BACKSTEPPING CONTROLLER DESIGN FOR NONLINEAR STOCHASTIC SYSTEMS UNDER A RISK-SENSITIVE COST CRITERION *

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Abstract. The paper develops a methodology for recursive construction of optimal and near-optimal controllers for strict-feedback stochastic nonlinear systems under a risk-sensitive cost function criterion. The design procedure follows the integrator backstepping methodology, and the controllers obtained guarantee any desired achievable level of long-term average cost, for a given risk-sensitivity parameter θ. Furthermore, they lead to closed-loop system trajectories that are bounded in probability, and in some cases asymptotically stable in the large. These results also generalize to nonlinear systems with strongly stabilizable zero dynamics. A numerical example included in the paper illustrates the analytical results.

Key words. stochastic differential equation, stochastic stability, risk-sensitive control, integrator backstepping, zero dynamics

AMS subject classifications. 93E15, 93E20, 90D25, 93C10, 90A46.

1. Introduction.

The topic of designing globally stabilizing controllers for nonlinear systems has been an intense area of research in recent years. A class of nonlinear systems that have attracted particular interest are those that can be transformed into linear time-invariant systems under a state diffeomorphism and state feedback — to so-called feedback linearizable nonlinear systems [10]. For this class of nonlinear systems, a general and flexible control design strategy has been introduced in the early 90's [15], which is now known as *integrator backstepping*. This methodology provides a general recursive constructive tool to design controllers for nonlinear systems that are given in the *strict-feedback* form, and for those that are feedback equivalent to such systems. Since the early 90's, several additional results have been obtained for strict feedback nonlinear systems using the integrator backstepping method, with some selective references in this area being [18], [26] and [12]. This methodology has also been used to design controllers for a class of nonholonomic nonlinear systems [14]. The recent book [19] contains an up-to-date coverage of this topic, with an extensive list of references.

The integrator backstepping methodology provides a considerable amount of flexibility in the design process. This is reflected in terms of the choice of the stabilizing control laws and the additive Lyapunov functions that can be prescribed at each step of the recursive construction. One of the interesting issues that arise in the applica-

*Paper first completed on July 15, 1996, submitted to *SIAM J. Control and Optimization*, and revised on September 30, 1997. Final version completed and submitted on March 15, 1998. Address for correspondence is the one for second author, given below.

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tion of backstepping methodology is the identification of appropriate choices of these design flexibilities at each step of the recursion, so that an improved control design is achieved at the end of the recursive construction. One approach toward this problem is the inverse optimal control design [6], [7]. In [7], for example, it has been shown that given a controlled Lyapunov function for a nonlinear system, one can construct a controller that is optimal for one out of a certain class of desirable cost functions.

Another approach toward reducing the choices of integrator backstepping design is to design the controller to guarantee a prespecified level of performance with respect to a given cost criterion. In a recent paper [23], we have introduced such an approach, and have used the backstepping design tool for guaranteed disturbance attenuation in the framework of the worst-case design methodology. This has involved obtaining controllers that achieve a zero value for a particular parametrized nonlinear differential game.

It has been known for some time that differential game problems are closely related to risk-sensitive stochastic control problems with exponentiated cost [11], [13], [25], [28], [29]. In particular, it has been shown that a general nonlinear/nonquadratic risk-sensitive stochastic control problem for continuous diffusions has an equivalent representation as a stochastic differential game [1]. If the noise intensity vanishes, then a particular (large deviation) limit of the stochastic differential game (and also that of the risk-sensitive stochastic control problem) yields a deterministic differential game. All these connections have motivated, in recent years, accelerated research on the topic of risk-sensitive stochastic control.

Because of the connections alluded to above between deterministic worst-case designs and risk-sensitive stochastic control designs for nonlinear systems, the question arises as to whether the counterpart of the results of [23] can be obtained under a risk-sensitive control formulation, using an integrator backstepping methodology. Such a study would of course also require the development of a "backstepping" tool for stochastic systems. This is precisely the problem addressed in this paper.

Toward this end, we introduce a class of strict-feedback stochastic nonlinear systems, along with an exponential cost function to be optimized, which is parametrized by a risk-sensitivity parameter θ. For this system, we seek to design a controller that would guarantee any desired achievable level of long term average cost. It turns out that the construction of a controller for the risk-sensitive cost function is quite different from that of the nonlinear H^∞ problem presented in [23], the reason being that the equivalence between the deterministic dynamic game and the exponential cost problem holds only as the intensity of the system noise diminishes to zero. We present here an explicit recursive construction for the nonlinear control laws that guarantee any prespecified level of long term average cost and lead to boundedness in probability of the closed-loop system trajectory. Three special cases are discussed in detail. One such case is when the vector fields for the disturbance vanish at the origin of the system; in this case, the control design can actually guarantee a zero long term average cost, and the closed-loop system becomes asymptotically stable in the large. Another special case we discuss in the paper pertains to the limit when the risk-sensitivity parameter converges to zero — the case of the risk-neutral performance index. Yet a third special case discussed is when the system noise level diminishes to zero, for which we show that the control design procedure converges (in some sense) to the H^∞ control design procedure of [23], uniformly on compact subsets of the entire state space. These results admit immediate extensions to a class of nonlinear systems with strongly stabilizable zero dynamics (Section 6). The theoretical findings are further illustrated in Section 7 on a second-order nonlinear system.

The balance of the paper is as follows. In the next section (Section 2), we present a precise formulation for the problem of guaranteed cost control design, and provide some general theorems regarding the uncontrolled version of the exponential cost problem. We discuss the recursive construction procedure for a guaranteed cost controller design in Sections 3 and 4, where the former deals with a single typical backstepping process for nonlinear systems, which serves as a building block for the more general derivation given in the latter. Extensive discussions of various limits and specializations of the results are included in Section 5. In Section 6, we extend the result to a class of more general nonlinear systems involving nontrivial zero dynamics. A numerical example is presented in Section 7. The paper concludes with some conclusions and remarks on future research directions, all included in Section 8, and an appendix of two results that are utilized in the main body of the paper.

2. Problem Formulation. We consider the following noise-prone strict-feedback nonlinear system, given by the Itô stochastic differential equation (SDE) [1]:

$$dx_1 = (x_2 + f_1(x_1))\,dt + h_1'(x_1)\,dw \tag{1a}$$

$$\vdots \qquad \vdots$$

$$dx_{n-1} = (x_n + f_{n-1}(x_1,\dots,x_{n-1}))\,dt + h_{n-1}'(x_1,\dots,x_{n-1})\,dw \tag{1b}$$

$$dx_n = (f_n(x_1,\dots,x_n) + b(x_1,\dots,x_n)u)\,dt + h_n'(x_1,\dots,x_n)\,dw \tag{1c}$$

$$y = x_1\,. \tag{1d}$$

Here, $x := (x_1,\dots,x_n)'$ is the n-dimensional state vector, with a fixed initial value $x(0)$; u is the scalar control input; w is the q-dimensional vector-valued standard Wiener process; and y is the scalar output. In compact form, the SDE (1) can be expressed as

$$dx = (f(x) + G(x)u)\,dt + H(x)\,dw \tag{2}$$

Note that, in addition to the strict-feedback form introduced in [15] in a deterministic setting, the above system incorporates additive stochastic disturbance inputs, where the nonlinear function multiplying the disturbance terms are also in triangular form. The underlying probability space is taken to be the triplet $(\Omega, \mathcal{F}, \boldsymbol{P})$, where Ω is the sample space, $\mathcal{F}$ is a filtration, and $\boldsymbol{P}$ is the probability measure.

For the nonlinear system (1), we make the following basic assumption as a starting point of our study:

Assumption A1: The nonlinear functions f_i and h_i are $\mathcal{C}^\infty$ in all their arguments (or simply are smooth), $i = 1,\dots,n$. The nonlinear functions b and $1/b$ are $\mathcal{C}^2$ in all their arguments. The functions f_i, $i = 1,\dots,n$, vanish at $x = 0$: $f_i(x_1,\dots,x_i)|_{x=0} = 0, \quad i = 1,\dots,n$ ◇

The first part of **A1** is a standard smoothness assumption for this class of nonlinear systems; the condition imposed on f_i at $x = 0$ is to assure that the origin is an equilibrium point of the deterministic (unperturbed) part of the system.

[1] In a deterministic form, systems having the structure (1) arise in several applications (see [19]), one of which is robotic manipulators ([27]). Here, we have taken the disturbance input to be a Wiener process instead of a deterministic unknown function. We should also note that stochastic systems in the state-space form (2) with general dynamics can be brought to the strict feedback form (1) via a diffeomorphic transformation as shown in [22]. This therefore justifies the study of stochastic systems in the form of (1).

The control input u depends the current value of the state, and hence is generated by

$$u(t) = \mu(x(t)) \tag{3}$$

where the mapping $\mu : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz. We denote the class of all such controllers by $\mathcal{M}$.

The objective of the controller design is to maintain a finite (and in fact arbitrarily small, positive) long term average value for the following *risk-sensitive* cost function:

$$J_\theta(\mu) = \limsup_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} y^2(t)\, dt \right) \right] \right\} \right\} \tag{4}$$

where θ is the risk-sensitivity parameter. When θ is positive, the cost functional weights heavily the large deviations of y (from a base zero) through the exponential operator, which leads to a risk-averse control design. The larger θ is, the more conservative is the controller. On the other hand, if θ is negative, the optimization will lead to a risk-seeking controller design. If we take $\theta \to 0$, then, the cost function converges to a standard integral cost, with the underlying stochastic control problem also known as a *risk-neutral* problem. In this paper, we will only study the case when θ is positive. The risk-seeking case (i. e., with $\theta < 0$) can be studied analogously, but will not be covered here.

An important point to note here is that in the performance function (4), there is no weighting on the control input, and hence control can be picked arbitrarily large without incurring any cost. This, coupled with the strict-feedback structure, allows arbitrarily small positive long term average value to be guaranteed for the closed-loop system, for any given risk-sensitivity parameter θ — a result that will actually be established in this paper. The smaller the long term average cost is, the larger the control effort will generally be. Similarly, the larger the value of θ is, the higher will be the induced weighting on large excursions of the output y, which implies a better systems response if the long term average cost is maintained at a constant value, but at the expense of a larger control effort. Motivated by these considerations, we embed the original problem of minimizing (4) in a larger class of "cost-level satisfaction" problems, where the objective is to find a controller that leads to satisfaction of a given bound on the risk-sensitive cost. This is captured in the following definition.

DEFINITION 1. *Given a risk-sensitivity parameter θ, a controller μ is said to achieve a guaranteed risk-sensitive cost $R_c \geq 0$, if the following inequality holds:*

$$J_\theta(\mu) = \limsup_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (y^2(t) + l(x(t)))\, dt \right) \right] \right\} \right\} \leq R_c \tag{5}$$

where the function $l(x(t))$ is nonnegative and is chosen by the designer.

Our objective, succinctly stated, is given $\theta > 0$ and any $R_c > 0$ ($R_c = 0$ under the additional assumption $H(0) = 0$), to construct a controller that achieves a risk sensitive cost of R_c. To achieve this, our approach to the problem is recursive construction of the value function using the *integrator backstepping* procedure. Toward this end, we first present a result that deals with the issues of existence, uniqueness, stability, and performance of a control free risk-sensitive problem. Such a risk-sensitive problem will correspond to the closed-loop system of (1) under a particular smooth control design.

We should note at this point that the nonlinear continuous-time risk sensitive stochastic control problem has been treated before by many authors, among which are [5], [3], [1], [4] [25], with the most recent paper in this area being [20], where the general risk-sensitive control problem has been treated in both finite- and infinite-horizon cases under some global Lipschitz conditions. Yet, the results presented in all these references are not completely suitable for the class of problems under consideration in this paper, since we are considering here the infinite horizon case with arbitrary (globally non-Lipschitzian) nonlinearities. In contradistinction with the early references, though, we are interested here in designing a suboptimal control strategy rather than an optimal one, which avoids the measure transformation that is commonly used for this type of problems.

Let us now consider the SDE:

$$dx = f(x)\,dt + H(x)\,dw \tag{6}$$

along with a risk-sensitive cost function:

$$J_\theta = \limsup_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \int_0^{t_f} q(x(t))\,dt \right] \right\} \right\} \tag{7}$$

Let us introduce the value function, $W(t_f - t; x)$, associated with the finite horizon version of this problem (on interval $[t, t_f]$):

$$W(t_f - t; x) = \frac{2}{\theta} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \int_t^{t_f} q(x(s))\,ds \right] \right\} \right\} \tag{8}$$

for a system with initial time-state pair (t, x). The HJB equation satisfied by W is [1]

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f(x) + \frac{\theta}{4} \frac{\partial W}{\partial x} H(x)H'(x) \left(\frac{\partial W}{\partial x} \right)' + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial^2 W}{\partial x^2} H(x)H'(x) \right) + q(x) = 0 \tag{9}$$

Generally (and in particular if $H(x)H'(x) > \delta I$, $\forall x \in \mathbb{R}^n$, for some $\delta > 0$), the value function $W(t_f - t; x)$ becomes unbounded as the terminal time t_f goes to infinity. The rate at which it goes to infinity corresponds to the long term average cost for the infinite horizon problem. Also in view of the results of [20], it is therefore reasonable to assume the following structure for W:

$$W(t_f - t; x) = V(x) + (t_f - t) R_c$$

where R_c is a desired long term average cost.

In terms of this constant, $V(x)$ satisfies the following HJB equation:

$$\frac{\partial V}{\partial x} f(x) + \frac{\theta}{4} \frac{\partial V}{\partial x} H(x)H'(x) \left(\frac{\partial V}{\partial x} \right)' + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial^2 V}{\partial x^2} H(x)H'(x) \right) + q(x) = R_c \tag{10}$$

Since we are interested in finding a controller that guarantees the desired long term average cost R_c, we can relax the equality in (10) to an inequality, and thus search for a $V(x)$ that satisfies the following HJB inequality:

$$\frac{\partial V}{\partial x} f(x) + \frac{\theta}{4} \frac{\partial V}{\partial x} H(x)H'(x) \left(\frac{\partial V}{\partial x} \right)' + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial^2 V}{\partial x^2} H(x)H'(x) \right) + q(x) = \Delta(x) \le R_c \tag{11}$$

where R_c is the desired long term average risk-sensitive cost, that uniformly bounds the function $\Delta(x)$. We will prove that the existence of a $\mathcal{C}^2$ solution to the above HJB inequality yields the desired existence, uniqueness, stability, and performance requirements for the original risk-sensitive control problem defined by (6) and (7).

First, we recall the stochastic process notions of "*bounded in probability*" and "*asymptotic stability in the large*," as introduced in the classical book [17] (Chapter 1 Section 4; Chapter 5 Section 4), which will be useful in our development.

DEFINITION 2. *A stochastic process* $\{x(t), t \geq 0\}$ *is said to be* bounded in probability *if*

$$\lim_{c\to\infty} \sup_{0\leq t<\infty} \boldsymbol{P}\{|x(t)| > c\} = 0 \tag{12}$$

DEFINITION 3. *Consider the SDE (6), with* $H(0) = 0$. *The solution* $x(t) \equiv 0$ *of such a SDE is said to be* asymptotically stable in the large *if for any* $\epsilon > 0$,

$$\lim_{|x_0|\to 0} \boldsymbol{P}\{\sup_{t\geq 0} |x(t)| \geq \epsilon\} = 0 \tag{13}$$

and for any initial condition x_0,

$$\boldsymbol{P}\{\lim_{t\to\infty} x(t) = 0\} = 1 \tag{14}$$

DEFINITION 4. *A scalar function* $V : \mathbb{R}^n \to \mathbb{R}$ *is* positive definite *if* $V(0) = 0$ *and* $V(x) > 0$ $\forall x \in \mathbb{R}^n \setminus \{0\}$. V *is* radially unbounded *if* $V(x) \to \infty$ *as* $|x| \to \infty$.

We are now in a position to state (and prove) the following theorem.

