

Adaptive Controller Design and Disturbance Attenuation for SISO Linear Systems with Zero Relative Degree under Noisy Output Measurements*

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Abstract

In this paper, we present robust adaptive controller design for SISO linear systems with zero relative degree under noisy output measurements. We formulate the robust adaptive control problem as a nonlinear H^∞ -optimal control problem under imperfect state measurements, and then solve it using game theory. By using the *a priori* knowledge of the parameter vector, we apply a soft projection algorithm, which guarantees the robustness property of the closed-loop system without any persistency of excitation assumption of the reference signal. Due to our formulation in state space, we allow the true system to be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and the uncontrollable modes on the $j\omega$ -axis are uncontrollable from the exogenous disturbance input. This assumption allows the adaptive controller to asymptotically cancel out, at the output, the effect of exogenous sinusoidal disturbance inputs with unknown magnitude, phase, and frequency. These strong robustness properties are illustrated by a numerical example.

Keywords Nonlinear H^∞ control; cost-to-come function analysis; robust adaptive control.

1 Introduction

The design of adaptive controllers has been an important research topic since 1970s. The classic adaptive control design, based on the certainty equivalence principle (Goodwin and Sin 1984, Goodwin and Mayne 1987), is to design the controller as if the system parameters are known and then in implementation to supply the controller with estimates of the parameters, using standard identifiers, as if the estimates are true values. This design method has been proven successful especially for the linear systems with or without stochastic disturbance inputs (Morse 1980, Kumar 1985, Narendra and Annaswamy 1989, Guo and Chen 1991, Ren and Kumar 1992, Naik *et al.* 1992, Guo 1996, Hsu *et al.* 1999). This approach leads to structurally simple adaptive controllers. Yet,

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early designs based on this approach has been shown to be nonrobust (Ioannou and Kokotović 1983, Rohrs *et al.* 1985) when the system is subject to exogenous disturbance inputs and unmodeled dynamics. Then, the stability and the performance of a system under disturbance and/or uncertainty becomes an important issue. This motivates the study of robust adaptive control which has attracted significant research attention since 1980s. Also, this approach fails to generalize to systems with severe nonlinearities. This motivates the study of nonlinear adaptive control in 1990s.

Robust adaptive control has been an important research topic in late 1980s and early 1990s. Various adaptive controllers were modified to render the closed-loop systems robust (Datta and Ioannou 1994, Ioannou and Sun 1996). Despite their successes, they fell short of directly addressing the disturbance attenuation property of the closed-loop system.

The topic of nonlinear adaptive control has been widely studied in the last decade after the celebrated characterization of feedback linearizable or partially feedback linearizable systems (Isidori 1995). The introduction of the integrator backstepping methodology (Kanellakopoulos *et al.* 1991) allow us to design adaptive controllers for parametric strict-feedback and parametric pure-feedback nonlinear systems systematically. Since then, a lot of important contributions were motivated by this approach, and a complete list of references can be found in the book (Krstić *et al.* 1995). More recently, systems with unknown sign of the high frequency gain have been studied using this approach (Ye 2001). Moreover, this approach has been applied to linear systems to compare performance with the certainty equivalence approach (Krstić *et al.* 1994). As to be expected, a systematically designed nonlinear adaptive control law leads to better closed-loop performance than that for the certainty equivalence based design when the system is free of disturbance. However, this approach has also been shown to be nonrobust (Başar *et al.* 1996) when the system is subject to exogenous disturbance inputs.

H^∞ -optimal control has been proposed as a solution to the robust control problem. It achieves the objectives of robust control, namely, improving transient response, accommodating unmodeled dynamics, and rejecting exogenous disturbance inputs, by studying only the disturbance attenuation property for the closed-loop system. The game-theoretic approach to H^∞ -optimal control (Başar and Bernhard 1995) developed for the linear quadratic problems, offers the most promising tool to generalize the results to nonlinear systems (van der Schaft 1991, 1992, Isidori and Astolfi 1992, Didinsky *et al.* 1993, Didinsky 1994, Marino *et al.* 1994, Isidori and Kang 1995). The worst-case analysis approach to adaptive control was proposed to address the disturbance attenuation properties of the closed-loop system directly. In this approach, the robust adaptive control problem is formulated as a nonlinear H^∞ -optimal control problem under imperfect state measurements. Using *cost-to-come* function analysis, it can be converted into a problem under full information measurements. This full information measurement problem is then solved for a suboptimal solution using the integrator backstepping methodology. This design paradigm has been applied to worst-case parameter identification problems (Didinsky *et al.* 1995, Pan and Başar 1996), which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems (Didinsky and Başar 1997, Pan and Başar 1998, Tezcan and Başar 1999, Pan and Başar 2000, Arslan and Başar 2001, Zeng and Pan 2003), which has led to new classes of parametrized robust adaptive controllers for linear and nonlinear systems. Adaptive control for strict-feedback systems was studied in Pan and Başar (1998) under noiseless full state measurements. More general class of nonlinear systems was studied in Arslan and Başar (2001), again with noiseless full state measurements. In Tezcan and Başar (1999), adaptive control for strict-feedback nonlinear systems was considered under noiseless output measurements. In Pan

and Başar (2000) and Zeng and Pan (2003), linear system was considered under noisy output measurements. Zeng and Pan (2003) generalizes the results of Pan and Başar (2000), by assuming part of the disturbance inputs are measured, which then lead to disturbance feedforward structure in the adaptive controller.

