal{D}_{id} \to \mathcal{W}$ are smooth and appropriately defined; $l_i(X_{io},X_{ia},X_{id}) \ge l_{i-1}(X_{i-1\,o},X_{i-1\,a},X_{i-1\,d}) + \left|z_i\right|^2_{\beta_i(\check{\theta})} \ge |\tilde{\eta}|^2_Y + \left|z_1\right|^2 + \sum_{j=1}^{i}\left|z_j\right|^2_{\beta_j(\check{\theta})} \ge 0$, $\forall(X_{io},X_{ia},X_{id})\in\mathcal{D}_{io}\times\mathcal{D}_{ia}\times\mathcal{D}_{id}$.

If $\check{x}_{i+1}$ is the actual control variable, we can choose the following virtual control law $\check{x}_{i+1}=\alpha_i$, which then guarantees the dissipation inequality with a supply rate of

$$-\left|\check{x}_1 - y_d\right|^2 - |\tilde{\eta}|^2_Y - \sum_{j=1}^{i}\left|z_j\right|^2_{\beta_j(\check{\theta})} + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\zeta^2\left|v\right|^2$$

This completes this step of the backstepping design.

Step r: Define

$$z_r := \check{x}_r - \alpha_{r-1}(X_{r-1\,o}, X_{r-1\,a}, X_{r-1\,d}) \tag{44}$$

Make use of Lemma 6 (or Lemma 7 for a simplified controller) to design the control function for u, by making the following substitutions:

$$\begin{gathered} X_{ro} := (X_{r-1\,o}, X_{r-1\,a}, X_{r-1\,d}) \to x_o; \quad X_{ra} := \check{x}_r \to x_a; \quad (\check{w}_2, v) \to w; \quad u_a \to u \\ X_{rd} := (y_d^{(r)}, \Phi_r, \check{x}_{r+1}, \ldots, \check{x}_n, \Phi_{r+1}, \ldots, \Phi_n, \check{w}_1) \to x_d; \quad \mathcal{D}_{ra} := \mathbb{R}^m \to D_a; \quad \mathcal{D}_w \to D_w \\ \mathcal{D}_{ro} := \mathcal{D}_{r-1\,o} \times \mathcal{D}_{r-1\,a} \times \mathcal{D}_{r-1\,d} \to D_o; \quad \left|\check{w}_2\right|^2 + \zeta^2 |v|^2 \to \|(\check{w}_2, v)\|_{\mathcal{W}}^2; \quad \mathcal{W} \to \mathcal{W} \\ \mathcal{D}_{rd} := \mathbb{R}^m \times \mathbb{R}^{m\times\sigma} \times \mathbb{R}^{(n-r)m} \times \mathbb{R}^{(n-r)m\times\sigma} \times \mathbb{R}^{\check{q}_1} \to D_d; \quad \mathcal{D}_{ro} \times \mathcal{D}_{rd} \to D_1; \quad V_{r-1} \to V_o \\ \mathbb{R}^m \to \mathcal{U}; \quad \bar{v}_{r-1} \to \sigma_o; \quad \alpha_{r-1} \to \alpha_o; \quad l_{r-1} \to l_o; \quad \infty \to k; \quad \mathcal{X}_{rd} := \mathcal{D}_{rd} \to \mathcal{X}_d \end{gathered}$$

and

$$\begin{aligned} f_o &\leftarrow \begin{bmatrix} f_{o\,\text{Step}\,r-1} \\ f_{a\,\text{Step}\,r-1} + g_{a\,\text{Step}\,r-1}\check{x}_r \\ y_d^{(r)} \\ \hat{a}_{r-1,1}\Phi_1 + a_{r-1,2}\Phi_2 + \cdots + a_{r-1,r}\Phi_r + A_{211,1,r-1}\check{x}_1 \end{bmatrix} \in \mathcal{X}_{r-1\,o} \times \mathcal{X}_{r-1\,a} \times \mathcal{X}_{r-1\,d} =: \mathcal{X}_{ro} \to \mathcal{X}_o \\ h_o &\leftarrow \begin{bmatrix} h_{o\,\text{Step}\,r-1} \\ h_{a\,\text{Step}\,r-1} \\ \mathbf{0}_{m\times(\check{q}_2+m)} \\ A_{211,3,2,r-1} \;\; A_{211,1,r-1} \end{bmatrix} \in \mathrm{B}\left(\mathcal{W}, \mathcal{X}_{ro}\right) \\ f_a &\leftarrow f_r(y_d, \check{x}_1, \ldots, \check{x}_r, \check{\theta}, \Phi_1, \Phi_r, \Sigma) + a_{r,r+1}\check{x}_{r+1} + (A_{211,3,1,r}^{T_{2,1}}\check{\theta} + \check{D}_{1,r})\check{w}_1 \in \mathbb{R}^m =: \mathcal{X}_{ra} \to \mathcal{X}_a \\ g_a &\leftarrow A_{212,r}^{T_{2,1}}\check{\theta} + B_r \in \mathrm{B}\left(\mathbb{R}^m, \mathcal{X}_{ra}\right) \\ h_a &\leftarrow \left[\, A_{211,3,2,r}^{T_{2,1}}\check{\theta} + \check{D}_{2,r} \;\; h_r(\check{\theta}, \Phi_1, \Phi_r, \Sigma) \,\right] \in \mathrm{B}\left(\mathcal{W}, \mathcal{X}_{ra}\right) \end{aligned}$$

Note that

$$X_{ro} := (y_d, \check{\theta}, \Sigma, s_\Sigma, \tilde{\eta}, \check{x}_1, y_d^{(1)}, \Phi_1, \ldots, \check{x}_{r-1}, y_d^{(r-1)}, \Phi_{r-1})$$

and $A_{212,r}^{T_{2,1}}\check{\theta} + B_r = B_{p0}(\check{\theta})$ is invertible since $\check{\theta} \in \Theta_o$. Choose two $\mathcal{C}_\infty$ mappings $\gamma_r : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$ and $\beta_r : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$. Next, in the application of Lemma 6 (or Lemma 7), we make the following substitutions:

$$\gamma_r \to Z; \quad \beta_r z_r \to \phi; \quad V_r \to V; \quad \mu_a \to \alpha$$

Then, $V_r = V_{r-1} + \left|z_r\right|^2_{\gamma_r(\check{\theta})}$, $V_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \to \overline{\mathbb{R}_+}$, and $\mu_a : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}^m$ are smooth and such that

$$\begin{gathered} \dot{V}_r\big|_{u_a=\mu_a(X_{ro},X_{ra},X_{rd})} = -l_r(X_{ro}, X_{ra}, X_{rd}) + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\zeta^2|v|^2 - \gamma^2\left|(\check{w}_2, v) - \bar{v}_r\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & N^{-1} \end{bmatrix}}; \\ \forall (X_{ro}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{rd},\ \forall X_{ra} \in \mathcal{D}_{ra},\ \forall (\check{w}_2, v) \in \mathcal{D}_w \end{gathered}$$

where $l_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \overline{\mathbb{R}_+}$ and $\bar{v}_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathcal{W}$ are smooth and appropriately defined; $l_r(X_{ro}, X_{ra}, X_{rd}) \geq l_{r-1}(X_{r-1\,o}, X_{r-1\,a}, X_{r-1\,d}) + \left|z_r\right|^2_{\beta_r(\check{\theta})} \geq |\tilde{\eta}|^2_Y + \left|z_1\right|^2 + \sum_{j=1}^r \left|z_j\right|^2_{\beta_j(\check{\theta})} \geq 0$, $\forall (X_{ro}, X_{ra}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd}$.

Hence, we have completed the design of the control function for u_a:

$$u_a = \mu_a(X_{ro}, X_{ra}, X_{rd}) \tag{45}$$

The corresponding upper bound of the value function is $V = V_r = |\tilde{\eta}|^2_Z + \sum_{j=1}^r \left|z_j\right|^2_{\gamma_j(\check{\theta})}$.

This completes the backstepping design procedure for the case $r > 1$.

Next, we consider the case of $r = 1$.

Step 1: Define the transformed variable

$$z_1 := \check{x}_1 - y_d \tag{46}$$

To apply Lemma 6 (or Lemma 7 for a computationally simplified controller), we make the following substitutions:

$$X_{1o} := (y_d, \check{\theta}, \Sigma, s_\Sigma, \tilde{\eta}) \to x_o; \quad \mathbb{R}^m \to \mathcal{U}; \quad \mathcal{D}_{1a} := \mathbb{R}^m \to D_a; \quad \left|\check{w}_2\right|^2 + \zeta^2|v|^2 \to \|(\check{w}_2, v)\|_{\mathcal{W}}^2$$

$$X_{1d} := (y_d^{(1)}, \Phi_1, \check{x}_2, \ldots, \check{x}_n, \Phi_2, \ldots, \Phi_n, \check{w}_1) \to x_d; \quad \mathcal{D}_{1o} := \mathbb{R}^m \times \Theta_o \times \mathcal{S}_{+\sigma} \times \mathbb{R}_+ \times \mathbb{R}^m \to D_o$$
$$\mathcal{D}_{1d} := \mathbb{R}^m \times \mathbb{R}^{m\times\sigma} \times \mathbb{R}^{(n-1)m} \times \mathbb{R}^{(n-1)m\times\sigma} \times \mathbb{R}^{\check{q}_1} \to D_d; \quad \mathcal{W} := (\mathcal{D}_w, \mathbb{R}, \|\cdot\|_{\mathcal{W}}) \to \mathcal{W}$$
$$\mathcal{D}_w := \mathbb{R}^{\check{q}_2+m} \to D_w; \quad V_0 \to V_o; \quad (\mathbf{0}_{\check{q}_2}, \tfrac{1}{\gamma^2\zeta^2} Z\tilde{\eta}) \to \sigma_o; \quad y_d \to \alpha_o; \quad |\tilde{\eta}|_Y^2 \to l_o; \quad \mathcal{D}_{1o} \times \mathcal{D}_{1d} \to D_1$$
$$X_{1a} := \check{x}_1 \to x_a; \quad u_a \to u; \quad (\check{w}_2, v) \to w; \quad \infty \to k; \quad \mathcal{X}_{1d} := \mathcal{D}_{1d} \to \mathcal{X}_d$$

and

$$f_o \leftarrow \begin{bmatrix} y_d^{(1)} \\ \delta(y_d, \check{x}_1, \Phi_1, \check{\theta}, \Sigma) \\ -(1 - K_c^{-1} s_\Sigma)(\gamma^2\zeta^2 - 1)\Sigma\Phi_1'\Phi_1\Sigma \\ (1 - K_c^{-1} s_\Sigma)(\gamma^2\zeta^2 - 1)\operatorname{Tr}(\Phi_1\Phi_1') \\ \lambda_m \tilde{\eta} + \check{x}_1 - y_d \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^\sigma \times \mathcal{S}_\sigma \times \mathbb{R} \times \mathbb{R}^m =: \mathcal{X}_{1o} \to \mathcal{X}_o$$

$$h_o \leftarrow \begin{bmatrix} \mathbf{0}_{m\times(\check{q}_2+m)} \\ \mathbf{0}_{\sigma\times\check{q}_2} \;\; h_\theta(\Phi_1, \Sigma) \\ \mathbf{0}_{\sigma\times\sigma\times(\check{q}_2+m)} \\ \mathbf{0}_{1\times(\check{q}_2+m)} \\ \mathbf{0}_{m\times\check{q}_2} \;\; I_m \end{bmatrix} \in \mathrm{B}\left(\mathcal{W}, \mathcal{X}_{1o}\right)$$

$$f_a \leftarrow f_1(y_d, \check{x}_1, \check{\theta}, \Phi_1, \Phi_1, \Sigma) + a_{1,2}\check{x}_2 + (A_{211,3,1,1}^{T_{2,1}}\check{\theta} + \check{D}_{1,1})\check{w}_1 \in \mathbb{R}^m =: \mathcal{X}_{1a} \to \mathcal{X}_a$$
$$g_a \leftarrow A_{212,1}^{T_{2,1}}\check{\theta} + B_1 \in \mathrm{B}\left(\mathbb{R}^m, \mathcal{X}_{1a}\right)$$
$$h_a \leftarrow \left[A_{211,3,2,1}^{T_{2,1}}\check{\theta} + \check{D}_{2,1} \;\; h_1(\check{\theta}, \Phi_1, \Phi_1, \Sigma) \right] \in \mathrm{B}\left(\mathcal{W}, \mathcal{X}_{1a}\right)$$

Note that $A_{212,1}^{T_{2,1}}\check{\theta} + B_1 = B_{p0}(\check{\theta})$ is invertible since $\check{\theta} \in \Theta_o$. Choose two $\mathcal{C}_\infty$ mappings $\gamma_1 : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$ and $\beta_1 : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$. Next, in the application of Lemma 6 (or Lemma 7), we make the following substitutions:

$$\gamma_1 \to Z; \quad (I_m + \beta_1)z_1 \to \phi; \quad V_1 \to V; \quad \mu_a \to \alpha$$

Then, $V_1 = V_0 + \left| z_1 \right|^2_{\gamma_1(\check{\theta})}$, $V_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \to \overline{\mathbb{R}_+}$, and $\mu_a : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathbb{R}^m$ are smooth and such that

$$\dot{V}_1\big|_{u_a=\mu_a(X_{1o}, X_{1a}, X_{1d})} = -l_1(X_{1o}, X_{1a}, X_{1d}) + \gamma^2 \left|\check{w}_2\right|^2 + \gamma^2\zeta^2 |v|^2 - \gamma^2 \left|(\check{w}_2, v) - \bar{v}_1\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & N^{-1} \end{bmatrix}};$$
$$\forall (X_{1o}, X_{1d}) \in \mathcal{D}_{1o} \times \mathcal{D}_{1d},\ \forall X_{1a} \in \mathcal{D}_{1a},\ \forall (\check{w}_2, v) \in \mathcal{D}_w$$

where $l_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \overline{\mathbb{R}_+}$ and $\bar{v}_1 : \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \to \mathcal{W}$ are smooth and appropriately defined; $l_1(X_{1o}, X_{1a}, X_{1d}) \geq |\tilde{\eta}|_Y^2 + \left|z_1\right|^2 + \left|z_1\right|^2_{\beta_1(\check{\theta})} \geq 0$, $\forall (X_{1o}, X_{1a}, X_{1d}) \in \mathcal{D}_{1o} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d}$.

Hence, we have completed the design of the control function for u_a:

$$u_a = \mu_a(X_{ro}, X_{ra}, X_{rd}) \tag{47}$$

The corresponding upper bound of the value function is $V = V_1 = |\tilde{\eta}|_Z^2 + \left|z_1\right|^2_{\gamma_1(\check{\theta})}$.

This completes the backstepping procedure for this case.

In summary, for both cases, we have obtained X_{jo}, X_{ja}, X_{jd}, $\mathcal{D}_{jo}$, $\mathcal{D}_{ja}$, $\mathcal{D}_{jd}$, $\gamma_j : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$, $\beta_j : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$, $l_j : \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \overline{\mathbb{R}_+}$, $j = 1, \ldots, r$, $\alpha_j : \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}^m$, $j = 1, \ldots, r-1$, $\bar{v}_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathcal{W}$, $V : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \to \overline{\mathbb{R}_+}$, and $\mu_a : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}^m$ such that α_j, γ_j, and β_j, $j = 0, \ldots, r-1$, $\bar{v}_r$, l_r, V, and μ_a are smooth and

$$V = |\tilde{\eta}|_Z^2 + \sum_{j=1}^{r} \left| \check{x}_j - \alpha_{j-1} \right|^2_{\gamma_j(\check{\theta})}$$

with

$$\dot{V}\big|_{u_a=\mu_a(X_{ro}, X_{ra}, X_{rd})} = -l_r(X_{ro}, X_{ra}, X_{rd}) + \gamma^2 \left|\check{w}_2\right|^2 + \gamma^2\zeta^2 |v|^2 - \gamma^2 \left|(\check{w}_2, v) - \bar{v}_r\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & N^{-1} \end{bmatrix}};$$
$$\forall (X_{ro}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{rd},\ \forall X_{ra} \in \mathcal{D}_{ra},\ \forall (\check{w}_2, v) \in \mathbb{R}^{\check{q}_2+m}$$

and $l_r(X_{ro}, X_{ra}, X_{rd}) \ge |\tilde{\eta}|_Y^2 + \left|\check{x}_1 - y_d\right|^2 + \sum_{j=1}^r \left|\check{x}_j - \alpha_{j-1}(X_{j-1\,o}, X_{j-1\,a}, X_{j-1\,d})\right|^2_{\beta_j(\check{\theta})} \ge 0, \forall (X_{ro}, X_{ra}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd}$. In the following, the control input u_a will always be set to $\mu_a(X_{ro}, X_{ra}, X_{rd})$.

Now that the (upper bound of the) value function for the control design has been chosen, we can optimize the choices for the controls u_b and ξ_c. Based on the dynamics for the observer (37), these signals enter the system in an affine manner. When, ξ_c and u_b are not vanishing, the derivative of V is given by

$$\dot{V} = -l_r(X_{ro}, X_{ra}, X_{rd}) + \gamma^2 \left|\check{w}_2\right|^2 + \gamma^2\zeta^2 |v|^2 + \varsigma_r'\bar{Q}\xi_c + \varsigma_b' u_b - \gamma^2 \left|(\check{w}_2, v) - \bar{v}_r\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & N^{-1} \end{bmatrix}};$$
$$\forall (X_{ro}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{rd},\ \forall X_{ra} \in \mathcal{D}_{ra},\ \forall (\check{w}_2, v) \in \mathbb{R}^{\check{q}_2+m},\ \forall u_b \in \mathbb{R}^{p-m},\ \forall \xi_c \in \mathbb{R}^{nm+\sigma}$$

where $\varsigma_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \bar{\mathcal{D}}_{rd} \to \mathbb{R}^{\sigma+mn}$ and $\varsigma_b : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \to \mathbb{R}^{p-m}$ are smooth and appropriately defined, $\bar{X}_{rd} := (\Phi_r, \check{x}_{r+1}, \ldots, \check{x}_n, \Phi_{r+1}, \ldots, \Phi_n) \in \bar{\mathcal{D}}_{rd} := \mathbb{R}^{m\times\sigma} \times \mathbb{R}^{(n-r)m} \times \mathbb{R}^{(n-r)m\times\sigma}$.

The closed-loop system admits the state vector

$$X := (x_{\bar{o}}, \theta, x, X_{ro}, X_{ra}, \Phi_r, \check{x}_{r+1}, \ldots, \check{x}_n, \Phi_{r+1}, \ldots, \Phi_n) = (x_{\bar{o}}, \theta, x, X_{ro}, X_{ra}, \bar{X}_{rd}) \tag{48}$$

which belongs to the set

$$\mathcal{D} := \left\{ X \;\middle|\; \Sigma \in \mathcal{S}_{+\sigma},\ s_\Sigma \in \mathbb{R}_+,\ \check{\theta} \in \Theta_o,\ \theta \in \Theta_o \right\} \tag{49}$$

The (upper bound of the) value function for the closed-loop system is

$$U := V + W = \left|\theta - \check{\theta}\right|^2_{\Sigma^{-1}} + \gamma^2 \left|x - \check{x} - \Phi(\theta - \check{\theta})\right|^2_{\Pi^{-1}} + |\tilde{\eta}|_Z^2 + \sum_{j=1}^r \left|\check{x}_j - \alpha_{j-1}(X_{j-1\,o}, X_{j-1\,a}, X_{j-1\,d})\right|^2_{\gamma_j(\check{\theta})} \tag{50}$$

which is the sum of (upper bounds of) the value functions for the identification design and control design, leading to $U : \mathcal{D} \to \overline{\mathbb{R}_+}$ being smooth. The derivative of this value function along the solution of the closed-loop dynamics is given by

$$\begin{aligned}
\dot{U} = & -\left|x_1 - y_d\right|^2 - \gamma^4 \left|x - \hat{x} - \Phi(\theta - \hat{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \hat{\theta}\right|^2_{\Phi'C'C\Phi} + 2(\theta - \check{\theta})' P_r(\check{\theta}) + \left|\xi_c\right|^2_{\bar{Q}} + \varsigma_r'\bar{Q}\xi_c + \varsigma_b' u_b - l_r \\
& + \left|z_1\right|^2 + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\left|w_b\right|^2 - \gamma^2\left|w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}, \hat{\xi}_{[0,t]})\right|^2 - \gamma^2\left|(\check{w}_2, v) - \bar{v}_r\right|^2_{\begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & \zeta^2 I_m \end{bmatrix}} \\
= & -\left|x_1 - y_d\right|^2 - \gamma^4 \left|x - \hat{x} - \Phi(\theta - \hat{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \hat{\theta}\right|^2_{\Phi'C'C\Phi} + 2(\theta - \check{\theta})' P_r(\check{\theta}) + \left|\xi_c + \varsigma_r/2\right|^2_{\bar{Q}} - \left|\varsigma_r\right|^2_{\bar{Q}}/4 \\
& + \varsigma_b' u_b - l_r + \left|z_1\right|^2 + \gamma^2\left|\check{w}_2\right|^2 + \gamma^2\left|w_b\right|^2 - \gamma^2\left|(\check{w}_2, w_b) - w_{opt}\right|^2; \\
& \forall X \in \mathcal{D},\ \forall y_d^{(r)} \in \mathbb{R}^m,\ \forall \hat{\xi} \in \mathbb{R}^{\sigma+nm},\ \forall w_b \in \mathbb{R}^{mq_b},\ \forall \check{w}_1 \in \mathbb{R}^{\check{q}_1},\ \forall \check{w}_2 \in \mathbb{R}^{\check{q}_2},\ \forall u_b \in \mathbb{R}^{p-m}
\end{aligned} \tag{51}$$

where the worst-case disturbance with respect to the value function U is given by

$$w_{opt} = \begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & E'N^{-1} \end{bmatrix} \bar{v}_r + \begin{bmatrix} \mathbf{0}_{\check{q}_2} \\ \gamma^{-2}(I_{mq_b} - E'N^{-1}E)\bar{D}'\bar{\Sigma}^{-1}(\xi - \check{\xi}) + E'N^{-1}C(\check{x} - x) \end{bmatrix} \tag{52}$$

The choice for u_b is to generate an additional negative drift for U while the magnitude of u_b remains bounded, since u_b enters the unknown system directly. A possible choice for u_b is

$$u_b = -\operatorname{SATF}(\varsigma_b) =: \mu_b(X_{ro}, X_{ra}) \tag{53}$$

where SATF is the smooth saturation function (see Definition 4) that applies element-wise on the vector ς_b, with each element given a possibly different saturation level $\bar{\varsigma}_{bi} \in \mathbb{R}_+$, $i = 1, \ldots, p - m$.

The optimal choice for the variable ξ_c is $\xi_{c*} = -\varsigma_r/2$, or equivalently, the optimal choice for the worst-case estimate $\hat{\xi}$ is

$$\hat{\xi}_*(X_{ro}, X_{ra}, \bar{X}_{rd}) = \check{\xi} - \varsigma_r/2 \tag{54}$$

This control design yields that the closed-loop system is dissipative with storage function U and supply rate

$$-\left|x_1 - y_d\right|^2 + \gamma^2\left|w_b\right|^2 + \gamma^2\left|\check{w}_2\right|^2$$

This optimal choice for $\hat{\xi}$, (54), results in the first proposed adaptive control law.

The optimal choice of ξ_{c*} is generally quite complicated, and leads to an identifier that is very different from the standard identifiers, such as least squares or least mean squares identifiers. On the other hand, the simple choice of $\xi_c = \mathbf{0}_{\sigma+nm}$, i. e.,

$$\hat{\xi} = \check{\xi} \tag{55}$$

results in a simplified identifier structure, which resembles the standard identifiers. In practical situations, this suboptimal choice of $\hat{\xi}$ may be preferable over the optimal one (54). This suboptimal choice of $\hat{\xi}$ results in the second proposed adaptive control law.

This completes the adaptive controller design step. Next, we turn to study the robustness and tracking properties of the proposed adaptive control laws.

6 | MAIN RESULT

In this section, we present the main result of this paper by stating a theorem and a corollary on the robustness and tracking properties of the two proposed adaptive control laws.