THEOREM 5. *Consider the SDE (6) with the risk-sensitive cost function (7). Let* $\theta > 0$ *and* $R_c \geq 0$ *be fixed. Assume that the vector field* $f(x)$ *and the matrix-valued function* $H(x)$ *are* $\mathcal{C}^1$ *in their arguments, and the function* $q(x)$ *is nonnegative definite and is* $\mathcal{C}^2$ *in its argument. Further assume that there exists a positive definite and radially unbounded* $\mathcal{C}^2$ *value function* $V(x)$ *satisfying the HJB inequality (11). Then,*

1. *There exists an a.s.* $\boldsymbol{P}$ *unique solution to the SDE (6) on* $[0, \infty)$.
2. *The cost (7) is upper bounded by* R_c.
3. *If, in addition,* $V(x)$ *and* $q(x)$ *satisfy the linear bound relationship* [2]

$$V(x) \leq c_1 q(x) + c_2 \qquad \forall x \in \mathbb{R}^n \tag{15}$$

for some positive constants c_1, c_2, *then the solution* $x(t)$ *to the SDE (6) is bounded in probability.*

4. *If* $R_c = 0$, $H(0) = 0$, *and the function* $q(x)$ *is positive definite, then the SDE (6) is asymptotically stable in the large.*

Proof. Consider the function $\tilde{W}(x) := V(x) + R_c$. By the HJB inequality (11), we have

$$\begin{aligned}\frac{\partial \tilde{W}}{\partial x} f(x) + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 \tilde{W}}{\partial x^2} H(x)H(x)'\right) &= -\frac{\theta}{4}\frac{\partial V}{\partial x} H(x)H(x)'\left(\frac{\partial V}{\partial x}\right)' - q(x) + \Delta(x)\\ &\leq \Delta(x) \leq R_c \leq \tilde{W}(x)\end{aligned}$$

[2]Later in the paper (see Section 4), we will provide an explicit construction for a function V that satisfies this bound.

for any $x \in \mathbb{R}^n$. Since the function $V(x)$ is radially unbounded, $\tilde{W}$ is also radially unbounded. Then, by Theorem 4.1 of Chapter 3 of [17], we conclude that there exists a solution to the SDE (6) on the infinite interval $[0, \infty)$, and that the solution is a.s. $\boldsymbol{P}$ unique. This completes the proof of the first statement.

For the second statement, we apply Itô's rule to the stochastic process $V(x(t))$ generated by composing $V(\cdot)$ with the stochastic process $x(t),\ t \geq 0$:

$$\begin{aligned} dV(x(t)) &= \left(\frac{\partial V}{\partial x} f(x) + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial^2 V}{\partial x^2} H(x) H'(x) \right) \right) dt + \frac{\partial V}{\partial x} H(x)\, dw \\ &= \frac{\partial V}{\partial x} H(x)\, dw - \left(\frac{\theta}{4} \frac{\partial V}{\partial x} H(x) H'(x) \left(\frac{\partial V}{\partial x} \right)' + q(x) - \Delta(x) \right) dt \end{aligned}$$

where in the second line we have made use of (11).

Integrating both sides of the above, we arrive at

$$\begin{aligned} \int_0^{t_f} q(x)\, dt &= \int_0^{t_f} \left(\frac{\partial V}{\partial x} H(x)\, dw - \left(\frac{\theta}{4} \frac{\partial V}{\partial x} H(x) H'(x) \left(\frac{\partial V}{\partial x} \right)' - \Delta(x) \right) dt \right) \\ &\quad + V(x(0)) - V(x(t_f)) \\ &\leq V(x(0)) + \int_0^{t_f} \left(\frac{\partial V}{\partial x} H(x)\, dw - \frac{\theta}{4} \frac{\partial V}{\partial x} H(x) H'(x) \left(\frac{\partial V}{\partial x} \right)' dt \right) + R_c t_f \end{aligned}$$

where in the second line we have used the bound: $\Delta(x) \leq R_c$, and replaced $V(x(t_f))$ by its lower bound 0. Using this expression in the cost function J_θ, we have

$$\begin{aligned} J_\theta \leq \limsup_{t_f \to \infty} \frac{2}{\theta t_f} \ln & \left\{ \boldsymbol{E} \left\{ \exp \left[\int_0^{t_f} \left(\frac{\theta}{2} \frac{\partial V}{\partial x} H(x)\, dw - \frac{\theta^2}{8} \left| H'(x) \left(\frac{\partial V}{\partial x} \right)' \right|^2 dt \right) \right] \right\} \right\} \\ & + R_c \end{aligned}$$

The stochastic process

$$\zeta(t) := \exp \left[\int_0^t \left(\frac{\theta}{2} \frac{\partial V}{\partial x} H(x)\, dw - \frac{\theta^2}{8} \frac{\partial V}{\partial x} H(x) H'(x) \left(\frac{\partial V}{\partial x} \right)' dt \right) \right] \tag{16}$$

is almost surely positive, and is a supermartingale (see the Appendix). Hence,

$$\boldsymbol{E}(\zeta(t_f)) \leq 1 \qquad \forall t_f > 0$$

Using this bound in the inequality for J_θ yields the desired bound:

$$J_\theta \leq R_c$$

This establishes the second statement of the theorem.

The third statement follows from Theorem 2 of Chapter 3, Section 13 of [8], which says that, for any $\delta > 0$,

$$\sup_{0 \leq t < \infty} \boldsymbol{P}\{V(x(t)) > \delta\} \leq \frac{1}{\delta} \left(V(x(0)) + \frac{c_2 + R_c}{c_1} \right)$$

because of the HJB inequality (11).

Since V is a positive definite and radially unbounded function, there exists a continuous strictly increasing scalar function $\alpha : [0, \infty) \to [0, \infty)$ with the property $\lim_{s\to\infty} \alpha(s) = \infty$ such that (see Lemma 3.5 in [16])

$$V(x) \geq \alpha(|x|)$$

Consequently,

$$\begin{aligned}\sup_{0\leq t<\infty} \boldsymbol{P}\{|x(t)| > \delta\} &= \sup_{0\leq t<\infty} \boldsymbol{P}\{\alpha(|x|) > \alpha(\delta)\} \leq \sup_{0\leq t<\infty} \boldsymbol{P}\{V(x(t)) > \alpha(\delta)\} \\ &\leq \frac{1}{\alpha(\delta)}\left(V(x(0)) + \frac{c_2 + R_c}{c_1}\right)\end{aligned}$$

The third statement follows by taking the limit $\delta \to \infty$.

When $R_c = 0$, the last statement is a direct consequence of Theorem 4.4 of Chapter 5 of [17].

This completes the proof of the theorem. □

In the development to follow, we will make extensive use of an equivalent form of the HJB equation (11), which is made precise by the following proposition.

PROPOSITION 6. *A $\mathcal{C}^2$ function $V(x)$ satisfies the HJB equation (11) if, and only if, the following algebraic relationship holds:*

$$\begin{aligned}&\frac{\partial V}{\partial x} f(x)\,dt + \frac{\partial V}{\partial x} H(x)\,dw + \frac{1}{2}\,Tr\left(\frac{\partial^2 V}{\partial x^2} H(x)H'(x)\right)\,dt \\ &\quad = \sigma(x)\,dw - \left(\frac{\theta}{4}\sigma(x)\sigma'(x) + q(x) - \Delta(x)\right)\,dt\end{aligned} \tag{17}$$

where $\sigma(x) = \dfrac{\partial V}{\partial x} H(x)$ for all $x \in \mathbb{R}^n$.

Proof. "Sufficiency" By the hypothesis of the proposition, we have

$$\begin{aligned}&\frac{\partial V}{\partial x} f(x)\,dt + \frac{\partial V}{\partial x} H(x)\,dw + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2} H(x)H'(x)\right)\,dt \\ &\quad = \sigma(x)\,dw - \left(\frac{\theta}{4}\sigma(x)\sigma'(x) + q(x) - \Delta(x)\right)\,dt\end{aligned} \tag{18}$$

Taking expectations on both sides of the equation, we get

$$\frac{\partial V}{\partial x} f(x)\,dt + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2} H(x)H'(x)\right)\,dt = -\left(\frac{\theta}{4}\sigma(x)\sigma'(x) + q(x) - \Delta(x)\right)\,dt$$

Therefore, by dividing both sides by dt, we have

$$\frac{\partial V}{\partial x} f(x) + \frac{\theta}{4}\sigma(x)\sigma'(x) + q(x) + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2} H(x)H'(x)\right) = \Delta(x)$$

Since this holds for any initial condition $x \in \mathbb{R}^n$, then the function $V(x)$ satisfies the HJB equation (11).

Substituting this into (18), we have

$$\frac{\partial V}{\partial x} H(x)\,dw = \sigma(x)\,dw$$

By Lemma 12 in the Appendix, we have that

$$\frac{\partial V}{\partial x} H(x) = \sigma(x)$$

Then, it holds for all $x \in \mathbb{R}^n$. Substitution of this equality into (18) leads to the HJB equation (11).

This completes the proof of the sufficiency part of the proposition.

"Necessity" Since the HJB equation (11) is satisfied by the $\mathcal{C}^2$ function $V(x)$, then

$$\begin{aligned}\left(\frac{\partial V}{\partial x} f(x) + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2} H(x) H'(x)\right)\right) dt + \frac{\partial V}{\partial x} H(x)\, dw \\ = \frac{\partial V}{\partial x} H(x)\, dw - \left(\frac{\theta}{4}\sigma(x)\sigma'(x) + q(x) - \Delta(x)\right) dt\end{aligned}$$

This completes the proof of the necessity part of the proposition. □

Remark 1. *We note that in Proposition 6, the left-hand-side of the equation (17) equals to the Itô differential of the stochastic process $V(x(t))$ with $x(t)$ generated by the stochastic differential equation*

$$dx = f(x)\, dt + H(x)\, dw \qquad x(0) = x \tag{19}$$

whenever it exists. As a consequence of the property that the function $V(x)$ satisfies the HJB equation (11), the Itô differential of $V(x(t))$ exists under additional growth conditions delineated in Theorem 5. These growth conditions will automatically be satisfied by the recursive construction procedure to be described later. For notational simplicity, we henceforth identify $dV(x(t))$ with

$$\left(\frac{\partial V}{\partial x} f(x) + \frac{1}{2} Tr\left(\frac{\partial^2 V}{\partial x^2} H(x) H'(x)\right)\right) dt + \frac{\partial V}{\partial x} H(x)\, dw$$

for a candidate $\mathcal{C}^2$ value function $V(x)$ without verifying the existence of the Itô differential. The validity of this correspondence, which is equivalent to the existence of the Itô differential of $V(x(t))$, will be justified by a later application of Theorem 5.

We now turn, in the next two sections, to the development of a recursive construction procedure for a risk-sensitive controller with a desired guaranteed cost. In the next section, we first study a typical step in the recursive design, which provides a technical tool in the construction of a cost bounding controller for the full-order system.

3. Recursive Design. In this section, we present a systematic recursive design methodology for constructing a guaranteed cost controller for the risk-sensitive stochastic control problem.

Consider the following two-level problem:

$$d\eta = (f_1(\eta) + g_1(\eta)\xi)\, dt + H_1(\eta)\, dw \tag{20a}$$

$$d\xi = (f_2(\eta, \xi) + v)\, dt + H_2(\eta, \xi)\, dw \tag{20b}$$

where η is of dimension $n_1 > 0$, and ξ is a scalar. The exogenous disturbance process w is as defined in Section 2. The scalar v is the control variable for this two-level system. All nonlinear functions involved are assumed to be smooth.

Suppose that the control design for the η system has been completed with ξ as the control input, and the first component of η as the output y. Further assume that this design has resulted in a smooth positive definite and radially unbounded value function $V_1(\eta)$ and a smooth control law $\alpha_1(\eta)$ (for ξ) when the running cost in (4) has an additional additive nonnegative term $q_1(\eta)$. Let a guaranteed time-average cost level be $R_1 \geq 0$, which can be picked as *zero* if $H_1(0) = 0$, and positive (but arbitrarily close to *zero*) otherwise. Further suppose that the following four induction hypotheses hold.

1. The equilibrium condition: $\alpha_1(0) = 0$.
2. If $\xi = \alpha_1(\eta)$, the value function V_1 for this system satisfies the HJB inequality

$$\frac{\partial V_1}{\partial \eta}(f_1(\eta) + g_1(\eta)\alpha_1(\eta)) + \frac{\theta}{4}\frac{\partial V_1}{\partial \eta}H_1(\eta)H_1'(\eta)\left(\frac{\partial V_1}{\partial \eta}\right)' + y^2 + q_1(\eta)$$
$$+\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V_1}{\partial \eta^2}H_1(\eta)H_1'(\eta)\right) =: \Delta_1(\eta) \leq R_1 \tag{21}$$

3. The weighting function $q_1(\eta)$ is smooth and positive definite, and there exist constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, and $c_4 > 0$, such that

$$q_1(\eta) \geq c_1\eta'\eta \qquad \forall|\eta| \leq c_2 \tag{22}$$

and

$$V_1(\eta) \leq c_3\, q_1(\eta) + c_4 \qquad \forall \eta \in \mathbb{R}^{n_1} \tag{23}$$

4. When $H_1(0) = 0$, the value function V_1 and control law α_1 can be chosen in such a way to make the long-term average cost *zero* (i.e., $R_1 = 0$).

These hypotheses lead to the conclusion that the η system admits a guaranteed risk-sensitive cost R_1 with risk-sensitivity parameter θ, i. e.,

$$\limsup_{t_f\to\infty} \frac{2}{\theta t_f} \ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2}\left(\int_0^{t_f}(y^2(t) + q_1(\eta(t)))\,dt\right)\right]\right\}\right\} \leq R_1$$

and is stochastically bounded, if we can set $\xi = \alpha_1(\eta)$. When $H_1(0) = 0$, the guaranteed cost R_1 is identically zero. In this case, the closed-loop system is asymptotically stable in the large.

Among the conditions above, we stress the importance of the quadratic growth condition on the function q_1 of the induction hypothesis 3. This growth condition is crucial for the function q_1 to bound the quadratic variation term that results from the ξ subsystem when R_2 is identically zero. This point will be elucidated shortly.

In the design of the control law α_1 and the value function V_1, we have taken ξ as a control input. In reality, ξ is the state of a SDE driven by the control input v. Hence, to complete the design process, we next consider the control design for the entire (η, ξ) system, based on the knowledge of the control design for the η subsystem just completed. Our objective for the design of the (η, ξ) system is to achieve the time-average cost bound:

$$\limsup_{t_f\to\infty} \frac{2}{\theta t_f} \ln\left\{\boldsymbol{E}\left\{\exp\left[\frac{\theta}{2}\left(\int_0^{t_f}(y^2 + (1-\lambda)q_1(\eta) + q_2(\eta,\xi))\,dt\right)\right]\right\}\right\} \leq R_1 + R_2$$

and assure stochastic boundedness of the closed-loop system, where the constant $\lambda \in (0, 1)$ is a design constant. In the above, q_2 is another arbitrary nonnegative functional chosen by the control designer, and R_2 is some nonnegative constant, reflecting the design specification. Note further that $q_1(\eta)$ has been rescaled by a factor of $1-\lambda$. When $H_1(0) = 0$ and $H_2(0,0) = 0$, we take R_2 to be zero, which reflects our desire to guarantee an asymptotically stable (in the sense of Definition 3) closed-loop system for this special case. Otherwise, R_2 is taken to be positive.

Toward obtaining a controller for the composite system, we consider the state transformation:

$$z := \xi - \alpha_1(\eta) \tag{24}$$

where z represents the deviation of the random process ξ from its desired trajectory $\alpha_1(\eta)$.

Let us now introduce a convention that is adopted for the rest of this section. Any function symbol with an "over bar" will denote a function defined in terms of the transformed state variables $(\eta', z)'$, such as $\bar{a}$ denoting the equivalent form of function a (of $(\eta', \xi)'$) in terms of transformed state variables, $(\eta', z)'$.

In terms of this transformed state variable, we can rewrite the SDE (20) as follows:

$$\begin{aligned}
d\eta &= (f_1 + g_1\alpha_1 + g_1 z)\,dt + H_1\,dw \\
dz &= \left(f_2 + v - \frac{\partial\alpha_1}{\partial\eta}(f_1 + g_1\alpha_1 + g_1 z) - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2\alpha_1}{\partial\eta^2}H_1H_1'\right)\right)dt \\
&\quad + \left(H_2 - \frac{\partial\alpha_1}{\partial\eta}H_1\right)dw
\end{aligned}$$

Introduce the following terms:

$$\begin{aligned}
\bar{a}_2(\eta, z) &:= f_2 - \frac{\partial\alpha_1}{\partial\eta}(f_1 + g_1\alpha_1 + g_1 z) - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2\alpha_1}{\partial\eta^2}H_1H_1'\right) \\
\bar{d}_2(\eta, z) &:= H_2 - \frac{\partial\alpha_1}{\partial\eta}H_1
\end{aligned}$$

We note here that the function $\bar{a}_2$ includes the quadratic variation of the α_1, and does not vanish at $\eta = 0$, in general. Since $\bar{d}_2$ is a smooth (nonlinear) function of (η, z), we can rewrite it as the follows, where $\bar{d}_{22}$ is also smooth:

$$\bar{d}_2(\eta, z) =: \bar{d}_{21}(\eta) + z\bar{d}_{22}(\eta, z)$$

Using the preceding notation, we have

$$dz = (\bar{a}_2 + v)\,dt + (\bar{d}_{21} + z\bar{d}_{22})\,dw$$

Consider the following smooth function $V_2(\eta, \xi)$ as a candidate value function for the (η, ξ) system:

$$\bar{V}_2(\eta, z) := V_1(\eta) + \Xi(\eta)z^2 \tag{25}$$

where Ξ is some positive smooth function yet to be determined. In forming the value function V_2, we have added to the previous value function V_1 the term Ξz^2, instead of a simple z^2. This is motivated by the fact that the quadratic variation term of the difference $V_2 - V_1$ does not in general vanish at $z = 0$. To guarantee the desired long term average performance, we have to introduce the Ξ function into the design, as will be clear shortly.