In this paper, we study the adaptive control design for SISO linear systems with zero relative degree under noisy output measurements using a similar approach as that of Pan and Başar (2000). We assume that the linear system admits a known upper bound for its dynamic order, is observable, has a strictly minimum phase transfer function with relative degree 0. The linear system may be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and all uncontrollable modes on the $j\omega$ -axis are uncontrollable from the disturbance input. Under these assumptions, the system may be transformed into the design model, which is linear in all of the unknown quantities. We formulate the robust adaptive control problem as a nonlinear H^∞ -optimal control problem under imperfect state measurements, where the objectives of asymptotic tracking, transient performance, and disturbance attenuation are incorporated into a single game theoretic cost function. To avoid singularity in the estimation step, we assume that the measurement is noisy. Then, we can apply *cost-to-come* function methodology to derive the estimator, which has a finite-dimensional structure. To relieve the persistency of excitation condition for the closed-loop system, we assume that the true value of the parameter vector belongs to a convex compact set characterized by a known smooth nonnegative radially unbounded and strictly convex function $P(\theta)$, and apply a soft projection algorithm for the estimator. Then, the closed-loop system is robust with or without the persistently exciting signals. After the estimator is determined, the original problem becomes a nonlinear H^∞ -optimal control problem under full-information measurement, and the controller can be obtained directly based on the cost function at that step. The closed-loop system admits a guaranteed disturbance attenuation level with respect to the exogenous disturbance inputs, where the ultimate attenuation lower bound for the achievable performance level is equal to the noise intensity in the measurement channel. All closed-loop signals are bounded for bounded disturbance input and bounded reference trajectory. Furthermore, it achieves asymptotic tracking of uniformly continuous and bounded reference trajectories for all bounded disturbance inputs that are of finite energy. This result has significant impact on active noise cancellation problems. That is, when the true system is subject to disturbances generated by an unknown exogenous linear system, we can extend our system model to include the states of the exogenous system as part of the model, and then asymptotically cancel out the effect of the noise at the output. This feature is illustrated by an example in the paper.

The balance of the paper is organized as follows. In Section 2, we list the notations to be used in this paper. In Section 3, we formulate adaptive control problem and discuss the general solution methodology. In Section 4, we present the estimation and control design using *cost-to-come* function methodology. In Section 5, we present the main result of the paper which states the robustness properties of the closed-loop system. The theoretical results are illustrated by a numerical example in Section 6. The paper ends with some concluding remarks in Section 7, and two appendices.

2 Notations

We denote \mathbb{R} to be the real line; \mathbb{N} to be the set of natural numbers; \mathbb{C} to be the set of complex numbers. For a function f , we say that it belongs to \mathcal{C} if it is continuous; we say that it belongs to

\mathcal{C}_k if it is k -times continuously (partial) differentiable. For any matrix A , A' denotes its transpose.

For any $b \in \mathbb{R}$, $\text{sgn}(b) = \begin{cases} -1 & b < 0 \\ 0 & b = 0 \\ 1 & b > 0 \end{cases}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{N}$, $|z|$ denotes

$(z'z)^{1/2}$. For any vector $z \in \mathbb{R}^n$, and any $n \times n$ -dimensional symmetric matrix M , where $n \in \mathbb{N}$, $|z|_M^2 = z'Mz$. For any matrix M , the vector \overline{M} is formed by stacking up its column vectors. For any symmetric matrix M , \underline{M} denotes the vector formed by stacking up the column vector of the lower triangular part of M . For $n \times n$ -dimensional symmetric matrices M_1 and M_2 , where $n \in \mathbb{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$ -dimensional positive definite matrices is denoted by \mathcal{S}_{+n} . For $n \in \mathbb{N} \cup \{0\}$, I_n denotes the $n \times n$ -dimensional identity matrix. For any matrix M , $\|M\|_p$ denotes its p -induced norm, $1 \leq p \leq \infty$. \mathcal{L}_2 denotes the set of square integrable functions and \mathcal{L}_∞ denotes the set of bounded functions. For any $n, m \in \mathbb{N}$, $\mathbf{0}_{n \times m}$ denotes the $n \times m$ -dimensional matrix whose elements are zeros.

3 Problem Formulation

We consider the adaptive control problem for single-input and single-output (SISO) linear time-invariant systems. We make the following assumption on the unknown system.

Assumption 1 *The linear system is known to be at most n dimensional, $n \in \mathbb{N}$.* \diamond

We consider the following true system dynamics:

$$\dot{x} = \dot{A}\dot{x} + \dot{B}u + \dot{D}\dot{w}; \quad \dot{x}(0) = \dot{x}_0 \quad (1a)$$

$$y = \dot{C}\dot{x} + b_0u + \dot{E}\dot{w} \quad (1b)$$

where \dot{x} is the \dot{n} -dimensional state vector, $\dot{n} \in \mathbb{N} \cup \{0\}$; u is the scalar control input; $b_0 \in \mathbb{R}$ and $b_0 \neq 0$; y is the scalar measurement output; \dot{w} is the \dot{q} -dimensional unmeasured disturbance input vector, $\dot{q} \in \mathbb{N}$; the state \dot{x} has initial condition \dot{x}_0 ; and all input and output signals y , u , and \dot{w} are continuous; the matrices \dot{A} , \dot{B} , \dot{C} , \dot{D} , and \dot{E} are of the appropriate dimensions, generally unknown or partially unknown. The transfer function from u to y is $H(s) = \dot{C}(sI_{\dot{n}} - \dot{A})^{-1}\dot{B} + b_0$.

