Locally Optimal and Robust Backstepping Design *†‡

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Abstract

For a class of nonlinear systems a new robust backstepping design achieves both local optimality and global inverse optimality. An analytic example illustrates the performance of the locally optimal control design.

Keywords: Backstepping, local optimality, inverse optimality, robust control, disturbance attenuation

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1 Introduction

While optimal designs, including H_∞-designs, are common for linear systems, optimal control of nonlinear systems is hampered by our inability to solve the Hamilton-Jacobi-Bellman (HJB) or the more general Hamilton-Jacobi-Isaacs (HJI) equation. When the nonlinear system admits a controllable linearization and the cost functional has a Taylor series with quadratic leading terms, then local approximations of the optimal control are computable [1, 2]. Although such local optimality properties are highly desirable, a deficiency of the existing locally optimal controllers is that they may adversely affect the system behavior in large regions of the state-space. In particular, the stability properties may deteriorate and the region of attraction may become intolerably small. Therefore, it is necessary to endow the locally optimal controllers with stronger stability properties, including a satisfactory region of attraction.

In this paper, we design locally optimal controllers which guarantee strong global stability properties. The new design modifies the backstepping procedure of [3] and [4] so that at each step the virtual control law and the Lyapunov function are chosen to be the linear-quadratic approximations of the H_∞-optimal control problem. In addition, by utilizing techniques similar to those of [5, 6] and [7], the control law is rendered globally inverse optimal with respect to a cost functional constructed during the recursive design procedure. In the absence of a disturbance, the equilibrium is globally asymptotically stable (GAS) and locally LQR optimal. A broad background for nonlinear concepts used in this paper is covered in the books [8, 9, 10, 11], while further details and extensions of the procedure presented here can be found in [12].

The organization of the paper is as follows. After the problem formulation in Section 2, we present in Section 3 our new nonlinear recursive design, which starts with a linear H_∞-design. A detailed example in Section 4 illustrates the stability properties and performance of the nonlinear controller designed by the new procedure.

2 Problem Formulation

Systems that lend themselves to backstepping designs are in strict-feedback form [3, 7]

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1) + h_1(x_1)w \\
\dot{x}_2 &= x_3 + f_2(x_1,x_2) + h_2(x_1,x_2)w \\
&\vdots \\
\dot{x}_n &= u + f_n(x_1,\cdots,x_n) + h_n(x_1,\cdots,x_n)w,
\end{aligned}$$

which can be written more compactly as

$$\dot{x} = f(x) + Bu + H(x)w, \tag{2.1}$$

where $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}'$ is the n-dimensional state; u is the scalar control input; and $w(t,x) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^q$ is a bounded unknown deterministic disturbance signal. We assume that the functions f_i, h_i are sufficiently smooth and $f_i(0) = 0$, $h_i(0) = d_i \in \mathbb{R}^{1\times q}$. For these systems it is well-known how to achieve global boundedness [4, 5, 6, 7] using robust backstepping. We rely on backstepping to guarantee global boundedness, but we extend the existing backstepping procedure to also meet two additional design objectives: one local, and the other global.

Our local objective is to achieve optimality in the region where the linear dynamics dominate. For this reason we separate the linear terms:

$$\begin{aligned}
\dot{x} &= Ax + Bu + Dw + \begin{bmatrix} \tilde{f}_1(x_1) \\ \vdots \\ \tilde{f}_n(x_1,\ldots,x_n) \end{bmatrix} + \begin{bmatrix} \tilde{h}_1(x_1) \\ \vdots \\ \tilde{h}_n(x_1,\ldots,x_n) \end{bmatrix} w \\
&= Ax + Bu + Dw + \tilde{f}(x) + \tilde{H}(x)w,
\end{aligned}$$

where $\tilde{f}(x) = f(x) - Ax$, $D = H(0)$, $\tilde{H}(x) = H(x) - D$, and

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} a_{11} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 1 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The pair (A, B) is controllable irrespective of the values of the coefficients a_{ij}. To meet our first objective we wish to find a locally stabilizing control law $u_l = \mu_l(x)$, which minimizes the L_2 -gain from the disturbance w to the controlled output $\left(x'Qx + Ru_l^2\right)^{1/2}$. As shown in [9], this disturbance attenuation problem is equivalent to the *dynamic game* $\min_{u_l} \max_{w_l} J_l(u_l, w_l)$ for the linearized system dynamics $\dot{x} = Ax + Bu_l + Dw_l$ and

$$J_l(u_l, w_l) = \int_0^\infty [x'Qx + Ru_l^2 - \gamma^2 w_l' w_l] dt, \tag{2.2}$$

where $Q = Q' > 0$ and $R > 0$. The optimal attenuation level $\gamma^* > 0$ exists because (A, B) is controllable and (A, Q) is observable. The desired attenuation level is $\gamma > \gamma^*$.

Our global objective is to achieve global inverse optimality with respect to the cost functional

$$J(u, w) = \int_0^\infty [q(x) + r(x)u^2 - \gamma^2 w'w] dt. \tag{2.3}$$

Our approach is inverse because we do not specify $q(x)$ and $r(x)$ beforehand. However, to assure that the cost (2.3) is *meaningful*, we require that

$$q(x) \text{ is positive definite, and } r(x) > 0 \ \ \forall x. \tag{2.4}$$

Furthermore, to satisfy the local optimality objective we impose the requirements

$$r(0) = R \qquad \text{and} \qquad q_{xx}(0) := \left.\frac{\partial^2 q(x)}{\partial x^2}\right|_{x=0} = Q. \tag{2.5}$$

We meet our local and global objectives with a significant extension of the backstepping procedure. At the i-th step of backstepping, we recursively construct a Lyapunov function $V_i(x_{[i]})$ and determine the i-th virtual control $\alpha_i(x_{[i]})$ such that if $x_{[i+1]} \equiv \alpha_i(x_{[i]})$

$$\dot{V}_i = -x_{[i]}' Q_{zi} x_{[i]} + \gamma^2 w'w - \gamma^2 \left| w - \nu_i(x_{[i]}) \right|^2, \tag{2.6}$$

where $x_{[i]} := \begin{bmatrix} x_1 & \dots & x_i \end{bmatrix}'$ and Q_{zi} is a positive definite matrix whose dependence on the matrix Q will be depicted later. The i-th worst-case disturbance is

$$\nu_i(x_{[i]}) := \frac{1}{2\gamma^2} H_{[i]}'(x_{[i]}) \frac{\partial V_i'}{\partial x_{[i]}}$$

with $H_{[i]} := \begin{bmatrix} h_1' & \cdots & h_i' \end{bmatrix}'$. The dissipation inequality

$$\dot{V}_i \leq -x_{[i]}' Q_{zi} x_{[i]} + \gamma^2 w'w$$

follows directly from (2.6). By construction, the final Lyapunov function $V(x) > 0$, obtained at $i = n$, is the value function of the dynamic game $\min_u \max_w J(u, w)$ for the cost (2.3) constrained by system (2.1). For a desired level of attenuation $\gamma > \gamma^*$, the satisfaction of the HJI equation

$$V_x(x) f(x) + \frac{1}{4} V_x(x) \left(\frac{1}{\gamma^2} H(x) H'(x) - B r^{-1}(x) B' \right) V_x'(x) + q(x) = 0$$

is achieved by selecting $q(x)$ and $r(x)$ which meet the positivity (2.4) and local optimality (2.5) requirements. The resulting optimal control law

$$u \ = \ \mu(x) := -\frac{1}{2} r^{-1}(x) B' V_x'(x), \tag{2.7}$$

and worst-case disturbance $w = \nu(x) := \dfrac{1}{2\gamma^2} H'(x) V_x'(x)$, satisfy

$$\min_u \max_w \left[q(x) + r(x)u^2 - \gamma^2 w'w + \dot{V}(x) \right] = 0$$

and the dissipation inequality

$$\dot{V} \quad \leq \quad -q(x) - r(x)u^2 + \gamma^2 w'w. \tag{2.8}$$

The satisfaction of (2.8) implies that the L_2-gain from the disturbance w to the controlled output $\left(q(x) + r(x)u^2\right)^{1/2}$ is bounded from above by γ.

Notation: The i-th linear subsystem matrix $A_{[i]}$ of the original system (2.1) is defined as

$$A_{[i]} := \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} \end{bmatrix}.$$

The same notation is used for the subsets of states, disturbance vectors and nonlinear vector fields. Quantities in the transformed z-coordinates are denoted by an "over bar," such as $\bar{A}$ corresponding to A. All higher-order functions in the x-coordinates we denote with a "tilde," such as $\tilde{f}(x)$. Similarly, higher-order functions in the transformed coordinates are denoted by a "hat," such as $\hat{f}(z)$. For the linearized version of the system the transformation is $z = Lx$. The same notation z is kept for the nonlinear system where $z = \Phi(x) = Lx + \tilde{\Phi}(x)$. Any further notational conventions will be clarified when introduced.