Using Itô's rule, the random process $\bar{V}_2(\eta(t), z(t))$ satisfies the following SDE [3]:

$$d\bar{V}_2(\eta(t), z(t)) = dV_1(\eta(t)) + \begin{bmatrix} z^2 \left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2z\Xi \end{bmatrix}' \begin{bmatrix} (f_1 + g_1\alpha_1 + g_1 z)\, dt + H_1\, dw \\ (\bar{a}_2 + v)\, dt + (\bar{d}_{21} + z\bar{d}_{22})\, dw \end{bmatrix}$$
$$+\frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} z^2 \frac{\partial^2 \Xi}{\partial \eta^2} & 2z\left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2z\frac{\partial \Xi}{\partial \eta} & 2\Xi \end{bmatrix} \begin{bmatrix} H_1 \\ \bar{d}_{21} + z\bar{d}_{22} \end{bmatrix} \begin{bmatrix} H_1 \\ \bar{d}_{21} + z\bar{d}_{22} \end{bmatrix}' \right) dt$$

Since the differential for $V_1(\eta(t))$ is:

$$dV_1(\eta(t)) = -\left(\frac{\theta}{4}\frac{\partial V_1}{\partial \eta} H_1 H_1' \left(\frac{\partial V_1}{\partial \eta}\right)' + y^2 + q_1 - \frac{\partial V_1}{\partial \eta} g_1 z - \Delta_1 \right) dt$$
$$+ \frac{\partial V_1}{\partial \eta} H_1\, dw$$

we further have:

$$d\bar{V}_2(\eta(t), z(t)) = -y^2\, dt - q_1\, dt + \bar{\sigma}_2\, dw - \frac{\theta}{4}\frac{\partial V_1}{\partial \eta} H_1 H_1' \frac{\partial V_1}{\partial \eta}'\, dt + \frac{\partial V_1}{\partial \eta} g_1 z\, dt \tag{26}$$
$$+\Delta_1\, dt + \begin{bmatrix} z^2 \left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2z\Xi \end{bmatrix}' \begin{bmatrix} f_1 + g_1\alpha_1 + g_1 z \\ \bar{a}_2 + v \end{bmatrix} dt$$
$$+\frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} z^2 \frac{\partial^2 \Xi}{\partial \eta^2} & 2z\left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2z\frac{\partial \Xi}{\partial \eta} & 2\Xi \end{bmatrix} \begin{bmatrix} H_1 H_1' & H_1 \bar{d}_2' \\ \bar{d}_2 H_1' & (\bar{d}_{21} + z\bar{d}_{22})(\bar{d}_{21} + z\bar{d}_{22})' \end{bmatrix} \right) dt$$

where

$$\bar{\sigma}_2(\eta, z) := \begin{bmatrix} \frac{\partial \bar{V}_2}{\partial \eta} & \frac{\partial \bar{V}_2}{\partial z} \end{bmatrix} \begin{bmatrix} H_1 \\ \bar{d}_2 \end{bmatrix} \equiv \frac{\partial V_1}{\partial \eta} H_1 + \begin{bmatrix} z^2 \left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2z\Xi \end{bmatrix}' \begin{bmatrix} H_1 \\ \bar{d}_2 \end{bmatrix}$$

We seek a control law α_2 such that $\bar{V}_2$ satisfies the following HJB inequality

$$\begin{bmatrix} \frac{\partial V_2}{\partial \eta} \\ \frac{\partial V_2}{\partial \xi} \end{bmatrix}' \begin{bmatrix} f_1(\eta) + g_1(\eta)\xi \\ f_2(\eta,\xi) + \alpha_2(\eta,\xi) \end{bmatrix} + y^2 + (1-\lambda) q_1(\eta) + q_2(\eta,\xi) \tag{27}$$
$$+\frac{\theta}{4} \begin{bmatrix} \frac{\partial V_2}{\partial \eta} \\ \frac{\partial V_2}{\partial \xi} \end{bmatrix}' \begin{bmatrix} H_1(\eta) \\ H_2(\eta,\xi) \end{bmatrix} \begin{bmatrix} H_1(\eta) \\ H_2(\eta,\xi) \end{bmatrix}' \begin{bmatrix} \frac{\partial V_2}{\partial \eta} \\ \frac{\partial V_2}{\partial \xi} \end{bmatrix}$$
$$+\frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} \frac{\partial^2 V_2}{\partial \eta^2} & \frac{\partial^2 V_2}{\partial \eta \partial \xi} \\ \frac{\partial^2 V_2}{\partial \eta \partial \xi} & \frac{\partial^2 V_2}{\partial \xi^2} \end{bmatrix} \begin{bmatrix} H_1(\eta) \\ H_2(\eta,\xi) \end{bmatrix} \begin{bmatrix} H_1(\eta) \\ H_2(\eta,\xi) \end{bmatrix}' \right) \le R_1 + R_2$$

[3] All these differentials exist as the explicit construction of V_1, $\bar{V}_2$, ... will reveal.

for some nonnegative function q_2, by making use of the equivalence given in Proposition 6. Toward this end, we add and subtract $\frac{\theta}{4}\bar{\sigma}_2(\eta,z)\bar{\sigma}_2'(\eta,z)\,dt$ and $\bar{\beta}(\eta,z)z^2\,dt$ to the right-hand-side of (26). This yields

$$
\begin{aligned}
d\bar{V}_2(\eta(t),z(t)) &= -y^2\,dt - q_1\,dt - \bar{\beta}z^2\,dt + \bar{\sigma}_2\,dw - \frac{\theta}{4}\bar{\sigma}_2\bar{\sigma}_2'\,dt + \Delta_1\,dt \\
&+\bar{\beta}z^2\,dt + \frac{\theta}{4}\bar{\sigma}_2\bar{\sigma}_2'\,dt + \frac{\partial V_1}{\partial\eta}g_1 z\,dt - \frac{\theta}{4}\frac{\partial V_1}{\partial\eta}H_1H_1'\frac{\partial V_1}{\partial\eta}'\,dt \\
&+\begin{bmatrix} z^2\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2z\Xi \end{bmatrix}'\begin{bmatrix} f_1+g_1\alpha_1+g_1z \\ \bar{a}_2+v \end{bmatrix}dt \\
&+\frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} z^2\frac{\partial^2\Xi}{\partial\eta^2} & 2z\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2z\frac{\partial\Xi}{\partial\eta} & 2\Xi \end{bmatrix}\begin{bmatrix} H_1H_1' & H_1\bar{d}_2' \\ \bar{d}_2H_1' & (\bar{d}_{21}+z\bar{d}_{22})(\bar{d}_{21}+z\bar{d}_{22})' \end{bmatrix}\right)dt
\end{aligned}
$$

where the function $\bar{\beta}(\eta,z)$ is to be determined shortly.

By multiplying out various terms above, we arrive at:

$$
\begin{aligned}
d\bar{V}_2(\eta(t),z(t)) &= -y^2\,dt - q_1\,dt + \bar{\sigma}_2\,dw - \frac{\theta}{4}\bar{\sigma}_2\bar{\sigma}_2'\,dt + \Delta_1\,dt - \bar{\beta}z^2\,dt \\
&+\bar{\beta}z^2\,dt + \frac{\theta}{4}\begin{bmatrix} z^2\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2z\Xi \end{bmatrix}'\begin{bmatrix} H_1 \\ \bar{d}_2 \end{bmatrix}H_1'\left(\frac{\partial V_1}{\partial\eta}\right)'dt \\
&+\frac{\theta}{4}\frac{\partial V_1}{\partial\eta}H_1\begin{bmatrix} H_1 \\ \bar{d}_2 \end{bmatrix}'\begin{bmatrix} z^2\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2z\Xi \end{bmatrix}dt + \frac{\partial V_1}{\partial\eta}g_1z\,dt \\
&+\frac{\theta}{4}\begin{bmatrix} z^2\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2z\Xi \end{bmatrix}'\begin{bmatrix} H_1' \\ \bar{d}_2 \end{bmatrix}\begin{bmatrix} H_1' \\ \bar{d}_2 \end{bmatrix}'\begin{bmatrix} z^2\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2z\Xi \end{bmatrix}dt \\
&+\begin{bmatrix} z^2\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2z\Xi \end{bmatrix}'\begin{bmatrix} f_1+g_1\alpha_1+g_1z \\ \bar{a}_2+v \end{bmatrix}dt + \frac{1}{2}\mathrm{Tr}\left(z^2\frac{\partial^2\Xi}{\partial\eta^2}H_1'H_1\right) \\
&+\frac{1}{2}\mathrm{Tr}\left(4z\left(\frac{\partial\Xi}{\partial\eta}\right)'\bar{d}_2H_1 + 2\Xi(z\bar{d}_{21}\bar{d}_{22}' + z\bar{d}_{22}\bar{d}_{21}' + z^2\bar{d}_{22}\bar{d}_{22}')\right)dt \\
&+\Xi\bar{d}_{21}\bar{d}_{21}'\,dt
\end{aligned}
$$

In the above SDE, we can select the control to cancel out all terms except those on the first and last lines. The control law that accomplishes this is:

$$
\begin{aligned}
(28)\,\bar{\alpha}_{2b}(\eta,z) = -\frac{1}{2\Xi}\Bigg(&\bar{\beta}z + \frac{\theta}{4}\begin{bmatrix} z\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2\Xi \end{bmatrix}'\begin{bmatrix} H_1 \\ \bar{d}_2 \end{bmatrix}H_1'\left(\frac{\partial V_1}{\partial\eta}\right)' \\
&+\frac{\theta}{4}\frac{\partial V_1}{\partial\eta}H_1\begin{bmatrix} H_1 \\ \bar{d}_2 \end{bmatrix}'\begin{bmatrix} z\left(\frac{\partial\Xi}{\partial\eta}\right)' \\ 2\Xi \end{bmatrix} + \frac{\partial V_1}{\partial\eta}g_1
\end{aligned}
$$

$$+\frac{\theta}{4}\left[\begin{array}{c} z\left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2\Xi \end{array}\right]'\left[\begin{array}{c} H_1 \\ \bar{d}_2 \end{array}\right]\left[\begin{array}{c} H_1 \\ \bar{d}_2 \end{array}\right]'\left[\begin{array}{c} z^2\left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2z\Xi \end{array}\right]$$
$$+\left[\begin{array}{c} z\left(\frac{\partial \Xi}{\partial \eta}\right)' \\ 2\Xi \end{array}\right]'\left[\begin{array}{c} f_1+g_1\alpha_1+g_1 z \\ \bar{a}_2 \end{array}\right]$$
$$\left.+\frac{1}{2}\mathrm{Tr}\left(z\frac{\partial^2 \Xi}{\partial \eta^2}H_1H_1'+4\left(\frac{\partial \Xi}{\partial \eta}\right)'\bar{d}_2H_1'+2\Xi(\bar{d}_{21}\bar{d}_{22}'+\bar{d}_{22}\bar{d}_{21}'+z\bar{d}_{22}\bar{d}_{22}')\right)\right)$$

Even though this controller leads to a guaranteed risk-sensitive cost design, in general it fails to preserve the equilibrium at $(\eta,\xi)=(0,0)$, except for the case when $H_1(0)=0$ and $H_2(0,0)=0$. To preserve the equilibrium at the origin, we take out the bias term from (28), to arrive at the modified controller:

$$v=\bar{\alpha}_2(\eta,z):=\bar{\alpha}_{2b}(\eta,z)-\frac{\Xi(0)}{\Xi(\eta)}\bar{\alpha}_{2b}(0,0) \tag{29}$$

which, by its construction, preserves the equilibrium at $(\eta,\xi)=(0,0)$.

Using this control law, the differential of $\bar{V}_2$ becomes

$$d\bar{V}_2(\eta(t),z(t))=-y^2\,dt-q_1\,dt+\bar{\sigma}_2\,dw-\frac{\theta}{4}\bar{\sigma}_2\bar{\sigma}_2'\,dt+\Delta_1\,dt-\bar{\beta}z^2\,dt$$
$$+\Xi\bar{d}_{21}\bar{d}_{21}'\,dt+2z\Xi(0)\bar{\alpha}_{2b}(0,0)\,dt$$

where the last term is due to the removal of the constant bias in the controller.

To establish the desired HJB inequality (27), we will first pick the function Ξ and then β in the following steps. The function $\bar{d}_{21}(\eta)$ is further decomposed into constant and η dependent parts:

$$\bar{d}_{21c}:=\bar{d}_{21}(0)$$
$$\bar{d}_{21v}(\eta):=\bar{d}_{21}(\eta)-\bar{d}_{21c}$$

We choose Ξ such that for any positive R_2 and $\forall\eta\in\mathbb{R}^{n_1}$,

$$\Xi(\eta)\bar{d}_{21}(\eta)\bar{d}_{21}'(\eta)\leq\frac{R_2}{2}+\lambda q_1(\eta) \tag{30}$$

and furthermore

$$\Xi(\eta)\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta)\leq\lambda q_1(\eta) \tag{31}$$

One possible choice of the function Ξ can be made in the following two steps. For the first step, we will select a function Ξ_1 such that the inequality (31) is satisfied with Ξ_1 instead of Ξ. Since $\bar{d}_{21v}(\eta)$ is a smooth function that vanishes at $\eta=0$, it can be written as

$$\bar{d}_{21v}(\eta)=\eta'\bar{D}_{21v}(\eta)$$

where $\bar{D}_{21v}$ is a smooth matrix-valued function of appropriate dimension. Let $\lambda_M=\lambda_{\max}(\bar{D}_{21v}(0)\bar{D}_{21v}'(0))$, where $\lambda_{\max}(M)$ denotes the maximum eigenvalue of a symmetric matrix M. Note that the following function is smooth:

$$\chi_1(\eta)=\frac{\lambda q_1(\eta)}{1+\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta)}$$

Then, by the smoothness of $\bar{D}_{21v}$, there exists a constant $c_5 \in (0, c_2]$ such that

$$\frac{\lambda c_1 \eta' \eta}{(0.5 + \lambda_M)\eta'\eta} - \chi_1(\eta) \geq \frac{\lambda c_1}{1 + \lambda_M} \quad \forall 0 < |\eta| \leq c_5$$

and

$$\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) = \eta' \bar{D}_{21v}(\eta)\bar{D}_{21v}'(\eta)\eta \leq (0.5 + \lambda_M)\eta'\eta \qquad \forall |\eta| \leq c_5$$

Define the following function

$$\chi_2(\eta) = \begin{cases} \exp\left(\dfrac{\eta'\eta}{\eta'\eta - c_5^2}\right) & |\eta| < c_5 \\ 0 & |\eta| \geq c_5 \end{cases}$$

which is clearly smooth. Then, Ξ_1 can be chosen as

$$\Xi_1(\eta) = \frac{\lambda c_1}{1 + \lambda_M}\chi_2(\eta) + \chi_1(\eta) \tag{32}$$

Then, Ξ_1 is smooth and

$$\begin{aligned} \Xi_1(\eta)\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) &\leq \left(\frac{\lambda c_1}{1 + \lambda_M} + \chi_1(\eta)\right) \bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) \\ &\leq \lambda c_1 \eta'\eta \frac{\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta)}{(0.5 + \lambda_M)\eta'\eta} \leq \lambda q_1(\eta) \qquad \forall 0 < |\eta| < c_5 \end{aligned}$$

and

$$\Xi_1(\eta)\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) \leq \chi_1(\eta)\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) \leq \lambda q_1(\eta) \qquad \forall |\eta| \geq c_5$$

Furthermore, $\Xi_1(\eta) > 0$, $\forall \eta \in \mathbb{R}^{n_1}$. Therefore, we have

$$\Xi_1(\eta)\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) \leq \lambda q_1(\eta) \qquad \forall \eta \in \mathbb{R}^{n_1}$$

For the second step, we will select the function Ξ based on Ξ_1 derived above. Let $C > 0$ be a constant such that

$$C\left(\bar{d}_{21c}\bar{d}_{21c}' + \sqrt{\bar{d}_{21c}\bar{d}_{21c}'}\right) \leq \frac{R_2}{2}$$

Given the above choice of the function Ξ_1, we can then select the function Ξ as

$$\Xi(\eta) = \frac{C\Xi_1(\eta)}{\sqrt{\Xi_1^2(\eta) + C^2\left(1 + \sqrt{\bar{d}_{21c}\bar{d}_{21c}'}\right)^2}} \tag{33}$$