Assumption 2¹ *The pair (\dot{A}, \dot{C}) is observable. The transfer function $H(s)$ is known to have relative degree 0, and is strictly minimum phase. Moreover, the uncontrollable part (with respect to u) of the unknown system is stable in the sense of Lyapunov. Any uncontrollable mode corresponding to the eigenvalues of the matrix \dot{A} on the $j\omega$ -axis are uncontrollable from \dot{w} .* \diamond

Remark 1 *If the true system is of order $\dot{n} < n$, we can add $(n - \dot{n})$ -dimensional dynamics as outlined in Lemma 6, such that the expanded system is of order n and satisfies the Assumption 2. Hence, without loss of generality, we will assume that the true system (1) is of order n .*

Since (\dot{A}, \dot{C}) is observable, there always exists a state diffeomorphism $x = \dot{T}\dot{x}$, and a disturbance transformation $w = \dot{M}\dot{w}$, where \dot{T} is an unknown real invertible matrix and \dot{M} is a real $q \times \dot{q}$ -dimensional unknown matrix with $q \in \mathbb{N}$, such that the system (1) can be transformed into the

¹When $\dot{n} = 0$, Assumption 2 is considered satisfied.

following form in the x coordinate with inputs u and w

$$\dot{x} = Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Bu + Dw; \quad x(0) = x_0 \quad (2a)$$

$$y = Cx + u\bar{C}_1\theta + b_{p0}u + Ew \quad (2b)$$

where θ is the σ -dimensional vector of unknown parameters of the system, $\sigma \in \mathbb{N}$; the matrices A , \bar{A}_{211} , \bar{A}_{212} , B , D , C , E , and \bar{C}_1 are of appropriate dimensions and completely known, and $b_{p0} \in \mathbb{R}$ is also known. In addition, the high frequency gain of the transfer function $H(s)$, b_0 , is equal to $b_{p0} + \bar{C}_1\theta$. We will design the adaptive controller based on system (2), which is called the design model. By Assumption 2, the pair (A, C) is observable.

We have the following assumptions about the design model.

Assumption 3 $EE' > 0$. ◇

Define $\zeta := (EE')^{-1/2}$ and $L := DE'$.

Remark 2 *The above transformation matrix \hat{T} always exists. One may choose the state diffeomorphism $x = \hat{T}\hat{x}$ to be the one which transforms the pair (\hat{A}, \hat{C}) into its observer canonical form. Next, we can choose a unknown matrix \hat{M} such that the matrices D and E are completely known.*

To guarantee the stability of the closed-loop system and the boundedness of the estimate of θ , we make the following assumption on the parameter vector θ .

Assumption 4 *The sign of the high-frequency gain b_0 is known. There exists a known smooth nonnegative radially-unbounded strictly convex function $P : \mathbb{R}^\sigma \rightarrow \mathbb{R}$, such that the true value of θ belongs to the set $\Theta := \{\bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) \leq 1\}$. Furthermore, for any $\bar{\theta} \in \Theta$, we have $\text{sgn}(b_0)(b_{p0} + \bar{C}_1\bar{\theta}) > 0$.* ◇

We make the following assumption about the reference signal, y_d .

Assumption 5 *The reference trajectory, y_d , is continuous, and available for the control design.* ◇

For the system (1), under Assumptions 1–5, the control law is generated by

$$u(t) = \mu(y_{[0,t]}, y_{d[0,t]}) \quad (3)$$

Furthermore, it must satisfy the following condition. For any uncertainty $(x_0, \theta, \dot{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \hat{\mathcal{W}} := \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathcal{C}$, which comprises the initial state, the true value of the unknown parameter vector, the unknown disturbance input waveform, and the reference trajectory, there must be a unique solution $\hat{x}_{[0,\infty)}$ for the closed-loop system, which results in a continuous control input waveform $u_{[0,\infty)}$. We denote the class of these admissible controllers by \mathcal{M}_u .

The objectives of our control design are to make the output of the system, $Cx + b_0u$, to asymptotically track the reference trajectory y_d , and guarantee the boundedness of all closed-loop signals, while rejecting the uncertainty $(x_0, \theta, \dot{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \hat{\mathcal{W}}$. For design purposes, instead of attenuating the effect of \dot{w} , we design the adaptive controller to attenuate the effect of w . We take the uncertainty $(x_0, \theta, w_{[0,\infty)}, y_{d[0,\infty)})$ to belong to the set $\mathcal{W} := \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathcal{C}$. All of these objectives can be captured by the optimization of a single game-theoretic cost function, defined as follows.

Definition 1 A controller $\mu \in \mathcal{M}_u$ is said to achieve disturbance attenuation level γ if there exist a nonnegative function $l(t, \theta, x, y_{[0,t]}, y_{d[0,t]})$ such that

$$\sup_{(x_0, \theta, \dot{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \dot{\mathcal{W}}} J_{\gamma t_f} \leq 0; \quad \forall t_f \geq 0 \quad (4)$$

where

$$\begin{aligned} J_{\gamma t_f} := & \int_0^{t_f} \left((Cx(\tau) + u(\tau)\bar{C}_1\theta + b_{p0}u(\tau) - y_d(\tau))^2 + l(\tau, \theta, x(\tau), y_{[0,\tau]}, y_{d[0,\tau]}) \right. \\ & \left. - \gamma^2 |w(\tau)|^2 \right) d\tau - \gamma^2 \left[\begin{array}{cc} \theta' - \check{\theta}'_0 & x'_0 - \check{x}'_0 \end{array} \right]' \Big|_{\bar{Q}_0}^2 \end{aligned} \quad (5)$$

where $\check{\theta}_0 \in \Theta$ is the initial guess of the unknown parameter vector θ ; \check{x}_0 is the initial guess of the unknown initial state x_0 ; the $(\sigma + n) \times (\sigma + n)$ -dimensional matrix \bar{Q}_0 is the quadratic weighting on the initial estimation error, quantifying the level of confidence in the estimate $\left[\begin{array}{cc} \check{\theta}'_0 & \check{x}'_0 \end{array} \right]'$; \bar{Q}_0^{-1} admits the structure $\left[\begin{array}{cc} Q_0^{-1} & Q_0^{-1}\Phi'_0 \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi'_0 \end{array} \right]$, where Q_0 and Π_0 are $\sigma \times \sigma$ - and $n \times n$ -dimensional positive definite matrices, respectively.