For the first adaptive control law (with the optimal choice of $\hat{\xi}$), the closed-loop system dynamics are

$$\dot{X} = F(X, y_d^{(r)}, \check{w}_1) + G(X)\begin{bmatrix}\check{w}_2\\ w_b\end{bmatrix} = F(X, y_d^{(r)}, \check{w}_1) + G(X)\begin{bmatrix}\check{w}_2\\ \grave{M}\grave{w}_b\end{bmatrix}; \qquad X(0) = X_0 \tag{56}$$

where F and G are smooth mappings on $\mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^{\check{q}_1}$ and $\mathcal{D}$, respectively; and

$$X_0 \in \mathcal{D}_0 := \Big\{ X_0 \in \mathcal{D} \,\Big|\, \theta \in \Theta,\ \check{\theta}_0 \in \Theta,\ \Sigma(0) = \gamma^{-2}Q_0^{-1} \in \mathcal{S}_{+\sigma},\ \mathrm{Tr}\left((\Sigma(0))^{-1}\right) \le K_c,$$
$$s_\Sigma(0) = \gamma^2\,\mathrm{Tr}\left(Q_0\right),\ \grave{T}(x_{\bar{\sigma}}(0), x_1(0), \dots, x_n(0)) \in \acute{\mathcal{D}}_0 \Big\}$$

Since (51) holds, then, by Lemma 8 of Appendix B, the value function U satisfies the following Hamilton-Jacobi-Isaacs equation:

$$\frac{\partial U}{\partial X}(X)F(X, y_d^{(r)}, \check{w}_1) + \frac{1}{4\gamma^2}\left\| \frac{\partial U}{\partial X}(X)G(X)\right\|^2_{\mathbb{R}^{\check{q}_2+mq_b}} + Q(X, y_d^{(r)}, \check{w}_1) = 0; \quad \forall X \in \mathcal{D},\ \forall y_d^{(r)} \in \mathbb{R},\ \forall \check{w}_1 \in \mathbb{R}^{\check{q}_1} \tag{57}$$

where $Q : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^{\check{q}_1} \to \mathbb{R}$ is smooth and given by

$$\begin{aligned} Q(X, y_d^{(r)}, \check{w}_1) =\ & \left|x_1 - y_d\right|^2 + \gamma^4\left|x - \hat{x} - \Phi(\theta - \hat{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \hat{\theta}\right|^2_{\Phi' C' C\Phi} \\ & -2(\theta - \check{\theta})' P_r(\check{\theta}) + \left|\varsigma_r\right|^2_{\bar{Q}}/4 + l_r(X_{ro}, X_{ra}, X_{rd}) - \left|z_1\right|^2 - \varsigma_b' \mu_b \\ \ge\ & \left|x_1 - y_d\right|^2 + \gamma^4\left|x - \hat{x} - \Phi(\theta - \hat{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \hat{\theta}\right|^2_{\Phi' C' C\Phi} \\ & -2(\theta - \check{\theta})' P_r(\check{\theta}) + \left|\varsigma_r\right|^2_{\bar{Q}}/4 + |\tilde{\eta}|^2_Y + \sum_{j=1}^{r}\left|z_j\right|^2_{\beta_j(\check{\theta})} - \varsigma_b'\mu_b \end{aligned}$$

Clearly, Q is nonnegative, $\forall X \in \mathcal{D}$ with $\theta \in \Theta$.

Since the value function U is not a positive-definite function for the entire closed-loop system state X, we cannot deduce stability properties of the closed-loop system directly from the value function U. As it turns out, the closed-loop adaptive system possesses a strong stability property: all closed-loop signals remain bounded under bounded disturbance $\grave{w}_{[0,\infty)} \in \grave{\mathcal{W}}_d$ and the initial condition $\acute{x}_0 \in \acute{\mathcal{D}}_0$ and bounded reference trajectory together with its derivatives up to rth order, in addition to the above stated attenuation (dissipation) property. This is made precise in the following theorem.

Remark 1. Assumptions 1 – 8 are standard as in the SISO case [5].

Theorem 1. Consider the robust adaptive control problem formulated in Section 3, with Assumptions 1 – 8 holding. Then, the robust adaptive controller μ given by (45) (or (47)) and (53), with the worst-case estimate $\hat{\xi}$ generated by the optimal policy (54), achieves the following strong robustness properties for the closed-loop system.

1. Given $c_w \in \overline{\mathbb{R}_+}$ and $c_d \in \overline{\mathbb{R}_+}$, there exists a constant $c_c \in \overline{\mathbb{R}_+}$ and a compact set $\Theta_c \subset \Theta_o$ such that for any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y^{(r)}_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$ with

$$\left|\acute{x}_0\right| \le c_w;\ \acute{x}_0 \in \acute{\mathcal{D}}_0; \quad |\grave{w}(t)| \le c_w;\ \grave{w}_{[0,\infty)} \in \grave{\mathcal{W}}_d; \quad \left|Y_d(t)\right| \le c_d; \qquad \forall t \in [0, \infty)$$

all closed-loop state variables $x_{\bar{o}}$, x, $\check{x}$, $\check{\theta}$, Σ, s_Σ, $\tilde{\eta}$, Φ, η_i, λ_{ai}, $i = 1, \dots, m$, λ_{bi}, $i = 1, \dots, p-m$, $\check{\eta}_{1i}$, $i = 1, \dots, \check{q}_1$, $\check{\eta}_{2i}$, $i = 1, \dots, \check{q}_2$, and λ_o exist and are bounded as follows, $\forall t \in \overline{\mathbb{R}_+}$:

$$\left|x_{\bar{o}}(t)\right| \le c_c, \quad |x(t)| \le c_c, \quad |\check{x}(t)| \le c_c, \quad \check{\theta}(t) \in \Theta_c, \quad |\tilde{\eta}(t)| \le c_c, \quad \|\Phi(t)\| \le c_c,$$
$$K_c^{-1} I_\sigma \le \Sigma(t) \le \gamma^{-2} Q_0^{-1}, \quad \gamma^2 \operatorname{Tr}(Q_0) \le s_\Sigma(t) \le K_c, \quad \left|\lambda_o(t)\right| \le c_c, \quad \left|\eta_i(t)\right| \le c_c, \ \left|\lambda_{ai}(t)\right| \le c_c, \ i = 1, \dots, m,$$
$$\left|\lambda_{bi}(t)\right| \le c_c, \ i = 1, \dots, p-m, \quad \left|\check{\eta}_{1i}(t)\right| \le c_c, \ i = 1, \dots, \check{q}_1, \quad \left|\check{\eta}_{2i}(t)\right| \le c_c, \ i = 1, \dots, \check{q}_2.$$

Therefore, there is a compact set $S \subset \mathcal{D}$ such that $X(t) \in S$, $\forall t \in \overline{\mathbb{R}_+}$. Hence, there exists a constant $c_u \in \overline{\mathbb{R}_+}$ such that $|u(t)| \le c_u$ and $\left|\hat{\xi}(t)\right| \le c_u$, $\forall t \in \overline{\mathbb{R}_+}$.

2. The controller μ belongs to $\mathcal{M}$ and achieves disturbance attenuation level 0 with respect to $\check{w}_1$ and disturbance attenuation level γ with respect to $\check{w}_2$ and w_b for any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}^{(r)}) \in \grave{\mathcal{W}}$.

3. For any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}^{(r)}) \in \grave{\mathcal{W}}$ with $\check{w}_{1[0,\infty)} \in \bar{\mathrm{L}}_\infty$, $\check{w}_{2[0,\infty)} \in \bar{\mathrm{L}}_2 \cap \bar{\mathrm{L}}_\infty$, $\grave{w}_{b[0,\infty)} \in \bar{\mathrm{L}}_2 \cap \bar{\mathrm{L}}_\infty$ and $Y_{d\,[0,\infty)} \in \bar{\mathrm{L}}_\infty$, the output of the system x_1 asymptotically tracks the reference trajectory y_d, i. e.,

$$\lim_{t \to \infty} (x_1(t) - y_d(t)) = \mathbf{0}_m$$

Proof. We consider the first statement. Fix an uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}^{(r)}) \in \grave{\mathcal{W}}$ with

$$\left|\acute{x}_0\right| \le c_w; \ \acute{x}_0 \in \acute{\mathcal{D}}_0; \quad |\grave{w}(t)| \le c_w; \ \grave{w}_{[0,\infty)} \in \grave{\mathcal{W}}_d; \quad \left|Y_d(t)\right| \le c_d; \qquad \forall t \in [0, \infty)$$

for some $c_w \in \overline{\mathbb{R}_+}$ and $c_d \in \overline{\mathbb{R}_+}$. With the controller μ and $\hat{\xi}$ designed, we have a fixed initial condition $X_0 \in \mathcal{D}_0$ for the closed-loop system (56). Consider the maximal interval $[0, T_f)$ where the differential equation (56) for the closed-loop system admits a solution that lies in $\mathcal{D}$, which is clearly an open set. Then, by the smoothness of the system, the solution $X(t)$ is unique on $[0, T_f)$. Note that the maximal length of the interval, T_f, may depend on the specific waveform for the disturbance $\grave{w}_{[0,\infty)}$ and the reference $y_{d\,[0,\infty)}^{(r)}$. We will show that the maximal length of the interval, T_f, is always ∞.

By Lemma 1, the covariance matrix Σ and the signal s_Σ are uniformly upper bounded and uniformly bounded away from 0, as depicted in the first statement of the theorem. By Proposition 3, Σ and s_Σ are inside compact subsets of $\mathcal{S}_{+\sigma}$ and $\mathbb{R}_+$, respectively. The reference trajectory and its derivatives up to rth order are uniformly bounded since $\left|Y_d(t)\right| \le c_d$, $\forall t \ge 0$.

Define the vector of variables

$$X_e := (\check{\theta}, \tilde{x} - \Phi\tilde{\theta}, \tilde{\eta}, z_1, \dots, z_r)$$

Clearly, $X_e : [0, T_f) \to \mathcal{D}_e := \Theta_o \times \mathbb{R}^{nm} \times \mathbb{R}^m \times \mathbb{R}^{rm}$, and the function U can be written as $U = \bar{U}(t, X_e(t))$, where $\bar{U} : [0, T_f) \times \mathcal{D}_e \to \overline{\mathbb{R}_+}$. Under the assumption that $\grave{w}$ is uniformly bounded on $[0, \infty)$, we have the following inequality for the derivative of U:

$$\begin{aligned}
\dot{U} \le\ & -\gamma^4 \left| x - \hat{x} - \Phi(\theta - \hat{\theta}) \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1) \left| \theta - \hat{\theta} \right|^2_{\Phi' C' C \Phi} \\
& + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|_Y^2 - \sum_{j=1}^r \left| z_j \right|^2_{\beta_j(\check{\theta})} - |\varsigma_r|^2_{\bar{Q}}/4 + \gamma^2 \left\| \begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & \grave{M} \end{bmatrix} \right\|^2 c_w^2 \\
=\ & -\left| \xi - \hat{\xi} \right|^2_{\bar{Q}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|_Y^2 - \sum_{j=1}^r \left| z_j \right|^2_{\beta_j(\check{\theta})} - \left| \hat{\xi} - \check{\xi} \right|^2_{\bar{Q}} + \bar{c}_w^2 \\
=\ & -\frac{1}{2} \left| \xi - \check{\xi} \right|^2_{\bar{Q}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|_Y^2 - \sum_{j=1}^r \left| z_j \right|^2_{\beta_j(\check{\theta})} - \frac{1}{2} \left| 2\hat{\xi} - \check{\xi} - \xi \right|^2_{\bar{Q}} + \bar{c}_w^2 \\
\le\ & -\gamma^4/2 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|_Y^2 - \sum_{j=1}^r \left| z_j \right|^2_{\beta_j(\check{\theta})} + \bar{c}_w^2
\end{aligned}$$

where $\bar{c}_w := \gamma c_w \left\| \begin{bmatrix} I_{\check{q}_2} & \mathbf{0} \\ \mathbf{0} & \grave{M} \end{bmatrix} \right\|$. Note that $\beta_j : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$ is smooth, then $\exists c_{\beta_j} \in \mathbb{R}_+$ such that $\beta_j(\check{\theta}) \ge c_{\beta_j} I_m$, $j = 1, \dots, r$, $\forall \check{\theta} \in \Theta_{\rho_o} \supset \Theta_o$. Then, there exists a compact set $\Omega_1(c_w) \subset \mathcal{D}_e$ such that, $\forall t \in [0, T_f)$, if $X_e \in \mathcal{D}_e \setminus \Omega_1(c_w)$ then $\dot{U} < 0$. Note that since $\gamma_j : \hat{\Theta}_{\rho_o} \to \mathcal{S}_{+m}$ is smooth, $\exists c_{\gamma_j m}, c_{\gamma_j M} \in \mathbb{R}_+$ such that $c_{\gamma_j m} I_m \le \gamma_j(\check{\theta}) \le c_{\gamma_j M} I_m$, $j = 1, \dots, r$, $\forall \check{\theta} \in \Theta_{\rho_o} \supset \Theta_o$. Let

$$U_M(X_e) := K_c \left| \theta - \check{\theta} \right|^2 + \gamma^2 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}} + |\tilde{\eta}|_Z^2 + \sum_{j=1}^r c_{\gamma_j M} \left| z_j \right|^2$$

$$U_m(X_e) := \gamma^2 \left|\theta - \check{\theta}\right|^2_{Q_0} + \gamma^2 \left|\tilde{x} - \Phi\tilde{\theta}\right|^2_{\Pi^{-1}} + |\tilde{\eta}|^2_Z + \sum_{j=1}^{r} c_{\gamma_j m} \left|z_j\right|^2$$

Then, clearly $U_m(X_e) \le \bar{U}(t, X_e) \le U_M(X_e)$, $\forall t \in [0, T_f)$, $\forall X_e \in \mathcal{D}_e = \Theta_o \times \mathbb{R}^{mn+m+rm}$. By Lemma 5 of Appendix A, there exists a constant $c_1 \in \overline{\mathbb{R}_+}$ such that $U_m(X_e(t)) \le c_1$, $\forall t \in [0, T_f)$.

Then, on the interval $[0, T_f)$, the vector X_e is uniformly bounded. Hence, we have that $\tilde{\theta}$, $\tilde{x} - \Phi\tilde{\theta}$, $\tilde{\eta}$ and $z_1, \ldots, z_r$ are uniformly bounded. ($\tilde{\theta}$ is bounded to begin with, since $\theta \in \Theta$ and $\check{\theta} \in \Theta_o$.)

To further conclude the uniform boundedness of the overall closed-loop system states, we distinguish 3 exhaustive and mutually exclusive cases: $r = 1$, $r = 2$, and $r \ge 3$. First consider Case 1: $r = 1$.

Note that the signal $\tilde{\eta}$ is uniformly bounded, and it has uniform vector relative degree 1 with respect to the input y. The linear system with input y and output $\tilde{\eta}$ is minimum phase with respect to $\mathcal{D}_{\tilde{\eta}_0} := \mathbb{R}^m$ and $\mathcal{C}$ according to [1], where the signal y_d is regarded as disturbance. Then, this signal $\tilde{\eta}$ has uniform vector relative degree $r+1$ with respect to the input u_a; and the composite system with states $\tilde{\eta}$ and $\acute{x}$, input u_a, and output $\tilde{\eta}$ is minimum phase with respect to $\mathcal{D}_{\tilde{\eta}_0} \times \acute{\mathcal{D}}_0$ and $\mathcal{C} \times \acute{\mathcal{W}}_d$ (by a straightforward vectorized version of Theorem 1 of [19]), where the signal y_d, u_b, and $\acute{w}$ are regarded as disturbances. It is easy to see that the $\tilde{\eta}$ dynamics with input y and output $\tilde{\eta}$ may serve as a reference system in the application of Proposition 2 of [20] (more precisely, a straightforward vectorized version of it). The composite system with control input u_a, output $\tilde{\eta}$, and disturbance inputs y_d and $\acute{w}_e$ may serve as a reference system in the application of Proposition 2 of [20] (more precisely, a straightforward vectorized version of it).

We need to conclude the boundedness of the variables Φ_1 in three steps. Define

$$\lambda_{ci} = (\lambda_{ci1}, \ldots, \lambda_{cin}); \quad i = 1, \ldots, m \tag{58a}$$

$$\dot{\lambda}_{ci} = A_{f1}\lambda_{ci} + e_{n,r}u_{ai}; \quad \lambda_{ci}(0) = \mathbf{0}_n; \ i = 1, \ldots, m \tag{58b}$$

$$\Phi_{u_{as}} = \begin{bmatrix} \Phi'_{u_{as}1} & \cdots & \Phi'_{u_{as}n} \end{bmatrix}' \tag{58c}$$

$$\dot{\Phi}_{u_{as}} = A_f \Phi_{u_{as}} + \begin{bmatrix} \mathbf{0}_{mr\times\sigma\times m} \\ A_{212,s} \end{bmatrix} u_a; \quad \Phi_{u_{as}}(0) = \mathbf{0}_{nm\times\sigma} \tag{58d}$$

$$\dot{\Phi}_y = A_f \Phi_y + A_{211,1}y + A_{211,2}u_b + A_{211,3}\acute{w}; \quad \Phi_y(0) = \Phi_0 \tag{58e}$$

where $A_{212,s}$ is a 2nd-order $\mathbb{R}^{(n-r)m}$-valued sub-tensor of A_{212} that consists of the $(mr+1)$st to mnth indices in the output dimension, all indices in the first dimension, and all indices in the second dimension; λ_{cij} is a scalar $i = 1, \ldots, m$, $j = 1, \ldots, n$, $\Phi_{u_{as}i}$ is a $m \times \sigma$-matrix, $i = 1, \ldots, n$. Then, we have

$$\Phi = \Phi_y + \Phi_{u_{as}} + \sum_{i=1}^{m} (\lambda_{ci} \otimes I_m) A_{212,r,:,:,i}$$

The relative degree for each of the elements of $\Phi_{u_{as}1}$ is at least $r+1$ with respect to the input u_a, and is the output of a stable linear system. By Proposition 2 of [20], this yields that $\Phi_{u_{as}1}$ is uniformly bounded, where the reference system has output $\tilde{\eta}$ and inputs u_a, $\acute{w}_e$, and y_d.

The relative degree for each of the elements of Φ_y is at least 1 with respect to the input y, and is the output of a stable linear system. By Proposition 2 of [20], this yields that Φ_y is uniformly bounded, where the reference system has output $\tilde{\eta}$ and input y, y_d. (Note that $\acute{w}$ and u_b are uniformly bounded.)

Because $\tilde{x} - \Phi\tilde{\theta}$, Φ_y, $\Phi_{u_{as}1}$, and $\tilde{\theta}$ are uniformly bounded, we have that the signal $\tilde{x}_1 - \sum_{i=1}^{m} \lambda_{ci1} A_{212,r,:,:,i}\tilde{\theta} = \tilde{x}_1 - (A_{212,r}\bar{\lambda}_{c1})\tilde{\theta}$ is uniformly bounded, where $\bar{\lambda}_{c1} := (\lambda_{c11}, \ldots, \lambda_{cm1})$. Furthermore, since $z_1 = \check{x}_1 - y_d$ and y_d are uniformly bounded, so is $\check{x}_1$.

Let $\bar{\lambda}_c := (\bar{\lambda}_{c1}, \ldots, \bar{\lambda}_{cn}) \in \mathbb{R}^{nm}$, where $\bar{\lambda}_{ci} = (\lambda_{c1i}, \ldots, \lambda_{cmi}) \in \mathbb{R}^m$, $i = 1, \ldots, n$. Then, $\bar{\lambda}_c$ satisfies the dynamics

$$\dot{\bar{\lambda}}_c = A_f \bar{\lambda}_c + \begin{bmatrix} \mathbf{0}_{(r-1)m\times m} \\ I_m \\ \mathbf{0}_{(n-r)m\times m} \end{bmatrix} u_a$$

We further define $\bar{\bar{\lambda}}_c := (I_n \otimes B_{p0}(\theta))\bar{\lambda}_c$, where $B_{p0}(\theta) = B_r + A^{T_{2,1}}_{212,r}\theta$ is the high-frequency gain matrix as defined in Assumption 5. Then, we have

$$\dot{\bar{\bar{\lambda}}}_c = (I_n \otimes B_{p0}(\theta))\dot{\bar{\lambda}}_c = (I_n \otimes B_{p0}(\theta))(A_f \bar{\lambda}_c + \begin{bmatrix} \mathbf{0}_{(r-1)m\times m} \\ I_m \\ \mathbf{0}_{(n-r)m\times m} \end{bmatrix} u_a)$$

$$= A_f(I_n \otimes B_{p0}(\theta))\bar{\lambda}_c + (I_n \otimes B_{p0}(\theta)) \begin{bmatrix} \mathbf{0}_{(r-1)m\times m} \\ I_m \\ \mathbf{0}_{(n-r)m\times m} \end{bmatrix} u_a = A_f \bar{\bar{\lambda}}_c + \begin{bmatrix} \mathbf{0}_{(r-1)m\times m} \\ B_{p0}(\theta) \\ \mathbf{0}_{(n-r)m\times m} \end{bmatrix} u_a$$

where we have made use of the structure of A_f that it commutes with $I_n \otimes B_{p0}(\theta)$.

Now a critical observation is that the signal $x_1 - (B_r + A_{212,r}^{T_{2,1}}\theta)\bar{\lambda}_{c1} =: x_1 - B_{p0}(\theta)\bar{\lambda}_{c1} =: x_1 - \bar{\bar{\lambda}}_{c1}$ is generated by the dynamics

$$\begin{aligned} \dot{x} - \dot{\bar{\bar{\lambda}}}_c &= A_f(x - \bar{\bar{\lambda}}_c) + \begin{bmatrix} \mathbf{0}_{rm\times m} \\ A_{212,s}^{T_{2,1}}\theta \end{bmatrix} u_a + (\zeta^2 L + \Pi C'(\zeta^2 - \gamma^{-2}))(y - E\grave{M}\grave{w}_b) + \check{A}y + \check{B}_b u_b \\ &\quad + \begin{bmatrix} \mathbf{0}_{m\times rm} & B_{r+1}' & \cdots & B_n' \end{bmatrix}' u_a + (A_{211,1}y + A_{211,2}u_b + A_{211,3}\check{w})\theta + \check{D}\check{w} + D\grave{M}\grave{w}_b \\ x_1 - \bar{\bar{\lambda}}_{c1} &= C(x - \bar{\bar{\lambda}}_c) \end{aligned}$$

To apply Proposition 2 of [20], the dynamics are separated into y dependent and u dependent parts using the linearity of the system, $x_1 - \bar{\bar{\lambda}}_{c1} =: x_{u1} + x_{y1}$. The dynamics of x_{u1} and x_{y1} are given by

$$\begin{aligned} \dot{x}_u &= A_f x_u + \begin{bmatrix} \mathbf{0}_{rm\times m} \\ A_{212,s}^{T_{2,1}}\theta \end{bmatrix} u_a + \begin{bmatrix} \mathbf{0}_{m\times rm} & B_{r+1}' & \cdots & B_n' \end{bmatrix}' u_a \\ x_{u1} &= Cx_u \\ \dot{x}_y &= A_f x_y + (\zeta^2 L + \Pi C'(\zeta^2 - \gamma^{-2}))(y - E\grave{M}\grave{w}_b) + \check{A}y + (A_{211,1}y + A_{211,2}u_b + A_{211,3}\check{w})\theta + \check{B}_b u_b + \check{D}\check{w} + D\grave{M}\grave{w}_b \\ x_{y1} &= Cx_y \end{aligned}$$

The signal x_{u1} has relative degree at least $r+1$ with respect to u_a. It is uniformly bounded by Proposition 2 of [20], where the reference system has inputs u_a, y_d, and $\grave{w}_e$, and output $\tilde{\eta}$. The signal x_{y1} has relative degree at least 1 with respect to y. It is uniformly bounded by Proposition 2 of [20], where the reference system has inputs y and y_d, and output $\tilde{\eta}$. Hence, $x_1 - B_{p0}(\theta)\bar{\lambda}_{c1}$ is uniformly bounded. It can further be concluded that $\check{x}_1 - B_{p0}(\check{\theta})\bar{\lambda}_{c1} = x_1 - B_{p0}(\theta)\bar{\lambda}_{c1} - (\tilde{x}_1 - (A_{212,r}^{T_{2,1}}\tilde{\theta})\bar{\lambda}_{c1}) = x_1 - B_{p0}(\theta)\bar{\lambda}_{c1} - (\tilde{x}_1 - (A_{212,r}\bar{\lambda}_{c1})\tilde{\theta}) = x_1 - B_{p0}(\theta)\bar{\lambda}_{c1} - (\tilde{x}_1 - \Phi_1\tilde{\theta}) - (C\Phi_y\tilde{\theta} + \Phi_{u_{as}1}\tilde{\theta})$ is uniformly bounded.