It is clear that Ξ is smooth, and it satisfies the inequalities (30) and (31) since

$$\begin{aligned} &\Xi(\eta) < \frac{\Xi_1(\eta)}{1 + \sqrt{\bar{d}_{21c}\bar{d}_{21c}'}} \quad \Xi(\eta) < C \quad \Xi(\eta) > 0 \qquad \forall \eta \in \mathbb{R}^{n_1} \\ &\bar{d}_{21}(\eta)\bar{d}_{21}'(\eta) = \bar{d}_{21c}\bar{d}_{21c}' + 2\bar{d}_{21c}\bar{d}_{21v}'(\eta) + \bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) \\ &\qquad \leq \bar{d}_{21c}\bar{d}_{21c}' + \frac{\bar{d}_{21c}\bar{d}_{21c}'}{\sqrt{\bar{d}_{21c}\bar{d}_{21c}'}} + \bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta)\sqrt{\bar{d}_{21c}\bar{d}_{21c}'} + \bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta) \\ &\Xi(\eta)\bar{d}_{21}(\eta)\bar{d}_{21}'(\eta) \leq \Xi(\eta)\left(\bar{d}_{21c}\bar{d}_{21c}' + \sqrt{\bar{d}_{21c}\bar{d}_{21c}'}\right) \\ &\qquad + \Xi(\eta)\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta)\left(1 + \sqrt{\bar{d}_{21c}\bar{d}_{21c}'}\right) < \frac{R_2}{2} + \lambda q_1(\eta) \qquad \forall \eta \in \mathbb{R}^{n_1} \end{aligned}$$

After fixing the choice of the function Ξ, we next choose the positive function β to be

$$\bar{\beta}(\eta, z) = \frac{2\Xi^2(0)\bar{\alpha}_{2b}^2(0,0)}{R_2} + \bar{\beta}_a(\eta, z) \tag{34}$$

$$\bar{\beta}_a(\eta, z) \geq c_6\Xi(\eta) + c_7 \tag{35}$$

where $c_6 > 0$ and $c_7 > 0$ are positive constants. Then, we have

$$2z\Xi(0)\bar{\alpha}_{2b}(0,0) \leq \frac{R_2}{2} + \bar{\beta}(\eta, z)z^2 - \bar{\beta}_a(\eta, z)z^2 \tag{36}$$

As a result of these steps,

$$\begin{aligned} d\bar{V}_2(\eta(t), z(t)) &= -y^2\,dt - q_1\,dt + \bar{\sigma}_2\,dw - \frac{\theta}{4}\bar{\sigma}_2\bar{\sigma}_2'\,dt + \Delta_1\,dt - \bar{\beta}z^2\,dt \\ &\quad + \Xi\bar{d}_{21}\bar{d}_{21}'\,dt + 2\Xi(0)\bar{\alpha}_{2b}(0,0)z\,dt \\ &=: -y^2\,dt - (1-\lambda)q_1\,dt - q_2\,dt + \bar{\sigma}_2\,dw - \frac{\theta}{4}\bar{\sigma}_2\bar{\sigma}_2'\,dt + \Delta_1\,dt + R_2\,dt \end{aligned}$$

where q_2 is defined by

$$\begin{aligned} \bar{q}_2(\eta, z) &:= \lambda q_1(\eta) + \bar{\beta}(\eta, z)z^2 + R_2 - \Xi(\eta)\bar{d}_{21}(\eta)\bar{d}_{21}'(\eta) - 2\Xi(0)\bar{\alpha}_{2b}(0,0)z \\ &\geq \bar{\beta}_a(\eta, z)z^2 \end{aligned}$$

By Proposition 6, we conclude that the function V_2 satisfies the HJB inequality (27).

REMARK 2. *Consider the special case $H_1(0) = 0$ and $H_2(0,0) = 0$. In this case, $\bar{d}_{21c} = 0$. Therefore, any choice of the function Ξ that satisfies the inequality (31) satisfies the inequality (30) with $R_2 = 0$. Fix such a choice of Ξ. In this special case, we observe that $\bar{\alpha}_{2b}(0,0) = 0$. This implies that the following choice for the function $\bar{\beta}$:*

$$\bar{\beta}(\eta, z) = \bar{\beta}_a(\eta, z) \tag{37}$$

leads to satisfaction of the inequality (36) with $R_2 = 0$. Hence, in this special case, the above choices of functions Ξ and $\bar{\beta}$ lead to a controller design with $R_2 = 0$.

We next check the satisfaction of the control law α_2 and value function V_2 with respect to the four induction hypotheses made at the beginning of this section.

1. The control law (29) is smooth and meets the equilibrium condition $\alpha_2(0,0) = 0$, by construction.
2. The function V_2 is smooth and satisfies the HJB inequality (27).
3. The following inequality holds

$$(1-\lambda)q_1(\eta) + q_2(\eta, \xi) \geq (1-\lambda)c_1\eta'\eta + c_7z^2 \geq c_1' \left| \left[\begin{array}{c} \eta \\ \xi \end{array} \right] \right|^2 \qquad \forall \left| \left[\begin{array}{c} \eta \\ \xi \end{array} \right] \right| \leq c_2' \tag{38}$$

for some constants $c_1' > 0$ and $c_2' > 0$, and the left hand side is a smooth function. We also have the following inequality:

$$\bar{V}_2(\eta, z) \leq c_3q_1(\eta) + c_4 + \bar{\beta}_a(\eta, z)z^2 \leq c_3'((1-\lambda)q_1(\eta) + q_2(\eta, z)) + c_4 \tag{39}$$

holds $\forall(\eta, z) \in \mathbb{R}^{n_1+1}$, for some $c_3' > 0$, because of (35).

4. In the special case of $H_1(0) = 0$ and $H_2(0,0) = 0$, by the induction hypothesis, there exist a control law α_1 and a corresponding value function V_1 to guarantee a desired long term average cost $R_1 = 0$ for the η dynamics. By Remark 2, we have found the design to yield $R_2 = 0$. Hence the pair α_2 and V_2 are smooth and guarantee the zero long term average cost (i.e., $R_1 + R_2 = 0$) for the two-level problem (20). Furthermore, the guaranteed incremental cost function for the two-level problem (20) is the smooth function $(1-\lambda)q_1 + q_2$.

Hence, the control law α_2 and value function V_2 meet all four induction hypotheses.

This completes the guaranteed cost design for this two-level problem. We summarize these findings in the following Lemma.

LEMMA 7. *Consider the stochastic differential equation (20), and assume the following:*

(i) *Given an arbitrary $R_1 > 0$, there exists a smooth positive definite and radially unbounded value function $V_1(\eta)$ and a smooth fictitious control law $\alpha_1(\eta)$ (for ξ), such that V_1 satisfies the HJB inequality (21), as well as the bounds (22) and (23) with a smooth incremental cost function $q_1(\eta)$, and $\alpha_1(0) = 0$;*

(ii) *When $H_1(0) = 0$, the guaranteed long term average cost level R_1 can be picked as 0.*

Then, the following hold:

1. *There exist a smooth value function $V_2(\eta,\xi)$ and a smooth control law $\alpha_2(\eta,\xi)$ as given by (25) and (29), respectively, where the former satisfies the HJB inequality (27) with the additional positive constant R_2;*
2. *The weighting function $(1-\lambda)q_1(\eta) + q_2(\eta,\xi)$ satisfies the bounds (38) and (39), and is smooth;*
3. *The control law $v = \alpha_2(\eta,\xi)$ has the property $\alpha_2(0,0) = 0$, and under it the closed-loop system trajectory is bounded in probability;*
4. *If $H_1(0) = 0$ and $H_2(0,0) = 0$, the constant $R_1 + R_2$ can be chosen as zero, which further implies that the closed-loop system is asymptotically stable in the large.*

4. Controller Design for the Full-order System. By using the backstepping Lemma 7 derived in the previous section, we can design a feedback controller that leads to a guaranteed risk-sensitive cost R_c, for an arbitrary positive R_c and risk-sensitivity parameter θ. The design follows n steps of integrator backstepping. As suggested by Lemma 7, at each step of the backstepping one has to accommodate some positive quadratic variation intrinsic to that particular step. Accordingly, we first split the desired risk-sensitive cost R_c into n pieces, each of which is positive. When $h_1(0) = \cdots = h_n(0,\ldots,0) = 0$, the control design can actually guarantee zero long term average, i.e., $R_c = 0$. Let

$$R_c = R_1 + \cdots + R_n$$

where $R_i > 0$, $i = 1,\ldots,n$. The controller design follows n steps of integrator backstepping, as elucidated below:

Step 1: Here, we are dealing with a scalar system:

$$dx_1 = (f_1(x_1) + x_2)\,dt + h_1'(x_1)\,dw$$

The controller design is to guarantee a risk-sensitive cost R_1.

For this purpose, we select the value function V_1 as:

$$V_1(x_1) = \frac{1}{1 + 2h_{1c}'h_{1c}/R_1}x_1^2 =: \Xi_1 x_1^2$$

where h_{1c} is the constant part of the function h_1:

$$h_1(x_1) = h_{1c} + h_{1v}(x_1)x_1$$

Then, using Itô's rule for $V_1(x_1(t))$, we obtain the following relationship:

$$dV_1(x_1(t)) = 2\Xi_1 x_1(f_1(x_1) + x_2)\,dt + 2\Xi_1 x_1 h_1'(x_1)\,dw + \mathrm{Tr}(\Xi_1 h_1'(x_1)h_1(x_1))\,dt$$

Choose the virtual control input to cancel out all the terms that involve a multiplier of x_1:

$$\begin{aligned}\alpha_{1b}(x_1) = {} & -\frac{1}{2\Xi_1}x_1 - \frac{1}{2\Xi_1}\beta x_1 - f_1(x_1) - \frac{\theta\Xi_1}{2}h_1'(x_1)h_1(x_1)x_1 \\ & -\frac{1}{2}h_{1v}'(x_1)h_{1v}(x_1)x_1 - h_{1c}'h_{1v}(x_1)\end{aligned}$$

where β is a constant to be chosen later.

To preserve the equilibrium at the origin, we define the control law as follows:

$$x_2 = \alpha_{1b}(x_1) - \alpha_{1b}(0) =: \alpha_1(x_1) \tag{40}$$

This choice of the virtual control input leads to

$$dV_1(x_1(t)) = 2\Xi_1 x_1 h_1'\,dw - (x_1^2 + \beta x_1^2 + \theta\Xi_1^2 h_1' h_1 x_1^2 + \Xi_1 h_{1c}' h_{1c} + 2\Xi_1\alpha_{1b}(0)x_1)\,dt$$

Choose

$$\beta = \frac{4\Xi_1^2\alpha_{1b}^2(0)}{R_1} + c_2$$

where $c_2 > 0$. Then, V_1 satisfies the HJB inequality (11) for the x_1 subsystem under α_1 with guaranteed cost R_1:

$$\frac{\partial V_1}{\partial x_1}(f_1 + \alpha_1) + \frac{\theta}{4}\frac{\partial V_1}{\partial x_1}h_1'h_1\left(\frac{\partial V_1}{\partial x_1}\right)' + x_1^2 + \frac{1}{2}\beta x_1^2 + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V_1}{\partial x_1^2}h_1 h_1'\right) \le R_1$$

It is easy to check to see that all four induction hypotheses are satisfied. This completes the first step of the integrator backstepping.

For steps 2, 3, *etc*, we can repeatedly apply the backstepping Lemma 7. At a typical step i, we can design the risk-sensitive controller with the increase in the long term average cost bounded by R_i. We then continue this process until the second to last step $n-1$, where we design the virtual control input

$$x_n = \alpha_{n-1}(x_1, \ldots, x_{n-1})$$

At step n, we let $v = bu$, and use the construction of Lemma 7 to lead to a smooth control law:

$$v = \alpha_n(x_1, \ldots, x_n)$$

Under the working assumption **A1**, we finally obtain

$$u = \frac{1}{b(x_1, \ldots, x_n)}\,\alpha_n(x_1, \ldots, x_n) \tag{41}$$

This completes the construction of a risk-sensitive controller, guaranteeing any desired level of positive long-term average cost. This construction is now made precise in the following theorem.

THEOREM 8. *Consider the strict-feedback stochastic nonlinear system described by the SDE (1), satisfying Assumption* **A1**. *For any risk-sensitivity parameter $\theta > 0$ and any desired long-term average cost level $R_c > 0$, there exists a smooth nonlinear feedback control (41) that achieves the level R_c. Such a controller can be constructed using the integrator backstepping procedure and by a repeated application of Lemma 7. The resulting closed-loop system trajectory is bounded in probability.*

If the functions h_i, $i = 1, \ldots, n$, vanish at $x = 0$, then, the controller (41) achieves a zero long-term average cost for the SDE (1). In this case, the closed-loop system is asymptotically stable in the large, in addition to being bounded in probability.

Proof. The result is an immediate application of Theorem 5, in view of the backstepping construction outlined above. Since the existence (and construction) of function V satisfying the HJB inequality is implied by the construction presented before the statement of the theorem, the boundedness in probability follows immediately.

When h_i vanishes at $x = 0$, the construction implies that the constant R_c can be chosen to be zero, by Remark 2. Since the weighting function is chosen to be positive definite, by Theorem 5 the system is asymptotically stable in the large, in addition to being bounded in probability. □

5. Two Special Cases. In this section, we discuss some implications of the results presented above as the design parameters approach specific limits. In particular, we are interested in two types of limiting processes. The first is known as the risk-neutral limit, with $\theta \downarrow 0$. The other one is known as the large deviation limit, where the noise intensity decreases to zero.

5.1. Risk-neutral case. If $\theta \downarrow 0$, the limiting problem is the one with a risk-neutral cost function:

$$J_0(\mu) = \limsup_{t_f \to \infty} \frac{1}{t_f} \boldsymbol{E} \left\{ \int_0^{t_f} (y^2(t) + l(x(t)))\, dt \right\} \leq R_c \tag{42}$$

where R_c is again the desired long term average cost bound.

The control design procedure presented in the previous two sections equally applies here, leading to a controller

$$u = \mu(x)$$

and a value function $V(x)$ such that

$$\frac{\partial V}{\partial x} f^u(x) + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial^2 V}{\partial x^2} H(x) H'(x) \right) + y^2 + l(x) = \Delta(x) \leq R_c \tag{43}$$

where

$$f^u(x) = \begin{bmatrix} f_1(x_1) + x_2 \\ \vdots \\ f_{n-1}(x_1, \ldots, x_{n-1}) + x_n \\ f_n(x_1, \ldots, x_n) + b(x_1, \ldots, x_n)\mu(x) \end{bmatrix} \quad H(x) = \begin{bmatrix} h_1'(x_1) \\ \vdots \\ h_n'(x_1, \ldots, x_n) \end{bmatrix}$$

This inequality immediately implies that the long term average cost for the risk-neutral cost function (42) is bounded by R_c.

Therefore, we have the following corollary to Theorem 8.

COROLLARY 9. *Consider the strict-feedback stochastic nonlinear system (1) satisfying Assumption* **A1**, *along with the risk-neutral cost function (42). For any desired long term average cost level* $R_c > 0$, *there exists a corresponding smooth nonlinear feedback control that guarantees it. Such a controller can be constructed using the integrator backstepping procedure presented in the previous two sections with the risk-sensitivity parameter set at* $\theta = 0$. *The resulting closed-loop system trajectories are bounded in probability.*

If the functions h_i, $i = 1, \ldots, n$ *vanish at* $x = 0$, *then the controller* μ *can be chosen to guarantee* $R_c = 0$. *In this case, the closed-loop system is asymptotically stable in the large, in addition to being bounded in probability.*

5.2. Large deviation limit. Another limiting scenario is when the noise intensity in the system dynamics asymptotically vanishes at the rate of ϵ, for some small positive parameter ϵ, while the cost function heavily penalizes any deviation of the output from zero at the rate of $1/\epsilon$. Letting $\theta = \dfrac{1}{\gamma^2\epsilon^2}$, for some positive design parameter γ (whose role and significance will become clear shortly), we now have the following problem formulation:

$$dx_1 = (x_2 + f_1(x_1))\, dt + \epsilon \tilde{h}_1'(x_1)\, dw \tag{44a}$$

$$\vdots \qquad \vdots$$

$$dx_{n-1} = (x_n + f_{n-1}(x_1, \ldots, x_{n-1}))\, dt + \epsilon \tilde{h}_{n-1}'(x_1, \ldots, x_{n-1})\, dw \tag{44b}$$

$$dx_n = (f_n(x_1, \ldots, x_n) + b(x_1, \ldots, x_n)u)\, dt + \epsilon \tilde{h}_n'(x_1, \ldots, x_n)\, dw \tag{44c}$$

$$y = x_1 \tag{44d}$$

$$J_\theta(\mu) = \limsup_{t_f \to \infty} \frac{2}{\theta t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (y^2 + l(x(t)))\, dt \right) \right] \right\} \right\} \le R_c \tag{45}$$

It is a well-known fact in the risk-sensitive optimal control literature [1] and [25] that the large deviation limit, as $\epsilon \to 0$, of the optimal risk-sensitive controller for this problem leads exactly to the H^∞ central controller with disturbance attenuation parameter γ, provided that the solution to either one exists. For the problem at hand, the corresponding H^∞ control problem involves the state dynamics:

$$\dot{x}_1 = x_2 + f_1(x_1) + \tilde{h}_1'(x_1)w \tag{46a}$$

$$\vdots \qquad \vdots$$

$$\dot{x}_{n-1} = x_n + f_{n-1}(x_1, \ldots, x_{n-1}) + \tilde{h}_{n-1}'(x_1, \ldots, x_{n-1})w \tag{46b}$$

$$\dot{x}_n = f_n(x_1, \ldots, x_n) + b(x_1, \ldots, x_n)u + \tilde{h}_n'(x_1, \ldots, x_n)w \tag{46c}$$

$$y = x_1 \tag{46d}$$

and the following worst-case (game) cost function:

$$J_\gamma(\mu, \nu) = \int_0^\infty (y^2(t) + l(x(t)) - \gamma^2 |w(t)|^2)\, dt \tag{47}$$

where w is generated by some adversary player through $w(t) = \nu(t, x(t))$.