Clearly, when the inequality (4) is achieved, the squared \mathcal{L}_2 norm of the output tracking error $Cx + u\bar{C}_1\theta + b_{p0}u - y_d$ is bounded by γ^2 times the squared \mathcal{L}_2 norm of the transformed disturbance input w plus some constant. When the \mathcal{L}_2 norm of \dot{w} is finite, the squared \mathcal{L}_2 norm of $Cx + u\bar{C}_1\theta + b_{p0}u - y_d$ is also finite, which implies $\lim_{t \rightarrow \infty} (Cx(t) + u(t)\bar{C}_1\theta + b_{p0}u(t) - y_d(t)) = 0$, under additional assumptions.

The following notation will be used throughout this paper. Let \check{x} denote the estimate of x , \tilde{x} denote the state estimation error $x - \check{x}$, $\check{\theta}$ denote the estimate of θ , $\tilde{\theta}$ denote the parameter estimation error $\theta - \check{\theta}$.

We intend to solve this robust adaptive control problem by formulating it as an H^∞ control problem with imperfect state measurements. To do this, we first expand the state space to include the parameter θ as part of the state. Let ξ denote the expanded state vector $\xi = [\theta' \ x']'$. Note that $\check{\theta} = 0$, we have the following expanded dynamics for system (2)

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} \mathbf{0}_{\sigma \times \sigma} & \mathbf{0}_{\sigma \times n} \\ y\bar{A}_{211} + u\bar{A}_{212} & A \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0}_{\sigma \times 1} \\ B \end{bmatrix} u + \begin{bmatrix} \mathbf{0}_{\sigma \times q} \\ D \end{bmatrix} w \\ &=: \bar{A}(u, y)\xi + \bar{B}u + \bar{D}w \end{aligned} \quad (6a)$$

$$y = \begin{bmatrix} u\bar{C}_1 & C \end{bmatrix} \xi + b_{p0}u + Ew =: \bar{C}(u)\xi + b_{p0}u + Ew \quad (6b)$$

The worst-case optimization of the cost function (5) can be carried out in two steps as depicted in the following inequality.

$$\begin{aligned} \sup_{(x_0, \theta, \dot{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \dot{\mathcal{W}}} J_{\gamma t_f} &= \sup_{y_{[0,\infty)} \in \mathcal{C}, y_{d[0,\infty)} \in \mathcal{C}} \sup_{(x_0, \theta, \dot{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \dot{\mathcal{W}}} J_{\gamma t_f} \\ &\leq \sup_{y_{[0,\infty)} \in \mathcal{C}, y_{d[0,\infty)} \in \mathcal{C}} \sup_{(x_0, \theta, w_{[0,\infty)}, y_{d[0,\infty)}) \in \mathcal{W}} J_{\gamma t_f} \end{aligned} \quad (7)$$

The inner supremum operator will be carried out first. It is the estimation design step, which will be presented in Section 4. Succinctly stated, in this step, we will calculate the maximum cost that is consistent with the given measurement waveform.

The outer supremum operator will be carried out second. It is the control design step, which will be discussed after the estimation design. In this step we design the control input u , which guarantees the robustness of the closed-loop system.

This completes the formulation of the robust adaptive control problem. Next, we turn to the estimation and control design in the next section.

4 Estimation and Control Design

In this section, we present the estimation and control design for the adaptive control problem formulated. The first step is estimation design. In this step, the measurement waveform $y_{[0,\infty)}$ and the reference trajectory $y_{d[0,\infty)}$ are assumed to be known. Since the control input is a causal function of y and y_d , then it is also known. We apply the cost-to-come function methodology. Set function l in (5) to $|\xi - \hat{\xi}|_{\bar{Q}}^2$, where $\hat{\xi}$ is the worst-case estimate for the expanded state ξ , $\hat{\xi} = [\check{\theta}' \ \check{x}']'$, and \bar{Q} is a matrix-valued weighting function to be introduced later. The cost function becomes

$$J_{\gamma t_f} = \int_0^{t_f} \left(|Cx(\tau) + u(\tau)\bar{C}_1\theta + b_{p0}u(\tau) - y_d(\tau)|^2 + |\xi(\tau) - \hat{\xi}(\tau)|_{\bar{Q}(\tau, y_{[0,\tau]}, y_{d[0,\tau]})}^2 - \gamma^2 |w(\tau)|^2 \right) d\tau - \gamma^2 \left| \begin{bmatrix} \theta' - \check{\theta}'_0 & x'_0 - \check{x}'_0 \end{bmatrix} \right|_{\bar{Q}_0}^2 \quad (8)$$

By the cost-to-come function analysis of Pan and Başar (2000), we have

$$\dot{\bar{\Sigma}} = (\bar{A}(u, y) - \zeta^2 \bar{L} \bar{C}(u)) \bar{\Sigma} + \bar{\Sigma} (\bar{A}(u, y) - \zeta^2 \bar{L} \bar{C}(u))' + \gamma^{-2} \bar{D} \bar{D}' - \gamma^{-2} \zeta^2 \bar{L} \bar{L}' - \bar{\Sigma} (\gamma^2 \zeta^2 (\bar{C}(u))' \bar{C}(u) - (\bar{C}(u))' \bar{C}(u) - \bar{Q}(t, y_{[0,t]}, y_{d[0,t]})) \bar{\Sigma}; \quad \bar{\Sigma}(0) = \gamma^{-2} \bar{Q}_0^{-1} \quad (9a)$$

$$\dot{\check{\xi}} = \bar{A}(u, y) \check{\xi} - \bar{\Sigma} \bar{Q}(t, y_{[0,t]}, y_{d[0,t]}) \xi_c + \bar{\Sigma} (\bar{C}(u))' (\bar{C}(u) \check{\xi} - (y_d - b_{p0}u)) + \bar{B}u + \zeta^2 (\gamma^2 \bar{\Sigma} (\bar{C}(u))' + \bar{L}) (y - b_{p0}u - \bar{C}(u) \check{\xi}); \quad \check{\xi}(0) = \begin{bmatrix} \check{\theta}'_0 & \check{x}'_0 \end{bmatrix}' \quad (9b)$$

where \bar{L} is defined as $\bar{L} = \begin{bmatrix} \mathbf{0}_{1 \times \sigma} & L' \end{bmatrix}'$ and $\xi_c := \hat{\xi} - \check{\xi}$.