Since $\check{x}_1$ is bounded, and $B_{p0}(\check{\theta})$ is uniformly bounded away from singularity due the $\check{\theta} \in \Theta_o$, $\forall t \in [0, T_f)$, (see Page 10) we have the uniform boundedness of the signal $\bar{\lambda}_{c1}$. This further implies the uniform boundedness of the signal Φ_1, and the uniform boundedness of the signals x_1 and y because of the boundedness of $x_1 - B_{p0}(\theta)\bar{\lambda}_{c1}$ and $\grave{w}_b$.

In order to show the existence of a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$, define the function

$$\Upsilon := U + P(\check{\theta})(\rho_o - P(\check{\theta}))^{-1}$$

Clearly, Υ can be written as $\Upsilon(t) = \bar{\Upsilon}(t, X_e(t))$, where $\bar{\Upsilon} : [0, T_f) \times \mathcal{D}_e \to \overline{\mathbb{R}}_+$. The total time derivative of Υ is given by

$$\begin{aligned} \dot{\Upsilon} &= \dot{U} + \rho_o\left(\rho_o - P(\check{\theta})\right)^{-2} \frac{\partial P}{\partial \theta}(\check{\theta})\dot{\check{\theta}} \\ &\le -\left|x_1 - y_d\right|^2 - \gamma^4 \left|x - \hat{x} - \Phi(\theta - \hat{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \hat{\theta}\right|^2_{\Phi' C' C\Phi} \\ &\quad + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r} \left|z_j\right|^2_{\beta_j} - \frac{1}{4}\left|\varsigma_r\right|^2_{\bar{Q}} + \bar{c}_w^2 \\ &\quad + \rho_o\left(\rho_o - P(\check{\theta})\right)^{-2} \Bigg(-\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma P_r(\check{\theta}) - \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\Phi' C'(y_d - \check{x}_1) \\ &\quad + \frac{1}{2}\frac{\partial P}{\partial \theta}(\check{\theta})\begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix}\bar{Q}\varsigma_r + \frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2\zeta^2\Sigma\Phi' C'(\tilde{x}_1 + E\grave{M}\grave{w}_b) \Bigg) \end{aligned}$$

Note the following partitioning of the matrix $\bar{\Sigma}$ and its inverse:

$$\bar{\Sigma} = \begin{bmatrix} \Sigma & \Sigma\Phi' \\ \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} \qquad \bar{\Sigma}^{-1} = \begin{bmatrix} \Sigma^{-1} + \gamma^2\Phi'\Pi^{-1}\Phi & -\gamma^2\Phi'\Pi^{-1} \\ -\gamma^2\Pi^{-1}\Phi & \gamma^2\Pi^{-1} \end{bmatrix}$$

By the special structure of $\bar{Q}$ prescribed by (25), the following equalities hold:

$$\bar{\Sigma}\bar{Q} = \begin{bmatrix} \epsilon\Sigma\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi & \mathbf{0} \\ \epsilon\Phi\Sigma\Phi' C'(\gamma^2\zeta^2 - 1)C\Phi - \gamma^2\Delta\Pi^{-1}\Phi & \gamma^2\Delta\Pi^{-1} \end{bmatrix} \qquad \Rightarrow$$

$$\frac{1}{2}\frac{\partial P}{\partial \theta}(\breve{\theta})\left[\,\Sigma\;\Sigma\Phi'\,\right]\bar{Q}\varsigma_r = \frac{1}{2}\frac{\partial P}{\partial \theta}(\breve{\theta})\left[\,\epsilon\Sigma\Phi'C'(\gamma^2\zeta^2-1)C\Phi\;\mathbf{0}_{\sigma\times n}\,\right]\varsigma_r = -\frac{\partial P}{\partial \theta}(\breve{\theta})\Sigma\Phi'C'\epsilon(\gamma^2\zeta^2-1)C\Phi(\hat{\theta}-\breve{\theta})$$

Therefore,

$$\begin{aligned}
\dot{\Upsilon} \le & -\left|\xi-\hat{\xi}\right|^2_{\bar{Q}} + 2(\theta-\breve{\theta})'P_r(\breve{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r}\left|z_j\right|^2_{\beta_j} - \left|\hat{\xi}-\breve{\xi}\right|^2_{\bar{Q}} + \bar{c}_w^2 \\
& -\rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2} p_r(\breve{\theta})\left|\frac{\partial P}{\partial \theta}(\breve{\theta})\right|^2_{\Sigma} - \rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\Sigma\Phi'C'(y_d-\breve{x}_1) \\
& -\rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\Sigma\Phi'C'\epsilon(\gamma^2\zeta^2-1)C\Phi(\hat{\theta}-\breve{\theta}) + \rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\gamma^2\zeta^2\Sigma\Phi'C'(\tilde{x}_1 + E\grave{M}\grave{w}_b) \\
\le & -\frac{1}{2}\left|\xi-\hat{\xi}\right|^2_{\bar{Q}} - \frac{1}{4}\left|\xi-\breve{\xi}\right|^2_{\bar{Q}} + 2(\theta-\breve{\theta})'P_r(\breve{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r}\left|z_j\right|^2_{\beta_j} - \frac{1}{2}\left|\hat{\xi}-\breve{\xi}\right|^2_{\bar{Q}} + \bar{c}_w^2 \\
& -\rho_o K_c^{-1}\left(\rho_o - P(\breve{\theta})\right)^{-2} p_r(\breve{\theta})\left|\frac{\partial P}{\partial \theta}(\breve{\theta})\right|^2 - \rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\Sigma\Phi'C'(y_d-\breve{x}_1) \\
& -\rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\Sigma\Phi'C'\epsilon(\gamma^2\zeta^2-1)C\Phi(\hat{\theta}-\breve{\theta}) + \rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\gamma^2\zeta^2\Sigma\Phi'C'(\tilde{x}_1 + E\grave{M}\grave{w}_b) \\
\le & -\gamma^4/4\left|\tilde{x}-\Phi\tilde{\theta}\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta-\breve{\theta})'P_r(\breve{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r}\left|z_j\right|^2_{\beta_j} - \frac{1}{2}\left|\hat{\theta}-\breve{\theta}\right|^2_{\Phi'C'\epsilon(\gamma^2\zeta^2-1)C\Phi} + \bar{c}_w^2 \\
& -\rho_o K_c^{-1}\left(\rho_o - P(\breve{\theta})\right)^{-2} p_r(\breve{\theta})\left|\frac{\partial P}{\partial \theta}(\breve{\theta})\right|^2 - \rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\Sigma\Phi'C'(y_d-\breve{x}_1) \\
& -\rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\Sigma\Phi'C'\epsilon(\gamma^2\zeta^2-1)C\Phi(\hat{\theta}-\breve{\theta}) + \rho_o\left(\rho_o - P(\breve{\theta})\right)^{-2}\frac{\partial P}{\partial \theta}(\breve{\theta})\gamma^2\zeta^2\Sigma\Phi'C'(\tilde{x}_1 + E\grave{M}\grave{w}_b) \\
\le & -\gamma^4/4\left|\tilde{x}-\Phi\tilde{\theta}\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta-\breve{\theta})'P_r(\breve{\theta}) - |\tilde{\eta}|^2_Y - \sum_{j=1}^{r}\left|z_j\right|^2_{\beta_j} \\
& -\rho_o/K_c p_r(\breve{\theta})\left(\rho_o - P(\breve{\theta})\right)^{-2}\left|\frac{\partial P}{\partial \theta}(\breve{\theta})\right|^2 + \left(\rho_o - P(\breve{\theta})\right)^{-4}\left|\frac{\partial P}{\partial \theta}(\breve{\theta})\right|^2 c_2 + c_2
\end{aligned}$$

for some constant $c_2 \in \overline{\mathbb{R}_+}$. This inequality follows from the uniform boundedness of y_d, x_1, $\breve{x}_1$, $C\Phi$, and $\grave{w}_e$, and a completion of squares with respect to $\hat{\theta}-\breve{\theta}$. Then, there exists a compact set $\Omega_2(c_2) \subset \mathcal{D}_e$ such that, $\forall t \in [0, T_f)$, if $X_e \in \mathcal{D}_e \setminus \Omega_2(c_2)$ then $\dot{\Upsilon} < 0$. Note that, $\forall (t, X_e) \in [0, T_f) \times \mathcal{D}_e$,

$$U_m(X_e) + P(\breve{\theta})(\rho_o - P(\breve{\theta}))^{-1} \le \bar{\Upsilon}(t, X_e) \le U_M(X_e) + P(\breve{\theta})(\rho_o - P(\breve{\theta}))^{-1}$$

By Lemma 5, there exists a constant $c_3 \in \overline{\mathbb{R}_+}$ such that $U_m(X_e(t)) + P(\breve{\theta}(t))(\rho_o - P(\breve{\theta}(t)))^{-1} \le c_3$, $\forall t \in [0, T_f)$. Hence, there exists a compact set $\Theta_c \subset \Theta_o$ such that $\breve{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$.

Now, use the true system $\acute{S}$ with inputs u_a and $\grave{w}_e$ and output x_1 as the reference system. Without loss of generality, assume that $\acute{S}$ is given in extended zero-dynamics canonical form (EZDCF) (see Lemma 2 or Lemma 3 of [1])

$$\dot{\acute{x}}_z = \acute{A}_z\acute{x}_z + \acute{A}_{z1}\acute{x}_1 + \acute{D}_{ez}\grave{w}_e \tag{59a}$$

$$\dot{\acute{x}}_i = \acute{A}_{i1}\acute{x}_1 + \acute{x}_{i+1} + \acute{D}_{ei}\grave{w}_e; \qquad i = 1, \ldots, r-1 \tag{59b}$$

$$\dot{\acute{x}}_r = \acute{A}_{rz}\acute{x}_z + \acute{A}_{r1}\acute{x}_1 + \acute{B}_0 u_a + \acute{D}_{er}\grave{w}_e \tag{59c}$$

$$y = \acute{x}_1 + \acute{E}\grave{w}_e \tag{59d}$$

Then, the entire state vector $\acute{x}$ is bounded on $[0, T_f)$ by the definition of minimum phase [1] since y is bounded. Then, η_i, $i = 1, \ldots, m$, $\breve{\eta}_{1i}$, $i = 1, \ldots, \breve{q}_1$, $\breve{\eta}_{2i}$, $i = 1, \ldots, \breve{q}_2$, λ_{bi}, $i = 1, \ldots, p-m$, and λ_o are bounded, since they are some stably filtered output signals of y or bounded signals. Then, λ_{ai}, $i = 1, \ldots, m$, are bounded since they are stably filtered signals of u_a with relative degree at least 1 with respect to u_a, where the reference system has the output y and input u_a and $\grave{w}_e$, in the application of the Proposition 2 of [20]. Then, the signal Φ is uniformly bounded. Further, x is bounded since it is a part of $\acute{x}$. Therefore, $\breve{x}$ is uniformly bounded, by the uniform boundedness of $\tilde{x} - \Phi\tilde{\theta}$.

The preceding analysis then leads to the conclusion that there exists a compact set $S \subset \mathcal{D}$ such that $X(t) \in S$, $\forall t \in [0, T_f)$. Thus, we conclude that $T_f = +\infty$. This further implies that the control inputs u and $\hat{\xi}$ are uniformly bounded. This establishes the first statement in this case.

Case 2: $r = 2$. In this case, using the same arguments as in the first eleven paragraphs in Case 1, we may conclude the boundedness of $\tilde{\eta}$, Φ_y, $\Phi_{u_{as}1}$, $\check{x}_1$, $x_1 - B_{p0}(\theta)\bar{\lambda}_{c1}$, $\bar{\lambda}_{c1}$, Φ_1, x_1, and y, on $[0, T_f)$, and the existence of a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$.

Note that X_{1o}, X_{1a}, and X_{1d} are inside compact subsets of their domains $\mathcal{D}_{1o}$, $\mathcal{D}_{1a}$, and $\mathcal{D}_{1d}$, respectively. Then, the virtual control signal $\alpha_1(X_{1o}, X_{1a}, X_{1d})$ is uniformly bounded. Now, use the true system with inputs u_a and $\acute{w}_e$ and output x_1 as the reference system as in EZDCF (59). By [1], $\acute{x}_z$ is bounded since y is bounded. The signal $x_1 \equiv \acute{x}_1$ is minimum phase with respect to $\acute{\mathcal{D}}_0$ and $\acute{\mathcal{W}}_d$, and admits uniform vector relative degree r with respect to the input u_a. By a similar bounding analysis as the one in the second through eighth paragraphs in Case 1, we can deduce the uniform boundedness of signals $\Phi_{u_{as}2}$, $\tilde{x}_2 - \Phi_2\tilde{\theta}$, $\check{x}_2$, $x_2 - B_{p0}(\theta)\bar{\lambda}_{c2}$, $\bar{\lambda}_{c2}$, Φ_2, and x_2.

Note that $x_1 \equiv \acute{x}_1$ and $\dot{x}_1 = A_{1,1}x_1 + a_{1,2}x_2 + B_{1,b}u_b + \acute{D}_1\acute{w} \equiv \acute{A}_{11}\acute{x}_1 + \acute{x}_2 + \acute{D}_{e,1}\acute{w}_e$. By the preceding analysis, we have that $\acute{x}_2$ is bounded on $[0, T_f)$. Using the true system (59) as reference system, with $\acute{x}_1$ and $\acute{x}_2$ being bounded, we can conclude that stably filtered signals of u_a with relative degree at least $r - 1 = 1$ are bounded. Thus, using the same arguments as in the last two paragraphs in Case 1, we can prove statement 1 in this case.

Case 3: $r \geq 3$. In this case, by the same arguments as in the first eleven paragraphs in Case 1, we may conclude the boundedness of $\tilde{\eta}$, Φ_y, $\Phi_{u_{as}1}$, $\check{x}_1$, $x_1 - B_{p0}(\theta)\bar{\lambda}_{c1}$, $\bar{\lambda}_{c1}$, Φ_1, x_1, and y, on $[0, T_f)$, and the existence of a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$.

Now, using the same arguments as in the second paragraph in Case 2, we may conclude the boundedness of $\Phi_{u_{as}2}$, $\tilde{x}_2 - \Phi_2\tilde{\theta}$, $\check{x}_2$, $x_2 - B_{p0}(\theta)\bar{\lambda}_{c2}$, $\bar{\lambda}_{c2}$, Φ_2, and x_2, on $[0, T_f)$.

By the same arguments as in the third paragraph in Case 2, we have that $\acute{x}_2$ of the true system (59) is bounded and (59) may serve as the reference system in the application of Proposition 2 of [20] to conclude the boundedness of outputs of stable systems with relative degree $r_2 \geq r - 1$ with respect to the input u_a.

By a line of reasoning that is similar to the one in the second paragraph in Case 2, we can conclude the boundedness of $\Phi_{u_{as}3}$, $\tilde{x}_3 - \Phi_3\tilde{\theta}$, $\check{x}_3$, $x_3 - B_{p0}(\theta)\bar{\lambda}_{c3}$, $\bar{\lambda}_{c3}$, Φ_3, and x_3, on $[0, T_f)$.

It is easy to see that we may conclude the boundedness of $\acute{x}_3$ in (59). Inductively, we can conclude the boundedness of $\check{x}_4$, $\bar{\lambda}_{c4}$, $x_4, \ldots, \check{x}_r$, $\bar{\lambda}_{cr}$, and x_r on $[0, T_f)$.

By a line of reasoning that is similar to the one in the last two paragraphs in Case 1, we can prove statement 1 in this case.

Thus, we have established statement 1 in all three cases. This completes the proof of statement 1.

Next, we prove the second statement. Fix any uncertainty quadruple $(\acute{x}_0, \theta, \acute{w}_{[0,\infty)}, y^{(r)}_{d\,[0,\infty)}) \in \acute{\mathcal{W}}$. For any $t_f \geq 0$, there exist constants $c_w \geq 0$ and $c_d \geq 0$ such that $|\acute{x}_0| \leq c_w$, $|\acute{w}(t)| \leq c_w$, and $|Y_d(t)| \leq c_d$, $\forall t \in [0, t_f]$, since $\acute{w}$ and Y_d are continuous. By the first statement and the causality of the closed-loop system, there exists a solution $X : [0, t_f] \to \mathcal{D}$ for the closed-loop system. Hence, the closed-loop system (56) admits a unique solution on $[0, \infty)$. This further implies that the proposed adaptive control law belongs to $\mathcal{M}$. Choose

$$
\begin{aligned}
l(t, \theta, x_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, Y_{d\,[0,t]}) &= \gamma^4 \left| x - \hat{x} - \Phi(\theta - \hat{\theta}) \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon(\gamma^2\zeta^2 - 1) \left| \theta - \hat{\theta} \right|^2_{\Phi'C'C\Phi} \\
&\quad -2(\theta - \check{\theta})' P_r(\check{\theta}) + \left| \varsigma_r \right|^2_{\bar{Q}}/4 + l_r - \left| z_1 \right|^2 - \varsigma_b'\mu_b \\
&\geq \gamma^4 \left| x - \hat{x} - \Phi(\theta - \hat{\theta}) \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon(\gamma^2\zeta^2 - 1) \left| \theta - \hat{\theta} \right|^2_{\Phi'C'C\Phi} - 2(\theta - \check{\theta})' P_r(\check{\theta}) + \left| \varsigma_r \right|^2_{\bar{Q}}/4 + |\tilde{\eta}|^2_Y + \sum_{j=1}^{r} \left| z_j \right|^2_{\beta_j(\check{\theta})} - \varsigma_b'\mu_b \\
l_0 &= V(X_{ro}(0), X_{ra}(0))
\end{aligned}
$$

The function l is clearly nonnegative as long as $X(t) \in \mathcal{D}$ with $\theta \in \Theta$, which is guaranteed by the first statement. Then, we have

$$
J_{\gamma t_f} = J_{\gamma t_f} + \int_0^{t_f} \dot{U}\,\mathrm{d}\tau + U(0) - U(t_f) \leq -U(t_f) \leq 0
$$

This shows that the controller μ, with the optimal choice $\hat{\xi}_*$, achieves the disturbance attenuation level 0 with respect to $\check{w}_1$ and disturbance attenuation level γ with respect to $\check{w}_2$ and w_b as prescribed by Definition 1. This establishes the second statement.

Last, we prove the third statement. For any uncertainty quadruple $(\acute{x}_0, \theta, \acute{w}_{[0,\infty)}, y^{(r)}_{d\,[0,\infty)}) \in \acute{\mathcal{W}}$ with $\check{w}_{1[0,\infty)} \in \bar{\mathrm{L}}_\infty$, $\check{w}_{2[0,\infty)} \in \bar{\mathrm{L}}_2 \cap \bar{\mathrm{L}}_\infty$, $\acute{w}_{b[0,\infty)} \in \bar{\mathrm{L}}_2 \cap \bar{\mathrm{L}}_\infty$ and $Y_{d\,[0,\infty)} \in \bar{\mathrm{L}}_\infty$, we have statements 1 and 2 hold. Then,

$$
\int_0^\infty |x_1(t) - y_d(t)|^2\,\mathrm{d}t \leq U(0) + \gamma^2 \int_0^\infty \left(\left| \acute{M}\acute{w}_b(t) \right|^2 + \left| \check{w}_2(t) \right|^2 \right) \mathrm{d}t < +\infty
$$

by the dissipation inequality (51) and the second statement. This implies that $x_1 - y_d \in \bar{\mathrm{L}}_2$ on the interval $[0, \infty)$. By the first statement, we have that $\dot{x}_1 - \dot{y}_d \in \bar{\mathrm{L}}_\infty$ on the interval $[0, \infty)$. Therefore,

$$\lim_{t \to \infty}(x_1(t) - y_d(t)) = \mathbf{0}_m$$

This completes the proof of the theorem. □

Consider the second adaptive control law where the choice for $\hat{\xi}$ is the suboptimal one. The closed-loop system dynamics are

$$\dot{X} = \hat{F}(X, y_d^{(r)}, \check{w}_1) + G(X)\begin{bmatrix}\check{w}_2\\ w_b\end{bmatrix} = \hat{F}(X, y_d^{(r)}, \check{w}_1) + G(X)\begin{bmatrix}\check{w}_2\\ \grave{M}\check{w}_b\end{bmatrix}; \qquad X(0) = X_0 \tag{60}$$

where $\hat{F}$ is a smooth mappings of $\mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^{\check{q}_1}$ and G is defined as in (56). Again, $X_0 \in \mathcal{D}_0$. Consider the value function U defined by (50), whose derivative is given by (51), where the two terms involving ς_r vanish since $\hat{\xi} = \check{\xi}$. By Lemma 8 of Appendix B, the value function U satisfies the following Hamilton-Jacobi-Isaacs equation:

$$\frac{\partial U}{\partial X}(X)\hat{F}(X, y_d^{(r)}, \check{w}_1) + \frac{1}{4\gamma^2}\left\| \frac{\partial U}{\partial X}(X)G(X)\right\|^2_{\mathbb{R}^{\check{q}_2+mq_b}} + \hat{Q}(X, y_d^{(r)}, \check{w}_1) = 0; \quad \forall X \in \mathcal{D},\ \forall y_d^{(r)} \in \mathbb{R}^m,\ \forall \check{w}_1 \in \mathbb{R}^{\check{q}_1} \tag{61}$$

where $\hat{Q} : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^{\check{q}_1} \to \mathbb{R}$ is smooth and given by

$$\begin{aligned}\hat{Q}(X, y_d^{(r)}, \check{w}_1) = &\left|x_1 - y_d\right|^2 + \gamma^4\left|x - \check{x} - \Phi(\theta - \check{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \check{\theta}\right|^2_{\Phi'C'C\Phi}\\ &-2(\theta - \check{\theta})'P_r(\check{\theta}) + l_r(X_{ro}, X_{ra}, X_{rd}) - \left|z_1\right|^2 - \varsigma_b'\mu_b\\ \geq &\left|x_1 - y_d\right|^2 + \gamma^4\left|x - \check{x} - \Phi(\theta - \check{\theta})\right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon(\gamma^2\zeta^2 - 1)\left|\theta - \check{\theta}\right|^2_{\Phi'C'C\Phi}\\ &-2(\theta - \check{\theta})'P_r(\check{\theta}) + |\tilde{\eta}|_Y^2 + \sum_{j=1}^r \left|z_j\right|^2_{\beta_j(\check{\theta})} - \varsigma_b'\mu_b\end{aligned}$$

Clearly, $\hat{Q}$ is nonnegative $\forall X \in \mathcal{D}$ with $\theta \in \Theta$.

This now leads to the following corollary to Theorem 1.

Corollary 1. Consider the robust adaptive control problem formulated in Section 3, under the same assumptions as those of Theorem 1. Then, the same results of Theorem 1 hold for the robust adaptive controller μ given by (45) (or (47)) and (53), with the worst-case estimate $\hat{\xi}$ generated by the suboptimal policy (55).

Proof. The proof follows essentially the same line of reasoning as that of Theorem 1, except one modification.

Following the same line of reasoning as in the first five paragraphs in the proof for Theorem 1, we may conclude that Σ and s_Σ are bounded as desired, Y_d and X_e are uniformly bounded on $[0, T_f)$, which is the maximum length interval such that (60) admits a solution. We again distinguish between three exhaustive cases. Case 1: $r = 1$. Following the same line of arguments as in the first eight paragraphs in the Case 1 of the proof of Theorem 1, it can be concluded that $\tilde{\eta}$, Φ_y, $\Phi_{u_{as}1}$, $\check{x}_1$, $x_1 - B_{p0}(\theta)\bar{\lambda}_{c1}$, $\bar{\lambda}_{c1}$, Φ_1, x_1, and y are bounded on $[0, T_f)$.

To show the existence of the compact set $\Theta_c \subset \Theta_o$, we consider the total time derivative of the function $P(\check{\theta})(\rho_o - P(\check{\theta}))^{-1}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(P(\check{\theta})(\rho_o - P(\check{\theta}))^{-1}\right) &= \rho_o\left(\rho_o - P(\check{\theta})\right)^{-2}\Bigg(-\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma P_r(\check{\theta}) - \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma\Phi'C'(y_d - \check{x}_1)\\ &\quad + \frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2\zeta^2\Sigma\Phi'C'(\tilde{x}_1 + E\grave{M}\check{w})\Bigg)\\ &\leq -\rho_o/K_c p_r(\check{\theta})\left(\rho_o - P(\check{\theta})\right)^{-2}\left|\frac{\partial P}{\partial \theta}(\check{\theta})\right|^2 + \left(\rho_o - P(\check{\theta})\right)^{-4}\left|\frac{\partial P}{\partial \theta}(\check{\theta})\right|^2 c_4 + c_4\end{aligned}$$

for some constant $c_4 \in \overline{\mathbb{R}_+}$. By Lemma 5, there exists a constant $c_5 \in \overline{\mathbb{R}_+}$ such that $P(\check{\theta}(t))(\rho_o - P(\check{\theta}(t)))^{-1} \leq c_5$, $\forall t \in [0, T_f)$. Then, there exists a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$ on this maximum length interval.