3 Locally Optimal Backstepping Design

The new locally optimal backstepping design is presented in four parts. First, we introduce a factorization of the solution to the *generalized algebraic Riccati equation* (GARE) for the linear H_∞-design. This factorization is instrumental in a recursive backstepping procedure resulting in the same linear control law as in the linear H_∞-design. We then augment this recursive backstepping procedure to incorporate the remaining nonlinearities at each step. Finally, we construct the state and control penalties $q(x)$ and $r(x)$, and the control law which achieves global inverse optimality.

3.1 Factorization of the Linear H_∞-Design

The objective of the H_∞-design for the linear part of system (2.1),

$$\dot{x} = Ax + Bu_l + Dw_l, \tag{3.1}$$

is to find the controller which minimizes the cost (2.2) for the worst-case disturbance. We are assured that the required unique positive definite solution $P = P' > 0$ of the GARE

$$A'P + PA + P\left(\frac{1}{\gamma^2}DD' - BR^{-1}B'\right)P + Q = 0 \tag{3.2}$$

exists for all $\gamma > \gamma^* > 0$ because (A, B) is controllable and (A, Q) is observable [9]. The control law which minimizes (2.2) for the worst-case disturbance $w_l = \nu_l(x) := \dfrac{1}{\gamma^2}D'Px$ is

$$u_l = \mu_l(x) := -R^{-1}B'Px. \tag{3.3}$$

A factorization of P which will make the H_∞-design suitable for backstepping is $P = L'\Delta L$, where L and Δ are

$$\begin{aligned} L \quad &:= \quad \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_{11} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\alpha_{n-1,1} & \cdots & -\alpha_{n-1,n-1} & 1 \end{bmatrix}, \\ \Delta \quad &:= \quad \text{diagonal of } (\delta_1, \ldots, \delta_n), \end{aligned} \tag{3.4}$$

with $\delta_i > 0$ for $i = 1, \ldots, n$. It is a matter of simple matrix algebra to show that $\Delta^{1/2}L = U'^{-1}$ where U is the unique Cholesky factor of $P^{-1} = U'U$. Therefore, the above pair (Δ, L) is unique. Furthermore, under the state transformation $z = Lx$, matrix L has the following property.

Property 3.1 *For* $1 \leq k \leq n$*:* $z_{[k]} = L_{[k]} x_{[k]}$ *where* $L_{[k]}$ *is invertible.* $\diamond$

Applying the transformation $z = Lx$ to the linear system (3.1) we obtain

$$\dot{z} = \bar{A}z + Bu_l + \bar{D}w_l, \quad \bar{A} = LAL^{-1}, \quad \bar{D} = LD. \tag{3.5}$$

Due to the structure of L, the structure of $\bar{A}$ is identical to that of A.

Property 3.2 *For* $1 \leq k < n$*:* $\dot{z}_{[k]} = \bar{A}_{[k]} z_{[k]} + \begin{bmatrix} 0 \\ \vdots \\ z_{k+1} \end{bmatrix} + \bar{D}_{[k]} w_l$. $\diamond$

In the GARE (3.2) we replace P with $L' \Delta L$ and, upon the pre-multiplication by ${L'}^{-1}$ and post-multiplication by L^{-1}, we obtain

$$\bar{A}' \Delta + \Delta \bar{A} + \Delta \left(\frac{1}{\gamma^2} \bar{D}\bar{D}' - BR^{-1}B' \right) \Delta + \bar{Q} = 0, \tag{3.6}$$

where $\bar{Q} = {L'}^{-1} Q L^{-1} > 0$. In this way we discover an important property of the transformed GARE (3.6).

Property 3.3 *For* $1 \leq k < n$*:* $\bar{A}'_{[k]} \Delta_{[k]} + \Delta_{[k]} \bar{A}_{[k]} + \frac{1}{\gamma^2} \Delta_{[k]} \bar{D}_{[k]} \bar{D}'_{[k]} \Delta_{[k]} + \bar{Q}_{[k]} = 0.$ $\diamond$

A consequence of Property 3.3 is that $\bar{A}_{[k]}$ is Hurwitz for $k = 1, \ldots, n-1$. Furthermore,

$$2z'_{[k]} \Delta_{[k]} \bar{A}_{[k]} z_{[k]} = -z'_{[k]} \bar{Q}_{[k]} z_{[k]} - \gamma^2 \bar{\nu}'_{lk} \bar{\nu}_{lk}, \qquad \forall k : \, 1 \leq k < n, \tag{3.7}$$

where $\bar{\nu}_{lk}(z_{[k]}) := \frac{1}{\gamma^2} \bar{D}'_{[k]} \Delta_{[k]} z_{[k]}$.

The transformed GARE (3.6) implies that $\bar{V} = z' \Delta z$ is the value function of the game

$$\min_{u_l} \max_{w_l} J_l(u_l, w_l)$$

for the linear system (3.5) and cost functional

$$J_l(u_l, w_l) \quad = \quad \int_0^\infty [z' \bar{Q} z + R u_l^2 - \gamma^2 w_l' w_l] dt. \tag{3.8}$$

The function $\bar{V}(z)$ can also be considered as a Lyapunov function and as a storage function. To derive the underlying dissipation inequality, we add $\gamma^2 w_l' w_l - \gamma^2 w_l' w_l + R u_l^2 - R u_l^2 \equiv 0$ to the derivative of $\bar{V}$, complete the squares with respect to both u_l and w_l, and using (3.6) obtain

$$\begin{aligned} \dot{\bar{V}} &= -z' \bar{Q} z - R u_l^2 + \gamma^2 w_l' w_l - \gamma^2 \left| w_l - \bar{\nu}_l \right|^2 + R \left(u_l - \bar{\mu}_l \right)^2 \\ &\leq -z' \bar{Q} z - R u_l^2 + \gamma^2 w_l' w_l, \end{aligned}$$

where the last inequality is guaranteed by the control law $u_l = \bar{\mu}_l(z) := -R^{-1} B' \Delta z$. The corresponding worst-case disturbance is given by $\bar{\nu}_l(z) := \frac{1}{\gamma^2} D' \Delta z$.

3.2 Linear Backstepping: Local Optimality

We now describe a linear backstepping procedure which is preparatory for the more difficult nonlinear design. In linear backstepping we restrict ourselves to the linear part (3.1) of the nonlinear system (2.1). We construct linear virtual controllers of the form $\alpha_i(x_{[i]}) = \alpha_{[i]} x_{[i]}$, where the $\alpha_{[i]}$'s are defined as

$$\alpha_{[i]} := \begin{bmatrix} \alpha_{i1} & \cdots & \alpha_{ii} \end{bmatrix} \qquad i = 1, \ldots, n-1,$$

with $\alpha_{[0]} = \alpha_{[n]} := 0$. Here the $\{\alpha_{ij}\}$'s are the elements of the matrix L specified in (3.4). Next we introduce the "error coordinates"

$$\begin{aligned} z_1 &= x_1 \\ z_i &= x_i - \alpha_{[i-1]} x_{[i-1]} \qquad i = 2, \ldots, n, \end{aligned}$$

given by the transformation $z = Lx$. At each step i of the backstepping procedure we select the positive definite Lyapunov function

$$\bar{V}_i(z_{[i]}) = \bar{V}_{i-1}(z_{[i-1]}) + \delta_i z_i^2 > 0, \quad \bar{V}_0 := 0, \quad i = 1, \ldots, n.$$

At step $i = n$ we get $\bar{V} = z'\Delta z$, the solution of the HJI equation corresponding to the GARE (3.6). To describe the steps of this recursive procedure we first rewrite system (3.1) as

$$\begin{aligned} \dot{x}_i &= a_{[i]} x_{[i]} + x_{i+1} + d_i w_l \qquad i = 1, \ldots, n-1, \\ \dot{x}_n &= a_{[n]} x_{[n]} + u_l + d_n w_l, \end{aligned}$$

where $a_{[i]} := \begin{bmatrix} a_{i1} & \ldots & a_{ii} \end{bmatrix}$. We also define the row vector $\bar{\alpha}_{[i]}$ as

$$\bar{\alpha}_{[i]} := \alpha_{[i]} L_{[i]}^{-1} \qquad i = 1, \ldots, n,$$

noting that $L_{[i]}^{-1} := (L_{[i]})^{-1}$.