Then, the cost function (8) can be equivalently written as, when $\bar{\Sigma}$ exists on $[0, t_f]$ and $\bar{\Sigma}(t)$ is positive definite $\forall t \in [0, t_f]$.

$$J_{\gamma t_f} = -|\xi(t_f) - \check{\xi}(t_f)|_{(\bar{\Sigma}(t_f))^{-1}}^2 + \int_0^{t_f} \left((\bar{C}(u(\tau)) \check{\xi}(\tau) + b_{p0}u(\tau) - y_d(\tau))^2 - \gamma^2 \zeta^2 (y(\tau) - b_{p0}u(\tau) - \bar{C}(u(\tau)) \check{\xi}(\tau))^2 + |\hat{\xi}(\tau) - \check{\xi}(\tau)|_{\bar{Q}(\tau, y_{[0,\tau]}, y_{d[0,\tau]})}^2 - \gamma^2 |w(\tau) - w_*(\xi(\tau), \check{\xi}(\tau), \bar{\Sigma}(\tau), u(\tau), w(\tau))|^2 \right) d\tau \quad (10)$$

where $w_* : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \times \mathbb{R} \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ is the worst-case disturbance for estimation step, given by

$$w_*(\xi, \check{\xi}, \bar{\Sigma}, u, w) = \zeta^2 E' (y - b_{p0}u - \bar{C}(u)\xi) + \gamma^{-2} (I_q - \zeta^2 E' E) \bar{D}' \bar{\Sigma}^{-1} (\xi - \check{\xi}) \quad (11)$$

The following steps of derivation for the estimator closely resembles that in Pan and Başar (2000). Partition $\bar{\Sigma}(t)$ as

$$\bar{\Sigma}(t) = \begin{bmatrix} \Sigma(t) & \bar{\Sigma}_{12}(t) \\ \bar{\Sigma}_{21}(t) & \bar{\Sigma}_{22}(t) \end{bmatrix} \quad (12)$$

where $\Sigma(t)$ is $\sigma \times \sigma$ -dimensional, and introduce $\Phi(t) := \bar{\Sigma}_{21}(t) (\Sigma(t))^{-1}$ and $\Pi(t) := \gamma^2 (\bar{\Sigma}_{22}(t) - \bar{\Sigma}_{21}(t) (\Sigma(t))^{-1} \bar{\Sigma}_{12}(t))$. Also partition $\check{\xi}$ compatibly as $\begin{bmatrix} \check{\theta}' & \check{x}' \end{bmatrix}$.

For the boundedness of Σ , the weighting matrix \bar{Q} in (8) admits the following structure

$$\begin{aligned} \bar{Q}(t, y_{[0,t]}, y_{d[0,t]}) &= (\bar{\Sigma}(t))^{-1} \begin{bmatrix} \mathbf{0}_{\sigma \times \sigma} & \mathbf{0}_{\sigma \times n} \\ \mathbf{0}_{n \times \sigma} & \Delta(t) \end{bmatrix} (\bar{\Sigma}(t))^{-1} \\ &+ \begin{bmatrix} \epsilon(t) (\bar{C}_1 u(t) + C\Phi(t))' (\gamma^2 \zeta^2 - 1) (\bar{C}_1 u(t) + C\Phi(t)) & \mathbf{0}_{\sigma \times n} \\ & \mathbf{0}_{n \times n} \end{bmatrix} \\ &= \begin{bmatrix} -(\Phi(t))' \\ I_n \end{bmatrix} \gamma^4 (\Pi(t))^{-1} \Delta(t) (\Pi(t))^{-1} \begin{bmatrix} -(\Phi(t))' \\ I_n \end{bmatrix}' \\ &+ \begin{bmatrix} \epsilon(t) (\bar{C}_1 u(t) + C\Phi(t))' (\gamma^2 \zeta^2 - 1) (\bar{C}_1 u(t) + C\Phi(t)) & \mathbf{0}_{\sigma \times n} \\ & \mathbf{0}_{n \times n} \end{bmatrix} \end{aligned} \quad (13)$$

where $\Delta(t) = \gamma^{-2} \beta_\Delta \Pi(t) + \Delta_1$, with $\beta_\Delta \geq 0$ being a constant and Δ_1 being an $n \times n$ positive-definite matrix, and ϵ is a scalar function defined by

$$\epsilon(t) := K_c^{-1} s_\Sigma(t) := \text{Tr}((\Sigma(t))^{-1}) / K_c \quad t \in [0, \infty) \quad (14a)$$

$$\text{or } \epsilon(t) := 1 \quad (14b)$$

and $K_c \geq \gamma^2 \text{Tr}(Q_0)$ is a constant. Because of this structure for \bar{Q} , we will later treat \bar{Q} as a function $\bar{Q} : \mathbb{R}^{n \times \sigma} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{(n+\sigma) \times (n+\sigma)}$, $\bar{Q}(\Phi, u, s_\Sigma)^2$.