This is precisely the problem that was addressed in [23]. The interesting question to ask here is whether the backstepping construction of risk-sensitive controllers for (44), (45), also converges to the backstepping construction for the H^∞ controller for (46), (47) presented in [23]. Since, both design procedures presented here and in [23] are suboptimal schemes, the convergence cannot be guaranteed by the well-established large deviation limit for risk-sensitive optimal control problems.

It turns out that the answer to this question is still in the affirmative. We examine again the single stage backstepping design presented in Section 3. The difference between the recursive construction presented in [23] and that of this paper is as follows. First of all, the choice of Ξ is solely determined by the inequalities (30) and (31). For any fixed compact set K, Ξ can be chosen to be constant, say $1/2$, on this compact set for sufficiently small $\epsilon > 0$, such that $\forall \eta \in K$

$$\operatorname{Tr}\left(\Xi(\eta)\bar{d}_{21}(\eta)\bar{d}_{21}'(\eta)\right) \leq \frac{1}{\epsilon^2}\left(\frac{R_2}{2} + \lambda q_1(\eta)\right)$$

and

$$\operatorname{Tr}\left(\Xi(\eta)\bar{d}_{21v}(\eta)\bar{d}_{21v}'(\eta)\right) \leq \frac{\lambda c_3 \eta' \eta}{\epsilon^2}$$

This compact set expands to the entire state space as $\epsilon \downarrow 0$. On this compact set, the added value function in this case is exactly the same as that for the H^∞ problem. Consider the controller (28); all the terms on the first three lines are exactly the same as in the controller of the H^∞ problem. The function $\bar{a}_2$ is within $O(\epsilon^2)$ of its counterpart in the H^∞ problem. The terms on the last line are of $O(\epsilon^2)$. Therefore, on this compact set, the risk-sensitive design provides a $O(\epsilon^2)$ approximation to the H^∞ solution.

Hence, we can state the following remark to capture the observation above.

REMARK 3. *Consider the strict-feedback stochastic nonlinear system (44) satisfying Assumption* **A1** *with the risk-sensitive cost function (45). For any desired long-term average cost level $R_c > 0$, there exists a corresponding nonlinear feedback control that guarantees it. Such a controller approximates the backstepping controller for the nonlinear H^∞ control problem [23] up to $O(\epsilon^2)$ on any given compact set $K \subset \mathbb{R}^n$.*

6. An Extension. We identify and present here an immediate (natural) extension of the results presented heretofore to a more general class of nonlinear systems, with nontrivial zero dynamics. Consider the class of nonlinear systems described by the following SDE:

$$\begin{aligned}
&(48a) & d\phi &= f_0(\phi, x_1)\,dt + H_0(\phi, x_1)\,dw \\
&(48b) & dx_1 &= (b_1(\phi, x_1)x_2 + f_1(\phi, x_1))\,dt + h_1'(\phi, x_1)\,dw \\
& & &\vdots \qquad \vdots \\
&(48c) & dx_{n-1} &= (b_{n-1}(\phi, x_1, \ldots, x_{n-1})x_n + f_{n-1}(\phi, x_1, \ldots, x_{n-1}))\,dt \\
& & &\quad + h_{n-1}'(\phi, x_1, \ldots, x_{n-1})\,dw \\
&(48d) & dx_n &= (f_n(\phi, x_1, \ldots, x_n) + b(\phi, x_1, \ldots, x_n)u)\,dt + h_n'(\phi, x_1, \ldots, x_n)\,dw \\
&(48e) & y &= p(\phi, x_1)\,.
\end{aligned}$$

where ϕ is an n_0-dimensional state vector, which can be viewed as the state of the zero dynamics of the system (see Chapter 9 of [10]). We note that these ϕ dynamics are affected by only x_1 and not the remaining components of the state x. Furthermore, the virtual control input variables $x_2, \ldots, x_n$ are premultiplied by nonlinear functions $b_1, \ldots, b_{n-1}$. For this class of nonlinear stochastic systems, we introduce the following two assumptions in addition to **A1**.

Assumption A2: The nonlinear functions f_0, H_0, b_i, and $1/b_i$ are $\mathcal{C}^\infty$ in all their arguments, $i = 1, \ldots, n-1$. The output function p is smooth. $\diamond$

Assumption A3: Given a risk-sensitivity parameter $\theta_0 > 0$, there exists a smooth virtual control law $\alpha_0(\phi)$, with $\alpha_0(0) = 0$, and a positive definite and radially unbounded smooth value function $V_0(\phi)$, such that the following PDE is satisfied for some positive definite function $q_0(\phi)$:

$$\begin{aligned}(49)\quad &\frac{\partial V_0}{\partial \phi} f_0(\phi, \alpha_0(\phi)) + \frac{\theta_0}{4}\frac{\partial V_0}{\partial \phi} H_0(\phi, \alpha_0(\phi)) H_0'(\phi, \alpha_0(\phi)) \left(\frac{\partial V_0}{\partial \phi}\right)' \\ &+ \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V_0}{\partial \phi^2} H_0(\phi, \alpha_0(\phi)) H_0'(\phi, \alpha_0(\phi))\right) + p^2(\phi, \alpha_0(\phi)) + q_0(\phi) = \Delta_0(\phi) \le R_0\end{aligned}$$

Furthermore, $q_0(\phi)$ is smooth and satisfies the assumptions of induction for the backstepping procedure, and the following growth conditions,

$$q_0(\phi) \ge c_1 \phi'\phi \qquad \forall |\phi| \le c_2 \tag{50}$$

and

$$V_0(\phi) \le c_3\, q_0(\phi) + c_4 \qquad \forall \phi \in \mathbb{R}^{n_0} \tag{51}$$

for some constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, and $c_4 > 0$. $\diamond$

We should note that **A2** is quite a standard assumption for this class of nonlinear systems, which guarantees that the state variables $x_1, \ldots, x_n$ are completely controllable; **A3**, on the other hand, is a strong stabilizability condition on the ϕ dynamics, which assures that there is a virtual control law for x_1 that guarantees a long term average value of R_0 for the risk-sensitive cost function

$$\limsup_{t_f \to \infty} \frac{2}{\theta_0 t_f} \ln\left\{ \boldsymbol{E}\left\{ \exp\left[\frac{\theta_0}{2}\left(\int_0^{t_f} y^2(t)\, dt\right)\right]\right\}\right\}$$

for a given value of θ_0. This is stated as an assumption, because for a given $\theta_0 > 0$ there may not exist a control law (for x_1) to return an arbitrarily small value for the risk-sensitive cost function above. This is partly due to the appearance of both ϕ and x_1 as arguments of p.

Under these assumptions, it is possible to construct recursively a nonlinear feedback controller that achieves any given long term average cost level $R_c > R_0$ for a given risk-sensitivity parameter $\theta \le \theta_0$. We note that, as stated earlier, we may no longer be able to achieve an arbitrarily small long term average cost for arbitrarily large values of the risk-sensitivity parameter. This performance limitation is solely due to the presence of the zero dynamics of the system.

This now brings us to the following theorem.

THEOREM 10. *Consider the class of stochastic nonlinear systems described by SDE (48), satisfying Assumptions* **A1**, **A2**, *and* **A3**. *For any risk-sensitivity parameter $\theta \le \theta_0$ and any desired long term average cost level $R_c > R_0$, there exists a* *nonlinear feedback control law μ, designed using the integrator backstepping procedure that achieves the level R_c. The resulting closed-loop system trajectory is bounded in probability.*

If $R_0 = 0$ is a feasible choice, and the nonlinear functions H_0, h_i, $i = 1, \dots, n$ all vanish at the origin of the system, then the nonlinear controller constructed with $R_0 = 0$ achieves a zero long term average cost for the SDE (48). In this case, the closed-loop system is asymptotically stable in the large, in addition to being bounded in probability.

Proof. We need only establish the result when $n = 1$. For the case when $n > 1$, the result follows directly from the recursive design procedure described in Section 3.

Consider the SDE:

$$\begin{aligned} d\phi &= f_0(\phi, x_1)\,dt + H_0(\phi, x_1)\,dw \\ dx_1 &= (b_1(\phi, x_1)u + f_1(\phi, x_1))\,dt + h_1'(\phi, x_1)\,dw \\ y &= p(\phi, x_1)\,. \end{aligned}$$

We will introduce the variable

$$v = b_1(\phi, x_1)u$$

and design a control for v, which, in turn, yields a control law for u, under the working Assumption **A2**.

By Assumption **A3**, the function V_0 satisfies the following HJB inequality for any $\theta \le \theta_0$:

$$\begin{aligned} (53)\quad & \frac{\partial V_0}{\partial \phi} f_0(\phi, \alpha_0(\phi)) + \frac{\theta}{4}\frac{\partial V_0}{\partial \phi} H_0(\phi, \alpha_0(\phi)) H_0(\phi, \alpha_0(\phi))' \left(\frac{\partial V_0}{\partial \phi}\right)' \\ & + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V_0}{\partial \phi^2} H_0(\phi, \alpha_0(\phi)) H_0(\phi, \alpha_0(\phi))'\right) + p^2(\phi, \alpha_0(\phi)) + q_0(\phi) = \Delta_0(\phi) \le R_0 \end{aligned}$$

Given this, we introduce the transformed state variable

$$z = x_1 - \alpha_0(\phi)$$

Then, by Itô's rule of differentiation, we have

$$dz = (\bar{a}_1 + v)\,dt + \bar{d}_1\,dw$$

where

$$\begin{aligned} \bar{a}_1(\phi, z) &= \bar{f}_1(\phi, z) - \frac{\partial \alpha_0}{\partial \phi}\bar{f}_0(\phi, z) - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 \alpha_0}{\partial \phi^2}\bar{H}_0(\phi, z)\bar{H}_0'(\phi, z)\right) \\ \bar{d}_1(\phi, z) &= \bar{h}_1'(\phi, z) - \frac{\partial \alpha_0}{\partial \phi}\bar{H}_0(\phi, z) \end{aligned}$$

Introduce a smooth function $\bar{V}_1$ as a candidate value function

$$\bar{V}_1(\phi, z) = V_0(\phi) + \Xi(\phi) z^2 \qquad (54)$$

where Ξ is a positive smooth function yet to be determined. We again set up the proper conditions to prove that $\bar{V}_1$ satisfies the desired HJB inequality by using the equivalent form of Proposition 6.

Define the smooth functions $\bar{f}_{01}$, $\bar{H}_{01}$, $\bar{d}_{11}$, and $\bar{d}_{12}$ by

$$f_0(\phi, x_1) = f_0(\phi, \alpha_0(\phi)) + \bar{f}_{01}(\phi, z)z \qquad H_0(\phi, x_1) = H_0(\phi, \alpha_0(\phi)) + \bar{H}_{01}(\phi, z)z$$
$$\bar{d}_{11}(\phi) = \bar{d}_1(\phi, 0) \qquad \bar{d}_1(\phi, z) = \bar{d}_1(\phi, 0) + \bar{d}_{12}(\phi, z)z$$

The Itô differential of $\bar{V}_1(\phi(t), z(t))$ is given by

$$\begin{aligned} d\bar{V}_1(\phi(t), z(t)) = &\left(\frac{\partial V_0}{\partial \phi} f_0(\phi, \alpha_0(\phi)) + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V_0}{\partial \phi^2} H_0(\phi, \alpha_0(\phi)) H_0'(\phi, \alpha_0(\phi))\right)\right) dt \\ &+ z(\bar{m}(\phi, z) + 2\Xi(\phi)v)\, dt + \bar{\sigma}(\phi, z)\, dw + \mathrm{Tr}(\Xi(\phi)\bar{d}_{11}(\phi)\bar{d}_{11}'(\phi))\, dt \end{aligned}$$

where

$$\begin{aligned} \bar{m}(\phi, z) &= \frac{\partial V_0}{\partial \phi}\bar{f}_{01} + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V_0}{\partial \phi^2}(2H_0\bar{H}_{01}' + z\bar{H}_{01}\bar{H}_{01}')\right) + z\frac{\partial \Xi}{\partial \phi}\bar{f}_0 + 2\Xi\bar{a}_1 \\ &\quad + \frac{1}{2}\mathrm{Tr}\left(z\frac{\partial^2 \Xi}{\partial \phi^2}\bar{H}_0\bar{H}_0' + 4\frac{\partial \Xi}{\partial \phi}\bar{H}_0\bar{d}_1' + 4\Xi\bar{d}_{11}\bar{d}_{12}' + 2\Xi\bar{d}_{12}\bar{d}_{12}'z\right) \\ \bar{\sigma}(\phi, z) &= \frac{\partial V_0}{\partial \phi}H_0 + z\bar{\sigma}_{11} \\ \bar{\sigma}_{11}(\phi, z) &= \frac{\partial V_0}{\partial \phi}\bar{H}_{01} + z\frac{\partial \Xi}{\partial \phi}\bar{H}_0 + 2\Xi\bar{d}_1 \end{aligned}$$

Using the HJB inequality (53), we arrive at the following stochastic differential inequality:

$$\begin{aligned} d\bar{V}_1(\phi(t), z(t)) = &\left(-p^2 - q_0 + \Delta_0 - \frac{\theta}{4}\bar{\sigma}\bar{\sigma}'\right) dt + \bar{\sigma}\, dw \\ &+ z\left(\bar{m} + \frac{\theta}{4}\left(2\frac{\partial V_0}{\partial \phi}H_0\bar{\sigma}_{11}' + z\bar{\sigma}_{11}\bar{\sigma}_{11}'\right) + 2\Xi v\right) dt + \mathrm{Tr}(\Xi\bar{d}_{11}\bar{d}_{11}')\, dt \end{aligned}$$

We next define a smooth nonlinear function $\bar{p}_1$ by

$$p(\phi, x_1) = p(\phi, \alpha_0(\phi)) + \bar{p}_1(\phi, z)z\,,$$

and introduce the control law

$$v = \bar{\alpha}_1(\phi, z) = \bar{\alpha}_{1b}(\phi, z) - \frac{\Xi(0)}{\Xi(\phi)}\bar{\alpha}_{1b}(0, 0) \tag{55}$$

where

$$\bar{\alpha}_{1b}(\phi, z) = \frac{1}{2\Xi}\left(-2p\bar{p}_1 - \bar{p}_1^2 z - \bar{\beta}z - \bar{m} - \frac{\theta}{4}\left(2\frac{\partial V_0}{\partial \phi}H_0\bar{\sigma}_{11}' + z\bar{\sigma}_{11}\bar{\sigma}_{11}'\right)\right)$$

and $\bar{\beta}(\phi, z)$ is a positive nonlinear function to be determined shortly.

Under this control law, the Itô differential of $\bar{V}_1(\phi(t), z(t))$ satisfies

$$\begin{aligned} d\bar{V}_1(\phi(t), z(t)) = &\left(-y^2 - q_0 - \bar{\beta}z^2 + \Delta_0 - \frac{\theta}{4}\bar{\sigma}\bar{\sigma}'\right) dt + \bar{\sigma}\, dw + 2z\Xi(0)\bar{\alpha}_{1b}(0, 0)\, dt \\ &+ \mathrm{Tr}(\Xi\bar{d}_{11}\bar{d}_{11}')\, dt \end{aligned}$$

Now, following a selection process similar to that of Section 3 for the functions Ξ and $\bar{\beta}$, we can show that

$$d\bar{V}_1(\phi(t), z(t)) \leq \left(-y^2 - q_1 + R_1 - \frac{\theta}{4}\bar{\sigma}\bar{\sigma}'\right) dt + \bar{\sigma}\, dw$$

for any $R_1 > R_0$ and the nonlinear weighting function q_1 is smooth and satisfies the induction assumptions of Section 3. This then completes the construction for this first step of backstepping procedure.

When $n > 1$, we can repeatedly apply Lemma 7 to construct the nonlinear cost-bounding controller for the overall system.