Step 1: Define $z_1 = x_1$. Select $\bar{\alpha}_1(z_{[1]}) = \bar{\alpha}_{[1]} z_{[1]}$ as the virtual control law for x_2, and $\bar{V}_1 = z_{[1]}' \Delta_{[1]} z_{[1]}$ as its value function. Since $L_{[1]} = 1$, the dynamics of z_1 are

$$\dot{z}_1 = (a_{[1]} + \bar{\alpha}_{[1]}) z_{[1]} + (x_2 - \bar{\alpha}_1) + d_1 w_l.$$

From Property 3.2, for $k = 1$, we see that the z_1-subsystem can be written as

$$\dot{z}_1 = \bar{a}_{[1]} z_{[1]} + z_2 + \bar{d}_1 w_l, \qquad \bar{a}_{[1]} = a_{[1]} + \bar{\alpha}_{[1]}, \qquad \bar{d}_1 = d_1.$$

Adding $\gamma^2 w_l' w_l - \gamma^2 w_l' w_l = 0$ to the derivative of $\bar{V}_1$, completing the squares with respect to w_l, and using (3.7) we obtain

$$\dot{\bar{V}}_1 = -z_{[1]}' \bar{Q}_{[1]} z_{[1]} + \gamma^2 w_l' w_l - \gamma^2 \left| w_l - \bar{\nu}_{l1} \right|^2 + 2 z_1 \delta_1 z_2,$$

where $\bar{\nu}_{l1}(z_{[1]}) := \dfrac{1}{\gamma^2} \bar{d}_1' \delta_1 z_1 = \dfrac{1}{\gamma^2} \bar{D}_{[1]}' \Delta_{[1]} z_{[1]}$. If $z_2 \equiv 0$, then $\bar{V}_1 \geq 0$ is a storage function with supply rate $-z_{[1]}' \bar{Q}_{[1]} z_{[1]} + \gamma^2 w_l' w_l$. In the absence of a disturbance the z_1-subsystem is GAS.

Step i: Define $z_i = x_i - \bar{\alpha}_{i-1}(z_{[i-1]})$ and select the value function for this step to be $\bar{V}_i = \bar{V}_{i-1} + \delta_i z_i^2 = z_{[i]}' \Delta_{[i]} z_{[i]}$. From the preceding step, the dynamics of the $z_{[i-1]}$-subsystem are described by Property 3.2 for $k = i - 1$. The derivative of $\bar{V}_{i-1}$ is expressed as

$$\dot{\bar{V}}_{i-1} = -z_{[i-1]}' \bar{Q}_{[i-1]} z_{[i-1]} + \gamma^2 w_l' w_l - \gamma^2 \left| w_l - \bar{\nu}_{l,i-1} \right|^2 + 2 z_{i-1} \delta_{i-1} z_i,$$

where $\bar{\nu}_{l,i-1}(z_{[i-1]}) := \dfrac{1}{\gamma^2} \bar{D}_{[i-1]}' \Delta_{[i-1]} z_{[i-1]}$. The dynamics of z_i are given by

$$\dot{z}_i = a_{[i]} L_{[i]}^{-1} z_{[i]} + x_{i+1} + d_i w_l - \frac{\partial \bar{\alpha}_{i-1}}{\partial z_{[i-1]}} \left[\bar{A}_{[i-1]} z_{[i-1]} + \begin{bmatrix} 0 \\ \vdots \\ z_i \end{bmatrix} + \bar{D}_{[i-1]} w_l \right].$$

Selecting for x_{i+1} the virtual control law $\bar{\alpha}_i(z_{[i]}) = \bar{\alpha}_{[i]} z_{[i]}$ we get

$$\dot{z}_i = \bar{a}_{[i]} z_{[i]} + z_{i+1} + \bar{d}_i w_l,$$

where we have used Properties 3.1 and 3.2 to establish the identities

$$\begin{aligned} \bar{a}_{[i]} &= a_{[i]}L_{[i]}^{-1} + \bar{\alpha}_{[i]} - \begin{bmatrix} \bar{\alpha}_{[i-1]}\bar{A}_{[i-1]} & \bar{\alpha}_{i-1,i-1} \end{bmatrix} && (3.9a) \\ \bar{d}_i &= d_i - \bar{\alpha}_{[i-1]}\bar{D}_{[i-1]}. && (3.9b) \end{aligned}$$

The term $\bar{\alpha}_{i-1,i-1}$ is defined as the $(i-1)$st element of $\bar{\alpha}_{[i-1]}$. The derivative of $\bar{V}_i$ at this step is

$$\dot{\bar{V}}_i = -z_{[i]}'\bar{Q}_{[i]}z_{[i]} + \gamma^2 w_l'w_l - \gamma^2 |w_l - \bar{\nu}_{li}|^2 + 2z_i\delta_i z_{i+1},$$

where $\bar{\nu}_{li}(z_{[i]}) := \bar{\nu}_{l,i-1} + \frac{1}{\gamma^2}\bar{d}_i'\delta_i z_i = \frac{1}{\gamma^2}\bar{D}_{[i]}'\Delta_{[i]}z_{[i]}$ and the dynamics of the $z_{[i]}$-subsystem are described by Property 3.2 for $k = i$. When $z_{i+1} \equiv 0$, $\bar{V}_i \geq 0$ is a storage function with supply rate $-z_{[i]}'\bar{Q}_{[i]}z_{[i]} + \gamma^2 w_l'w_l$. In the absence of a disturbance the $z_{[i]}$-subsystem is GAS.

Step n: At the final step of the linear backstepping procedure, we define $z_n = x_n - \bar{\alpha}_{n-1}(z_{[n-1]})$ and choose

$$\bar{V} = \bar{V}_{n-1} + \delta_n z_n^2 = z'\Delta z$$

as the value function. The dynamics of z_n are given by

$$\dot{z}_n = a_{[n]}L^{-1}z + u_l + d_n w_l - \frac{\partial\bar{\alpha}_{n-1}}{\partial z_{[n-1]}}\left[\bar{A}_{[n-1]}z_{[n-1]} + \begin{bmatrix} 0 \\ \vdots \\ z_n \end{bmatrix} + \bar{D}_{[n-1]}w_l\right].$$

Using the identity (3.9) with $i = n$, we rewrite the z_n-dynamics as $\dot{z}_n = \bar{a}_{[n]}z + u_l + \bar{d}_n w_l$. Therefore, the transformed system is now described by (3.5). Finally, after a completion of squares with respect to both u_l and w_l, we get

$$\dot{\bar{V}} = -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l'w_l - \gamma^2 |w_l - \bar{\nu}_l|^2 + R(u_l - \bar{\mu}_l)^2,$$

where $\bar{\nu}_l(z) := \bar{\nu}_{l,n-1} + \frac{1}{\gamma^2}\bar{d}_n'\delta_n z_n = \frac{1}{\gamma^2}\bar{D}'\Delta z$ is the worst-case disturbance. The control law

$$\bar{\mu}_l(z) := -R^{-1}B'\Delta z \tag{3.10}$$

is the H_∞ -controller (3.3) in the z-coordinates. When $u_l = \bar{\mu}_l(z)$, we obtain

$$\begin{aligned} \dot{\bar{V}} &= -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l'w_l - \gamma^2 |w_l - \bar{\nu}_l|^2 \\ &\leq -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l'w_l. \end{aligned}$$

By this construction, $\bar{V}(z)$ is the solution to the HJI equation corresponding to the GARE (3.6). We have therefore successfully completed the linear backstepping procedure which stabilizes system (3.1) while minimizing the cost functional (2.2) for the worst-case disturbance $\bar{\nu}_l(z)$.

3.3 Nonlinear Backstepping: Global Properties

The nonlinear backstepping design procedure builds upon the linear procedure described in the previous section. At each step of the design we select as our virtual control the same terms as in the linear case and add a term to cancel the nonlinearities and to attenuate the disturbance. The cancelations are only employed to simplify the presentation. The procedure can be modified to avoid canceling beneficial nonlinearities. With the completion of each step, a new state $z_i = \phi_i(x_{[i]})$ is defined by our choice of the virtual control law from the previous step. At the last step, this construction results in the lower-triangular diffeomorphism $z = \Phi(x)$ and control Lyapunov function (CLF) $\bar{V}(z)$.