Then, we have the following differential equation for Σ , Φ , and Π

$$\dot{\Sigma} = -(1 - \epsilon) \Sigma (\bar{C}_1 u + C\Phi)' (\gamma^2 \zeta^2 - 1) (\bar{C}_1 u + C\Phi) \Sigma; \quad \Sigma(0) = \gamma^{-2} Q_0^{-1} \quad (15a)$$

$$\begin{aligned} \dot{\Phi} &= (A - \zeta^2 LC - \Pi C' (\zeta^2 - \gamma^{-2}) C) \Phi + y \bar{A}_{211} + u (\bar{A}_{212} - \zeta^2 L \bar{C}_1 - \Pi C' (\zeta^2 - \gamma^{-2}) \bar{C}_1); \\ \Phi(0) &= \Phi_0 \end{aligned} \quad (15b)$$

$$\begin{aligned} \dot{\Pi} &= (A - \zeta^2 LC + \beta_\Delta / 2 I_n) \Pi + \Pi (A - \zeta^2 LC + \beta_\Delta / 2 I_n)' - \Pi C' (\zeta^2 - \gamma^{-2}) C \Pi + D D' \\ &\quad - \zeta^2 L L' + \gamma^2 \Delta_1; \quad \Pi(0) = \Pi_0 \end{aligned} \quad (15c)$$

The matrix Σ will play the role of worst-case covariance matrix of the parameter estimation error. The choice of \bar{Q} guarantees that Σ is bounded from above and bounded from below away from 0 as depicted in the following Lemma, whose proof is given in Pan and Başar (2000).

Lemma 1 *Consider the dynamic equation (15a) for the covariance matrix Σ . Let $K_c \geq \gamma^2 \text{Tr}(Q_0)$, $Q_0 > 0$, and $\gamma \geq \zeta^{-1}$. Then, the matrix Σ is upper and lower bounded as follows; for either choice of $\epsilon(t)$ as in (14), whenever Σ is defined on $[0, t_f]$, and Φ and u are continuous on $[0, t_f]$.*

$$K_c^{-1} I_\sigma \leq \Sigma(t) \leq \Sigma(0) = \gamma^{-2} Q_0^{-1}; \quad \gamma^2 \text{Tr}(Q_0) \leq \text{Tr}((\Sigma(t))^{-1}) \leq K_c; \quad \forall t \in [0, t_f]$$

² $\Pi(t)$ will be set as a constant by Assumption 7.

To avoid the inversion of Σ on-line, we define $s_\Sigma(t) := \text{Tr}((\Sigma(t))^{-1})$, and its time derivative is given by

$$\dot{s}_\Sigma = (\gamma^2 \zeta^2 - 1)(1 - \epsilon)(\bar{C}_1 u + C\Phi)(\bar{C}_1 u + C\Phi)'; \quad s_\Sigma(0) = \gamma^2 \text{Tr}(Q_0) \quad (16)$$

Then, $\epsilon(t) = K_c^{-1} s_\Sigma(t)$, which does not require the inversion of $\Sigma(t)$, when ϵ is defined by (14a).

Based on Lemma 1, we note that $\gamma \geq \zeta^{-1}$. This means that the quantity ζ^{-1} is the ultimate lower bound on the achievable performance level for the adaptive system, using the design method proposed in this paper.

Assumption 6 *If the matrix $A - \zeta^2 LC$ is Hurwitz, then the desired disturbance attenuation level $\gamma \geq \zeta^{-1}$. In case $\gamma = \zeta^{-1}$, choose $\beta_\Delta \geq 0$ such that $A - \zeta^2 LC + \beta_\Delta/2I_n$ is Hurwitz. If the matrix $A - \zeta^2 LC$ is not Hurwitz, then the desired disturbance attenuation level $\gamma > \zeta^{-1}$. \diamond*

Assumption 7 *The initial weighting matrix Π_0 in (15c) is chosen as the unique positive definite solution to the algebraic Riccati equation:*

$$(A - \zeta^2 LC + \beta_\Delta/2I_n)\Pi + \Pi(A - \zeta^2 LC + \beta_\Delta/2I_n)' - \Pi C'(\zeta^2 - \gamma^{-2})C\Pi + DD' - \zeta^2 LL' + \gamma^2 \Delta_1 = \mathbf{0} \quad (17)$$

\diamond

Then, we note that the unique positive-definite solution of (15c) is time-invariant and equal to the initial value Π_0 , and the matrix $A_f := A - \zeta^2 LC - \Pi C'(\zeta^2 - \gamma^{-2})C$ is Hurwitz.

To guarantee the boundedness of estimated parameters without persistently exciting signals, we introduce soft projection design on the parameter estimate, which is based on the *a priori* information on the bounds of the true value of the parameter vector θ , i.e., Assumption 4.

Define

$$\rho := \inf\{P(\bar{\theta}) \mid \bar{\theta} \in \mathbb{R}^\sigma \text{ and } b_{p0} + \bar{C}_1 \bar{\theta} = 0\}$$

By Assumption 4, we have $1 < \rho \leq \infty$. Fix any $\rho_o \in (1, \rho)$, and define the open set $\Theta_o := \{\bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) < \rho_o\}$. Our control design will guarantee the estimate $\check{\theta}$ lies in Θ_o , which immediately implies $\check{b}_0 := b_{p0} + \bar{C}_1 \check{\theta} \geq c_0 > 0$ (See Zeng and Pan 2003). Moreover, the convexity of P implies the following inequality

$$\frac{\partial P}{\partial \theta}(\check{\theta})(\theta - \check{\theta}) < 0 \quad \forall \check{\theta} \in \mathbb{R}^\sigma \setminus \Theta$$

Add the term $-\bar{\Sigma} \left[\left(P_r(\check{\theta}) \right)' \quad \mathbf{0}_{1 \times n} \right]'$ to the right-hand-side of the dynamics (9b), where

$$P_r(\check{\theta}) := \begin{cases} \frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{(\rho_o - P(\check{\theta}))^3} \left(\frac{\partial P}{\partial \theta}(\check{\theta}) \right)' & \forall \check{\theta} \in \Theta_o \setminus \Theta \\ \mathbf{0}_{\sigma \times 1} & \forall \check{\theta} \in \Theta \end{cases} \quad (18)$$

$$=: p_r(\check{\theta}) \left(\frac{\partial P}{\partial \theta}(\check{\theta}) \right)' \quad (19)$$

and $P_r(\check{\theta})$ and $p_r(\check{\theta})$ are smooth functions on Θ_o . Then, we have

$$\begin{aligned}\check{\xi} &= -\bar{\Sigma} \left[(P_r(\check{\theta}))' \quad \mathbf{0}_{1 \times n} \right]' + \bar{A}(u, y)\check{\xi} - \bar{\Sigma}\bar{Q}\xi_c + \bar{\Sigma}(\bar{C}(u))'(\bar{C}(u)\check{\xi} \\ &\quad - (y_d - b_{p0}u)) + \bar{B}u + \zeta^2(\gamma^2\bar{\Sigma}(\bar{C}(u))' + \bar{L})(y - b_{p0}u - \bar{C}(u)\check{\xi}); \\ \check{\xi}(0) &= \left[\check{\theta}'_0 \quad \check{x}'_0 \right]'\end{aligned}\quad (20)$$