This completes the proof of the theorem. □

7. An Example. In this section, we illustrate the design procedure presented in the main body of the paper by a second order example, and compare the resulting control design with that of [23] and the risk neutral case as $\theta \downarrow 0$. The second order system is taken as:

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\, dt + \begin{bmatrix} 0 & 1 \\ 1 & x_1 \end{bmatrix} dw \tag{56a}$$

$$y = x_1 \tag{56b}$$

where u is the control input and w is a two-dimensional standard Wiener process. For this SDE, we will consider two types of cost functions, namely, the risk-sensitive cost function and the risk-neutral cost function. We will take the risk-sensitivity parameter $\theta = 1$ and a desired guaranteed long-term average cost of $R_c = 4$. This corresponds to the following choice of design criterion for the risk-sensitive case:

$$J_{rs}(\mu) = \limsup_{t_f \to \infty} \frac{2}{t_f} \ln \left\{ \boldsymbol{E} \left\{ \exp \left[\frac{1}{2} \left(\int_0^{t_f} (y^2(t) + l(x(t)))\, dt \right) \right] \right\} \right\} \leq 4 \tag{57}$$

and the following one for the risk-neutral case:

$$J_{rn}(\mu) = \limsup_{t_f \to \infty} \frac{1}{t_f} \boldsymbol{E} \left\{ \int_0^{t_f} (y^2(t) + l(x(t)))\, dt \right\} \leq 4 \tag{58}$$

The corresponding nonlinear H^∞-control problem of [23] (see also (46) and (47) in Subsection 5.2), on the other hand, is described by thefollowing second-order dynamics:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 1 & x_1 \end{bmatrix} w \tag{59a}$$

$$y = x_1 \tag{59b}$$

where w is to be chosen by some adversary player to disrupt the control objective. The associated (game-theoretic) cost function is

$$J_{rb}(\mu, \nu) = \int_0^\infty (y^2(t) + l(x(t)) - w'(t)w(t))\, dt \tag{60}$$

Here we have picked $\epsilon = 1$ and $\gamma = 1$ — a choice that is consistent with the specific choice $\theta = 1$ for the risk-sensitivity parameter, because of the relationship $\theta = \dfrac{1}{\epsilon^2\gamma^2}$.

Carrying out the risk-sensitive backstepping design procedure, we arrive at the following value function and control law:

$$V_{rs}(x) = 0.5x_1^2 + 0.0391251\left(x_2 + 2.25x_1 + x_1^2\right)^2 \tag{61a}$$

$$\begin{aligned}\mu_{rs}(x) = &-4.93429x_2 - 19.9442x_1 - 7.02849x_1^2 - 2.26409x_1x_2 - 2.66023x_1^3 \\ &-0.176063x_1^2x_2 - 0.176063x_1^4\end{aligned} \tag{61b}$$

An important observation to make here is that the constant multiplying the second term in the value function V_{rs} above is much smaller as compared with that of the first term. This constant is determined by the bounds (30) and (31) for the recursive design.

Now, for the risk-neutral case, the counterparts of (61a) and (61b) are, respectively:

$$V_{rn}(x) = 0.5x_1^2 + 0.0406138\left(x_2 + 2x_1 + x_1^2\right)^2 \tag{62a}$$

$$\mu_{rn}(x) = -4.47237x_2 - 17.2558x_1 - 4.47237x_1^2 - 2x_1x_2 - 2x_1^3 \tag{62b}$$

We observe that the assignment of the constant multipliers in the value function above is quite similar to the risk-sensitive case. The control law, however, is somewhat different; it does not include the high order nonlinear terms that are part of the risk-sensitive design. This result corroborates the earlier statement that the risk neutral control design does not put weight on the higher (than second) order moments in the system output. Apart from the missing high order nonlinear terms, the control design for this risk-neutral case is quite similar to that of the risk-sensitive case.

Finally, we design a control law for the worst-case disturbance attenuation problem (59a), (59b) and (60). The value function and the control law are given in this case as follows:

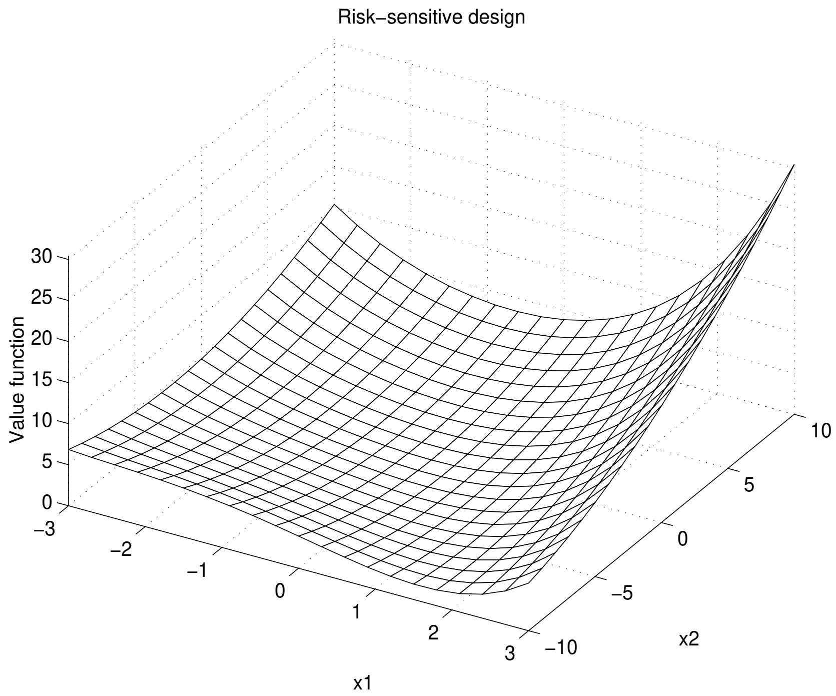
$$V_{rb}(x) = 0.5x_1^2 + 0.5\left(x_2 + 1.75x_1 + x_1^2\right)^2 \tag{63a}$$

$$\begin{aligned}\mu_{rb}(x) = &-3.26563x_2 - 4.52734x_1 - 9.35938x_1^2 - 2.25x_1^2x_2 - 4.62500x_1x_2 \\ &-8.56250x_1^3 - 2.25x_1^4\end{aligned} \tag{63b}$$

We observe that both the value function and the controller are quite different from their counterparts in the risk-sensitive and risk-neutral cases. Here, the gains of the linear control terms are relatively small as compared with their stochastic counterparts, yet the gains on high-order nonlinear terms are significantly larger than those of the stochastic controllers.

The mesh plots of the value functions and the control laws for the three designs are depicted in Figures 1–3. Comparing these figures, we observe that both stochastic designs lead to smaller value functions as opposed to the worst-case design. The stochastic control laws are much softer than the worst-case control laws. In particular, in the region of the state space $x_1 < 0$, the stochastic control laws are much softer than the worst-case (nonlinear H^∞) control law.

Next, we simulated the closed-loop system under these three controllers subject to two different types of disturbance inputs. The simulations were carried out using the Matlab Simulink software package. We chose the disturbance inputs to be sampled Gaussian white noise inputs with sample rate 10Hz. Each of the noise channels had power 1. The simulation results are plotted in Figures 4–6. We observe from these figures that the stochastic control designs lead to smaller system outputs than the



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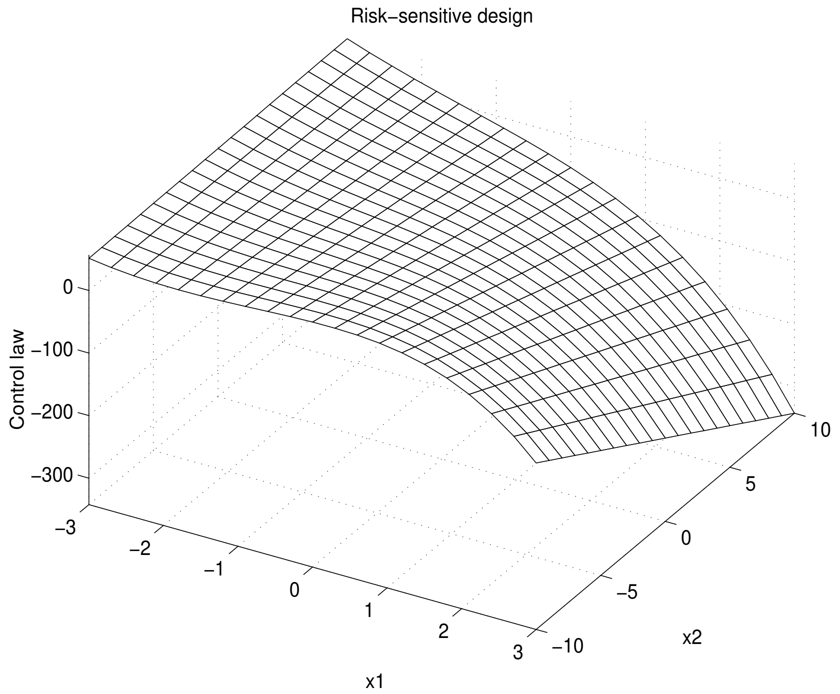

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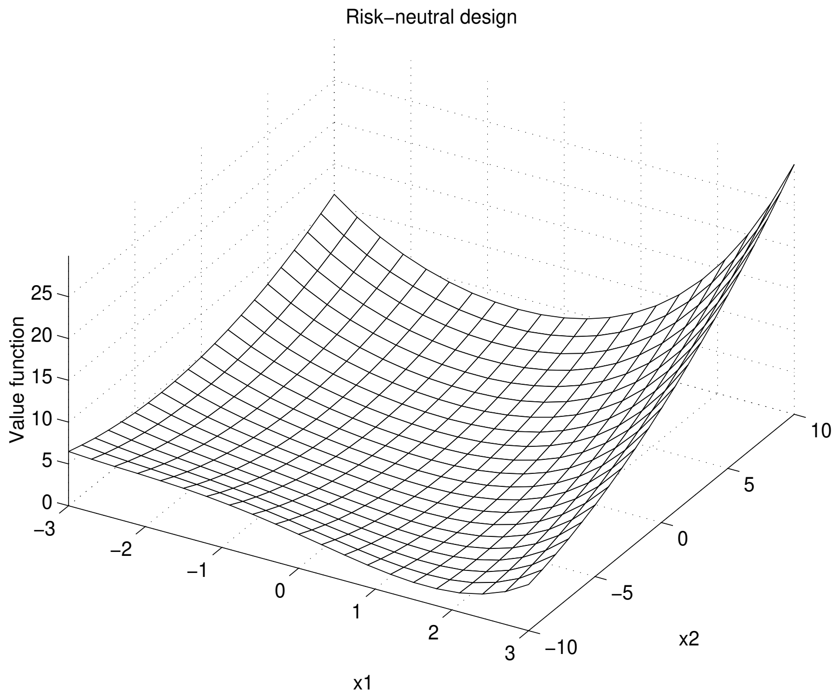
FIG. 1. *Value function and control law for risk-sensitive design.*



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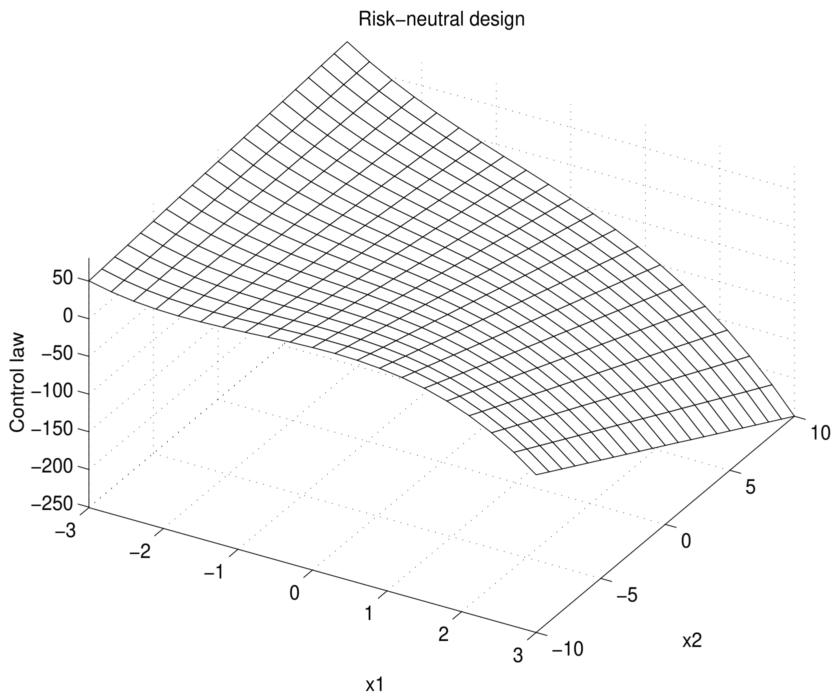

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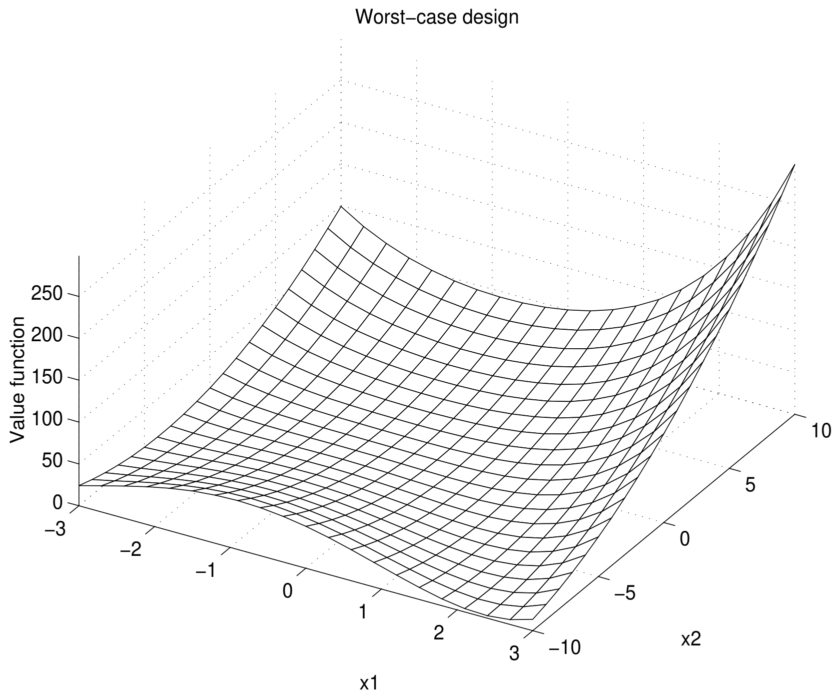
FIG. 2. *Value function and control law for risk-neutral design.*



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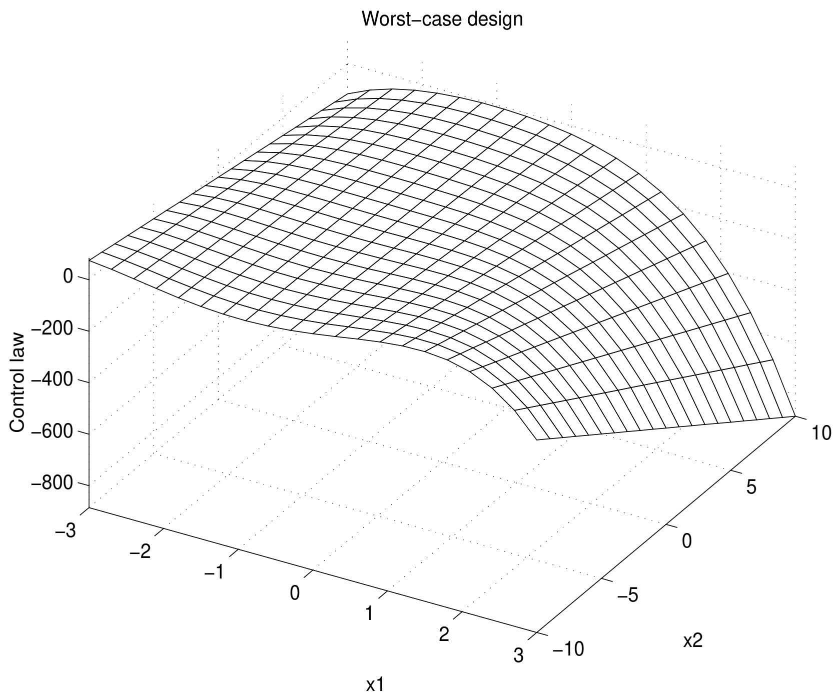

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FIG. 3. *Value function and control law for worst-case design.*

worst-case controller. A simple calculation yields that the $\mathcal{L}_2$ norm of the output for the risk-sensitive control design is the smallest among the three, which is 5.0058. Second best is the risk-neutral controller, whose output norm is 6.0840. The worst-case controller yields a norm of 7.0492 for its output. Regarding the control effort, we observe that the stochastic controllers use more control effort when the system states are small in norm. Yet, the worst-case controller leads to larger peak control actions. A simple calculation yields that the $\mathcal{L}_2$ norms of the stochastic controllers are bounded by twice of that for the worst-case controller.