Step 1: Define $z_1 = \phi_1(x_1) := x_1$ and select $\bar{V}_1 = z_{[1]}'\Delta_{[1]}z_{[1]} > 0$ as the value function. Since $x_{[1]} = \Phi_{[1]}^{-1}(z_{[1]}) = z_{[1]}$, the $z_{[1]}$-subsystem is

$$\dot{z}_1 = a_{[1]}z_{[1]} + \tilde{f}_1(z_{[1]}) + x_2 + h_1(z_{[1]})w.$$

The recursive design of the nonlinear control law begins by adding a nonlinear term $\hat{\alpha}_1(z_{[1]})$ to the linear virtual control law for x_2, that is $\bar{\alpha}_1(z_{[1]}) = \bar{a}_{[1]}z_{[1]} + \hat{\alpha}_1(z_{[1]})$. Using (3.9) with $i = 1$ we get

$$\begin{aligned}
\dot{z}_1 &= \bar{a}_{[1]}z_{[1]} + \hat{f}_1(z_{[1]}) + (x_2 - \bar{\alpha}_1(z_{[1]})) + \bar{h}_1(z_{[1]})w \\
\dot{\bar{V}}_1 &= 2z'_{[1]}\Delta_{[1]}\left(\bar{a}_{[1]}z_{[1]} + \hat{f}_1 + (x_2 - \bar{\alpha}_1) + \bar{h}_1 w\right),
\end{aligned}$$

where $\hat{f}_1(z_{[1]}) := \hat{\alpha}_1(z_{[1]}) + \tilde{f}_1(z_{[1]})$, and $\bar{h}_1(z_{[1]}) := h_1(z_{[1]})$. Adding $\gamma^2 w'w - \gamma^2 w'w \equiv 0$ to $\dot{\bar{V}}_1$, completing the squares with respect to w, and using (3.7) with $k = 1$, we find

$$\dot{\bar{V}}_1 = -z'_{[1]}\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w - \gamma^2 |w - \bar{\nu}_1|^2 + 2z'_{[1]}\Delta_{[1]}\left[\hat{f}_1 + \frac{1}{2\gamma^2}\left(\bar{h}_1\bar{h}'_1 - \bar{d}_1\bar{d}'_1\right)\Delta_{[1]}z_{[1]}\right] + 2z_1\delta_1(x_2 - \bar{\alpha}_1),$$

where $\bar{\nu}_1(z_{[1]}) := \frac{1}{\gamma^2}\bar{h}'_1(z_1)\delta_1 z_1 = \frac{1}{\gamma^2}\bar{H}'_{[1]}(z_{[1]})\Delta_{[1]}z_{[1]}$. Now we select $\hat{\alpha}_1$ to cancel the term in the brackets:

$$\hat{\alpha}_1 = -\tilde{f}_1 - \frac{1}{2\gamma^2}\left(\bar{h}_1\bar{h}'_1 - \bar{d}_1\bar{d}'_1\right)\Delta_{[1]}z_{[1]},$$

which brings $\dot{\bar{V}}_1$ to the desired dissipativity form

$$\begin{aligned}
\dot{\bar{V}}_1 &= -z'_{[1]}\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w - \gamma^2 |w - \bar{\nu}_1|^2 + 2z_1\delta_1(x_2 - \bar{\alpha}_1) \\
&= -z'_{[1]}\bar{Q}_{[1]}z_{[1]} - \gamma^2\bar{\nu}'_1\bar{\nu}_1 + 2\gamma^2\bar{\nu}'_1 w + 2z_1\delta_1(x_2 - \bar{\alpha}_1).
\end{aligned}$$

Indeed, the dissipation inequality $\dot{\bar{V}}_1 \le -z'_{[1]}\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w$ is satisfied when $x_2 \equiv \bar{\alpha}_1(z_{[1]})$. Therefore, $\bar{V}_1 \ge 0$ is a storage function with supply rate $-z'_{[1]}\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w$ and the $z_{[1]}$-subsystem has been stabilized. Furthermore, in the absence of a disturbance, the equilibrium $z_{[1]} = 0$ of the first nonlinear subsystem is GAS.

Step i: Define $z_i = \phi_i(x_{[i]}) := x_i - \bar{\alpha}_{i-1}(z_{[i-1]})$ where $\bar{\alpha}_{i-1}(z_{[i-1]}) = \bar{a}_{[i-1]}z_{[i-1]} + \hat{\alpha}_{i-1}(z_{[i-1]})$, and select the value function $\bar{V}_i = \bar{V}_{i-1} + \delta_i z_i^2 = z'_{[i]}\Delta_{[i]}z_{[i]} > 0$. In the recursion, we assume that the preceding step results in

$$\dot{z}_{[i-1]} = \bar{A}_{[i-1]}z_{[i-1]} + \hat{f}_{[i-1]}(z_{[i-1]}) + \begin{bmatrix} 0 \\ \vdots \\ z_i \end{bmatrix} + \bar{H}_{[i-1]}(z_{[i-1]})w,$$

and

$$\begin{aligned}
\dot{\bar{V}}_{i-1} &= -z'_{[i-1]}\bar{Q}_{[i-1]}z_{[i-1]} + \gamma^2 w'w - \gamma^2 |w - \bar{\nu}_{i-1}|^2 + 2z_{i-1}\delta_{i-1}z_i \\
&= -z'_{[i-1]}\bar{Q}_{[i-1]}z_{[i-1]} - \gamma^2\bar{\nu}'_{i-1}\bar{\nu}_{i-1} + 2\gamma^2\bar{\nu}'_{i-1}w + 2z_{i-1}\delta_{i-1}z_i \\
&= 2z'_{[i-1]}\Delta_{[i-1]}\bar{A}_{[i-1]}z_{[i-1]} + \gamma^2\bar{\nu}'_{l,i-1}\bar{\nu}_{l,i-1} - \gamma^2\bar{\nu}'_{i-1}\bar{\nu}_{i-1} + 2\gamma^2\bar{\nu}'_{i-1}w + 2z_{i-1}\delta_{i-1}z_i,
\end{aligned}$$

where $\bar{\nu}_{i-1}(z_{[i-1]}) := \frac{1}{\gamma^2}\bar{H}'_{[i-1]}(z_{[i-1]})\Delta_{[i-1]}z_{[i-1]}$. The z_i-dynamics are given by

$$\begin{aligned}
\dot{z}_i = {} & a_{[i]}\Phi_{[i]}^{-1}(z_{[i]}) + \tilde{f}_i(\Phi_{[i]}^{-1}(z_{[i]})) + x_{i+1} + h_i(\Phi_{[i]}^{-1}(z_{[i]}))w \\
& -\frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \hat{f}_{[i-1]}(z_{[i-1]}) + \begin{bmatrix} 0 \\ \vdots \\ z_i \end{bmatrix} + \bar{H}_{[i-1]}(z_{[i-1]})w\right],
\end{aligned}$$

where we have used $x_{[i]} = \Phi_{[i]}^{-1}(z_{[i]})$. We now select $\bar{\alpha}_i(z_{[i]}) = \bar{a}_{[i]}z_{[i]} + \hat{\alpha}_i(z_{[i]})$ as the virtual control for x_{i+1} and, using (3.9), obtain the z_i-dynamics

$$\dot{z}_i = \bar{a}_{[i]}z_{[i]} + \hat{f}_i(z_{[i]}) + (x_{i+1} - \bar{\alpha}_i(z_{[i]})) + \bar{h}_i(z_{[i]})w,$$

where

$$\hat{f}_i(z_{[i]}) := \hat{\alpha}_i(z_{[i]}) + a_{[i]}\hat{\Psi}_{[i]}(z_{[i]}) + \tilde{f}_i(\Phi_{[i]}^{-1}(z_{[i]})) - \frac{\partial\hat{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0\\ \vdots\\ z_i\end{bmatrix}\right] - \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\hat{f}_{[i-1]}(z_{[i-1]})$$

$$\bar{h}_i(z_{[i]}) := h_i(\Phi_{[i]}^{-1}(z_{[i]})) - \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\bar{H}_{[i-1]}(z_{[i-1]}),$$

and $\hat{\Psi}_{[i]}(z_{[i]}) := \Phi_{[i]}^{-1}(z_{[i]}) - L_{[i]}^{-1}z_{[i]}$ contains only higher-order terms. Therefore, the $z_{[i]}$-dynamics is given by

$$\dot{z}_{[i]} = \bar{A}_{[i]}z_{[i]} + \hat{f}_{[i]}(z_{[i]}) + \bar{H}_{[i]}(z_{[i]})w + \begin{bmatrix}0\\ \vdots\\ x_{i+1} - \bar{\alpha}_i(z_{[i]})\end{bmatrix}.$$

Using the identity $2z'_{[i]}\Delta_{[i]}\bar{A}_{[i]}z_{[i]} = 2z'_{[i-1]}\Delta_{[i-1]}\bar{A}_{[i-1]}z_{[i-1]} + 2z_{i-1}\delta_{i-1}z_i + 2z_i\delta_i\bar{a}_{[i]}z_{[i]}$, we obtain

$$\dot{\bar{V}}_i = 2z'_{[i]}\Delta_{[i]}\bar{A}_{[i]}z_{[i]} + \gamma^2\bar{\nu}'_{l,i-1}\bar{\nu}_{l,i-1} - \gamma^2\bar{\nu}'_{i-1}\bar{\nu}_{i-1} + 2\gamma^2\bar{\nu}'_i w + 2z_i\delta_i\left(\hat{f}_i + (x_{i+1} - \bar{\alpha}_i)\right), \tag{3.11}$$

where $\bar{\nu}_i(z_{[i]}) := \bar{\nu}_{i-1} + \frac{1}{\gamma^2}\bar{h}'_i\delta_i z_i = \frac{1}{\gamma^2}\bar{H}'_{[i]}(z_{[i]})\Delta_{[i]}z_{[i]}$. Using the GARE identity (3.7),

$$\dot{\bar{V}}_i = -z'_{[i]}\bar{Q}_{[i]}z_{[i]} - \gamma^2\bar{\nu}'_{i-1}\bar{\nu}_{i-1} - \gamma^2\left(\bar{\nu}'_{li}\bar{\nu}_{li} - \bar{\nu}'_{l,i-1}\bar{\nu}_{l,i-1}\right) + 2\gamma^2\bar{\nu}'_i w + 2z_i\delta_i\left(\hat{f}_i + (x_{i+1} - \bar{\alpha}_i)\right).$$