Partition $\check{\xi}$ into $[\check{\theta}' \ \check{x}']'$ to obtain the dynamics for these individual estimates. We summarize the equations below.

$$(A - \zeta^2 LC + \beta_\Delta/2I_n)\Pi + \Pi(A - \zeta^2 LC + \beta_\Delta/2I_n)' - \Pi C'(\zeta^2 - \gamma^{-2})C\Pi + DD' - \zeta^2 LL' + \gamma^2 \Delta_1 = \mathbf{0} \quad (21a)$$

$$\dot{\Sigma} = -(1 - \epsilon)\Sigma(\bar{C}_1 u + C\Phi)'(\gamma^2 \zeta^2 - 1)(\bar{C}_1 u + C\Phi)\Sigma; \quad \Sigma(0) = \gamma^{-2}Q_0^{-1} \quad (21b)$$

$$\dot{s}_\Sigma = (\gamma^2 \zeta^2 - 1)(1 - \epsilon)(\bar{C}_1 u + C\Phi)(\bar{C}_1 u + C\Phi)'; \quad s_\Sigma(0) = \gamma^2 \text{Tr}(Q_0) \quad (21c)$$

$$A_f = A - \zeta^2 LC - \Pi C'(\zeta^2 - \gamma^{-2}) \quad (21d)$$

$$\dot{\Phi} = A_f \Phi + y\bar{A}_{211} + u(\bar{A}_{212} - \zeta^2 L\bar{C}_1 - \Pi C'(\zeta^2 - \gamma^{-2})\bar{C}_1); \quad \Phi(0) = \Phi_0 \quad (21e)$$

$$\begin{aligned}\dot{\check{\theta}} &= -\Sigma P_r(\check{\theta}) - \left[\Sigma \quad \Sigma\Phi' \right] \bar{Q}\xi_c - (\Sigma\bar{C}'_1 u + \Sigma\Phi'C')(y_d - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}) \\ &\quad + \gamma^2 \zeta^2 (\Sigma\bar{C}'_1 u + \Sigma\Phi'C')(y - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}); \quad \check{\theta}(0) = \check{\theta}_0\end{aligned}\quad (21f)$$

$$\begin{aligned}\dot{\check{x}} &= -\Phi\Sigma P_r(\check{\theta}) + A\check{x} + (y\bar{A}_{211} + u\bar{A}_{212})\check{\theta} + Bu - \left[\Phi\Sigma \quad \gamma^{-2}\Pi + \Phi\Sigma\Phi' \right] \bar{Q}\xi_c \\ &\quad + \zeta^2(\gamma^2(\Phi\Sigma\bar{C}'_1 u + \gamma^{-2}\Pi C' + \Phi\Sigma\Phi'C') + L)(y - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}) \\ &\quad - (\Phi\Sigma\bar{C}'_1 u + \gamma^{-2}\Pi C' + \Phi\Sigma\Phi'C')(y_d - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}); \quad \check{x}(0) = \check{x}_0\end{aligned}\quad (21g)$$

To simplify the controller structure, the dynamics for Φ can be implemented with $3n$ integrators instead of the σn integrators. First, we observe that the pair (A_f, C) is observable. Then we introduce the matrix

$$M_f := \begin{bmatrix} A_f^{n-1} p_n & \cdots & A_f p_n & p_n \end{bmatrix} \quad (22)$$

where p_n is an n -dimensional vector such that the pair (A_f, p_n) is controllable³, which implies that M_f is invertible. Then the following $3n$ -dimensional prefiltering system for y and u generates the Φ online:

$$\dot{\eta} = A_f \eta + p_n y; \quad \eta(0) = \eta_0 \quad (23a)$$

$$\dot{\lambda} = A_f \lambda + p_n u; \quad \lambda(0) = \lambda_0 \quad (23b)$$

$$\dot{\lambda}_o = A_f \lambda_o; \quad \lambda_o(0) = p_n \quad (23c)$$

$$\begin{aligned}\Phi &= \left[A_f^{n-1} \eta \quad \cdots \quad A_f \eta \quad \eta \right] M_f^{-1} \bar{A}_{211} + \left[A_f^{n-1} \lambda \quad \cdots \quad A_f \lambda \quad \lambda \right] M_f^{-1} (\bar{A}_{212} \\ &\quad - \zeta^2 L\bar{C}_1 - \Pi C'(\zeta^2 - \gamma^{-2})\bar{C}_1) + \left[A_f^{n-1} \lambda_o \quad \cdots \quad A_f \lambda_o \quad \lambda_o \right] M_f^{-1} \Phi_{o0}\end{aligned}\quad (23d)$$

where $\eta_0 \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{R}^n$, and $\Phi_{o0} \in \mathbb{R}^{n \times \sigma}$ are such that (23d) holds at $t = 0$.