Finally, we simulated the system response under sinusoidal disturbance inputs:

$$w_1(t) = 8sin(t) \qquad w_2(t) = 8sin(2t+1)$$

We tried various levels for the magnitudes of the sine waves. For smaller-magnitude sine waves, the stochastic controllers have shown better performance than the worst-case controller. When the magnitudes were fixed at 8, the performances of the two stochastic controllers were very different. These simulation results are plotted in Figures 7–9. We observe from these figures that the risk-sensitive controller again leads to the best overall system performance. A simple calculation yields that the $\mathcal{L}_2$ norm of the output for the risk-sensitive control design is the smallest among the three, which is 107.3. Second best is the worst-case controller, whose output norm is 122.3. The risk-neutral controller yields a norm of 479.4 for its output. Regarding the control effort, we observe that the worst-case controller uses the least amount of control effort. The risk-sensitive controller uses 33% more control effort when measured in $\mathcal{L}_2$ norm. The system performance for the risk-neutral controller is not satisfactory under this disturbance.

Overall, we observe that the risk-sensitive controller and the worst-case controller are quite satisfactory for this simple example. The risk-sensitive controller leads to smaller output $\mathcal{L}_2$ norm in both of the simulations, at the expense of more control effort. This does not contradict with the observation that the risk-sensitive controller has a smaller controller gain. It is generally observed that the behavior of the state variable x_2 is much better for the worst-case controller. Smaller control gain guarantees a smaller control effort only if the system states behave the same way.

8. Conclusion. In this paper, we have presented a recursive control design for strict-feedback stochastic nonlinear systems. Under an exponential (risk-sensitive) cost function, we have obtained controllers that guarantee any desired positive level of long term average cost for a fixed risk-sensitivity parameter. Such an objective was feasible because (i) no weighting was attached to the control variable, (ii) the system was in the strict-feedback form, and (iii) there was weighting only on the output of the system. The construction presented here for the risk-sensitive control problem differs substantially from that of the H^∞ problem [23], which is to be expected since the equivalence between the two problems holds only as the noise intensity diminishes to zero. As a special case, the solution to the risk neutral control problem has also been obtained, by simply taking the limit as the risk-sensitivity parameter approaches zero. Another special case treated was when the vector fields for the disturbance vanish at the origin of the system; in this case, the control design can actually guarantee a *zero* long term average cost. Furthermore, the closed-loop system becomes asymptotically stable in the large. These results have then been generalized to a class of nonlinear systems involving zero dynamics that are strongly stabilizable. The theoretical findings of the paper were further illustrated in Section 7 on a second-order nonlinear system.

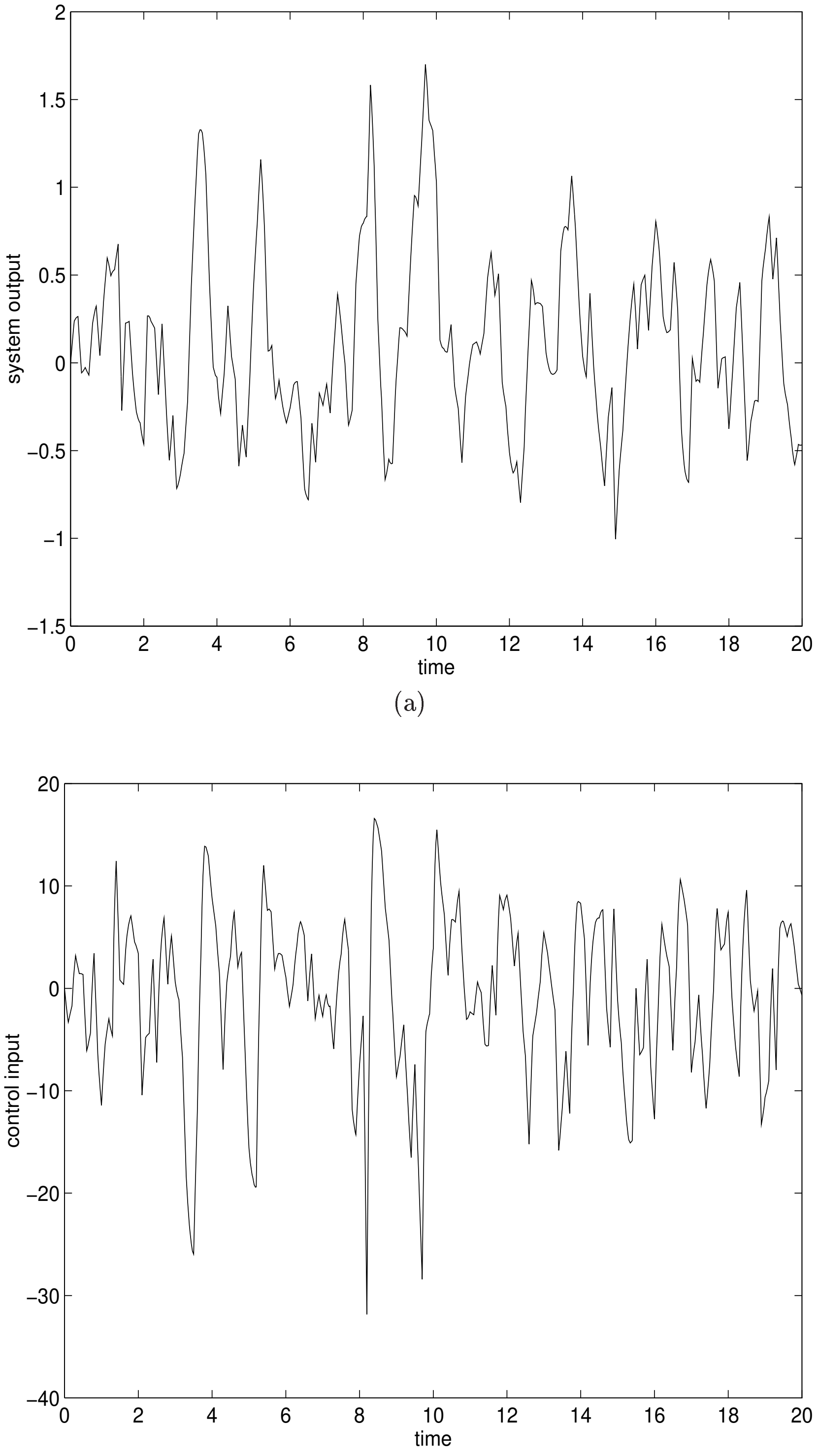

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FIG. 4. *Risk-sensitive controller under white noise input.*

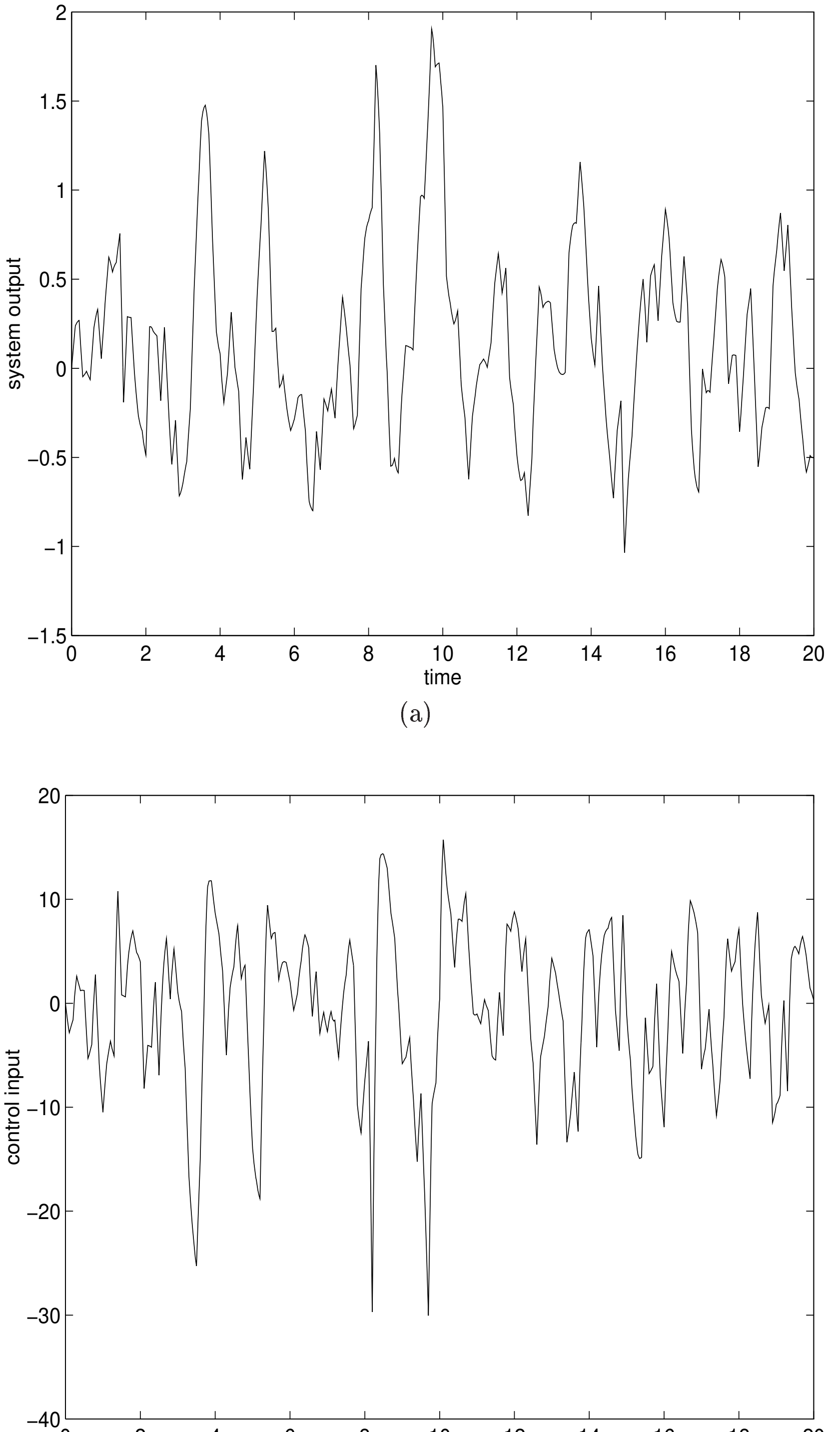

FIG. 5. *Risk-neutral controller under white noise input.*

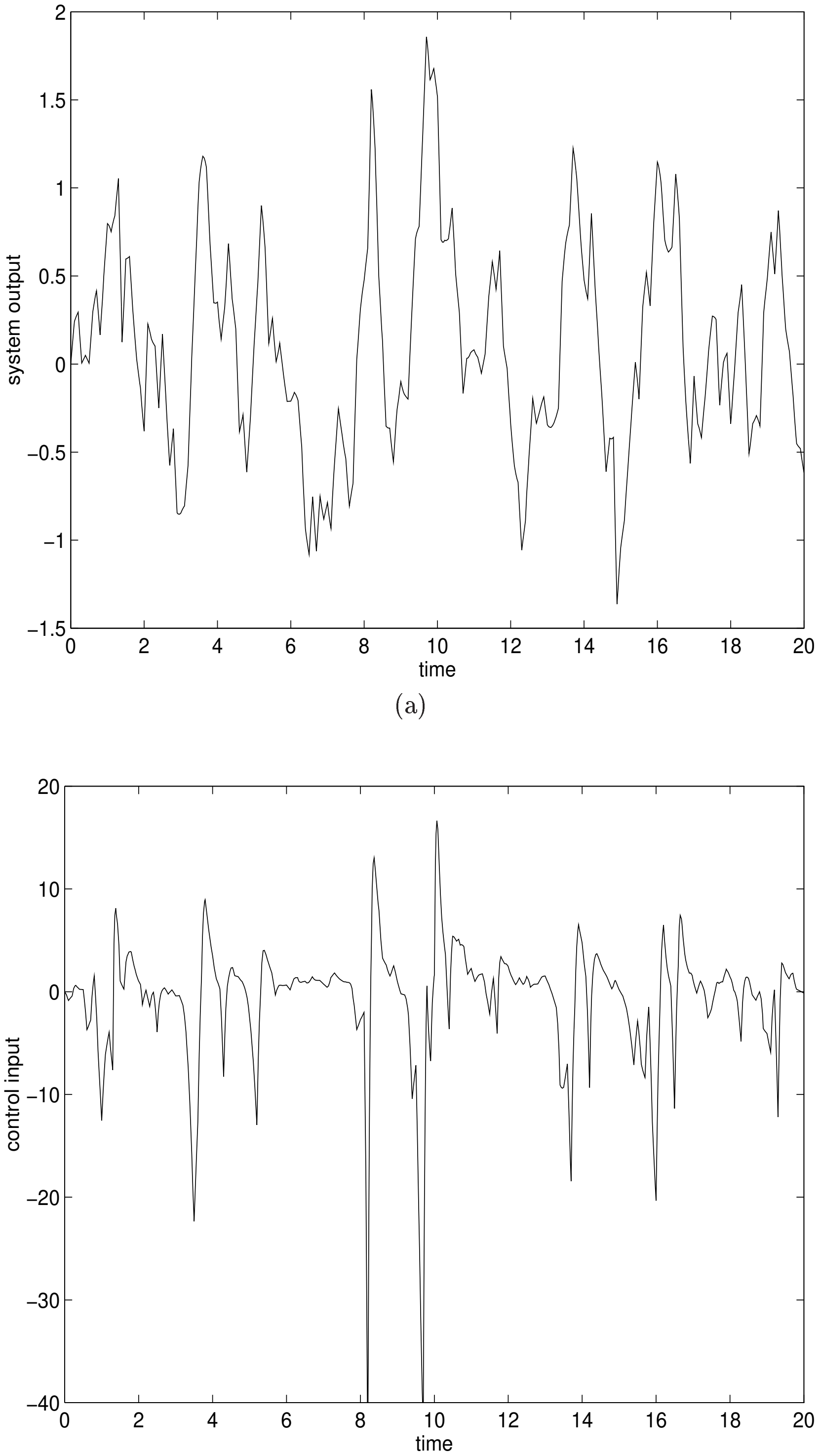

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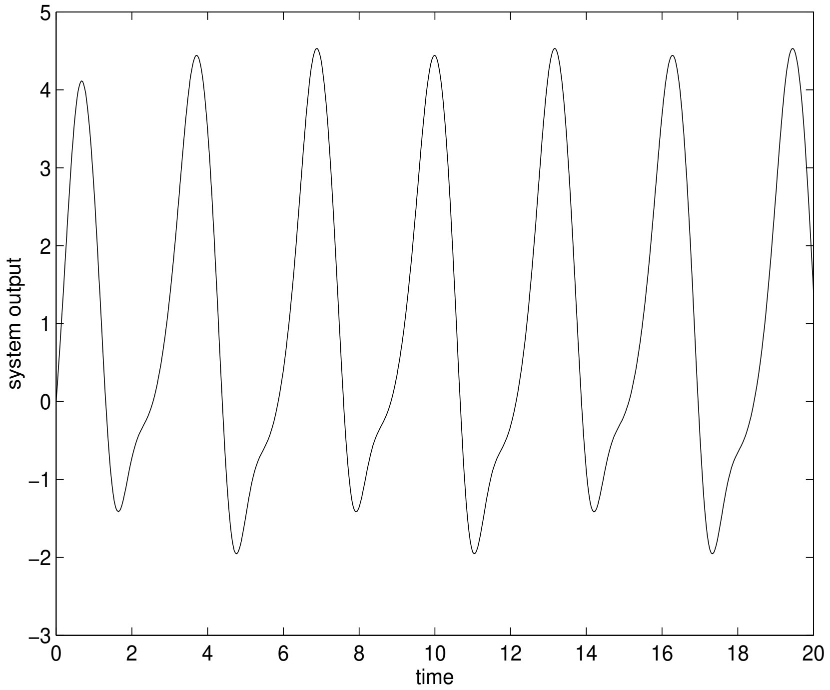
FIG. 6. *Worst-case controller under white noise input.*