We now add $\gamma^2 w'w - \gamma^2 w'w \equiv 0$, and complete the squares with respect to w,

$$\begin{aligned}\dot{\bar{V}}_i &= -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w - \gamma^2|w - \bar{\nu}_i|^2 + \gamma^2\left(\bar{\nu}'_i\bar{\nu}_i - \bar{\nu}'_{i-1}\bar{\nu}_{i-1}\right) - \gamma^2\left(\bar{\nu}'_{li}\bar{\nu}_{li} - \bar{\nu}'_{l,i-1}\bar{\nu}_{l,i-1}\right)\\ &\quad + 2z_i\delta_i\hat{f}_i + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i)\\ &= -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w - \gamma^2|w - \bar{\nu}_i|^2\\ &\quad + 2z_i\delta_i\left[\hat{f}_i + \left(\bar{h}_i\bar{\nu}_{i-1} - \bar{d}_i\bar{\nu}_{l,i-1}\right) + \frac{1}{2\gamma^2}\left(\bar{h}_i\bar{h}'_i - \bar{d}_i\bar{d}'_i\right)\delta_i z_i\right] + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i).\end{aligned}$$

Then we select $\hat{\alpha}_i$ to cancel the term in the brackets:

$$\begin{aligned}\hat{\alpha}_i(z_{[i]}) &= -a_{[i]}\hat{\Psi}_{[i]}(z_{[i]}) - \tilde{f}_i(\Phi_{[i]}^{-1}(z_{[i]})) + \frac{\partial\hat{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0\\ \vdots\\ z_i\end{bmatrix}\right] + \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\hat{f}_{[i-1]}(z_{[i-1]})\\ &\quad - \left(\bar{h}_i\bar{\nu}_{i-1} - \bar{d}_i\bar{\nu}_{l,i-1}\right) - \frac{1}{2\gamma^2}\left(\bar{h}_i\bar{h}'_i - \bar{d}_i\bar{d}'_i\right)\delta_i z_i.\end{aligned}$$

This results in the dissipative form of $\dot{\bar{V}}_i$:

$$\begin{aligned}\dot{\bar{V}}_i &= -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w - \gamma^2|w - \bar{\nu}_i|^2 + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i)\\ &\le -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i).\end{aligned}$$

Indeed, when $x_{i+1} \equiv \bar{\alpha}_i(z_{[i]})$, $\bar{V}_i \ge 0$ is a storage function with supply rate $-z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w$. Therefore, $z_{[i]}$ is globally stable, and, in the absence of a disturbance, the equilibrium $z_{[i]} = 0$ is GAS.

Step n: For the final step we define $z_n = \phi_n(x) := x_n - \bar{\alpha}_{n-1}(z_{[n-1]})$, and select $\bar{V} = \bar{V}_{n-1} + \delta_n z_n^2 = z'\Delta z > 0$. Then

$$\begin{aligned}\dot{z}_n &= a_{[n]}\Phi^{-1}(z) + \tilde{f}_n(\Phi^{-1}(z)) + u + h_n(\Phi^{-1}(z))w\\ &\quad - \frac{\partial\bar{\alpha}_{n-1}}{\partial z_{[n-1]}}\left[\bar{A}_{[n-1]}z_{[n-1]} + \hat{f}_{[n-1]}(z_{[n-1]}) + \begin{bmatrix}0\\ \vdots\\ z_n\end{bmatrix} + \bar{H}_{[n-1]}(z_{[n-1]})w\right],\end{aligned}$$

which can be rewritten as

$$\dot{z}_n = \bar{a}_{[n]}z + \hat{f}_n(z) + u + \bar{h}_n(z)w,$$

where $\bar{a}_{[n]}$ is defined by formula (3.9) for $i = n$, and

$$\begin{aligned}\hat{f}_n(z) &:= a_{[n]}\hat{\Psi}(z) + \tilde{f}_n(\Phi^{-1}(z)) - \frac{\partial\hat{\alpha}_{n-1}}{\partial z_{[n-1]}}\left[\bar{A}_{[n-1]}z_{[n-1]} + \begin{bmatrix} 0 \\ \vdots \\ z_n \end{bmatrix}\right] - \frac{\partial\bar{\alpha}_{n-1}}{\partial z_{[n-1]}}\hat{f}_{[n-1]}(z_{[n-1]}) \\ \bar{h}_n(z) &:= h_n(\Phi^{-1}(z)) - \frac{\partial\bar{\alpha}_{n-1}}{\partial z_{[n-1]}}\bar{H}_{[n-1]}(z_{[n-1]}).\end{aligned}$$

$\hat{\Psi}(z) := \Phi^{-1}(z) - L^{-1}z$ contains only higher-order terms. With this procedure we have completed the construction of a diffeomorphism $z = \Phi(x)$ and transformed the original system (2.1) into the form

$$\dot{z} = \bar{A}z + \hat{f}(z) + Bu + \bar{H}(z)w. \tag{3.12}$$

The linear part of this diffeomorphism is Lx, that is $z = Lx + \tilde{\Phi}(x)$, where $\tilde{\Phi}(x) := \Phi(x) - Lx$ contains only higher-order terms. Thus, in both z and x coordinates, the properties of the linearized system are the same.

3.4 Inverse Optimal Design

We now complete the design by constructing a nonlinear control law which attenuates the effects of the harmful nonlinearities and disturbance, and achieves global inverse optimality. The properties of the constructed control law are summarized in the following theorem.

Theorem 3.1 *There exists a positive definite function $\bar{q}(z)$ and a strictly positive function $\bar{r}(z)$ with properties $\bar{q}_{zz}(0) = \bar{Q} = \bar{Q}' > 0$ and $\bar{r}(0) = R > 0$, such that the feedback control law*

$$u = \bar{\mu}(z) := -\bar{r}^{-1}(z)B'\Delta z, \tag{3.13}$$

applied to the system (3.12) achieves local optimality with respect to the cost functional (3.8) and global inverse optimality with respect to the cost functional of the form

$$J(u, w) = \int_0^\infty \left[\bar{q}(z) + \bar{r}(z)u^2 - \gamma^2 w'w\right] dt, \tag{3.14}$$

for the worst-case disturbance $w = \bar{\nu}(z) := \frac{1}{\gamma^2}\bar{H}'(z)\Delta z$. Furthermore, in the absence of a disturbance, $w \equiv 0$, the control law (3.13) renders the equilibrium $z = 0$ globally asymptotically stable (GAS) and locally exponentially stable (LES).

Proof: The CLF $\bar{V} = z'\Delta z > 0$ for system (3.12), constructed through the backstepping procedure of Section 3.3, has the derivative given by the formula (3.11) with $i = n$ and $x_{n+1} := u$, $\bar{\alpha}_n := 0$, namely

$$\dot{\bar{V}} = 2z'\Delta\bar{A}z + \gamma^2\bar{\nu}'_{l,n-1}\bar{\nu}_{l,n-1} - \gamma^2\bar{\nu}'_{n-1}\bar{\nu}_{n-1} + 2\gamma^2\bar{\nu}'w + 2z_n\delta_n\left(\hat{f}_n + u\right),$$

where $\bar{\nu}(z) := \bar{\nu}_{n-1} + \frac{1}{\gamma^2}\bar{h}'_n\delta_n z_n = \frac{1}{\gamma^2}\bar{H}'(z)\Delta z$. Using the GARE identity (3.6), we reduce $\dot{\bar{V}}$ to

$$\dot{\bar{V}} = -z'\bar{Q}z + R^{-1}\delta_n^2 z_n^2 - \gamma^2\bar{\nu}'_{n-1}\bar{\nu}_{n-1} - \gamma^2\left(\bar{\nu}'_l\bar{\nu}_l - \bar{\nu}'_{l,n-1}\bar{\nu}_{l,n-1}\right) + 2\gamma^2\bar{\nu}'w + 2z_n\delta_n\left(\hat{f}_n + u\right).$$