By a line of reasoning that is the same as in the last two paragraphs in Case 1 of the proof of Theorem 1, statement 1 is established in this case.

Case 2: $r = 2$ and Case 3: $r \geq 3$ can be similarly handled as those in the proof of Theorem 1 with the above modified proof for the fact that $\check{\theta}(t) \in \Theta_c \subset \Theta_o$, $\forall t \in [0, T_f)$. This completes the proof for statement 1.

By a line of reasoning that is similar to that of the proof of Theorem 1, we conclude that the adaptive controller (45) (or (47)) and (53), with the suboptimal policy $\hat{\check{\xi}} = \check{\xi}$, belongs to $\mathcal{M}$ and achieves the disturbance attenuation level 0 with respect to $\check{w}_1$ and disturbance attenuation level γ with respect to $\check{w}_2$ and w_b for any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y^{(r)}_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$.

Furthermore, the asymptotic tracking of the state variable x_1 to the reference trajectory y_d follows from the same argument as that of the proof for Theorem 1.

This completes the proof of this corollary. □

7 | AN EXAMPLE

In this section, we present a numerical example that serves to illustrate the robust adaptive control design presented in this paper. The designs for the example were carried out using MATHEMATICA.

We consider the following adaptive noise cancellation problem. The uncertain linear system is given as below, where $\theta_1 \in \bar{r}_{1,4}$ and $\theta_2 \in \bar{r}_{1,4}$ are unknown parameters,

$$\dot{\acute{x}} = \begin{bmatrix} -\theta_1 & 0 & 1 & 0 \\ -\theta_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\theta_2 & 0 \end{bmatrix} \acute{x} + \begin{bmatrix} 0 & \theta_1 \\ 0 & \theta_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \acute{u} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \grave{w}_b; \qquad \acute{x}_0 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{10} \\ \frac{1}{10} \end{bmatrix} \tag{62a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \acute{x} + \begin{bmatrix} \theta_2 & 0 \\ -\theta_1 & 0 \end{bmatrix} \acute{u} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \grave{w}_b \tag{62b}$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \acute{x} + \begin{bmatrix} \theta_2 & 0 \\ -\theta_1 & 0 \end{bmatrix} \acute{u} \tag{62c}$$

This uncertain system does not have vector relative degree, but with one step of dynamic extension, it can be made to have uniform vector relative degree of 1. The dynamic extension is independent of the unknown parameters θ_1 and θ_2:

$$\dot{\imath} = \begin{bmatrix} 1 & 0 \end{bmatrix} u; \qquad \imath_0 = 0 \tag{63a}$$

$$\acute{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \imath + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} u \tag{63b}$$

The composite system of (62) and (63) has the extended zero dynamics canonical form (after state transformations)

$$\dot{\acute{x}} = \begin{bmatrix} 0 & -\theta_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\theta_2}{\theta_1} & \frac{\theta_1(\theta_1-\theta_2)\theta_2}{\theta_1^2+\theta_2^2} & \frac{\theta_1^2(-\theta_1+\theta_2)}{\theta_1^2+\theta_2^2} & \frac{\theta_1\theta_2(-\theta_1+\theta_2)}{\theta_1^2+\theta_2^2} \\ 0 & 1 & \frac{\theta_1^2\theta_2}{\theta_1^2+\theta_2^2} & -\frac{\theta_1^3}{\theta_1^2+\theta_2^2} & - -\frac{\theta_1^2\theta_2}{\theta_1^2+\theta_2^2} \\ 1 & 0 & \frac{\theta_1^2\theta_2}{\theta_1^2+\theta_2^2} & -\frac{\theta_1^3}{\theta_1^2+\theta_2^2} & -\frac{\theta_1^2\theta_2}{\theta_1^2+\theta_2^2} \end{bmatrix} \acute{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \theta_2 & \theta_1 \\ -\theta_1 & \theta_2 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\theta_2}{\theta_1} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \grave{w}_b$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \acute{x} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \grave{w}_b$$

This implies that the extended zero dynamics is of third order and the system is minimum phase with respect to $\mathbb{R}^5$ and C if $0 < \theta_1 < \theta_2$ according to [1]. Then, we add a dummy state variable to make the system have uniform observability indices, and subsequently transform it into strict observer canonical form. We thus arrive at the following design model

$$\dot{x} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes I_2\right) x + \begin{bmatrix} -\theta_1 - \theta_3 & -\theta_{12} \\ -\theta_1 - \theta_2 + \theta_9 & \theta_3 \\ -\theta_3 + \theta_4 & \theta_3 \\ -\theta_8 + \theta_9 - \theta_{10} & -\theta_9 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} y + \begin{bmatrix} \theta_2 & \theta_1 \\ -\theta_1 & \theta_2 \\ \theta_4 + \theta_6 + 3\theta_8 + \theta_9 + \theta_{10} & -\theta_2 + \theta_3 + \theta_8 \\ \theta_6 + \theta_7 + 3\theta_8 + \theta_9 + \theta_{10} & 2\theta_8 - \theta_9 + \theta_{10} \\ \theta_9 - \theta_{10} & \theta_8 - \theta_9 \\ -\theta_5 + \theta_{11} & \theta_5 - \theta_6 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} y$$

$$
+(\begin{bmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \otimes I_2) w_b \tag{64a}
$$

$$
y = (\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \otimes I_2) x + (\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \otimes I_2) w_b \tag{64b}
$$

where we have defined

$$
\begin{aligned}
\theta &:= (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10}, \theta_{11}, \theta_{12}) \\
&:= \Big(\theta_1, \theta_2, \frac{\theta_1^2 \theta_2}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_1^3 + \theta_1 \theta_2^2}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_1^2 \theta_2^3}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_1^3 \theta_2^2}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_2^4}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_1^3 \theta_2}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \\
&\qquad \frac{\theta_1^2 \theta_2^2}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_1 \theta_2^3}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_1 \theta_2^4}{\theta_2^2 + \theta_1^2(2+\theta_2)}, \frac{\theta_1^2}{\theta_2^2 + \theta_1^2(2+\theta_2)} \Big)
\end{aligned}
$$

and introduced the disturbance transformation $w_b = \grave{M} \grave{w}_b$ with

$$
\grave{M} := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{\theta_1+\theta_3}{6} & -\frac{1-\theta_{12}}{6} & \frac{1}{6} & 0 \\
\frac{\theta_1+\theta_2-\theta_9}{6} & -\frac{\theta_3}{6} & 0 & \frac{1}{6} \\
\frac{\theta_3+\theta_4}{4} & -\frac{\theta_3}{4} & \frac{\theta_3}{4} & \frac{\theta_{12}-1}{4} \\
\frac{\theta_8-\theta_9+\theta_{10}}{4} & \frac{\theta_9}{4} & \frac{\theta_2-\theta_9}{4} & -\frac{\theta_3}{4} \\
0 & 0 & \frac{\theta_3}{4} & -\frac{\theta_3}{4} \\
0 & 0 & -\frac{\theta_9}{4} & \frac{\theta_9}{4}
\end{bmatrix}
$$

The true values of the parameters are $(\theta_1, \theta_2) = (1, 2)$. This corresponds to the true values of $\theta = (1, 2, \frac{1}{4}, \frac{5}{8}, 1, \frac{1}{2}, 2, \frac{1}{4}, \frac{1}{2}, 1, 2, \frac{1}{8})$. The compact set for θ is given by $\theta_1 \in \bar{\mathrm{r}}_{1,\theta_2}$, $\theta_2 \in \bar{\mathrm{r}}_{1,4}$, $\theta_3 \in \bar{\mathrm{r}}_{\frac{2}{11},\frac{4}{7}}$, $\theta_4 \in \bar{\mathrm{r}}_{\frac{1}{2},\frac{8}{7}}$, $\theta_5 \in \bar{\mathrm{r}}_{\frac{1}{4},\frac{64}{7}}$, $\theta_6 \in \bar{\mathrm{r}}_{\frac{1}{4},\frac{64}{7}}$, $\theta_7 \in \bar{\mathrm{r}}_{\frac{1}{4},\frac{128}{11}}$, $\theta_8 \in \bar{\mathrm{r}}_{\frac{2}{11},\frac{16}{7}}$, $\theta_9 \in \bar{\mathrm{r}}_{\frac{1}{4},\frac{16}{7}}$, $\theta_{10} \in \bar{\mathrm{r}}_{\frac{1}{4},\sqrt{\frac{32}{3}}}$, $\theta_{11} \in \bar{\mathrm{r}}_{\frac{1}{4},\sqrt{\frac{512}{3}}}$, and $\theta_{12} \in \bar{\mathrm{r}}_{\frac{1}{22},\frac{1}{4}}$. The initial estimates for the parameters are selected to be $\check{\theta}_0 = (2, 2, \frac{2}{5}, \frac{4}{5}, \frac{8}{5}, \frac{8}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{8}{5}, \frac{1}{5})$. The initial estimate for the state vector is selected to be $\check{x}_0 = \mathbf{0}_6$. The reference trajectory is set to be identically zero.

The first set of simulations is aimed to demonstrate the asymptotic cancellation of the sinusoidal noise capability of the controller. The disturbance input $\grave{w}_b$ is fixed to be identically zero. The simulation results are shown in Figure 1. We observe that the tracking errors converge to zero as predicted and control inputs are bounded in magnitude by 0.15 and the transient of the system response is well behaved. The parameter estimation errors do not converge to zero since there is no persistant excitation in the system (only one sinusoidal for twelve parameters). The integral performance index seems to grow from zero to some positive constant. These simulation results corroborate our theoretical results. We observe that the parameter estimation errors are well behaved.

The second set of simulations is aimed to demonstrate the disturbance rejection properties of the controller. The disturbance input is set to be

$$
\grave{w}_b(t) = (\frac{1}{50}\cos(2t), -\frac{1}{25}\sin(2t), \frac{1}{50}\cos(\pi t), -\frac{\pi}{50}\sin(\pi t))
$$

The simulation results are shown in Figure 2. We see that the tracking errors are bounded in magnitude by 0.08 and are asymptotically bounded by 0.065; the control inputs are bounded in magnitude by 0.2 and asymptotically by 0.2; and the transient of the system response is well behaved. Further, the parameter estimation errors are well behaved. The integral performance index is upper bounded by 0 and shows a negative slope of 0.0038 converging to negative infinity. These simulation results corroborate our theoretical results.

8 | CONCLUSIONS

In this paper, we have presented a systematic design procedure for robust adaptive controllers for minimum phase uncertain MIMO linear systems that are right invertible and can be dynamically extended to a linear system with vector relative degree using a known dynamic compensator. For this class of systems, it is always possible to dynamically extend them [1], and/or integrate a select set of output channels [15], and padding dummy state variable [15] to arrive at a system model that has uniform vector relative degree $r \in \mathbb{Z}_+$ and uniform observability indices $\nu \in \mathbb{N}$ ($r \leq \nu$) that is minimum phase according to [1]. We assumed that $r \in \mathbb{N}$ is known and an upper bound n for ν is known ($r = 0$ case will be treated in another paper). Thus, the system admits the extended zero dynamics canonical form and the strict observer canonical form. The observable part of the system is then the design model for the system, which is further restricted to be in a block diagonal structure for the backbone of the system that is independent of the unknown parameter vector and the control inputs and measurement outputs of the system. The design procedure closely resembles that for the SISO case [5]. This design procedure has led to a recursive design scheme for two classes of robust adaptive controllers for the minimum phase uncertain MIMO linear system (each one parametrized by the desired disturbance attenuation level γ). The controller actively incorporates the covariance information on the parameter estimates into the design, and exhibits (in principle) the asymptotic certainty equivalence property, if the worst case covariance matrix converges to zero. However, to guarantee the boundedness of all closed-loop signals under any admissible bounded exogenous disturbance inputs, any bounded reference trajectory together with its derivatives up to rth order, and any admissible bounded initial conditions, an appropriate cost functional was selected to keep the covariance matrix bounded away from zero. Hence, the asymptotic certainty equivalence structure is in fact never realized. But, when the covariance matrix is close to zero, the controller behaves as a certainty equivalent one. The adaptive controller also achieves the desired disturbance attenuation level for all admissible initial conditions and all admissible continuous exogenous disturbance input waveforms on the infinite horizon. Furthermore, it is proved that the control law guarantees boundedness of all closed-loop signals under any admissible bounded exogenous disturbance inputs, any bounded reference trajectory together with its derivatives up to rth order, and any admissible bounded initial conditions without the need for any persistency of excitation condition or any stochastic noise assumptions. Asymptotic tracking is achieved when the initial condition is admissible, the reference trajectory together with its derivatives up to rth order are bounded, the admissible disturbance inputs are bounded, and those disturbance inputs with positive attenuation level are of finite energy. A numerical example was worked out and illustrates the steps involved in designing a robust adaptive controller for a minimum phase uncertain MIMO linear system with two inputs and two outputs. The simulation results corroborate our theoretical findings.

A number of future research directions stand out as promising. One fruitful direction of research pertains to the study of the counterpart of the theory developed here to MIMO nonlinear systems with noiseless output measurements or with noiseless output measurements and noisy output derivative measurements. Another interesting topic is to study the robustness of the adaptive control scheme presented here with respect to unmodeled fast dynamics. Another interesting direction of research lies in the study of networked robust adaptive control systems. It has been observed and proved that robust adaptive control systems designed according to [5] can be networked in a feedback loop fashion, and under the satisfaction of the small gain condition for the L_2-gains of the closed-loop system, the closed-loop signals will remain bounded for any admissible bounded exogeneous disturbance inputs and any admissible bounded initial conditions that are further convergent (that is, the tracking errors converge to zeros) when the exogeneous disturbance inputs are L_2 and vanishing. This result paves the way for the application of the robust adaptive control system theory in practical use. Another fruitful research direction lies in the case when the given MIMO LTI system is comprised of multiple square MIMO LTI subsystems in parallel interconnection satisfying an interconnection property, where the subsystems are assumed to be robust adaptive control ready (i. e., with uniform vector relative degree and uniform observability indices) but the composite system may have nonuniform vector relative degree and/or nonuniform observability indices. In this case, we envision that a centralized controller can be designed without requiring any dynamic extension or adding dummy state variables to the design model.

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Conflict of interest

The authors declare no potential conflict of interests.

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APPENDIX

A MATHEMATICAL PRELIMINARIES

In this section, we introduce some mathematical preliminaries.

Fix any real normed linear space $\mathcal{X}$; $\mathcal{S}_{\mathcal{X}} = \mathrm{B}_{\mathrm{S}\,2}(\mathcal{X}, \mathbb{R})$ is a closed subspace of $\mathrm{B}(\mathcal{X}, \mathcal{X}^*)$, and therefore a real Banach space. $\forall M \in \mathcal{S}_{\mathcal{X}}$, we will write $M \in \mathcal{S}_{+\,\mathcal{X}}$ if $\exists \alpha \in \mathbb{R}_+$, $\forall x \in \mathcal{X}$, we have $M(x)(x) \geq \alpha \| x \|^2$; and we will write $M \in \mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$ if $\forall x \in \mathcal{X}$, we have $M(x)(x) \geq 0$. Letting $\mathcal{S}_{-\,\mathcal{X}} := -\mathcal{S}_{+\,\mathcal{X}}$ and $\mathcal{S}_{\mathrm{nsd}\,\mathcal{X}} := -\mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$. $\forall M_1, M_2 \in \mathcal{S}_{\mathcal{X}}$, we write $M_1 < M_2$ if $M_2 - M_1 \in \mathcal{S}_{+\,\mathcal{X}}$; and $M_1 \leq M_2$ if $M_2 - M_1 \in \mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$.

Proposition 1. Let $\mathcal{X}$ be a real normed linear space, and $M_0 \in \mathcal{S}_{\mathcal{X}}$. Then, we have that $\mathcal{S}_{+\,\mathcal{X}}$, $\mathcal{S}_{-\,\mathcal{X}}$, $\mathcal{M}_1 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M > M_0 \}$, $\mathcal{M}_2 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M < M_0 \}$ are open sets in $\mathcal{S}_{\mathcal{X}}$; $\mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$, $\mathcal{S}_{\mathrm{nsd}\,\mathcal{X}}$, $\mathcal{M}_3 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M \geq M_0 \}$, and $\mathcal{M}_4 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M \leq M_0 \}$ are closed sets in $\mathcal{S}_{\mathcal{X}}$. Furthermore, $\mathcal{S}_{+\,\mathcal{X}} \subseteq \mathcal{S}^{\circ}_{\mathrm{psd}\,\mathcal{X}}$ and $\mathcal{S}_{-\,\mathcal{X}} \subseteq \mathcal{S}^{\circ}_{\mathrm{nsd}\,\mathcal{X}}$, $\mathcal{M}_1 \subseteq \mathcal{M}_3^{\circ}$ and $\mathcal{M}_2 \subseteq \mathcal{M}_4^{\circ}$.

Proof. This follows directly from Proposition 10.4 of [21].[2] □

Next, we specialize the above result to real Hilbert spaces.

Proposition 2. Let $\mathcal{X}$ be a real Hilbert space, and $M_0 \in \mathcal{S}_{\mathcal{X}}$. Then we have that $\mathcal{S}_{+\,\mathcal{X}}$, $\mathcal{S}_{-\,\mathcal{X}}$, $\mathcal{M}_1 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M > M_0 \}$, $\mathcal{M}_2 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M < M_0 \}$ are open sets in $\mathcal{S}_{\mathcal{X}}$; the closures of $\mathcal{S}_{+\,\mathcal{X}}$ and $\mathcal{S}_{-\,\mathcal{X}}$ are $\mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$ and $\mathcal{S}_{\mathrm{nsd}\,\mathcal{X}}$, respectively, and the closures of $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{M}_3 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M \geq M_0 \}$ and $\mathcal{M}_4 := \{ M \in \mathcal{S}_{\mathcal{X}} \mid M \leq M_0 \}$, respectively.

Proof. By Proposition 1, all we need to show is that $\overline{\mathcal{S}_{+\,\mathcal{X}}} = \mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$. Then, $\overline{\mathcal{S}_{-\,\mathcal{X}}} = \overline{-\mathcal{S}_{+\,\mathcal{X}}} = -\overline{\mathcal{S}_{+\,\mathcal{X}}} = -\mathcal{S}_{\mathrm{psd}\,\mathcal{X}} = \mathcal{S}_{\mathrm{nsd}\,\mathcal{X}}$, where the second equality follows from Proposition 7.102 of [21].[3] Furthermore, $\overline{\mathcal{M}_1} = \overline{M_0 + \mathcal{S}_{+\,\mathcal{X}}} = M_0 + \overline{\mathcal{S}_{+\,\mathcal{X}}} = M_0 + \mathcal{S}_{\mathrm{psd}\,\mathcal{X}} = \mathcal{M}_3$, where the second equality follows from Proposition 7.16 of [21],[4] and $\overline{\mathcal{M}_2} = \overline{M_0 + \mathcal{S}_{-\,\mathcal{X}}} = M_0 + \overline{\mathcal{S}_{-\,\mathcal{X}}} = M_0 + \mathcal{S}_{\mathrm{nsd}\,\mathcal{X}} = \mathcal{M}_4$, where the second equality follows from Proposition 7.16 of [21].[5]

$\forall M \in \mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$, $\forall \delta \in (0, \infty) \subset \mathbb{R}$, $M + \frac{\delta}{2}\Phi_{\mathrm{inv}} \in \mathcal{B}_{\mathcal{S}_{\mathcal{X}}}(M, \delta) \cap \mathcal{S}_{+\,\mathcal{X}}$, where $\Phi : \mathcal{X}^* \to \mathcal{X}$ is defined as in Riesz-Fréchet Theorem 13.15 of [21].[6] By the arbitrariness of δ, we have $M \in \overline{\mathcal{S}_{+\,\mathcal{X}}}$. Hence, by the arbitrariness of M, $\mathcal{S}_{\mathrm{psd}\,\mathcal{X}} \subseteq \overline{\mathcal{S}_{+\,\mathcal{X}}}$. Hence, $\overline{\mathcal{S}_{+\,\mathcal{X}}} = \mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$, by Proposition 1.

This completes the proof of the proposition. □

When $\mathcal{X}$ is $\mathbb{R}^n$, we can obtain the following result.

Proposition 3. Let $M_1, M_2 \in \mathcal{S}_n$, where $n \in \mathbb{N}$. Let $\mathcal{M} := \{ M \in \mathcal{S}_n \mid M_1 \leq M \leq M_2 \}$. Then, $\mathcal{M}$ is compact.

Proof. In case that $M_1 \not\leq M_2$, then, $\mathcal{M} = \emptyset$. Clearly, $\mathcal{M}$ is compact. In the following, we will consider only the case where $M_1 \leq M_2$. Then, $\mathcal{M} = \{ M \in \mathcal{S}_n \mid M \geq M_1 \} \cap \{ M \in \mathcal{S}_n \mid M \leq M_2 \}$. By Proposition 2, $\mathcal{M}$ is a closed set. Clearly, $\mathcal{M}$ is nonempty. Now, we will show that $\mathcal{M}$ is bounded. Denote the elements of M_l by $(m_{l,ij})_{n\times n}$, $l = 1, 2$. $\forall M = (m_{ij})_{n\times n} \in \mathcal{M}$, we have $M - M_1 = (m_{ij} - m_{1,ij})_{n\times n} \in \mathcal{S}_{\mathrm{psd}\,n}$. Then, we have $m_{ii} \geq m_{1,ii}$, $i = 1, \ldots, n$. By the fact that $M_2 - M \in \mathcal{S}_{\mathrm{psd}\,n}$, we have $m_{ii} \leq m_{2,ii}$, $i = 1, \ldots, n$. Hence, m_{ii} is bounded inside the closed interval $[m_{1,ii}, m_{2,ii}]$, $i = 1, \ldots, n$. $\forall i, j \in \{1, \ldots, n\}$ with $i < j$, the 2×2 matrix

$$\begin{bmatrix} m_{ii} - m_{1,ii} & m_{ij} - m_{1,ij} \\ m_{ji} - m_{1,ji} & m_{jj} - m_{1,jj} \end{bmatrix} \in \mathcal{S}_{\mathrm{psd}\,2}$$

[2]For the convenience of the reader, this proposition has been reproduced as Proposition 7 in Appendix C.
[3]For the convenience of the reader, this proposition has been reproduced as Proposition 8 in Appendix C.
[4]For the convenience of the reader, this proposition has been reproduced as Proposition 9 in Appendix C.
[5]For the convenience of the reader, this proposition has been reproduced as Proposition 9 in Appendix C.
[6]For the convenience of the reader, this theorem has been reproduced as Theorem 2 in Appendix C.

since $M \geq M_1$. Then, $(m_{ij} - m_{1,ij})^2 \leq (m_{ii} - m_{1,ii})(m_{jj} - m_{1,jj})$. Hence, m_{ij} is bounded. Therefore, all elements of M are bounded. Hence, $\mathcal{M}$ is a closed and bounded subset of $\mathcal{S}_n$ and $\mathcal{S}_n$ is a finite dimensional real normed linear space. Therefore, $\mathcal{M}$ is compact by Proposition 7.42 of [21].[7]

This completes the proof of the proposition. □

Definition 2. Define functions $\kappa_1 : \mathbb{R} \to \mathbb{R}$, $\kappa_2 : \mathbb{R} \to \mathbb{R}$, $\kappa_3 : \mathbb{R} \to \mathbb{R}$, and $\kappa : \mathbb{R} \to \mathbb{R}$ by, $\forall x \in \mathbb{R}$,

$$\kappa_1(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{A1a}$$

$$\kappa_2(x) = 1 - e\kappa_1(1 - x) \tag{A1b}$$

$$\kappa_3(x) = \kappa_2(e\kappa_1(x)) \tag{A1c}$$

$$\kappa(x) = \frac{\kappa_3(x) + 1 - \kappa_3(1 - x)}{2} \tag{A1d}$$

Then, we have the following result concerning the properties of the above functions.