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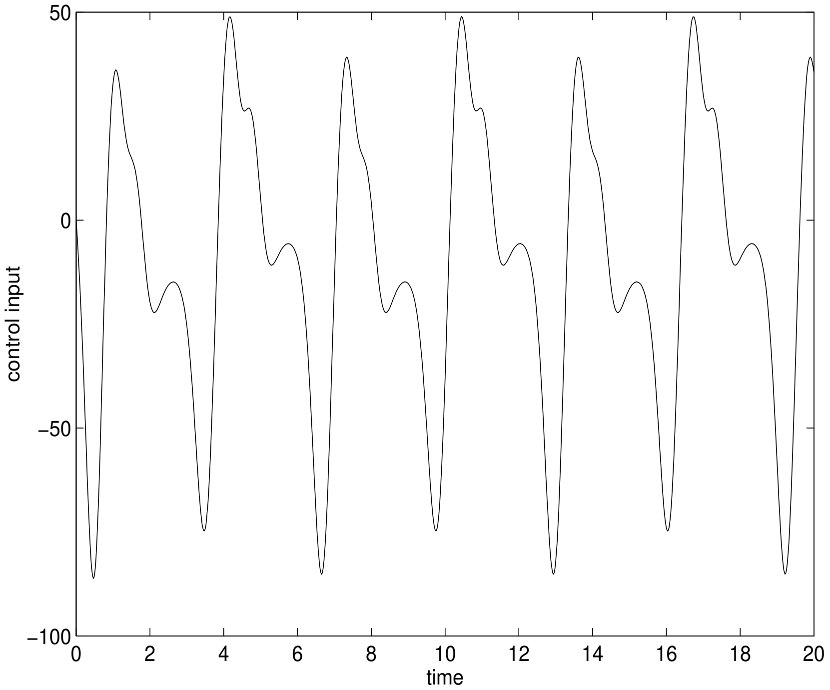

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FIG. 7. *Risk-sensitive controller under sinusoidal noise input.*

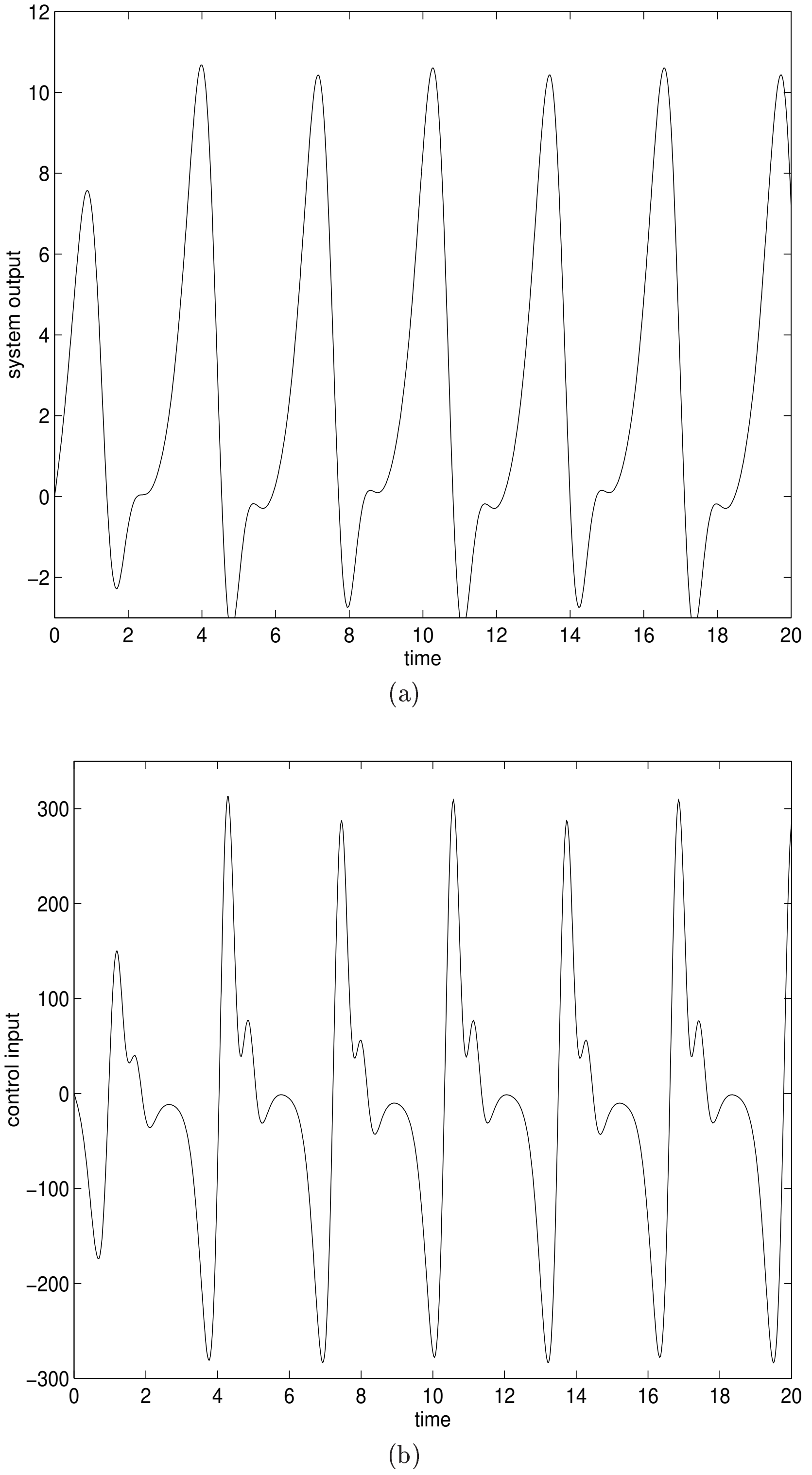
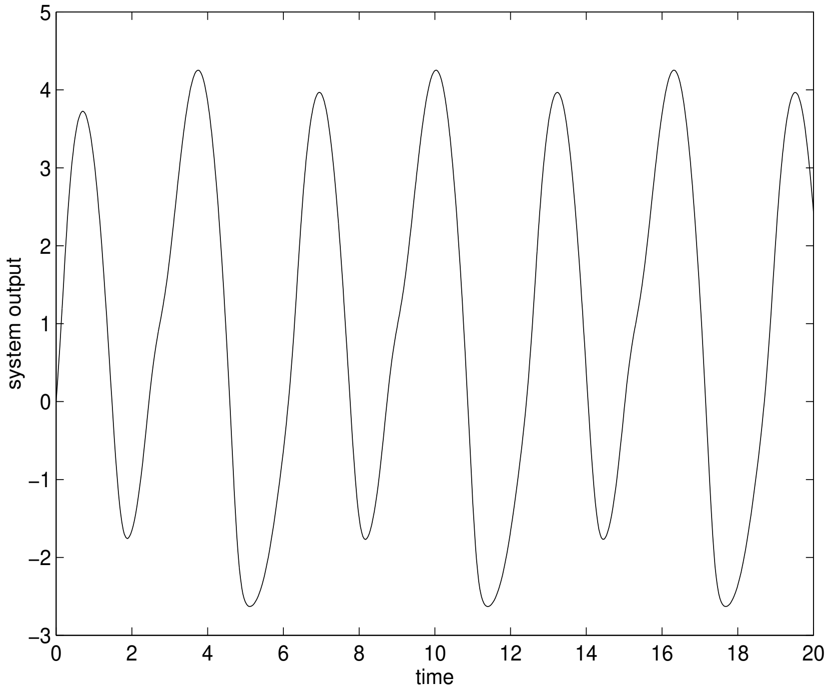


FIG. 8. *Risk-neutral controller under sinusoidal noise input.*



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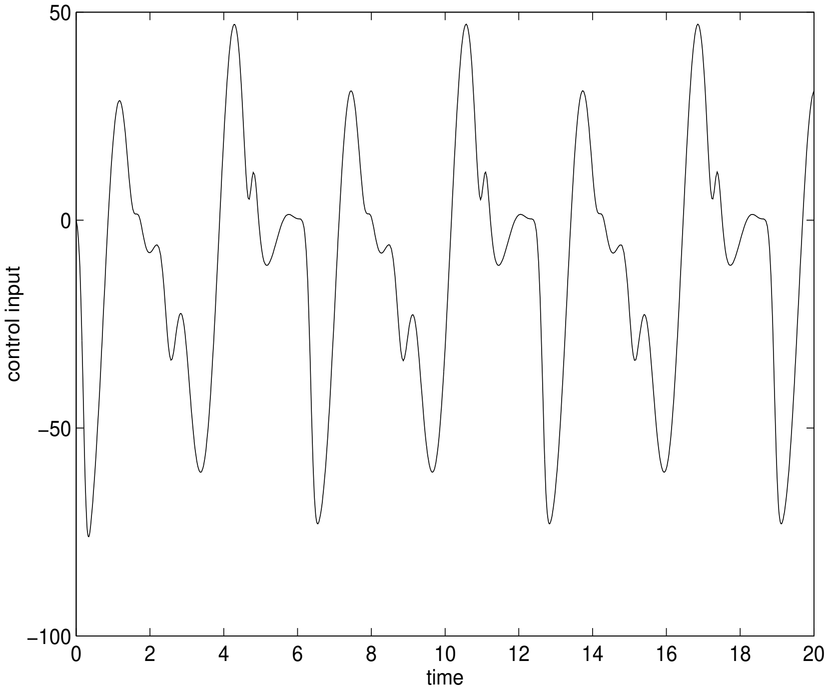

(b)

FIG. 9. *Worst-case controller under sinusoidal noise input.*

Future research on this topic lies in several directions. One of these pertains to the parameter identification problem under a risk-sensitive performance criterion, so as to generalize the results of [2] and [24] to the stochastic case. Another interesting topic is the stochastic adaptive control design for strict-feedback systems, so as to achieve a desired risk-sensitive performance level for a given risk-sensitivity parameter, which would be a generalization of the results of [23] to the stochastic case.

Appendix.

In this appendix, we provide a proof of the fact (see Girsanov [9]) that the stochastic process $\zeta(t)$ defined in the proof of Theorem 5 is a supermartingale. The precise statement is as follows.

THEOREM 11. *Let $w(t)$ be an m-dimensional vector-valued standard Wiener process on the time interval $[0, t_f]$ adapted to a filtration $\mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Further let a vector-valued stochastic process $f : [0, t_f] \times \Omega \to \mathbb{R}^m$ be adapted to the filtration $\mathcal{F}$. Define the scalar process*

$$\zeta_0^t(f) := \exp\left(\int_0^t f'(s)\, dw - \frac{1}{2}\int_0^t |f(s)|^2\, ds\right)$$

Then, we have $\boldsymbol{E}\left(\zeta_0^t(f)\right) \le 1, \ \ \forall t \in [0, t_f]$.

Proof. Let each component of the mapping f be given by

$$f = \left[\begin{array}{ccc} f^1 & \cdots & f^m \end{array}\right]'$$

For each $n = 1, 2, \ldots$, define the function $f_n : [0, t_f] \times \Omega \to \mathbb{R}^m$ by

$$f_n = \left[\begin{array}{ccc} \max\{\min\{f^1, n\}, -n\} & \cdots & \max\{\min\{f^m, n\}, -n\} \end{array}\right]'$$

Clearly, we have

$$\lim_{n\to\infty} f_n = f \qquad \text{a.e.}$$

It is easy to check that

$$\boldsymbol{E}\left(\exp\left(\frac{1}{2}\int_0^t |f_n(s)|^2\, ds\right)\right) \le \exp\left(\frac{1}{2}n^2 t\right) < \infty$$

By the result of [21], we have that

$$\boldsymbol{E}\left(\zeta_0^t(f_n)\right) = 1 \qquad \forall t \in [0, t_f];\ n = 1, 2, \ldots$$

By Fatou's Lemma, we have

$$1 = \liminf_{n\to\infty} \boldsymbol{E}\left(\zeta_0^t(f_n)\right) \ge \boldsymbol{E}\left(\liminf_{n\to\infty} \zeta_0^t(f_n)\right) = \boldsymbol{E}\left(\zeta_0^t(f)\right)$$

This completes the proof of the theorem. ◻

Next, we present a lemma which is used in the proof of Proposition 6.

LEMMA 12. *Let $w(t)$ be an m-dimensional vector-valued standard Wiener process on the time interval $[0, t_f]$ adapted to a filtration $\mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Let x and z be two constant m-dimensional vectors satisfying*

$$x'\, dw = z'\, dw$$

for some time instance $t_0 \geq 0$. Then, $x = z$.

Proof. Let x, z, and $w(t)$ be given by

$$x = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}' \quad z = \begin{bmatrix} z_1 & \cdots & z_m \end{bmatrix}' \quad w(t) = \begin{bmatrix} w_1(t) & \cdots & w_m(t) \end{bmatrix}'$$

At t_0, we have

$$\sum_{i=1}^{m} x_i \, dw_i(t_0) = \sum_{i=1}^{m} z_i \, dw_i(t_0)$$

For each $i = 1, 2, \ldots, m$, multiply both sides of the above equation by $dw_i(t_0)$, and then take expectations on both sides of the equation. This leads to

$$x_i \, dt = z_i \, dt \qquad i = 1, 2, \ldots, m \qquad \Rightarrow \qquad x_i = z_i \qquad i = 1, 2, \ldots, m$$

From which it follows that $x = z$. This completes the proof of the lemma. □

REFERENCES

[1] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, Birkhäuser, Boston, MA, 2nd ed., 1995.

[2] G. Didinsky, Z. Pan, and T. Başar, *Parameter identification for uncertain plants using H^∞ methods*, Automatica, 31 (1995), pp. 1227–1250.

[3] W. H. Fleming and M. R. James, *The risk-sensitive index and the H_2 and H^∞ norms for nonlinear systems.* To appear in *Math. Control Signals and Systems.*

[4] W. H. Fleming and W. M. McEneaney, *Risk sensitive control with ergodic cost criteria*, in Proceedings of the 31st Conference on Decision and Control, Tucson, AZ, 1992, pp. 2048–2052.

[5] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, vol. 25, Springer-Verlag, 1993.

[6] R. A. Freeman and P. V. Kokotović, *Design of 'softer' robust control laws*, Automatica, 29 (1993), pp. 1425–1437.

[7] ———, *Inverse optimality in robust stabilization.* To appear in *SIAM J. Control and Optimization*, 1996.

[8] I. I. Gihman and A. V. Skorohod, *Stochastic Differential Equations*, Springer-Verlag, New York, NY, 1972.

[9] I. V. Girsanov, *On transforming a certain class of stochastic processes by absolutely continuous substitution of measures*, Theory Prob. App., 5 (1960), pp. 285–301.

[10] A. Isidori, *Nonlinear control systems*, Springer-Verlag, London, 3rd ed., 1995.

[11] D. H. Jacobson, *Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games*, IEEE Transactions on Automatic Control, 18 (1973), pp. 167–172.

[12] S. Jain and F. Khorrami, *Application of a decentralized adaptive output feedback based on backstepping to power systems*, in Proceedings of the the 34rd IEEE Conference on Decision and Control, New Orleans, LA, 1995, pp. 1585–1590.

[13] M. R. James, J. Baras, and R. J. Elliott, *Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems*, IEEE Transactions on Automatic Control, 39 (1994), pp. 780–792.

[14] Z. P. Jiang and J.-B. Pomet, *Backstepping-based adaptive controllers for uncertain nonholonomic systems*, in Proceedings of the 34rd IEEE Conference on Decision and Control, New Orleans, LA, 1995, pp. 1573–1578.

[15] I. Kanellakopoulos, P. V. Kokotović, and A. S. Morse, *Systematic design of adaptive controllers for feedback linearizable systems*, IEEE Transactions on Automatic Control, 36 (1991), pp. 1241–1253.

[16] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, Upper Saddle River, NJ, 2nd ed., 1996.

[17] R. Z. Khas'minskii, *Stochastic Stability of Differential Equations*, S& N International publisher, Rockville, Maryland, 1980.

[18] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear design of adaptive controllers for linear systems*, IEEE Transactions on Automatic Control, 39 (1994), pp. 738–752.

[19] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*, Wiley, New York, NY, 1995.
[20] H. Nagai, *Bellman equations of risk-sensitive control*, SIAM J. Control and Optimization, 34 (1996), pp. 74–101.
[21] A. A. Novikov, *On an identity for stochastic integrals*, Theory Prob. App., 16 (1972), pp. 717–720.
[22] Z. Pan, *Canonical forms for stochastic nonlinear systems.* Submitted to *IEEE Transactions on Automatic Control*, March 1997.
[23] Z. Pan and T. Başar, *Adaptive controller design for tracking and disturbance attenuation in parametric-strict-feedback nonlinear systems*, in Proceedings of the 13th IFAC Congress of Automatic Control, San Francisco, CA, USA, June 30 – July 5 1996.
[24] ———, *Parameter identification for uncertain linear systems with partial state measurements under an H^∞ criterion*, IEEE Transactions on Automatic Control, (1996 (to appear)).
[25] T. Runolfsson, *The equivalence between infinite horizon control of stochastic systems with exponential-of-integral performance index and stochastic differential games*, IEEE Transactions on Automatic Control, 39 (1994), pp. 1551–1563.
[26] D. Seto, A. M. Annaswamy, and J. Baillieul, *Adaptive control of nonlinear systems with triangular structure*, IEEE Transactions on Automatic Control, 39 (1994), pp. 1411–1428.
[27] M. W. Spong and Vidyasagar, *Robot Dynamics and Control*, John Wiley and Sons, New York, NY, 1989.
[28] P. Whittle, *Risk-sensitive linear-quadratic-Gaussian control*, Adv. Appl. Prob., 13 (1981), pp. 764–777.
[29] ———, *Risk-Sensitive Optimal Control*, John Wiley and Sons, Chichester, NY, 1990.