Adding $\bar{r}(z)u^2 - \bar{r}(z)u^2 + \gamma^2 w'w - \gamma^2 w'w \equiv 0$, and completing the squares with respect to both u and w we obtain

$$\begin{aligned}\dot{\bar{V}} &= -z'\bar{Q}z - \bar{r}(z)u^2 + \gamma^2 w'w - \gamma^2|w - \bar{\nu}|^2 + \bar{r}(z)(u - \bar{\mu})^2 - \left(\bar{r}^{-1}(z) - R^{-1}\right)\delta_n^2 z_n^2 \\ &\quad + \gamma^2\left(\bar{\nu}'\bar{\nu} - \bar{\nu}'_{n-1}\bar{\nu}_{n-1}\right) - \gamma^2\left(\bar{\nu}'_l\bar{\nu}_l - \bar{\nu}'_{l,n-1}\bar{\nu}_{l,n-1}\right) + 2z_n\delta_n\hat{f}_n, \\ &= -z'\bar{Q}z - \bar{r}(z)u^2 + \gamma^2 w'w - \gamma^2|w - \bar{\nu}|^2 + \bar{r}(z)(u - \bar{\mu})^2 - \left(\bar{r}^{-1}(z) - R^{-1}\right)\delta_n^2 z_n^2 + 2z_n\delta_n\bar{\eta},\end{aligned}$$

where $\bar{\mu}(z)$ is defined by (3.13), and $\bar{\eta}(z) := \hat{f}_n + \left(\bar{h}_n \bar{\nu}_{n-1} - \bar{d}_n \bar{\nu}_{l,n-1}\right) + \frac{1}{2\gamma^2}\left(\bar{h}_n \bar{h}_n' - \bar{d}_n \bar{d}_n'\right) \delta_n z_n$.

The functions $\bar{r}(z)$ and $\bar{q}(z)$ are to be selected to satisfy the dissipation inequality

$$\dot{\bar{V}} \le -\bar{q}(z) - \bar{r}(z)u^2 + \gamma^2 w'w, \tag{3.15}$$

and to achieve global inverse optimality with respect to cost functional (3.14). For a meaningful cost $J(u,w)$, we restrict $\bar{q}(z)$ to be positive definite and $\bar{r}(z)$ to be positive for all z. Furthermore, we achieve local optimality by letting $\bar{r}(0) = R > 0$ and $\bar{q}_{zz}(0) = \bar{Q} = \bar{Q}' > 0$. One such choice of $\bar{q}(z)$ is

$$\bar{q}(z) := z'\bar{Q}z + \left(\bar{r}^{-1}(z) - R^{-1}\right)\delta_n^2 z_n^2 - 2z_n\delta_n\bar{\eta}, \tag{3.16}$$

where $\bar{r}(z) > 0$ is to be constructed such that $\bar{q}(z)$ is positive definite and $\bar{r}(0) = R$.

We begin to construct $\bar{r}(z)$ by factoring out the terms of $\bar{\eta}(z)$ and $z'\bar{Q}z$ that depend on z_n:

$$\begin{aligned} \bar{\eta}(z) &= \bar{\eta}_1(z_{[n-1]})z_{[n-1]} + \bar{\eta}_2(z)z_n \\ z'\bar{Q}z &= \begin{bmatrix} z_{[n-1]}' & z_n \end{bmatrix} \begin{bmatrix} \bar{Q}_{[n-1]} & \bar{q}_1 \\ \bar{q}_1' & \bar{q}_2 \end{bmatrix} \begin{bmatrix} z_{[n-1]} \\ z_n \end{bmatrix}, \end{aligned}$$

where $\bar{\eta}_1(0) = 0$ and $\bar{\eta}_2(0) = 0$. Next, we complete the squares in (3.16) with respect to $z_{[n-1]}$ and select $\bar{r}(z)$ to dominate only those terms in $\bar{q}(z)$ which are negative and which have not already been dominated by other positive terms:

$$\bar{q}(z) = \left| z_{[n-1]} + \bar{Q}_{[n-1]}^{-1}(\bar{q}_1 - \bar{\eta}_1'\delta_n)z_n \right|^2_{\bar{Q}_{[n-1]}} + \left(\bar{q}_2 - \bar{q}_1'\bar{Q}_{[n-1]}^{-1}\bar{q}_1\right) z_n^2 + (\bar{r}^{-1}(z) - R^{-1} - \bar{\sigma}(z))\delta_n^2 z_n^2,$$

where

$$\bar{\sigma}(z) := 2\delta_n^{-1}\bar{\eta}_2 - 2\bar{q}_1\bar{Q}_{[n-1]}^{-1}\delta_n^{-1}\bar{\eta}_1' + \bar{\eta}_1\bar{Q}_{[n-1]}^{-1}\bar{\eta}_1'. \tag{3.17}$$

Clearly $\bar{\sigma}(0) = 0$. If $\bar{r}^{-1}(z) \ge R^{-1} + \bar{\sigma}(z)$, then $\bar{q}(z)$ is guaranteed to be positive for all $z \ne 0$. Moreover, if $\bar{r}(0) = R$ then $\bar{q}_{zz}(0) = \bar{Q}$. Therefore, any positive function $\bar{r}(z) > 0$ such that $\bar{r}(0) = R$ and $\bar{r}^{-1}(z) \ge R^{-1} + \bar{\sigma}(z)$ will achieve our dual objective of local optimality and global inverse optimality.

We identify two particular choices of $\bar{r}(z)$. The first choice is

$$\bar{r}(z) := \left((R^{-1} + \bar{\sigma}(z))^2 + \bar{\epsilon}(z)\right)^{-1/2},$$

where $\bar{\epsilon}(z)$ is positive definite and guarantees that $\bar{r}(0) = R$ and that $\bar{r}(z)$ is positive. One such function for $\bar{\epsilon}(z)$ is $\bar{\epsilon}(z) := \lambda \frac{\bar{\sigma}^2(z)}{1 + \bar{\sigma}^2(z)} > 0$, where $\lambda \in (0, 1]$. The second choice,

$$\bar{r}(z) := \begin{cases} \left(R^{-1} + \bar{\sigma}(z)\right)^{-1} & : \ \bar{\sigma}(z) \ge 0 \\ R & : \ \bar{\sigma}(z) < 0, \end{cases} \tag{3.18}$$

requires $\bar{r}^{-1}(z)$ to dominate $R^{-1} + \bar{\sigma}(z)$ only when necessary, and is therefore *leaner* than the first. However, it may not be differentiable everywhere, but it is smooth at the origin. For either of the two choices,

$$\dot{\bar{V}} = -\bar{q}(z) - \bar{r}(z)u^2 + \gamma^2 w'w - \gamma^2 |w - \bar{\nu}|^2 + \bar{r}(z)(u - \bar{\mu})^2. \tag{3.19}$$

It is obvious from (3.19) that the optimal control $u = \bar{\mu}(z)$ and the worst-case disturbance $w = \bar{\nu}(z)$ satisfy

$$\min_u \max_w \left[\bar{q}(z) + \bar{r}(z)u^2 - \gamma^2 w'w + \dot{\bar{V}}(z)\right] = 0,$$

and that $\bar{V}(z)$ satisfies the dissipation inequality (3.15) and the HJI equation

$$\bar{V}_z(z)\left(\bar{A}z + \hat{f}(z)\right) + \frac{1}{4}\bar{V}_z(z)\left(\frac{1}{\gamma^2}\bar{H}(z)\bar{H}'(z) - B\bar{r}^{-1}(z)B'\right)\bar{V}_z'(z) + \bar{q}(z) = 0.$$

Therefore, $\bar{V}(z)$ is the value function for the cost functional (3.14). Likewise, $V(x) =: \bar{V}(\Phi(x))$ is the value function in the x-coordinates. $\square$

4 Example

We illustrate the steps of the design procedure on the simple nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 + w \\ \dot{x}_2 &= u, \end{aligned} \tag{4.1}$$

and cost functional

$$J = \int_0^\infty \left[x_1^2 + x_2^2 + u^2 - 5^2 w^2\right] dt.$$

Let us first examine the stability properties of the H_∞-optimal controller for the linearized system

$$\begin{aligned} \dot{x}_1 &= x_2 + w \\ \dot{x}_2 &= u. \end{aligned} \tag{4.2}$$

The optimal attenuation level for this problem is $\gamma^* = 1.27$. Since $\gamma = 5 > \gamma^*$, the GARE (3.2) has a unique positive definite solution $P = \begin{bmatrix} 1.82 & 1.06 \\ 1.06 & 1.78 \end{bmatrix} > 0$, and the optimal controller (3.3) for (4.2) is

$$u = -B'Px = -1.06x_1 - 1.78x_2. \tag{4.3}$$

When controller (4.3) is applied to the nonlinear system (4.1), the resulting feedback system is

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 + w \\ \dot{x}_2 &= -1.06x_1 - 1.78x_2. \end{aligned}$$

The disturbance-free phase-portrait of this system is represented by the dashed trajectories in Figure 1. The key instability property is the presence of an unstable equilibrium (saddle-point) at $x_1 = 0.60$, $x_2 = -0.35$. As shown in Figure 1, the stability boundary of the desired equilibrium $x_1 = x_2 = 0$ is the stable manifold of the saddle-point equilibrium denoted by M_s. The region of attraction of $x = 0$ is to the left of this manifold. All trajectories to the right of M_s are unbounded. For initial conditions in the region of attraction which are close to the manifold M_s, the transient performance is poor. Under the action of the worst-case disturbance, the region of attraction would be much smaller.