Proposition 4. Let κ_1, κ_2, κ_3, and κ be defined in Definition 2. Then,

1. κ_1 is C_∞, monotonically nondecreasing, strictly increasing on $[0, \infty)$, and $\lim_{x\to+\infty} \kappa_1(x) = 1$.
2. κ_2 is C_∞, monotonically nondecreasing, strictly increasing on $(-\infty, 1]$, $\kappa_2(x) = 1$, $\forall x \geq 1$, and $\kappa_2(0) = 0$.
3. κ_3 is C_∞, monotonically nondecreasing, strictly increasing on $[0, 1]$, $\kappa_3(x) = 1$, $\forall x \geq 1$, and $\kappa_3(x) = 0$, $\forall x \leq 0$.
4. κ is C_∞, monotonically nondecreasing, strictly increasing on $[0, 1]$, $\kappa(x) = 1$, $\forall x \geq 1$, $\kappa(x) = 0$, $\forall x \leq 0$ and $-\frac{1}{2} + \kappa(x + \frac{1}{2}) = \frac{1}{2} - \kappa(\frac{1}{2} - x)$, $\forall x \in \mathbb{R}$.

Proof. Statement 1 is standard from analysis.

For statement 2, κ_2 is C_∞ since it is the composition of C_∞ functions. $\forall x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$, we have $1 - x_1 \geq 1 - x_2$, which implies that $\kappa_1(1 - x_1) \geq \kappa_1(1 - x_2)$, and hence, $\kappa_2(x_1) \leq \kappa_2(x_2)$. This proves that κ_2 is monotonically nondecreasing. $\forall x_1, x_2 \in (-\infty, 1] \subset \mathbb{R}$ with $x_1 < x_2$, we have $1 - x_1 > 1 - x_2 \geq 0$, which implies that $\kappa_1(1 - x_1) > \kappa_1(1 - x_2)$, and hence, $\kappa_2(x_1) < \kappa_2(x_2)$. This proves that κ_2 is strictly increasing on $(-\infty, 1]$. $\forall x \geq 1$, we have $1 - x \leq 0$, which implies that $\kappa_1(1 - x) = 0$, and hence, $\kappa_2(x) = 1$. Note that $\kappa_2(0) = 1 - e\kappa_1(1) = 1 - ee^{-1} = 0$. This completes the proof of statement 2.

For statement 3, κ_3 is clearly C_∞ since it is a composition of C_∞ functions. $\forall x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$, we have $e\kappa_1(x_1) \leq e\kappa_1(x_1)$, which implies that $\kappa_3(x_1) = \kappa_2(e\kappa_1(x_1)) \leq \kappa_2(e\kappa_1(x_2)) = \kappa_3(x_2)$. This proves that κ_3 is monotonically nondecreasing. $\forall x_1, x_2 \in [0, 1] \subset \mathbb{R}$ with $x_1 < x_2$, we have $0 \leq e\kappa_1(x_1) < e\kappa_1(x_2) \leq 1$, which implies that $0 \leq \kappa_2(e\kappa_1(x_1)) = \kappa_3(x_1) < \kappa_3(x_2) = \kappa_2(e\kappa_1(x_2)) \leq 1$. This proves that κ_3 is strictly increasing on $[0, 1]$. $\forall x \geq 1$, we have $e\kappa_1(x) \geq e\kappa_1(1) = 1$, which implies that $\kappa_3(x) = \kappa_2(e\kappa_1(x)) = 1$. $\forall x \leq 0$, we have $e\kappa_1(x) = 0$, which implies that $\kappa_3(x) = \kappa_2(0) = 0$. This completes the proof of statement 3.

For statement 4, κ is clearly C_∞. $\forall x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$, we have $\kappa_3(x_1) \leq \kappa_3(x_2)$, $1 - x_1 \geq 1 - x_2$, and $\kappa_3(1 - x_1) \geq \kappa_3(1 - x_2)$, which further implies that $\kappa(x_1) \leq \kappa(x_2)$. This proves that κ is monotonically nondecreasing. $\forall x_1, x_2 \in [0, 1] \subset \mathbb{R}$ with $x_1 < x_2$, we have $\kappa_3(x_1) < \kappa_3(x_2)$, $1 \geq 1 - x_1 > 1 - x_2 \geq 0$, and $\kappa_3(1 - x_1) > \kappa_3(1 - x_2)$, which implies that $\kappa(x_1) < \kappa(x_2)$. This proves that κ is strictly increasing on $[0, 1]$. $\forall x \geq 1$, we have $\kappa_3(x) = 1$, $1 - x \leq 0$, and $\kappa_3(1 - x) = 0$, which further implies that $\kappa(x) = 1$. $\forall x \leq 0$, we have $\kappa_3(x) = 0$, $1 - x \geq 1$, $\kappa_3(x) \geq 1$, which implies that $\kappa(x) = 0$. $\forall x \in \mathbb{R}$, we have

$$\kappa(x + \frac{1}{2}) + \kappa(\frac{1}{2} - x) = \frac{\kappa_3(x + \frac{1}{2}) + 1 - \kappa_3(\frac{1}{2} - x)}{2} + \frac{\kappa_3(\frac{1}{2} - x) + 1 - \kappa_3(\frac{1}{2} + x)}{2} = 1$$

This completes the proof of statement 4.

This completes the proof of the proposition. □

Remark 2. The statement 4 of the previous proposition shows that the graph of κ is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$.

Definition 3. Define $\rho_1 : \mathbb{R} \to \mathbb{R}$ by, $\forall x \in \mathbb{R}$,

$$\rho_1(x) = x(1 - \kappa_1(x)) \tag{A2}$$

[7] For the convenience of the reader, this proposition has been reproduced as Proposition 10 in Appendix C.

Define $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\psi(a,b) = \begin{cases} 0 & b = 0 \\ \frac{a+\sqrt{(\rho_1(a))^2+\rho_1(b^2)}}{b} & b \neq 0 \end{cases} \qquad \forall (a,b) \in \mathbb{R} \times \mathbb{R} \tag{A3}$$

Then, we have the following results for ρ_1 and ψ.

Proposition 5. ρ_1 is C_∞ and strictly increasing. $\rho_1(x) = x$, $\forall x \leq 0$. $0 < \rho_1(x) < x$, $\forall x > 0$. $\lim_{x \to +\infty} \rho_1(x) = 1$.

Proof. Clearly, ρ_1 is C_∞. $\forall x \leq 0$, we have $\rho_1(x) = x(1-0) = x$. $\forall x > 0$, we have $0 < \rho_1(x) = x(1 - e^{-\frac{1}{x}}) < x$. Note that

$$\lim_{x\to+\infty} \rho_1(x) = \lim_{x\to+\infty} x(1-e^{-\frac{1}{x}}) = \lim_{y\to 0^+} \frac{1-e^{-y}}{y} = \lim_{y \to 0^+} e^{-y} = 1$$

where we have applied L'Hospital's rule in the second to last equality. Now, all we need to show is that ρ_1 is strictly increasing. By the facts that we have proved above, we only need to show that ρ_1 is strictly increasing on $(0, +\infty)$. We will show that $\rho_1^{(1)}(x) > 0$, $\forall x > 0$. $\forall x > 0$, we have

$$\rho_1^{(1)}(x) = 1 - \kappa_1(x) - x\kappa_1^{(1)}(x) = 1 - e^{-\frac{1}{x}} - xe^{-\frac{1}{x}}\frac{1}{x^2} = 1 - e^{-\frac{1}{x}} - e^{-\frac{1}{x}}\frac{1}{x}$$

Note that $e^y \geq 1 + y$, $\forall y \in \mathbb{R}$, with equality holding if, and only if, $y = 0$. Then, we have

$$1 > (1+y)e^{-y}; \quad \forall y > 0; \qquad 1 > e^{-\frac{1}{x}} + e^{-\frac{1}{x}}\frac{1}{x}; \quad \forall x > 0$$

Hence, we have $\rho_1^{(1)}(x) > 0$. This completes the proof of the proposition. □

Lemma 2. Let $D_\psi := \left\{(a,b) \in \mathbb{R}^2 \;\middle|\; b \neq 0 \text{ or } a < 0\right\}$. Then, D_ψ is open in $\mathbb{R}^2$ and ψ is C_∞ on D_ψ.

Proof. $\forall (a_0, b_0) \in D_\psi$, we will distinguish between two exhaustive cases: Case 1: $b_0 \neq 0$; Case 2: $a_0 < 0$.

Case 1: $b_0 \neq 0$. Let $O := \mathcal{B}_{\mathbb{R}^2}\left((a_0, b_0), |b_0|/2\right)$. $\forall (a,b) \in O$, we have $|b| > |b_0|/2 > 0$. Then, $(a,b) \in D_\psi$. Hence, we have $O \subseteq D_\psi$. We will then show that ψ is C_∞ on O. Note that $\forall (a,b) \in O$, we have $b \neq 0$, which implies that $\rho_1(b^2) > 0$ and $(\rho_1(a))^2 + \rho_1(b^2) > 0$. Since the square root function is C_∞ on $(0, +\infty)$ and the inverse function is C_∞ on $\mathbb{R} \setminus \{0\}$, then, $\psi(a,b) = \frac{a+\sqrt{(\rho_1(a))^2+\rho_1(b^2)}}{b}$, $\forall (a,b) \in O$, is C_∞ on O.

Case 2: $a_0 < 0$. Let $O := \mathcal{B}_{\mathbb{R}^2}\left((a_0, b_0), |a_0|/2\right)$. $\forall (a,b) \in O$, we have $a < a_0/2 < 0$. Hence, $(a,b) \in D_\psi$. Therefore, we have $O \subseteq D_\psi$. Fix any $(a,b) \in O$. Note that $\rho_1(a) = a < 0$. Then, we have

$$\begin{aligned} \psi(a,b) &= \begin{cases} 0 & b = 0 \\ \frac{a+\sqrt{(\rho_1(a))^2+\rho_1(b^2)}}{b} & b \neq 0 \end{cases} = \begin{cases} 0 & b = 0 \\ \frac{a+\sqrt{a^2+\rho_1(b^2)}}{b} & b \neq 0 \end{cases} \\ &= \begin{cases} 0 & b = 0 \\ \frac{\rho_1(b^2)}{b(\sqrt{a^2+\rho_1(b^2)}-a)} & b \neq 0 \end{cases} = \begin{cases} 0 & b = 0 \\ \frac{b(1-\kappa_1(b^2))}{\sqrt{a^2+\rho_1(b^2)}-a} & b \neq 0 \end{cases} = \frac{b(1-\kappa_1(b^2))}{\sqrt{a^2+\rho_1(b^2)}-a} \end{aligned}$$

Notice that $a^2 + \rho_1(b^2) \geq a^2 > 0$ and $\sqrt{a^2+\rho_1(b^2)} - a \geq 2|a| > 0$. Then, ψ is C_∞ on O.

Thus, in both cases, we have found an open ball O centered at (a_0, b_0) which is a subset of D_ψ and ψ is C_∞ on O. Hence, we have D_ψ is open in $\mathbb{R}^2$ and ψ is C_∞ on D_ψ. This completes the proof of the lemma. □

Remark 3. We make the following observations on the function ψ.

(a) If $a > 0$ and $b \neq 0$, we have

$$\left|\psi(a,b) - \frac{a}{b}\right| = \frac{\sqrt{(\rho_1(a))^2 + \rho_1(b^2)}}{|b|} \leq \frac{\sqrt{2}}{|b|}$$

(b) If $a < 0$, we have, by the proof of Lemma 2,

$$\psi(a,b) = \frac{b(1-\kappa_1(b^2))}{\sqrt{a^2+\rho_1(b^2)}-a} \quad \Rightarrow \quad |\psi(a,b)| \leq \frac{|b|(1-\kappa_1(b^2))}{|b|\sqrt{1-\kappa_1(b^2)}+|a|} < 1$$

Definition 4. Define $\kappa_6 : \mathbb{R}^2 \to \mathbb{R}$ and $\kappa_4 : \mathbb{R} \times (1, \infty) \to \mathbb{R}$ by

$$\kappa_6(x,p) := x(1 - \kappa_1(px)) \tag{A4}$$

$$\kappa_4(x,p) := \begin{cases} \kappa_6(x-1,\frac{1}{p-1})+1 & \text{if } x \geq 0 \\ -1-\kappa_6(-1-x,\frac{1}{p-1}) & \text{if } x < 0 \end{cases} \tag{A5}$$

We define the saturation function SATF : $\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\mathrm{SATF}(x,p) := \frac{10p}{11} * \kappa_4(\frac{11}{10p}x, \frac{11}{10}) \tag{A6}$$

Proposition 6. The following statements holds for κ_6, κ_4, and SATF functions, respectively.

(i) κ_6 is $\mathcal{C}_\infty$; and $\kappa_6(x,p) = x$ if $xp \leq 0$.

(ii) If $p > 0$, then $\kappa_6(x,p) = \frac{1}{p}\rho_1(px)$, and $\kappa_6(x,p)$ is strictly increasing with respect to $x \in \mathbb{R}$, and $\lim_{x\to\infty} \kappa_6(x,p) = \frac{1}{p}$.

(iii) $\forall p > 1$, $\kappa_4(x,p)$ is strictly increasing in $x \in \mathbb{R}$, $\lim_{x\to-\infty} \kappa_4(x,p) = -p$, $\lim_{x\to\infty} \kappa_4(x,p) = p$, and $\kappa_4(x,p) = x$, $\forall x \in [-1,1] \subset \mathbb{R}$; and κ_4 is $\mathcal{C}_\infty$.

(iv) SATF is $\mathcal{C}_\infty$; and $\forall p > 0$, SATF(x,p) is strictly increasing in $x \in \mathbb{R}$, $\lim_{x\to-\infty}$ SATF$(x,p) = -p$, $\lim_{x\to\infty}$ SATF$(x,p) = p$, and SATF$(x,p) = x$, $\forall x \in [-\frac{10p}{11}, \frac{10p}{11}] \subset \mathbb{R}$.

Proof. (i) This follows directly from (i) of Proposition 4.
(ii) This follows directly from Definition 3 and Proposition 5.
(iii) This follows directly from (i) and (ii).
(iv) This follows directly from (iii). This completes the proof of the proposition. □

The plot of functions κ_1, κ_3, κ, ρ_1, SATF, and ψ are illustrated in Figure A1.

Lemma 3. Let $\mathcal{X}$ and $\mathcal{Y}$ be real normed linear spaces, $q : A \to \mathbb{R}$ be $\mathcal{C}_k$, where $A \subseteq \mathcal{X}$ is open, and $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Let $A_1 := \{ x \in A \mid q(x) < c_1 \}$ and $A_2 := \{ x \in A \mid q(x) > c_2 \}$, where $c_1, c_2 \in \mathbb{R}$ and $c_1 > c_2$. Let $f_i : A_i \to \mathcal{Y}$, $i = 1,2$, be $\mathcal{C}_k$. Clearly, $A = A_1 \cup A_2$. Define $f : A \to \mathcal{Y}$ by

$$f(x) = \begin{cases} f_1(x) & \forall x \in A_1 \setminus A_2 \\ f_1(x) + \kappa_3\left(\frac{q(x)-c_2}{\varrho_1(c_1-c_2)} - \varrho_2\right)(f_2(x) - f_1(x)) & \forall x \in A_1 \cap A_2 \\ f_2(x) & \forall x \in A_2 \setminus A_1 \end{cases}$$

where $\varrho_1, \varrho_2 \in (0,1) \subset \mathbb{R}$ and $(1+\varrho_2)\varrho_1 < 1$, and κ_3 is defined in Definition 2. Then, f is $\mathcal{C}_k$ on A.

Proof. Clearly, A_1 and A_2 are open sets in $\mathcal{X}$, since q is continuous. Define $A_3 := \{ x \in A \mid q(x) < c_2 + \varrho_1\varrho_2(c_1 - c_2) \}$ and $A_4 := \{ x \in A \mid q(x) > c_2 + (1+\varrho_2)\varrho_1(c_1 - c_2) \}$. Clearly, $A_3 \subseteq A_1$ and $A_4 \subseteq A_2$ are open sets in $\mathcal{X}$. $\forall x_0 \in A$, we will show that $\exists O \subseteq \mathcal{X}$, which is open, such that $x_0 \in O \subseteq A$ and f is $\mathcal{C}_k$ on O. Then, f is $\mathcal{C}_k$ on A. We will distinguish between three exhaustive and mutually exclusive cases: Case 1: $x_0 \in A_1 \setminus A_2$; Case 2: $x_0 \in A_1 \cap A_2$; Case 3: $x_0 \in A_2 \setminus A_1$.

Case 1: $x_0 \in A_1 \setminus A_2$. Then, $q(x_0) \leq c_2$ and $x_0 \in A_3$. $\forall x \in A_3$, we have either $x \in A_1 \setminus A_2$, which implies that $f(x) = f_1(x)$; or $x \in A_1 \cap A_2$, which implies that $\frac{q(x)-c_2}{\varrho_1(c_1-c_2)} < \varrho_2$ and $f(x) = f_1(x)$ by Proposition 4. Hence, $f(x) = f_1(x)$, $\forall x \in A_3$. Hence, f is $\mathcal{C}_k$ on $A_3 \ni x_0$.

Case 2: $x_0 \in A_1 \cap A_2$. Note that $A_1 \cap A_2$ is open in $\mathbb{R}^n$. Then, f is $\mathcal{C}_k$ on $A_1 \cap A_2 \ni x_0$ by Proposition 4 and Proposition 9.45 of [21].[8]

Case 3: $x_0 \in A_2 \setminus A_1$. Then, $q(x_0) \geq c_1$ and $x_0 \in A_4$. $\forall x \in A_4$, we have either $x \in A_2 \setminus A_1$, which implies that $f(x) = f_2(x)$; or $x \in A_2 \cap A_1$, which implies that $\frac{q(x)-c_2}{\varrho_1(c_1-c_2)} > 1 + \varrho_2$ and $f(x) = f_2(x)$ by Proposition 4. Hence, $f(x) = f_2(x)$, $\forall x \in A_4$, and f is $\mathcal{C}_k$ on $A_4 \ni x_0$.

This completes the proof of the lemma. □

Next, we present a lemma that factors a nonlinear function. This result is useful in integrator backstepping designs.

Lemma 4. Let $\mathcal{X}$ and $\mathcal{Y}$ be real normed linear spaces, and $\mathcal{Z}$ be a real Banach space, $f : D \to \mathcal{Z}$, where $D \subseteq \mathcal{X} \times \mathcal{Y}$ is open. Let $D_1 := \{ x \in \mathcal{X} \mid \exists y \in \mathcal{Y}$ such that $(x,y) \in D \}$, which is the projection of D onto $\mathcal{X}$ and is open in $\mathcal{X}$. Let $\alpha : D_1 \to \mathcal{Y}$ be such that $(x,\alpha(x)) \in D$, $\forall x \in D_1$, and $\forall (x,y) \in D$, the line segment connecting (x,y) and $(x,\alpha(x))$ is entirely in D, i. e.,

[8]For the convenience of the reader, this proposition has been reproduced as Proposition 11 in Appendix C.

$(x, s\alpha(x)+(1-s)y) \in D, \forall s \in [0,1] \subset \mathbb{R}$. Assume that f, $\frac{\partial f}{\partial y}$, and α are $\mathcal{C}_k$, where $k \in \{0\}\cup\mathbb{N}\cup\{\infty\}$. Then, $\exists \tilde{f} : D \to \mathrm{B}(\mathcal{Y}, \mathcal{Z})$, which is $\mathcal{C}_k$ and satisfies

$$f(x,y) - f(x,\alpha(x)) = \tilde{f}(x,y)(y-\alpha(x)), \quad \forall (x,y) \in D \tag{A7}$$

Proof. Define $\tilde{f} : D \to \mathrm{B}(\mathcal{Y}, \mathcal{Z})$ by

$$\tilde{f}(x,y) := \int_0^1 \frac{\partial f}{\partial y}(x, \alpha(x) + s(y-\alpha(x)))\,\mathrm{d}s; \qquad \forall (x,y) \in D$$

Note that $(x,y) \in D$ implies that $(x, \alpha(x) + s(y-\alpha(x))) \in D$, $\forall s \in [0,1]$. By Theorem 12.112 of [21],[9] we have $\tilde{f}$ is $\mathcal{C}_k$. Note also that

$$\tilde{f}(x,y)(y-\alpha(x)) = \int_0^1 \frac{\partial f}{\partial y}(x, \alpha(x) + s(y-\alpha(x)))\,\mathrm{d}s\,(y-\alpha(x))$$
$$= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(f(x, \alpha(x) + s(y-\alpha(x))))\,\mathrm{d}s = f(x,y) - f(x,\alpha(x)); \qquad \forall (x,y) \in D$$

where the second equality follows from Propositions 11.92 and 7.126 of [21];[10] and the last equality follows from Theorem 12.83 of [21].[11] Hence, $\tilde{f}$ satisfies (A7) on D.

This completes the proof of the lemma. □

Lemma 5. Let $n \in \mathbb{N}$, $D \subseteq \mathbb{R}^n$ be nonempty, $[t_0, t_1) \subset \mathbb{R}$ be a nonempty interval, and $K \subseteq D$ be compact. Let $\xi : [t_0, t_1) \to D$ be continuous, $V : [t_0, t_1) \times D \to \mathbb{R}$ be nonnegative and continuous, and $W_i : D \to \mathbb{R}$ be nonnegative and continuous, $i = 1, 2$. Assume that

(i) $W_1(x) \le V(t,x) \le W_2(x)$, $\forall (t,x) \in [t_0,t_1) \times D$;

(ii) $\forall t \in [t_0, t_1)$, with $\xi(t) \in D \setminus K$ implies $\limsup_{h\to 0^+} (V(t+h, \xi(t+h)) - V(t, \xi(t)))/h < 0$.

Then, there exists a constant $\eta \in \overline{\mathbb{R}_+}$, such that $V(t, \xi(t)) \le \eta$, $\forall t \in [t_0, t_1)$. Furthermore, $W_1(\xi(t)) \le \eta$, $\forall t \in [t_0, t_1)$.

Proof. Define $\eta := \max\{V(t_0, \xi(t_0)), \sup_{x\in K} \sup_{t\in[t_0,t_1)} V(t,x)\}$. Note that $\sup_{x\in K} \sup_{t\in[t_0,t_1)} V(t,x) \le \sup_{x\in K} W_2(x) < +\infty$, which implies that $\eta \in \mathbb{R}$.

Fix any $\bar{\eta} > \eta$. We will show that $V(t, \xi(t)) \le \bar{\eta}$, $\forall t \in [t_0, t_1)$. Define

$$T_{\bar{\eta}} = \left\{ t \in [t_0, t_1) \;\middle|\; V(\bar{t}, \xi(\bar{t})) \le \bar{\eta}, \quad \forall \bar{t} \in [t_0, t] \right\}$$

Clearly, $V(t_0, \xi(t_0)) \le \eta < \bar{\eta}$. Then, $t_0 \in T_{\bar{\eta}}$. Define $t_f = \sup T_{\bar{\eta}}$. Then, we have $t_0 \le t_f \le t_1$.

We will next show $V(t, \xi(t)) \le \bar{\eta}$, $\forall t \in [t_0, t_1)$. Consider 2 exhaustive and mutually exclusive cases. Case 1: $t_f = t_1$. In this case, $\forall t \in [t_0, t_1)$, there exists $\bar{t} \in (t, t_1)$ such that $\bar{t} \in T_{\bar{\eta}}$. Then, $V(t, \xi(t)) \le \bar{\eta}$ by the definition of $T_{\bar{\eta}}$. This case is thus proven.