When a linear optimal design fails as in the above example, the design presented in this paper provides a systematic way to improve both the performance and stability. We now proceed with the new design in which we first recover (4.3) as the linear part of the nonlinear controller. As shown in Section 3.1 the key to achieving local optimality in backstepping designs is the factorization $P = L'\Delta L$ of the positive definite solution of the GARE (3.2). The matrices L and Δ uniquely define the constants α_{11}, δ_1 and δ_2 which are needed in the locally optimal backstepping design:

$$L = \begin{bmatrix} 1 & 0 \\ -\alpha_{11} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.6 & 1 \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = \begin{bmatrix} 1.18 & 0 \\ 0 & 1.78 \end{bmatrix}.$$

Under the transformation $z = Lx$, that is $z_1 = x_1$, $z_2 = x_2 + 0.6x_1$, the optimal control law (3.10) for the linear system is $u = -B'\Delta z = -1.78z_2$, or equivalently, $u = -1.78\,(0.6x_1 + x_2) = -1.06x_1 - 1.78x_2$, which is the same as (4.3).

Carrying out the first step of the locally optimal backstepping procedure of Section 3.3, we define $z_1 = \phi_1(x_1) := x_1$. Then, as dictated by the linear design and the factorization of P, we select $\bar{V}_1 = 1.18z_1^2$ as the value function, and $\bar{\alpha}_1(z_1) := -0.6z_1 + \hat{\alpha}_1(z_1)$ as the virtual control law for x_2. Adding $25w^2 - 25w^2 \equiv 0$ to the derivative of $\bar{V}_1$, and completing the squares with respect to w, we compute

$$\dot{\bar{V}}_1 = -1.36z_1^2 + 25w^2 - 25\,(w - \bar{\nu}_1)^2 + 2.36z_1\left[z_1^2 + \hat{\alpha}_1\right] + 2.36z_1\,(x_2 - \bar{\alpha}_1)\,,$$

where $\bar{\nu}_1(z_1) := 0.05z_1$. The nonlinear virtual control term $\hat{\alpha}_1(z_1)$ is selected to cancel the nonlinearity in brackets, that is, $\hat{\alpha}_1(z_1) = -z_1^2$. Now,

$$\dot{\bar{V}}_1 = -1.36z_1^2 + 25w^2 - 25\,(w - \bar{\nu}_1)^2 + 2.36z_1\,(x_2 - \bar{\alpha}_1)\,,$$

and when $x_2 \equiv \bar{\alpha}_1(z_1)$, $\bar{V}_1$ satisfies the dissipation inequality

$$\dot{\bar{V}}_1 \leq -1.36 z_1^2 + 25 w^2.$$

At the second step, we define $z_2 = \phi_2(x) := x_2 - \bar{\alpha}_1(z_1)$ and select $\bar{V} = \bar{V}_1 + 1.78 z_2^2$ as the value function. With this final step we have constructed the diffeomorphism

$$z := \begin{bmatrix} x_1 \\ 0.6x_1 + x_2 + x_1^2 \end{bmatrix} = \Phi(x) = Lx + \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}. \tag{4.4}$$

Because the linear part of the (4.4) is $z = Lx$, the local properties of the linear design are preserved. Using (4.4), we obtain the nonlinear system dynamics in the z-coordinates:

$$\begin{aligned} \dot{z}_1 &= -0.6 z_1 + z_2 + w \\ \dot{z}_2 &= -0.36 z_1 + 0.6 z_2 - 1.2 z_1^2 + 2 z_1 z_2 + u + (2 z_1 + 0.6) w. \end{aligned}$$

The derivative of the value function $\bar{V}$ is

$$\dot{\bar{V}} = 2z'\Delta \bar{A} z + 50 \bar{\nu} w + 3.56 \left(-1.19 z_1^2 + 2 z_1 z_2 + u\right) z_2,$$

where $\bar{A} = LAL^{-1} = \begin{bmatrix} -0.6 & 1 \\ -0.36 & 0.6 \end{bmatrix}$ and the worst-case disturbance is

$$\bar{\nu}(z) := \bar{\nu}_1 + 0.07 (2 z_1 + 0.6) z_2 = 0.05 z_1 + 0.04 z_2 + 0.14 z_1 z_2.$$

Using the GARE identity (3.6) with $\bar{Q} = L'^{-1} Q L^{-1} = \begin{bmatrix} 1.36 & -0.6 \\ -0.6 & 1 \end{bmatrix}$, and completing the squares with respect to both w and u, we obtain

$$\dot{\bar{V}} = -\bar{q}(z) - \bar{r}(z) u^2 + 25 w^2 - 25 (w - \bar{\nu})^2 + \bar{r}(z) (u - \bar{\mu})^2,$$

where

$$\bar{\mu}(z) := -1.78 \bar{r}^{-1}(z) z_2, \tag{4.5}$$

and

$$\begin{aligned} \bar{q}(z) &:= z'\bar{Q}z + 3.17 \left(\bar{r}^{-1}(z) - 1\right) z_2^2 - 3.56 \bar{\eta}_1(z_1) z_1 z_2 - 3.56 \bar{\eta}_2(z) z_2^2 \\ \bar{\eta}_1(z_1) &:= -1.10 z_1 \\ \bar{\eta}_2(z) &:= 2.08 z_1 + 0.09 z_1^2. \end{aligned}$$

After completing the squares in $\bar{q}(z)$ with respect to z_1, we define $\bar{\sigma}(z)$ as prescribed by (3.17): $\bar{\sigma}(z) = 1.8 z_1 + 1.05 z_1^2$. We now rewrite $\bar{q}(z)$ as

$$\bar{q}(z) = 1.36 (z_1 - 0.44 z_2 + 1.44 z_1 z_2)^2 + 0.74 z_2^2 + 3.17 (\bar{r}^{-1}(z) - 1 - \bar{\sigma}(z)) z_2^2.$$

The feedback law (4.5) will stabilize the system with either of the two choices for $\bar{r}(z)$ introduced in Section 3.4. We select $\bar{r}(z)$ as defined by (3.18) and the feedback control law (4.5) becomes

$$u = \begin{cases} -1.78 (1 + \bar{\sigma}(z)) z_2 & : \ \bar{\sigma}(z) \geq 0 \\ -1.78 z_2 & : \ \bar{\sigma}(z) < 0. \end{cases} \tag{4.6}$$

Since $\sigma(x) := \bar{\sigma}(\Phi(x))$, we can write (4.6) in the x-coordinates (2.7) as

$$u = \begin{cases} -1.78 (1 + \sigma(x)) \left(0.6 x_1 + x_2 + x_1^2\right) & : \ \sigma(x) \geq 0 \\ -1.78 \left(0.6 x_1 + x_2 + x_1^2\right) & : \ \sigma(x) < 0. \end{cases} \tag{4.7}$$

Solid trajectories in Figure 2 represent the phase-portrait of the disturbance-free system (4.1) controlled by the nonlinear controller (4.7). Unlike the linear controller, trajectories of the nonlinear controller remain bounded for all initial conditions. The nonlinear controller achieves not only global asymptotic stability, but it also improves the performance without an increase in control effort. This is illustrated in Figure 3 for trajectory A which is close to the stability boundary M_s. The control effort required by both the linear and nonlinear controllers is comparable. As shown in Figure 4, the states of the system driven by the nonlinear controller converge to zero much faster than those of the system driven by the linear controller. This difference is more pronounced when the initial condition is closer to M_s. The local optimality is illustrated in Figure 2 by the fact that the trajectories of both controllers are almost identical for initial conditions close to the origin.

In the presence of a bounded disturbance, the nonlinear controller (4.7) guarantees boundedness and the L_2-gain from the disturbance w to the controlled output $\left(x_1^2 + x_2^2 + u^2\right)^{1/2}$ is bounded from above by $\gamma = 5$. Thus, the nonlinear controller has preserved the good local properties of the linear controller while attaining robust global stability.

5 Conclusions

The new backstepping design procedure presented in this paper simultaneously achieves several objectives: local optimality with respect to a prescribed quadratic cost, global inverse optimality with respect to a cost functional constructed during the design, and a specified L_2 -gain. In the absence of a disturbance, it also achieves global asymptotic stability of the equilibrium. In this way, the design freedoms of earlier robust backstepping procedures [4, 6, 7] are now exploited to systematically improve both the performance and stability properties of nonlinear feedback systems.