Case 2: $t_f < t_1$. We will show that this case leads to contradiction. We first claim that $t_f \in T_{\bar{\eta}}$. Suppose $t_f \in \mathbb{R} \setminus T_{\bar{\eta}}$. Then, there exists $t_2 \in (t_0, t_f]$ such that $V(t_2, \xi(t_2)) > \bar{\eta}$. By continuity of ξ and V, this implies $\exists t_3 \in (t_0, t_2)$ such that $V(t_3, \xi(t_3)) > \bar{\eta}$. Hence, $\forall \bar{t} \in [t_3, t_1)$, $\bar{t} \in \mathbb{R} \setminus T_{\bar{\eta}}$. This leads to the contradiction $t_f = \sup T_{\bar{\eta}} \le t_3 < t_2 \le t_f$. Therefore, $t_f \in T_{\bar{\eta}}$, which implies that $V(t_f, \xi(t_f)) \le \bar{\eta}$. We further claim that $V(t_f, \xi(t_f)) = \bar{\eta}$. Suppose $V(t_f, \xi(t_f)) < \bar{\eta}$. By continuity of V and ξ, there exists $t_2 \in (t_f, t_1)$ such that $V(t, \xi(t)) \le \bar{\eta}$, $\forall t \in [t_f, t_2]$. This, coupled with $t_f \in T_{\bar{\eta}}$, implies that $t_2 \in T_{\bar{\eta}}$. This fact contradicts with the definition of t_f. Hence, $V(t_f, \xi(t_f)) = \bar{\eta} > \eta$. By the definition of η, we have $\xi(t_f) \in D \setminus K$, which implies that $\limsup_{h\to 0^+} (V(t_f + h, \xi(t_f + h)) - V(t_f, \xi(t_f)))/h < 0$. By the definition of lim sup, there exists $t_2 \in (t_f, t_1)$ such that, $\forall t \in (t_f, t_2]$, $\frac{V(t,\xi(t)) - V(t_f, \xi(t_f))}{t - t_f} \le 0$. This implies that $V(t, \xi(t)) \le V(t_f, \xi(t_f)) = \bar{\eta}$, $\forall t \in (t_f, t_2]$. This, coupled with the fact that $t_f \in T_{\bar{\eta}}$, implies that $t_2 \in T_{\bar{\eta}}$. This fact contradicts with the definition of t_f. This shows that Case 2 is impossible.

In the above, we have shown $V(t, \xi(t)) \le \bar{\eta}$, $\forall t \in [t_0, t_1)$. By the arbitrariness of $\bar{\eta} > \eta$, we have $V(t, \xi(t)) \le \eta$, $\forall t \in [t_0, t_1)$. This further implies that $W_1(\xi(t)) \le \eta$, $\forall t \in [t_0, t_1)$.

This completes the proof of the lemma. □

[9]For the convenience of the reader, this theorem has been reproduced as Theorem 3 in Appendix C.
[10]For the convenience of the reader, these propositions has been reproduced as Propositions 12 and 13 in Appendix C.
[11]For the convenience of the reader, this theorem has been reproduced as Theorem 4 in Appendix C.

B BACKSTEPPING LEMMAS

In this section, we first present a backstepping lemma based on cancellation and Arztan's formula.

Lemma 6. Consider the following system

$$\dot{x}_o = f_o(x_o, x_a, x_d) + h_o(x_o, x_a, x_d)w \tag{B8a}$$

$$\dot{x}_a = f_a(x_o, x_a, x_d) + g_a(x_o, x_a, x_d)u + h_a(x_o, x_a, x_d)w \tag{B8b}$$

where x_o is a state vector, $x_o \in D_o \subseteq \mathcal{X}_o$, D_o is nonempty and open, and $\mathcal{X}_o$ is a real Banach space; x_a is a state vector, $x_a \in D_a \subseteq \mathcal{X}_a$, D_a is nonempty open and convex, $\mathcal{X}_a$ is a real Hilbert space; x_d is some signal, $x_d \in D_d \subseteq \mathcal{X}_d$, D_d is nonempty and open, and $\mathcal{X}_d$ is a real normed linear space; u is the control input, $u \in \mathcal{U}$, and $\mathcal{U}$ is a real Hilbert space; w is the disturbance input, $w \in D_w \subseteq \mathcal{W}$, D_w contain a nonempty open subset of $\mathcal{W}$, and $\mathcal{W}$ is a real Hilbert space; $D_1 := D_o \times D_d$; f_o, h_o, f_a, g_a, and h_a be mappings of $D_o \times D_a \times D_d$ into $\mathcal{X}_o$, $\mathrm{B}(\mathcal{W}, \mathcal{X}_o)$, $\mathcal{X}_a$, $\mathrm{B}(\mathcal{U}, \mathcal{X}_a)$, and $\mathrm{B}(\mathcal{W}, \mathcal{X}_a)$, respectively; f_o and h_o be $\mathcal{C}_k$ and all of their partial derivatives of kth order are further continuously partial differentiable with respect to x_a, $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$; f_a, g_a, and h_a be $\mathcal{C}_k$, $g_a(x_o, x_a, x_d) \in \mathrm{B}(\mathcal{U}, \mathcal{X}_a)$ is bijective, $\forall (x_o, x_a, x_d) \in D_o \times D_a \times D_d$.

Assume that we are given $V_o : D_o \to \mathbb{R}$, which is $\mathcal{C}_{k+1}$, and $\alpha_o : D_o \to D_a$, which is $\mathcal{C}_{k+1}$ such that the derivative of $V_o(x_o(t))$ along a solution of the dynamics (B8a) with $x_a(t) = \alpha_o(x_o(t))$ can be written as

$$\dot{V}_o(x_o, x_a, x_d, w)\big|_{x_a=\alpha_o(x_o)} = -l_o(x_o, x_d) + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \| w - \sigma_o(x_o, x_d) \|_{\mathcal{W}}^2 ; \quad \forall (x_o, x_d) \in D_1, \quad \forall w \in D_w \tag{B9}$$

where $l_o : D_1 \to \mathbb{R}$ is continuous, $\gamma \in \mathbb{R}_+$, $\sigma_o : D_o \times D_d \to \mathcal{W}$ be $\mathcal{C}_k$ and defined by

$$\sigma_o(x_o, x_d) := \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V_o}{\partial x_o}(x_o) h_o(x_o, \alpha_o(x_o), x_d) \right) \tag{B10}$$

where $\Phi_{\mathcal{W}} : \mathcal{W}^* \to \mathcal{W}$ is the isometrical isomorphism defined in Riesz-Fréchet Theorem 13.15 of [21].[12] Let $\phi : D_o \times D_a \times D_d \to \mathcal{X}_a$ be a $\mathcal{C}_k$ design function, $Z : D_o \times D_a \to \mathcal{S}_{+\mathcal{X}_a} \subseteq \mathcal{S}_{\mathcal{X}_a} = \mathrm{B}_{\mathrm{S}2}(\mathcal{X}_a, \mathbb{R})$ be a $\mathcal{C}_{k+1}$ design function, and $R : D_o \times D_a \times D_d \to \mathcal{S}_{+\mathcal{U}}$ be a $\mathcal{C}_k$ design function. Assume that Z satisfies the following two conditions.

(i) $Z(x_o, x_a) \in \mathcal{S}_{+\mathcal{X}_a}$, $\forall (x_o, x_a) \in D_o \times D_a$.

(ii) $\pi_Z(x_o, x_a) := 2Z(x_o, x_a) + \left(\frac{\partial Z}{\partial x_a}(x_o, x_a) \right)^{T_{2,1}} (x_a - \alpha_o(x_o)) \in \mathrm{B}(\mathcal{X}_a, \mathcal{X}_a^*)$ is bijective, $\forall (x_o, x_a) \in D_o \times D_a$.

Let $V : D_o \times D_a \to \mathbb{R}$ be defined by $V(x_o, x_a) := V_o(x_o) + Z(x_o, x_a)(x_a - \alpha_o(x_o))(x_a - \alpha_o(x_o))$, $\forall (x_o, x_a) \in D_o \times D_a$, which is $\mathcal{C}_{k+1}$. Let $\varrho_1, \varrho_2 \in (0,1) \subset \mathbb{R}$ and $\varrho_3, \varrho_4 \in (0, \infty) \subset \mathbb{R}$ with $(1+\varrho_2)\varrho_1 < 1$. Then, there exists a $\mathcal{C}_k$ function $\alpha : D_o \times D_a \times D_d \to \mathcal{U}$ given by (B15) such that the derivative of $V(x_o(t), x_a(t))$ along a solution of the dynamics (B8) with $u(t) = \alpha(x_o(t), x_a(t), x_d(t))$ can be written as

$$\begin{aligned} &\dot{V}(x_o, x_a, x_d, u, w)\big|_{u=\alpha(x_o,x_a,x_d)} = -l(x_o, x_a, x_d) + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \| w - \sigma(x_o, x_a, x_d) \|_{\mathcal{W}}^2 \\ &\le -l_o(x_o, x_d) - \langle \phi(x_o, x_a, x_d), x_a - \alpha_o(x_o) \rangle_{\mathcal{X}_a} + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \| w - \sigma(x_o, x_a, x_d) \|_{\mathcal{W}}^2 ; \\ &\quad \forall (x_o, x_d) \in D_1, \quad \forall x_a \in D_a, \quad \forall w \in D_w \end{aligned} \tag{B11}$$

where $l : D_o \times D_a \times D_d \to \mathbb{R}$ is continuous; $l - l_o : D_o \times D_a \times D_d \to \mathbb{R}$ is $\mathcal{C}_k$; and $\sigma : D_o \times D_a \times D_d \to \mathcal{W}$ is $\mathcal{C}_k$ and given by

$$\sigma(x_o, x_a, x_d) := \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V}{\partial (x_o, x_a)}(x_o, x_a) \begin{bmatrix} h_o(x_o, x_a, x_d) \\ h_a(x_o, x_a, x_d) \end{bmatrix} \right) \in \mathcal{W} \tag{B12}$$

If, in addition, there exists $(x_{o0}, x_{a0}, x_{d0}) \in D_o \times D_a \times D_d$, such that $\frac{\partial V_o}{\partial x_o}(x_{o0}) = \vartheta_{\mathcal{X}_o^*}$, $f_o(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_o}$, $f_a(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, $\alpha_o(x_{o0}) = x_{a0}$, and $\phi(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, then $\alpha(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{U}}$.

Proof. $\forall (x_o, x_d) \in D_1$, $\forall x_a \in D_a$, $\forall w \in D_w$, we have

$$\begin{aligned} &\dot{V}_o(x_o, x_a, x_d, w) = \frac{\partial V_o}{\partial x_o}(x_o)(f_o(x_o, x_a, x_d) + h_o(x_o, x_a, x_d)w) \\ &= \frac{\partial V_o}{\partial x_o}(x_o)(f_o(x_o, \alpha_o(x_o), x_d) + h_o(x_o, \alpha_o(x_o), x_d)w) + \frac{\partial V_o}{\partial x_o}(x_o)(\tilde{f}_o(x_o, x_a, x_d) + (\tilde{h}_o(x_o, x_a, x_d))^{T_{2,1}}(w))(x_a - \alpha_o(x_o)) \end{aligned}$$

[12] For the convenience of the reader, this theorem has been reproduced as Theorem 2 in Appendix C.

where $\tilde{f}_o$ and $\tilde{h}_o$ are C_k functions of $D_o \times D_a \times D_d$ to $\mathrm{B}\left(\mathcal{X}_a, \mathcal{X}_o\right)$ and $\mathrm{B}\left(\mathcal{X}_a, \mathrm{B}\left(\mathcal{W}, \mathcal{X}_o\right)\right)$, respectively, by Lemma 4 since $D_a \subseteq \mathcal{X}_a$ is convex. By the assumption, we have

$$\dot{V}_o(x_o, x_a, x_d, w) = -l_o(x_o, x_d) + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \| w - \sigma_o(x_o, x_d) \|_{\mathcal{W}}^2 + \left\langle\left\langle \frac{\partial V_o}{\partial x_o}(x_o), (\tilde{f}_o(x_o, x_a, x_d) + (\tilde{h}_o(x_o, x_a, x_d))^{T_{2,1}}(w))(x_a - \alpha_o(x_o)) \right\rangle\right\rangle_{\mathcal{X}_o}$$

Let $z = x_a - \alpha_o(x_o)$. Then, $\forall u \in \mathcal{U}$,

$$\begin{aligned}\dot{V}(x_o, x_a, x_d, u, w) &= \dot{V}_o + 2Z(\dot{z})(z) + \dot{Z}(z)(z) = \dot{V}_o + \left\langle\left\langle \pi_Z \dot{x}_a + \left(\left(\frac{\partial Z}{\partial x_o} \right)^{T_{2,1}} (z) - 2Z \frac{\partial \alpha_o}{\partial x_o} \right) \dot{x}_o, z \right\rangle\right\rangle_{\mathcal{X}_a} \\ &=: -l_o(x_o, x_d) + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \| w - \sigma_o(x_o, x_d) \|_{\mathcal{W}}^2 + \left\langle\left\langle \chi_1 + 2\gamma^2 \chi_2 w + \chi_3 u, z \right\rangle\right\rangle_{\mathcal{X}_a}\end{aligned}$$

where χ_1, χ_2, and χ_3 are C_k functions on $D_o \times D_a \times D_d$ and given by

$$\begin{aligned}\chi_1(x_o, x_a, x_d) &= \left(\left(\frac{\partial Z}{\partial x_o}(x_o, x_a) \right)^{T_{2,1}} (x_a - \alpha_o(x_o)) - 2Z(x_o, x_a) \frac{\partial \alpha_o}{\partial x_o}(x_o) \right) f_o(x_o, x_a, x_d) + \\ &\quad \pi_Z(x_o, x_a) f_a(x_o, x_a, x_d) + (\tilde{f}_o(x_o, x_a, x_d))' \frac{\partial V_o}{\partial x_o}(x_o) \in \mathcal{X}_a^* \\ \chi_2(x_o, x_a, x_d) &= \frac{1}{2\gamma^2} \left(\left(\frac{\partial Z}{\partial x_o}(x_o, x_a) \right)^{T_{2,1}} (x_a - \alpha_o(x_o)) - 2Z(x_o, x_a) \frac{\partial \alpha_o}{\partial x_o}(x_o) \right) \cdot \\ &\quad h_o(x_o, x_a, x_d) + \frac{1}{2\gamma^2} \pi_Z(x_o, x_a) h_a(x_o, x_a, x_d) + \frac{1}{2\gamma^2} \frac{\partial V_o}{\partial x_o}(x_o) (\tilde{h}_o(x_o, x_a, x_d))^{T_{2,1}} \in \mathrm{B}\left(\mathcal{W}, \mathcal{X}_a^*\right) \\ \chi_3(x_o, x_a, x_d) &= \pi_Z(x_o, x_a) g_a(x_o, x_a, x_d) \in \mathrm{B}\left(\mathcal{U}, \mathcal{X}_a^*\right)\end{aligned}$$

Define $\alpha_1 : D_o \times D_a \times D_d \to \mathcal{U}$ by

$$\begin{aligned}\alpha_1(x_o, x_a, x_d) &= (\chi_3(x_o, x_a, x_d))^{-1}(-\chi_1(x_o, x_a, x_d) - 2\gamma^2 \chi_2(x_o, x_a, x_d)\sigma_o(x_o, x_d) \\ &\quad -\gamma^2 \chi_2(x_o, x_a, x_d)(\chi_2(x_o, x_a, x_d))^* \Phi_{\mathcal{X}_{a\mathrm{inv}}}(x_a - \alpha_o(x_o)) - \Phi_{\mathcal{X}_{a\mathrm{inv}}}(\phi(x_o, x_a, x_d)))\end{aligned} \tag{B13}$$

where $\Phi_{\mathcal{X}_a} : \mathcal{X}_a^* \to \mathcal{X}_a$ is the isometrical isomorphism defined in Riesz-Fréchet Theorem 13.15 of [21].[13] Clearly, α_1 is C_k. α_1 is the cancellation control law. Then, it implies that

$$\begin{aligned}\dot{V}(x_o, x_a, x_d, u, w)\big|_{u=\alpha_1(x_o,x_a,x_d)} &= -l_o(x_o, x_d) - \langle \phi(x_o, x_a, x_d), z \rangle_{\mathcal{X}_a} + \gamma^2 \| w \|_{\mathcal{W}}^2 \\ &\quad -\gamma^2 \left\| w - \sigma_o(x_o, x_d) - (\chi_2(x_o, x_a, x_d))^* \Phi_{\mathcal{X}_{a\mathrm{inv}}}(z) \right\|_{\mathcal{W}}^2\end{aligned}$$

Define $\alpha_2 : D_o \times D_a \times D_d \to \mathcal{U}$ by

$$\alpha_2(x_o, x_a, x_d) = -\psi\left(a(x_o, x_a, x_d), b(x_o, x_a, x_d)\right) (R(x_o, x_a, x_d))^{-1} (\chi_3(x_o, x_a, x_d))' (x_a - \alpha_o(x_o)) \in \mathcal{U} \tag{B14a}$$

$$\begin{aligned}a(x_o, x_a, x_d) &= \left\langle x_a - \alpha_o(x_o), \phi(x_o, x_a, x_d) \right\rangle_{\mathcal{X}_a} + \left\langle\left\langle 2\gamma^2 \chi_2(x_o, x_a, x_d)\sigma_o(x_o, x_d) \right.\right. \\ &\quad \left.\left. +\chi_1(x_o, x_a, x_d) + \gamma^2 \chi_2(x_o, x_a, x_d) (\chi_2(x_o, x_a, x_d))^* \Phi_{\mathcal{X}_{a\mathrm{inv}}}(x_a - \alpha_o(x_o)), x_a - \alpha_o(x_o) \right\rangle\right\rangle_{\mathcal{X}_a} \in \mathbb{R}\end{aligned} \tag{B14b}$$

$$b(x_o, x_a, x_d) = \left\langle\left\langle (\chi_3(x_o, x_a, x_d))'(x_a - \alpha_o(x_o)), (R(x_o, x_a, x_d))^{-1} (\chi_3(x_o, x_a, x_d))' (x_a - \alpha_o(x_o)) \right\rangle\right\rangle_{\mathcal{U}} \in \mathbb{R} \tag{B14c}$$

$$a_1(x_o, x_a, x_d) = -a(x_o, x_a, x_d) + \varrho_3 b(x_o, x_a, x_d) \in \mathbb{R} \tag{B14d}$$

where ψ is as defined in Definition 3. Clearly α_2 is C_k if $a_1(x_o, x_a, x_d) > 0$. α_2 is the Arztan's formula based control law. Then, the derivative of V is given by, $\forall (x_o, x_a, x_d) \in D_o \times D_a \times D_d$ with $a_1(x_o, x_a, x_d) > 0$,

$$\begin{aligned}\dot{V}(x_o, x_a, x_d, u, w)\big|_{u=\alpha_2(x_o,x_a,x_d)} &= -l_o(x_o, x_d) - \langle x_a - \alpha_o(x_o), \phi(x_o, x_a, x_d) \rangle_{\mathcal{X}_a} + \gamma^2 \| w \|_{\mathcal{W}}^2 \\ &\quad -((\rho_1(a(x_o, x_a, x_d)))^2 + \rho_1((b(x_o, x_a, x_d))^2))^{1/2} - \gamma^2 \left\| w - \sigma_o(x_o, x_d) - (\chi_2(x_o, x_a, x_d))^* \Phi_{\mathcal{X}_{a\mathrm{inv}}} z \right\|_{\mathcal{W}}^2 \\ &\le -l_o(x_o, x_d) - \langle \phi(x_o, x_a, x_d), z \rangle_{\mathcal{X}_a} + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \left\| w - \sigma_o(x_o, x_d) - (\chi_2(x_o, x_a, x_d))^* \Phi_{\mathcal{X}_{a\mathrm{inv}}} z \right\|_{\mathcal{W}}^2\end{aligned}$$

Let $A_1 := \{ (x_o, x_a, x_d) \in D_o \times D_a \times D_d \mid a_1(x_o, x_a, x_d) < \varrho_4 \}$, and $A_2 := \{ (x_o, x_a, x_d) \in D_o \times D_a \times D_d \mid a_1(x_o, x_a, x_d) > 0 \}$.

[13] For the convenience of the reader, this theorem has been reproduced as Theorem 2 in Appendix C.

Define $\alpha : D_o \times D_a \times D_d \to \mathbb{R}^{n_a}$ according to Lemma 3 by

$$\alpha(\bar{x}) = \left\{ \begin{array}{ll} \alpha_1(\bar{x}) & \forall \bar{x} \in A_1 \setminus A_2 \\ \alpha_1(\bar{x}) + \kappa_3 \left(\frac{a_1(\bar{x})}{\varrho_1 \varrho_4} - \varrho_2 \right) (\alpha_2(\bar{x}) - \alpha_1(\bar{x})) & \forall \bar{x} \in A_1 \cap A_2 \\ \alpha_2(\bar{x}) & \forall \bar{x} \in A_2 \setminus A_1 \end{array} \right\} \in \mathcal{U} \tag{B15}$$

By Lemma 3, α is C_k on $D_o \times D_a \times D_d$.

Then, the derivative of V is

$$\dot{V}(x_o, x_a, x_d, u, w)\big|_{u=\alpha(x_o,x_a,x_d)} =: -l(x_o, x_a, x_d) + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \| w - \sigma(x_o, x_a, x_d) \|_{\mathcal{W}}^2$$
$$\leq -l_o(x_o, x_d) - \langle \phi(x_o, x_a, x_d), z \rangle_{\mathcal{X}_a} + \gamma^2 \| w \|_{\mathcal{W}}^2 - \gamma^2 \| w - \sigma(x_o, x_a, x_d) \|_{\mathcal{W}}^2$$

where

$$\begin{aligned}
&\sigma(x_o, x_a, x_d) = \sigma_o(x_o, x_d) + (\chi_2(x_o, x_a, x_d))^* \Phi_{\mathcal{X}_{a\,\mathrm{inv}}}(x_a - \alpha_o(x_o)) \\
&= \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V_o}{\partial x_o}(x_o)(h_o(x_o, \alpha_o(x_o), x_d)) \right) + \Phi_{\mathcal{W}}(\chi_2(x_o, x_a, x_d))' \Phi_{\mathcal{X}_a} \Phi_{\mathcal{X}_{a\,\mathrm{inv}}}(x_a - \alpha_o(x_o)) \\
&= \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V_o}{\partial x_o}(x_o)(h_o(x_o, \alpha_o(x_o), x_d)) \right) + \Phi_{\mathcal{W}}(\chi_2(x_o, x_a, x_d))'(x_a - \alpha_o(x_o)) \\
&= \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V_o}{\partial x_o}(x_o)(h_o(x_o, \alpha_o(x_o), x_d)) + \left(\left(\left(\frac{\partial Z}{\partial x_o}(x_o, x_a) \right)^{T_{2,1}} (x_a - \alpha_o(x_o)) - 2Z(x_o, x_a) \frac{\partial \alpha_o}{\partial x_o}(x_o) \right) (h_o(x_o, x_a, x_d)) \right)' \cdot \right. \\
&\quad \left. (x_a - \alpha_o(x_o)) + (\pi_Z(x_o, x_a) h_a(x_o, x_a, x_d))'(x_a - \alpha_o(x_o)) + \left(\frac{\partial V_o}{\partial x_o}(x_o)(\tilde{h}_o(x_o, x_a, x_d))^{T_{2,1}} \right)' (x_a - \alpha_o(x_o)) \right) \\
&= \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V_o}{\partial x_o}(x_o)(h_o(x_o, \alpha_o(x_o), x_d)) + (h_o(x_o, x_a, x_d))' \left(\left(\frac{\partial Z}{\partial x_o}(x_o, x_a) \right)^{T_{2,1}} (x_a - \alpha_o(x_o)) - 2Z(x_o, x_a) \frac{\partial \alpha_o}{\partial x_o}(x_o) \right)' \cdot \right. \\
&\quad \left. (x_a - \alpha_o(x_o)) + (h_a(x_o, x_a, x_d))'(\pi_Z(x_o, x_a))'(x_a - \alpha_o(x_o)) + \frac{\partial V_o}{\partial x_o}(x_o) \tilde{h}_o(x_o, x_a, x_d)(x_a - \alpha_o(x_o)) \right) \\
&= \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V_o}{\partial x_o}(x_o)(h_o(x_o, x_a, x_d)) + (h_o(x_o, x_a, x_d))' \left(\left(\frac{\partial Z}{\partial x_o}(x_o, x_a) \right)^{T_{2,1}} (x_a - \alpha_o(x_o)) \right. \right. \\
&\quad \left. \left. -2Z(x_o, x_a) \frac{\partial \alpha_o}{\partial x_o}(x_o) \right)' (x_a - \alpha_o(x_o)) + (h_a(x_o, x_a, x_d))'(\pi_Z(x_o, x_a))'(x_a - \alpha_o(x_o)) \right) \\
&= \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V}{\partial (x_o, x_a)}(x_o, x_a) \begin{bmatrix} h_o(x_o, x_a, x_d) \\ h_a(x_o, x_a, x_d) \end{bmatrix} \right)
\end{aligned}$$

This proves (B12), which is C_k on $D_o \times D_a \times D_d$.

If, in addition, there exists $(x_{o0}, x_{a0}, x_{d0}) \in D_o \times D_a \times D_d$, such that $\frac{\partial V_o}{\partial x_o}(x_{o0}) = \vartheta_{\mathcal{X}_o^*}$, $f_o(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_o}$, $f_a(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, $\alpha_o(x_{o0}) = x_{a0}$, and $\phi(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, then $\chi_1(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, $\sigma_o(x_{o0}, x_{d0}) = \vartheta_{\mathcal{W}}$, and $x_{a0} - \alpha_o(x_{o0}) = \vartheta_{\mathcal{X}_a}$. This further implies that $\alpha_1(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{U}}$. Furthermore, $a(x_{o0}, x_{a0}, x_{d0}) = 0$, $b(x_{o0}, x_{a0}, x_{d0}) = 0$, and $a_1(x_{o0}, x_{a0}, x_{d0}) = 0$. Then, $\alpha(x_{o0}, x_{a0}, x_{d0}) = \alpha_1(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{U}}$.

This completes the proof of the lemma. □

The preceding lemma yields a controller that is sufficiently complex, which may not be desired computationally if we just use only cancellation but not Arztan's formula. Below, we present a backstepping lemma that only uses cancellation, which yields a (computationally) much simpler controller.

Lemma 7. Consider the following system

$$\dot{x}_o = f_o(x_o, x_a, x_d) + h_o(x_o, x_a, x_d) w \tag{B16a}$$
$$\dot{x}_a = f_a(x_o, x_a, x_d) + g_a(x_o, x_a, x_d) u + h_a(x_o, x_a, x_d) w \tag{B16b}$$

where x_o is a state vector, $x_o \in D_o \subseteq \mathcal{X}_o$, D_o is nonempty and open, and $\mathcal{X}_o$ is a real Banach space; x_a is a state vector, $x_a \in D_a \subseteq \mathcal{X}_a$, D_a is nonempty open and convex, $\mathcal{X}_a$ is a real Hilbert space; x_d is some signal, $x_d \in D_d \subseteq \mathcal{X}_d$, D_d is nonempty and open, and $\mathcal{X}_d$ is a real normed linear space; u is the control input, $u \in \mathcal{U}$, and $\mathcal{U}$ is a real Hilbert space; w is the disturbance input, $w \in D_w \subseteq \mathcal{W}$, D_w contain a nonempty open subset of $\mathcal{W}$, and $\mathcal{W}$ is a real Hilbert space; $D_1 := D_o \times D_d$; f_o, h_o, f_a, g_a, and h_a be mappings of $D_o \times D_a \times D_d$ into $\mathcal{X}_o$, $\mathrm{B}(\mathcal{W}, \mathcal{X}_o)$, $\mathcal{X}_a$, $\mathrm{B}(\mathcal{U}, \mathcal{X}_a)$, and $\mathrm{B}(\mathcal{W}, \mathcal{X}_a)$, respectively; f_o and h_o be C_k and all of their partial derivatives of kth order are further continuously partial differentiable with respect to x_a, $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$; f_a, g_a, and h_a be $\mathcal{C}_k$, $g_a(x_o, x_a, x_d) \in \mathrm{B}\left(\mathcal{U}, \mathcal{X}_a\right)$ is bijective, $\forall (x_o, x_a, x_d) \in D_o \times D_a \times D_d$.

Assume that we are given $V_o : D_o \to \mathbb{R}$, which is $\mathcal{C}_{k+1}$, and $\alpha_o : D_o \to D_a$, which is $\mathcal{C}_{k+1}$ such that the derivative of $V_o(x_o(t))$ along a solution of the dynamics (B8a) with $x_a(t) = \alpha_o(x_o(t))$ can be written as

$$\dot{V}_o(x_o, x_a, x_d, w)\big|_{x_a=\alpha_o(x_o)} = -l_o(x_o, x_d) + \gamma^2 \| w \|^2_{\mathcal{W}} - \gamma^2 \left\| w - \sigma_o(x_o, x_d) \right\|^2_{\mathcal{W}}; \quad \forall (x_o, x_d) \in D_1, \quad \forall w \in D_w \tag{B17}$$

where $l_o : D_1 \to \mathbb{R}$ is continuous, $\gamma \in \mathbb{R}_+$, $\sigma_o : D_o \times D_d \to \mathcal{W}$ is $\mathcal{C}_k$ and defined by

$$\sigma_o(x_o, x_d) := \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V_o}{\partial x_o}(x_o) h_o(x_o, \alpha_o(x_o), x_d) \right) \tag{B18}$$

where $\Phi_{\mathcal{W}} : \mathcal{W}^* \to \mathcal{W}$ is the isometrical isomorphism defined in Riesz-Fréchet Theorem 13.15 of [21].[14] Let $\phi : D_o \times D_a \times D_d \to \mathcal{X}_a$ be a $\mathcal{C}_k$ design function and $Z : D_o \times D_a \to \mathcal{S}_{+\mathcal{X}_a} \subseteq \mathcal{S}_{\mathcal{X}_a} = \mathrm{B}_{\mathrm{S}2}\left(\mathcal{X}_a, \mathbb{R}\right)$ be a $\mathcal{C}_{k+1}$ design function. Assume that Z satisfies the following two conditions.

(i) $Z(x_o, x_a) \in \mathcal{S}_{+\mathcal{X}_a}$, $\forall (x_o, x_a) \in D_o \times D_a$.

(ii) $\pi_Z(x_o, x_a) := 2Z(x_o, x_a) + \left(\frac{\partial Z}{\partial x_a}(x_o, x_a) \right)^{T_{2,1}} (x_a - \alpha_o(x_o)) \in \mathrm{B}\left(\mathcal{X}_a, \mathcal{X}_a^*\right)$ is bijective, $\forall (x_o, x_a) \in D_o \times D_a$.

Let $V : D_o \times D_a \to \mathbb{R}$ be defined by $V(x_o, x_a) := V_o(x_o) + Z(x_o, x_a)(x_a - \alpha_o(x_o))(x_a - \alpha_o(x_o))$, $\forall (x_o, x_a) \in D_o \times D_a$, which is $\mathcal{C}_{k+1}$. Then, there exists a $\mathcal{C}_k$ function $\alpha : D_o \times D_a \times D_d \to \mathcal{U}$ given by (B13) such that the derivative of $V(x_o(t), x_a(t))$ along a solution of the dynamics (B8) with $u(t) = \alpha(x_o(t), x_a(t), x_d(t))$ can be written as

$$\begin{aligned} &\dot{V}(x_o, x_a, x_d, u, w)\big|_{u=\alpha(x_o,x_a,x_d)} = -l(x_o, x_a, x_d) + \gamma^2 \| w \|^2_{\mathcal{W}} - \gamma^2 \left\| w - \sigma(x_o, x_a, x_d) \right\|^2_{\mathcal{W}} \\ &= -l_o(x_o, x_d) - \langle \phi(x_o, x_a, x_d), x_a - \alpha_o(x_o) \rangle_{\mathcal{X}_a} + \gamma^2 \| w \|^2_{\mathcal{W}} - \gamma^2 \left\| w - \sigma(x_o, x_a, x_d) \right\|^2_{\mathcal{W}}; \\ &\quad \forall (x_o, x_d) \in D_1, \quad \forall x_a \in D_a, \quad \forall w \in D_w \end{aligned} \tag{B19}$$

where $l : D_o \times D_a \times D_d \to \mathbb{R}$ is continuous; $l - l_o : D_o \times D_a \times D_d \to \mathbb{R}$ is $\mathcal{C}_k$; and $\sigma : D_o \times D_a \times D_d \to \mathcal{W}$ is $\mathcal{C}_k$ and given by

$$\sigma(x_o, x_a, x_d) := \frac{1}{2\gamma^2} \Phi_{\mathcal{W}} \left(\frac{\partial V}{\partial (x_o, x_a)}(x_o, x_a) \begin{bmatrix} h_o(x_o, x_a, x_d) \\ h_a(x_o, x_a, x_d) \end{bmatrix} \right) \in \mathcal{W} \tag{B20}$$

If, in addition, there exists $(x_{o0}, x_{a0}, x_{d0}) \in D_o \times D_a \times D_d$, such that $\frac{\partial V_o}{\partial x_o}(x_{o0}) = \vartheta_{\mathcal{X}_o^*}$, $f_o(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_o}$, $f_a(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, $\alpha_o(x_{o0}) = x_{a0}$, and $\phi(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, then $\alpha(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{U}}$.

Proof. Follow the proof of Lemma 6 to the design of α_1 and let $\alpha = \alpha_1$. The result then follows immediately. □

Remark 4. We note here that the preceding two backstepping lemmas are very generally stated, where the states of the systems are in general abstract spaces. Therefore, we no longer need to convert all states into column vectors for the result to be applied, which is done in the SISO paper [5] due to the limitation of the backstepping lemmas there.

Lemma 8. Let $\mathcal{X}$ be a real Banach space, $\mathcal{X}_d$ be a real normed linear space, and $\mathcal{W}$ be a real Hilbert space, $D \subseteq \mathcal{X}$ be a nonempty open set, $D_d \subseteq \mathcal{X}_d$ be nonempty, $D_w \subseteq \mathcal{W}$ which contains a nonempty open subset of $\mathcal{W}$, and $D_1 \subseteq D \times D_d$ be nonempty. Let $V : D \to \mathbb{R}$ be $\mathcal{C}_1$, f and g be continuous mappings of $D \times D_d$ into $\mathcal{X}$ and $\mathrm{B}(\mathcal{W}, \mathcal{X})$, respectively, $l : D_1 \to \mathbb{R}$ be continuous, $\sigma : D_1 \to \mathcal{W}$, and $\gamma \in \mathbb{R}_+$ be a constant. Consider the dynamics

$$\dot{x}(t) = f(x(t), x_d(t)) + g(x(t), x_d(t)) w(t) \tag{B21}$$

where $w(\cdot)$ is any $\mathcal{B}_{\mathrm{B}}(\mathbb{R})$-measurable signal taking values in D_w, and $x_d(\cdot)$ is a continuous signal taking values in D_d. Then, the following statements are equivalent.

(i) The function V satisfies the Hamilton-Jacobi-Isaacs equation:

$$\frac{\partial V}{\partial x}(x)(f(x, x_d)) + \frac{1}{4\gamma^2} \left\| \frac{\partial V}{\partial x}(x) g(x, x_d) \right\|^2_{\mathcal{W}^*} + l(x, x_d) = 0; \quad \forall (x, x_d) \in D_1 \tag{B22}$$

[14] For the convenience of the reader, this theorem has been reproduced as Theorem 2 in Appendix C.

(ii) The derivative of $V(x(t))$ along a solution of (B21) can be written as

$$\dot{V}(x, x_d, w) = \frac{\mathrm{d}}{\mathrm{d}t}(V(x(t)))\Big|_{x(t)=x, x_d(t)=x_d, w(t)=w} = -l(x, x_d) + \gamma^2 \|w\|_{\mathcal{W}}^2 - \gamma^2 \|w - \sigma(x, x_d)\|_{\mathcal{W}}^2; \quad \forall (x, x_d) \in D_1, \ \forall w \in D_w \tag{B23}$$

Furthermore, statement (ii) implies that

$$\sigma(x, x_d) = \frac{1}{2\gamma^2}\Phi_{\mathcal{W}}(\frac{\partial V}{\partial x}(x) g(x, x_d)); \qquad \forall (x, x_d) \in D_1 \tag{B24}$$

where $\Phi_{\mathcal{W}} : \mathcal{W}^* \to \mathcal{W}$ is the isometrical isomorphism defined in Riesz-Fréchet Theorem 13.15 of [21].[15]

Proof. We first show "(ii) ⇒ (i)." $\forall (x, x_d) \in D_1$, $\forall w \in D_w$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(V(x(t)))\Big|_{x(t)=x, x_d(t)=x_d, w(t)=w} &= \left\langle\left\langle \frac{\partial V}{\partial x}(x), f(x, x_d) \right\rangle\right\rangle_{\mathcal{X}} + \left\langle\left\langle \frac{\partial V}{\partial x}(x), g(x, x_d) w \right\rangle\right\rangle_{\mathcal{X}} \\
&= \left\langle\left\langle \frac{\partial V}{\partial x}(x), f(x, x_d) \right\rangle\right\rangle_{\mathcal{X}} + \left\langle\left\langle \frac{\partial V}{\partial x}(x) g(x, x_d), w \right\rangle\right\rangle_{\mathcal{W}} = \left\langle\left\langle \frac{\partial V}{\partial x}(x), f(x, x_d) \right\rangle\right\rangle_{\mathcal{X}} + \left\langle \Phi_{\mathcal{W}}(\frac{\partial V}{\partial x}(x) g(x, x_d)), w \right\rangle_{\mathcal{W}} \\
&= -l(x, x_d) + \gamma^2 \|w\|_{\mathcal{W}}^2 - \gamma^2 \|w - \sigma(x, x_d)\|_{\mathcal{W}}^2 = -l(x, x_d) + 2\gamma^2 \langle \sigma(x, x_d), w \rangle_{\mathcal{W}} - \gamma^2 \langle \sigma(x, x_d), \sigma(x, x_d) \rangle_{\mathcal{W}}
\end{aligned} \tag{B25}$$

Since the above holds for all $w \in D_w$, which contains a nonempty open set subset of $\mathcal{W}$, then, $2\gamma^2 \sigma(x, x_d) = \Phi_{\mathcal{W}}(\frac{\partial V}{\partial x}(x) g(x, x_d))$, $\forall (x, x_d) \in D_1$. This proves (B24). Substituting this equality into (B25) yields the Hamilton-Jacobi-Isaacs equation (B22).

Next, we show "(i) ⇒ (ii)." $\forall (x, x_d) \in D_1$, $\forall w \in D_w$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(x(t)))\Big|_{x(t)=x, x_d(t)=x_d, w(t)=w} = \left\langle\left\langle \frac{\partial V}{\partial x}(x), f(x, x_d) \right\rangle\right\rangle_{\mathcal{X}} + \left\langle\left\langle \frac{\partial V}{\partial x}(x), g(x, x_d) w \right\rangle\right\rangle_{\mathcal{X}}$$

Since V satisfies the Hamilton-Jacobi-Isaacs equation on D_1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(x(t)))\Big|_{x(t)=x, x_d(t)=x_d, w(t)=w} = -l(x, x_d) + \gamma^2 \|w\|_{\mathcal{W}}^2 - \gamma^2 \left\| w - \frac{1}{2\gamma^2}\Phi_{\mathcal{W}}(\frac{\partial V}{\partial x}(x) g(x, x_d)) \right\|_{\mathcal{W}}^2$$

Then, equation (B23) holds with $\sigma(x, x_d) = \frac{1}{2\gamma^2}\Phi_{\mathcal{W}}(\frac{\partial V}{\partial x}(x) g(x, x_d))$, $\forall (x, x_d) \in D_1$.

This completes the proof of this lemma. □

Lemma 6 is useful in backstepping controller design. The significance of the function V and control law α can be be demonstrated by a Hamilton-Jacobi-Isaacs equation as described in the following lemma, which presents essentially the same result as Lemma 6 based on the equivalence relationship of Lemma 8.

Lemma 9. Let $\mathcal{X}_o$ be a real Banach space, $\mathcal{X}_a$, $\mathcal{U}$, and $\mathcal{W}$ be real Hilbert spaces, $\mathcal{X}_d$ be a real normed linear space; $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$; $D_o \subseteq \mathcal{X}_o$ be nonempty and open, $D_a \subseteq \mathcal{X}_a$ be nonempty open and convex, $D_d \subseteq \mathcal{X}_d$ be nonempty and open, and $D_1 \subseteq D_o \times D_d$ be nonempty; f_o, h_o, f_a, g_a, and h_a be mappings of $D_o \times D_a \times D_d$ into $\mathcal{X}_o$, $\mathrm{B}(\mathcal{W}, \mathcal{X}_o)$, $\mathcal{X}_a$, $\mathrm{B}(\mathcal{U}, \mathcal{X}_a)$, and $\mathrm{B}(\mathcal{W}, \mathcal{X}_a)$, respectively; f_o and h_o be $\mathcal{C}_k$ and all of their partial derivatives of kth order are further continuously differentiable with respect to $x_a \in D_a$; f_a, g_a, and h_a be $\mathcal{C}_k$, $g_a(x_o, x_a, x_d) \in \mathrm{B}(\mathcal{U}, \mathcal{X}_a)$ be bijective, $\forall (x_o, x_a, x_d) \in D_o \times D_a \times D_d$, $l_o : D_1 \to \mathbb{R}$ be continuous, $\gamma \in \mathbb{R}_+$, $V_o : D_o \to \mathbb{R}$ be $\mathcal{C}_{k+1}$, $\alpha_o : D_o \to D_a$ be $\mathcal{C}_{k+1}$, $\phi : D_o \times D_a \times D_d \to \mathcal{X}_a$ be $\mathcal{C}_k$, $Z : D_o \times D_a \to \mathcal{S}_{+\mathcal{X}_a}$ be a $\mathcal{C}_{k+1}$, and $R : D_o \times D_a \times D_d \to \mathcal{S}_{+\mathcal{U}}$ be a $\mathcal{C}_k$. ϕ, R, and Z are design functions. Assume that Z satisfies the following two conditions.

(i) $Z(x_o, x_a) \in \mathcal{S}_{+\mathcal{X}_a}$, $\forall (x_o, x_a) \in D_o \times D_a$.

(ii) $\pi_Z(x_o, x_a) := 2Z(x_o, x_a) + \left(\frac{\partial Z}{\partial x_a}(x_o, x_a)\right)^{T_{2,1}} (x_a - \alpha_o(x_o)) \in \mathrm{B}(\mathcal{X}_a, \mathcal{X}_a^*)$ is bijective, $\forall (x_o, x_a) \in D_o \times D_a$.

Let $V : D_o \times D_a \to \mathbb{R}$ be defined by $V(x_o, x_a) := V_o(x_o) + Z(x_o, x_a)(x_a - \alpha_o(x_o))(x_a - \alpha_o(x_o))$, $\forall (x_o, x_a) \in D_o \times D_a$, which is $\mathcal{C}_{k+1}$. Let $\varrho_1, \varrho_2 \in (0, 1) \subset \mathbb{R}$ and $\varrho_3, \varrho_4 \in (0, \infty) \subset \mathbb{R}$ with $(1 + \varrho_2)\varrho_1 < 1$.

Assume that V_o satisfies the Hamilton-Jacobi-Isaacs equation

$$\left\langle\left\langle \frac{\partial V_o}{\partial x_o}(x_o), f_o(x_o, \alpha_o(x_o), x_d) \right\rangle\right\rangle_{\mathcal{X}_o} + \frac{1}{4\gamma^2}\left\| \frac{\partial V_o}{\partial x_o}(x_o) h_o(x_o, \alpha_o(x_o), x_d) \right\|_{\mathcal{W}^*}^2 + l_o(x_o, x_d) = 0; \qquad \forall (x_o, x_d) \in D_1 \tag{B26}$$

[15]For the convenience of the reader, this theorem has been reproduced as Theorem 2 in Appendix C.

Then, there exists a $\mathcal{C}_k$ function $\alpha : D_o \times D_a \times D_d \to \mathcal{U}$ given by (B15), and a continuous function $l : D_o \times D_a \times D_d \to \mathbb{R}$, such that V satisfies the following Hamilton-Jacobi-Isaacs equation, with $x := (x_o, x_a)$, $\forall (x_o, x_d) \in D_1$, $\forall x_a \in D_a$,

$$\left\langle\left\langle \frac{\partial V}{\partial x}(x_o, x_a), \begin{bmatrix} f_o(x_o, x_a, x_d) \\ f_a(x_o, x_a, x_d) + g_a(x_o, x_a, x_d)\alpha(x_o, x_a, x_d) \end{bmatrix} \right\rangle\right\rangle_{\mathcal{X}_o \times \mathcal{X}_a} + \frac{1}{4\gamma^2} \left\| \frac{\partial V}{\partial x}(x_o, x_a) \begin{bmatrix} h_o(x_o, x_a, x_d) \\ h_a(x_o, x_a, x_d) \end{bmatrix} \right\|_{\mathcal{W}^*}^2 + l(x_o, x_a, x_d) = 0 \tag{B27}$$

where $l(x_o, x_a, x_d) \geq l_o(x_o, x_d) + \langle \phi(x_o, x_a, x_d), x_a - \alpha_o(x_o) \rangle_{\mathcal{X}_a}$, $\forall (x_o, x_d) \in D_1$, $\forall x_a \in D_a$.

If, in addition, there exists $(x_{o0}, x_{a0}, x_{d0}) \in D_o \times D_a \times D_d$, such that $\frac{\partial V_o}{\partial x_o}(x_{o0}) = \vartheta_{\mathcal{X}_o^*}$, $f_o(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_o}$, $f_a(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, $\alpha_o(x_{o0}) = x_{a0}$, and $\phi(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{X}_a}$, then $\alpha(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{U}}$.

Proof. First, we will apply Lemma 8 to show that the assumptions of this lemma implies the assumptions of Lemma 6. To apply Lemma 8, we make the following substitutions:

$$\mathcal{X}_o \to \mathcal{X},\ \mathcal{X}_d \to \mathcal{X}_d,\ \mathcal{W} \to \mathcal{W},\ D_o \to D,\ x_o \to x,\ D_d \to D_d,\ x_d \to x_d,\ f_o(x_o, \alpha_o(x_o), x_d) \to f(x, x_d)$$
$$h_o(x_o, \alpha_o(x_o), x_d) \to g(x, x_d),\ l_o \to l,\ D_1 \to D_1,\ \gamma \to \gamma,\ V_o \to V, (B26) \to (B22)$$

and choose D_w to be some subset of $\mathcal{W}$ satisfying the condition of Lemma 8. Then, the derivative of $V_o(x_o(t))$ along a solution of the dynamics (B21) can be written as (B9).

By Lemma 6, there exists a $\mathcal{C}_k$ function α satisfying (B11). We will again apply Lemma 8 to show the desired result (B27). Toward that end, make the following substitutions:

$$\mathcal{X}_o \times \mathcal{X}_a \to \mathcal{X},\ \mathcal{X}_d \to \mathcal{X}_d,\ \mathcal{W} \to \mathcal{W},\ D_o \times D_a \to D,\ (x_o, x_a) \to x,\ D_d \to D_d,\ x_d \to x_d,\ D_w \to D_w$$
$$\begin{bmatrix} f_o(x_o, x_a, x_d) \\ f_a(x_o, x_a, x_d) + g_a(x_o, x_a, x_d)\alpha(x_o, x_a, x_d) \end{bmatrix} \to f(x, x_d),\ \begin{bmatrix} h_o(x_o, x_a, x_d) \\ h_a(x_o, x_a, x_d) \end{bmatrix} \to g(x, x_d),\ l(x_o, x_a, x_d) \to l(x, x_d)$$
$$w \to w,\ \sigma \to \sigma,\ \gamma \to \gamma,\ V \to V,\ (B11) \to (B23),\ \{ (x_o, x_a, x_d) \in D_o \times D_a \times D_d \mid (x_o, x_d) \in D_1,\ x_a \in D_a \} \to D_1$$

Then, V satisfies (B27).

With α defined by (B15), by Lemma 6, we have that $\alpha(x_{o0}, x_{a0}, x_{d0}) = \vartheta_{\mathcal{U}}$ under the additional assumption on (x_{o0}, x_{a0}, x_{d0}).

This completes the proof of the lemma. ☐

C CITED RESULTS OF [21]

Proposition 7. Let $\mathcal{X}$ be a real normed linear space. Then,

(i) $\mathcal{S}_{-\mathcal{X}} = -\mathcal{S}_{+\mathcal{X}}$ and $\mathcal{S}_{\mathrm{nsd}\,\mathcal{X}} = -\mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$;

(ii) $\mathcal{S}_{+\mathcal{X}}$ and $\mathcal{S}_{-\mathcal{X}}$ are open sets in $\mathrm{B}_{\mathrm{S}\,2}(\mathcal{X}, \mathbb{R}) = \mathcal{S}_{\mathcal{X}}$;

(iii) $\mathcal{S}_{\mathrm{psd}\,\mathcal{X}}$ and $\mathcal{S}_{\mathrm{nsd}\,\mathcal{X}}$ are closed convex cones in $\mathcal{S}_{\mathcal{X}}$;

(iv) $\mathcal{S}_{+\mathcal{X}} \subseteq \mathcal{S}^{\circ}_{\mathrm{psd}\,\mathcal{X}}$ and $\mathcal{S}_{-\mathcal{X}} \subseteq \mathcal{S}^{\circ}_{\mathrm{nsd}\,\mathcal{X}}$.