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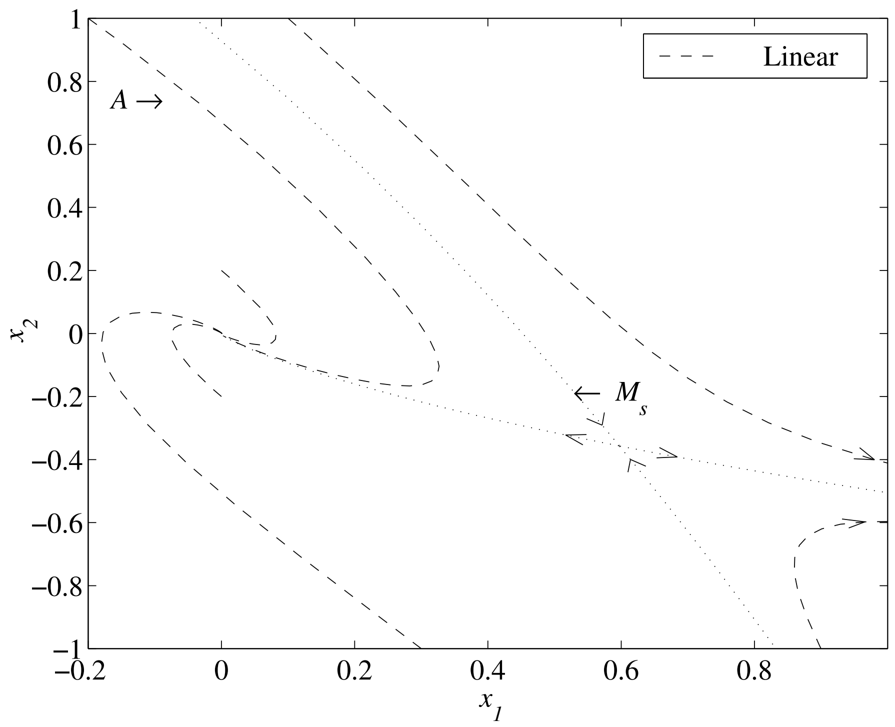

Figure 1: Linear Phase-Plane Trajectories.

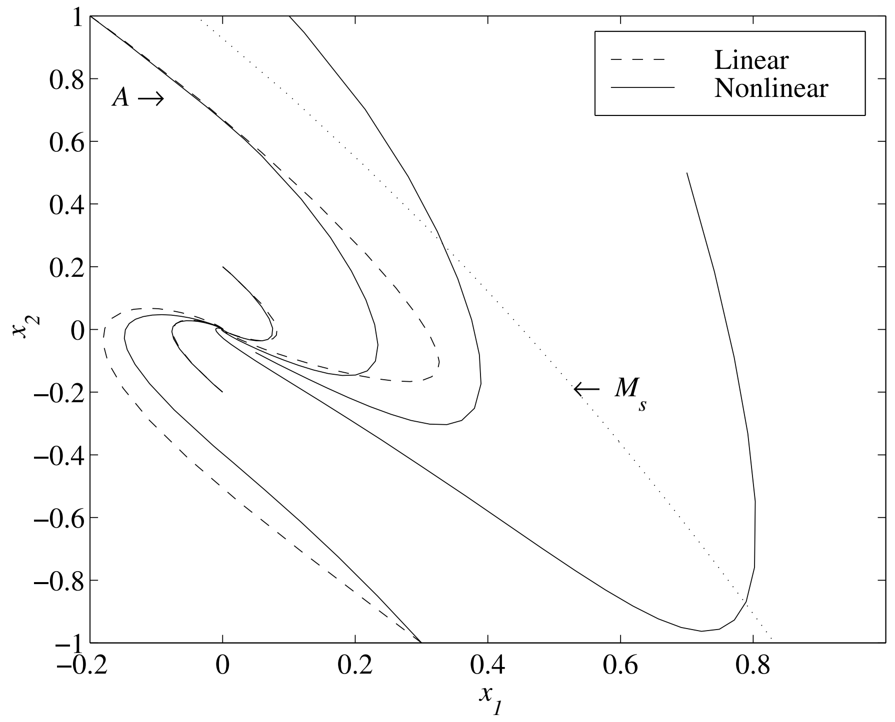

Figure 2: Phase-Plane Trajectories.

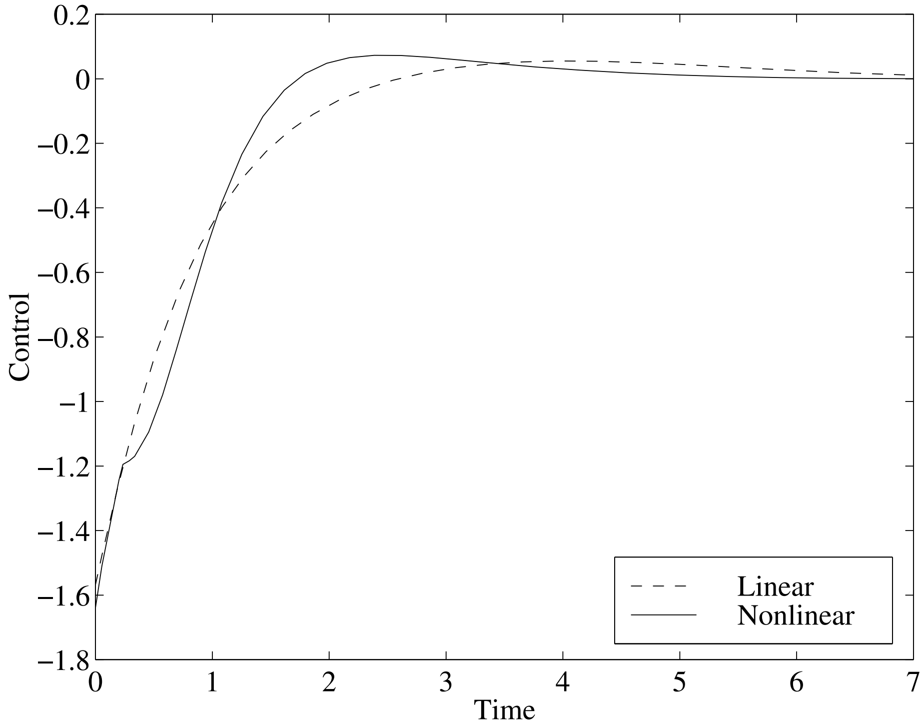

Figure 3: Control history for trajectory A.

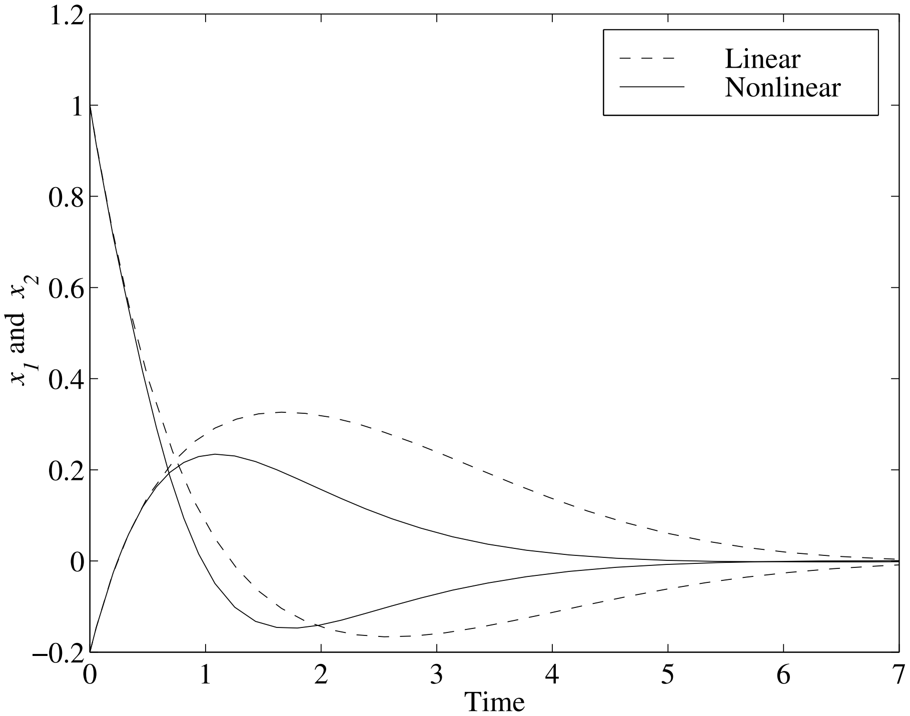

Figure 4: State history for trajectory A.