Proposition 8. Let $\mathcal{X}$ be a normed linear space over the field $\mathbb{K}$, $S, T \subseteq \mathcal{X}$, and $\alpha \in \mathbb{K}$. Then, the following statements hold.

(i) $\overline{\alpha S} = \alpha \overline{S}$.

(ii) If $\alpha \neq 0$, then $\widetilde{\alpha S} = \alpha \widetilde{S}$.

(iii) If $\alpha \neq 0$, then $(\alpha S)^{\circ} = \alpha S^{\circ}$.

(iv) $\overline{S} + \overline{T} \subseteq \overline{S + T}$.

(v) $S^{\circ} + T^{\circ} \subseteq (S + T)^{\circ}$.

Proposition 9. Let $\mathcal{X}$ be a normed linear space, $x_0 \in \mathcal{X}$, $S \subseteq \mathcal{X}$, and $P = x_0 + S$. Then, $\overline{P} = x_0 + \overline{S}$ and $P^{\circ} = x_0 + S^{\circ}$.

Theorem 2 (Riesz-Fréchet)**.** Let $\mathcal{X}$ be a Hilbert space over $\mathbb{K}$. Then, the following statements hold.

(i) $\forall f \in \mathcal{X}^*$, there exists a unique $y_0 \in \mathcal{X}$ such that $f(x) = \langle x, y_0 \rangle$, $\forall x \in \mathcal{X}$, and $\| f \|_{\mathcal{X}^*} = \| y_0 \|_{\mathcal{X}}$. Therefore, we may define a mapping $\Phi : \mathcal{X}^* \to \mathcal{X}$ by $\Phi(f) = y_0$.

(ii) $\forall y \in \mathcal{X}$, define $g : \mathcal{X} \to \mathbb{K}$ by $g(x) = \langle x, y \rangle$, $\forall x \in \mathcal{X}$, then $g \in \mathcal{X}^*$.

(iii) The mapping Φ is bijective, uniformly continuous, norm preserving, and conjugate linear (that is $\Phi(\alpha f_1 + \beta f_2) = \overline{\alpha}\Phi(f_1) + \overline{\beta}\Phi(f_2)$, $\forall f_1, f_2 \in \mathcal{X}^*$, $\forall \alpha, \beta \in \mathbb{K}$).

(iv) If $\mathbb{K} = \mathbb{R}$, then Φ is a isometrical isomorphism between $\mathcal{X}^*$ and $\mathcal{X}$.

(v) If $\mathbb{K} = \mathbb{C}$, let $\phi : \mathcal{X} \to \mathcal{X}^{**}$ be the natural mapping as defined in Remark 7.88 of [21], then ϕ is surjective and $\mathcal{X}$ is reflexive.

(vi) If $\mathbb{K} = \mathbb{C}$, then $\mathcal{X}^*$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}^*}$, defined by $\langle f, g \rangle_{\mathcal{X}^*} := \langle \Phi(g), \Phi(f) \rangle$, $\forall f, g \in \mathcal{X}$, is a Hilbert space, and it is reflexive.

Henceforth, we will denote $\Phi_{\text{inv}}(x) =: x^*$, $\forall x \in \mathcal{X}$. Furthermore, the following statement hold.

(vii) When $\mathbb{K} = \mathbb{C}$, let $\Phi_* : \mathcal{X}^{**} = \mathcal{X} \to \mathcal{X}^*$ be the mapping of Φ if $\mathcal{X}$ is replaced by $\mathcal{X}^*$. Then, $\Phi_* = \Phi_{\text{inv}}$. This leads to the identity $(x^*)^* = x$, $\forall x \in \mathcal{X}$.

(viii) If $\mathcal{X}$ is separable, then $\mathcal{X}^*$ is separable.

Proposition 10. Let $\mathcal{X}$ be a finite-dimensional normed linear space over the field $\mathbb{K}$. $K \subseteq \mathcal{X}$ is compact if, and only if, K is closed and bounded.

Proposition 11. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be normed linear spaces over $\mathbb{K}$, $D_1 \subseteq \mathcal{X}$, $D_2 \subseteq \mathcal{Y}$, $f : D_1 \to D_2$, $g : D_2 \to \mathcal{Z}$, $x_0 \in D_1$, and $y_0 := f(x_0) \in D_2$. Then, the following statements hold.

(i) Assume that f is C_k at x_0 and g is C_k at y_0, for some $k \in \mathbb{N} \cup \{\infty\}$. Then, $h := g \circ f$ is C_k at x_0.

(ii) Let $k \in \mathbb{N}$. Assume that f is k-times differentiable and g is k-times differentiable. Then, h is k-times differentiable.

Theorem 3. Let $f : D \times \mathcal{Y} \to \mathrm{B}(\mathcal{Z}, \mathcal{W})$, where $D \subseteq \mathcal{X}$, $\mathcal{X}$ is a normed linear space over $\mathbb{K}$, $\mathcal{Y}$ is a compact metric space, $\mathcal{Z}$ is a normed linear space over $\mathbb{K}$, $\mathcal{W}$ is a Banach space over $\mathbb{K}$, and $\mathcal{J} := (J, \mathcal{B}, \mu)$ be a finite $\mathcal{Z}$-valued measure space. Assume that the following conditions hold.

(i) $\forall x_0 \in D$, we have $\overline{\operatorname{span}\left(\mathrm{A}_D\left(x_0\right)\right)} = \mathcal{X}$.

(ii) $\forall x_0 \in D$, $\exists \delta_{x_0} \in \mathbb{R}_+$, such that the set $(D \cap \mathcal{B}_{\mathcal{X}}\left(x_0, \delta_{x_0}\right)) - x_0$ is a conic segment.

(iii) $\frac{\partial f}{\partial x} : D \times \mathcal{Y} \to \mathrm{B}(\mathcal{X}, \mathrm{B}(\mathcal{Z}, \mathcal{W}))$ exists, f and $\frac{\partial f}{\partial x}$ are continuous.

(iv) $w : \mathcal{J} \to \mathcal{Y}$ is $\mathcal{B}$-measurable.

Define $F : D \to \mathcal{W}$ by $F(x) := \int_J f(x, w(t))\, \mathrm{d}\mu(t) \in \mathcal{W}$, $\forall x \in D$. Then, F is continuously Fréchet differentiable and $\mathrm{D}F(x) = \int_J \left(\frac{\partial f}{\partial x}(x, w(t))\right)^{T_{2,1}} \mathrm{d}\mu(t) \in \mathrm{B}(\mathcal{X}, \mathcal{W})$, $\forall x \in D$.

Proposition 12. Let $\mathcal{X} := (X, \mathcal{B}, \mu)$ be a measure space, $\mathcal{Y}$ be a Banach space over $\mathbb{K}$, $\mathcal{W}$ be a separable subspace of $\mathcal{Y}$, $\mathcal{Z}$ be a Banach space over $\mathbb{K}$, $f_i : X \to \mathcal{W}$ be absolutely integrable over $\mathcal{X}$, $i = 1, 2$. Then, the following statements hold.

(i) f_i is integrable over $\mathcal{X}$ and $\int_X f_i \,\mathrm{d}\mu \in \mathcal{Y}$, $i = 1, 2$.

(ii) $f_1 + f_2$ is absolutely integrable over $\mathcal{X}$ and $\int_X (f_1 + f_2)\,\mathrm{d}\mu = \int_X f_1 \,\mathrm{d}\mu + \int_X f_2 \,\mathrm{d}\mu \in \mathcal{Y}$.

(iii) $\forall A \in \mathrm{B}(\mathcal{Y}, \mathcal{Z})$, Af_1 is absolutely integrable over $\mathcal{X}$ and $\int_X (Af_1)\,\mathrm{d}\mu = A\int_X f_1 \,\mathrm{d}\mu \in \mathcal{Z}$.

(iv) $\forall c \in \mathbb{K}$, cf_1 is absolutely integrable over $\mathcal{X}$ and $\int_X (cf_1)\,\mathrm{d}\mu = c\int_X f_1 \,\mathrm{d}\mu \in \mathcal{Y}$.

(v) $\forall H \in \mathcal{B}$, $f_1|_H$ is absolutely integrable over $\mathcal{H}$ and $\int_X (f_1 \chi_{H,X})\,\mathrm{d}\mu = \int_H f_1|_H \,\mathrm{d}\mu_H \in \mathcal{Y}$, where $\mathcal{H} := (H, \mathcal{B}_H, \mu_H)$ is the measure subspace of $\mathcal{X}$ as defined in Proposition 11.13 of [21]. We will henceforth denote $\int_H f_1|_H \,\mathrm{d}\mu_H$ by $\int_H f_1 \,\mathrm{d}\mu$.

(vi) If $f_1 = f_2$ a.e. in $\mathcal{X}$ then $\int_X f_1 \, \mathrm{d}\mu = \int_X f_2 \, \mathrm{d}\mu \in \mathcal{Y}$.

(vii) $\forall$ pairwise disjoint $\left(E_i\right)_{i=1}^{\infty} \subseteq \mathcal{B}$, $\sum_{i=1}^{\infty} \int_{E_i} f_1 \, \mathrm{d}\mu = \int_{\bigcup_{i=1}^{\infty} E_i} f_1 \, \mathrm{d}\mu \in \mathcal{Y}$.

(viii) $0 \leq \left\| \int_X f_1 \, \mathrm{d}\mu \right\| \leq \int_X \mathcal{P} \circ f_1 \, \mathrm{d}\mu < +\infty$.

(ix) If $\mathcal{Y}$ admits a positive cone P and $f_1 \stackrel{\lessdot}{=} f_2$ a.e. in $\mathcal{X}$, then $\int_X f_1 \, \mathrm{d}\mu \stackrel{\lessdot}{=} \int_X f_2 \, \mathrm{d}\mu$.

Proposition 13. Let $\mathcal{X} := (X, \mathcal{O})$ be a separable topological space, $\mathcal{Y}$ be a Banach space, and $f : X \to \mathcal{Y}$ be continuous. Then, $\mathcal{W} := \mathrm{span}\,(f(\mathcal{X})) \subseteq \mathcal{Y}$ is a separable normed linear subspace of $\mathcal{Y}$, and $\overline{\mathcal{W}} \subseteq \mathcal{Y}$ is a separable Banach subspace of $\mathcal{Y}$.

Theorem 4. Let $I := [a, b] \subset \mathbb{R}$ with $a, b \in \mathbb{R}$ and $a < b$, $\mathbb{I} := ((I, |\cdot|), \mathcal{B}, \mu)$ be the finite complete metric measure subspace of $\mathbb{R}$, $\mathcal{Y}$ be a Banach space over $\mathbb{K}$, and $F : I \to \mathcal{Y}$ be $\mathcal{C}_1$. (Note that, when $\mathbb{K} = \mathbb{C}$, I is viewed as a subset of $\mathbb{C}$ in calculations of $F^{(1)}$.) Then, $F^{(1)} : I \to \mathcal{Y}$ is absolutely integrable over $\mathbb{I}$ and $F(b) - F(a) = \int_a^b F^{(1)}(t) \, \mathrm{d}t = \int_a^b F^{(1)} \, \mathrm{d}\mu_{\mathrm{B}}$.

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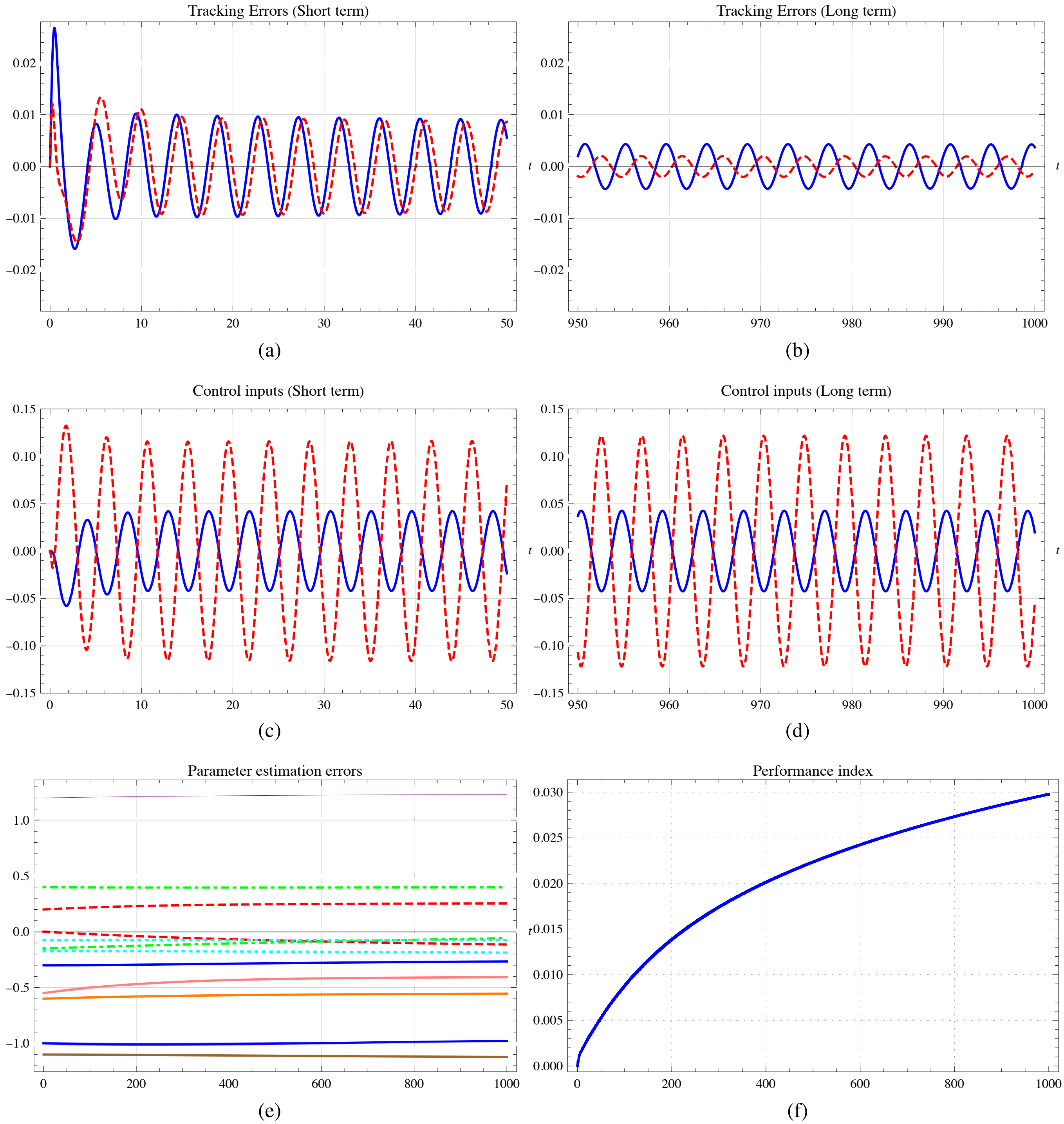

FIGURE 1 System response under no exogeneous disturbances.

(a) Tracking errors (Short term); (b) Tracking errors (Long term); (c) Control inputs (Short term); (d) Control inputs (Long term); (e) Parameter estimation errors ; (f) $\int_0^t (|z(\tau)|^2 - \gamma^2 |w(\tau)|^2)\, d\tau$

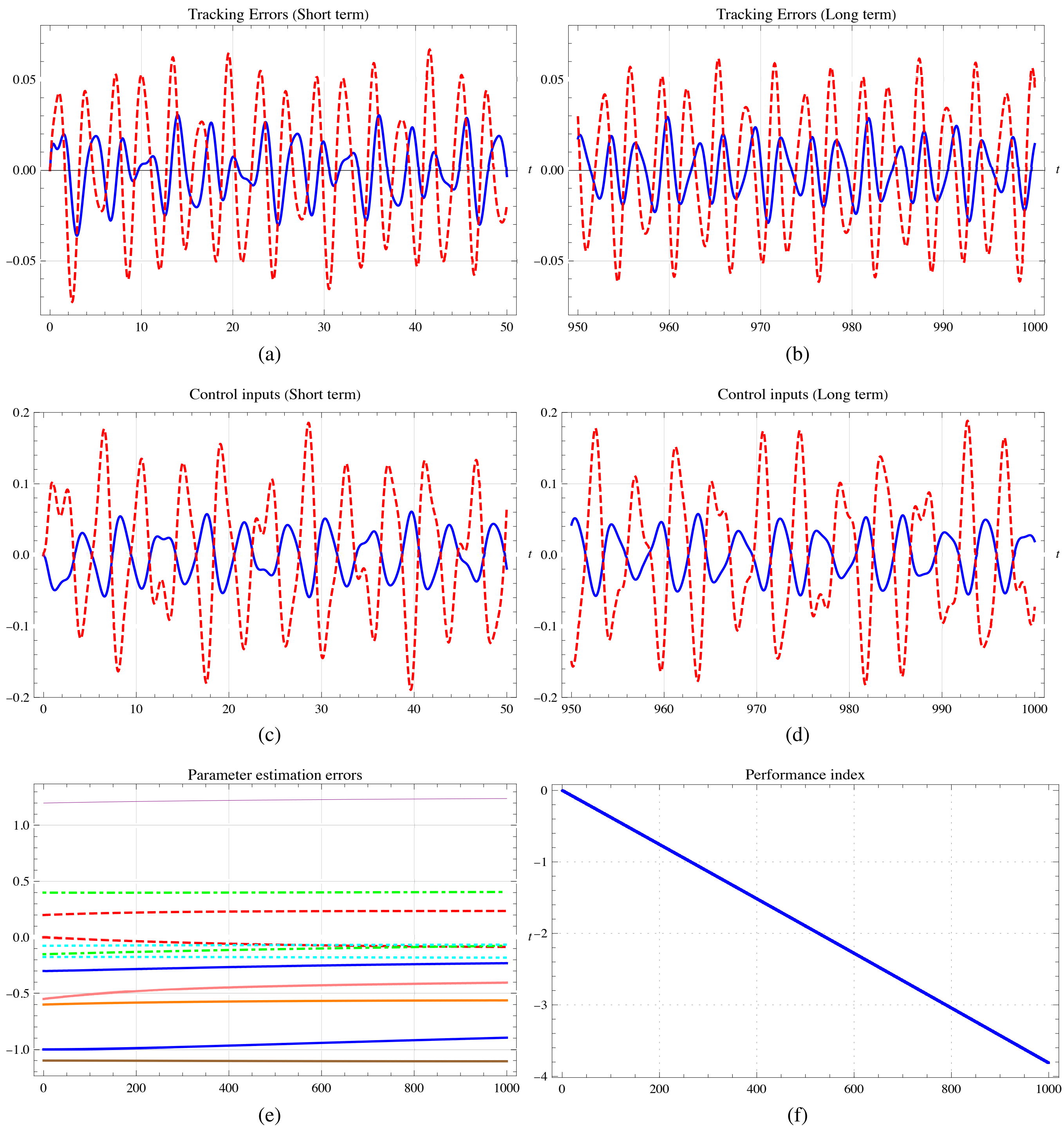

FIGURE 2 System response under sinusoidal exogeneous disturbances.

(a) Tracking errors (Short term); (b) Tracking errors (Long term); (c) Control inputs (Short term); (d) Control inputs (Long term); (e) Parameter estimation errors ; (f) $\int_0^t (|z(\tau)|^2 - \gamma^2 |w(\tau)|^2)\, d\tau$

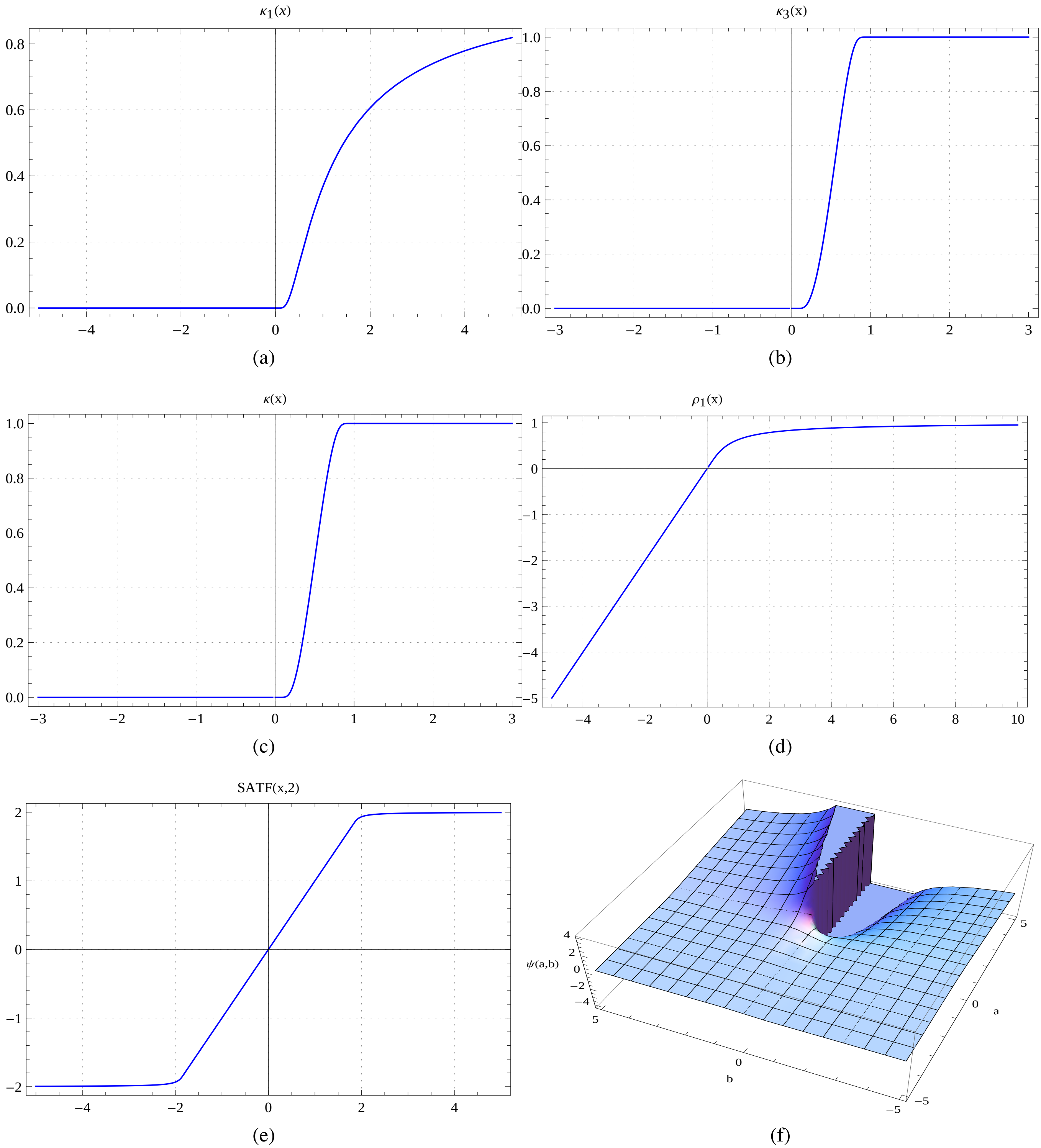

FIGURE A1 The basic smooth nonlinear functions.
(a) $\kappa_1(x)$; (b) $\kappa_3(x)$; (c) $\kappa(x)$; (d) $\rho_1(x)$; (e) $\mathrm{SATF}(x, 2)$; (f) $\psi(a, b)$

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He was a coauthor of a paper (with T. Başar) that received the 1995 George Axelby Best Paper Award. He is a member of IEEE and AMS, and an affiliate of IFAC.

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Tamer Başar. He received the B.S.E.E. degree from Robert College, Istanbul, and the M.S., M.Phil., and Ph.D. degrees from Yale University. He has been with the University of Illinois at Urbana-Champaign since 1981, where he is currently Swanlund Endowed Chair Emeritus of Center for Advanced Study (CAS) Professor Emeritus of Electrical and Computer Engineering, with also affiliations with the Coordinated Science Laboratory and the Information Trust Institute. At Illinois, he has also served as Director of CAS (2014–2020), Interim Dean of Engineering (2018), and Interim Director of the Beckman Institute (2008–2010). His current research interests include stochastic teams, games, and networks; multiagent systems and learning; data-driven distributed optimization; epidemics modeling and control over networks; security and trust; energy systems; and cyber-physical systems.

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