Associated with the above identifier, introduce the value function, $W : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \rightarrow \mathbb{R}$,

$$\begin{aligned}W(\xi, \check{\xi}, \bar{\Sigma}) &= |\xi - \check{\xi}|_{\bar{\Sigma}^{-1}}^2 \\ &= |\theta - \check{\theta}|_{\bar{\Sigma}^{-1}}^2 + \gamma^2 |x - \check{x} - \Phi(\theta - \check{\theta})|_{\Pi^{-1}}^2\end{aligned}\quad (24)$$

³The existence of p_n is proven in Zhao and Pan (2003).

whose time derivative along the dynamics of ξ , $\check{\xi}$, and $\bar{\Sigma}$ are given by

$$\begin{aligned} \dot{W}(\xi, \check{\xi}, \bar{\Sigma}, s_\Sigma, y_d, \hat{\xi}, u, w) &= -|Cx + \bar{C}_1\theta u - (y_d - b_{p0}u)|^2 + |C\check{x} + \bar{C}_1\check{\theta}u - (y_d - b_{p0}u)|^2 \\ &\quad -|\xi - \hat{\xi}|_{\bar{Q}}^2 + |\check{\xi} - \hat{\xi}|_{\bar{Q}}^2 + \gamma^2|w|^2 - \gamma^2|w - w_*|^2 \\ &\quad -\gamma^2\zeta^2|y - b_{p0}u - C\check{x} - \bar{C}_1\check{\theta}u|^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) \end{aligned} \quad (25)$$

which holds as long as $\Sigma > 0$, $\check{\theta} \in \Theta_o$. We note that the last term in \dot{W} is nonpositive, zero on the set Θ and approaches $-\infty$ as $\check{\theta}$ approaches the boundary of the set Θ_o , which guarantees the boundness of $\check{\theta}$.

Then the cost function (8) can be equivalently written as, assuming $\Sigma(t) > 0$ and $\check{\theta}(t) \in \Theta_o$, $\forall t \in [0, t_f]$,

$$\begin{aligned} J_{\gamma t_f} &= J_{\gamma t_f} + W(\xi(0), \check{\xi}(0), \bar{\Sigma}(0)) - W(\xi(t_f), \check{\xi}(t_f), \bar{\Sigma}(t_f)) + \int_0^{t_f} \dot{W} d\tau \\ &= -|\xi(t_f) - \check{\xi}(t_f)|_{(\bar{\Sigma}(t_f))^{-1}}^2 + \int_0^{t_f} \left((C\check{x}(\tau) + b_{p0}u(\tau) + \bar{C}_1\check{\theta}(\tau)u(\tau) - y_d(\tau))^2 \right. \\ &\quad -\gamma^2\zeta^2 (y(\tau) - C\check{x}(\tau) - b_{p0}u(\tau) - \bar{C}_1\check{\theta}(\tau)u(\tau))^2 + |\xi_c(\tau)|_{\bar{Q}(\Phi(\tau), u(\tau), s_\Sigma(\tau))}^2 \\ &\quad \left. -\gamma^2|w(\tau) - w_*(\xi(\tau), \check{\xi}(\tau), \bar{\Sigma}(\tau), u(\tau), w(\tau))|^2 + 2(\theta - \check{\theta}(\tau))'P_r(\check{\theta}(\tau)) \right) d\tau \end{aligned} \quad (26)$$

This completes the estimation design step. Next, we consider the control design step.

Based on the inequality (7) in Section 3, the controller design is to guarantee that the following supremum is less than or equal to zero for all measurement waveforms,

$$\begin{aligned} &\sup_{(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{\mathcal{W}}} J_{\gamma t_f} \\ &\leq \sup_{y_{[0, \infty)} \in \mathcal{C}, y_{d[0, \infty)} \in \mathcal{C}} \sup_{(x_0, \theta, w_{[0, \infty)}, y_{d[0, \infty)}) \in \mathcal{W} | y_{[0, \infty)}, y_{d[0, \infty)}} J_{\gamma t_f} \\ &\leq \sup_{y_{[0, \infty)} \in \mathcal{C}, y_{d[0, \infty)} \in \mathcal{C}} \left\{ \int_0^{t_f} \left((C\check{x}(\tau) + (b_{p0} + \bar{C}_1\check{\theta}(\tau))u(\tau) - y_d(\tau))^2 \right. \right. \\ &\quad \left. \left. + |\xi_c(\tau)|_{\bar{Q}(\Phi(\tau), u(\tau), s_\Sigma(\tau))}^2 - \gamma^2\zeta^2 (y(\tau) - C\check{x}(\tau) - (b_{p0} + \bar{C}_1\check{\theta}(\tau))u(\tau))^2 \right) d\tau \right\} \end{aligned} \quad (27)$$

By inequality (27), we observe that the cost function is expressed in terms of signals that we can measure or construct. This is then a nonlinear H^∞ -optimal control problem under full information measurements. Instead of considering y as the maximizing variable, we can equivalently deal with the transformed variable:

$$v := \zeta(y - C\check{x} - (b_{p0} + \bar{C}_1\check{\theta})u)$$

A clear choice for control input u , and the worst-case estimate $\hat{\xi}$, which guarantees that the right-hand-side of (27) is nonpositive, is

$$u := \bar{\mu}(\check{\theta}, \check{x}, y_d) = \frac{y_d - C\check{x}}{\bar{C}_1\check{\theta} + b_{p0}} \quad (28a)$$

$$\hat{\xi} = \check{\xi} \quad (28b)$$

where $\bar{\mu} : \Theta_o \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth. The value function for the closed-loop system is simply W , whose time derivative along solutions of the dynamics for ξ , $\check{\xi}$, and $\bar{\Sigma}$ is

$$\begin{aligned}\dot{W} &= -(Cx + \bar{C}_1\theta u + b_{p0}u - y_d)^2 - |\xi - \hat{\xi}|_Q^2 + \gamma^2|w|^2 - \gamma^2|v|^2 - \gamma^2|w - w_*|^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) \\ &= -(Cx + \bar{C}_1\theta u + b_{p0}u - y_d)^2 - |\xi - \hat{\xi}|_Q^2 + \gamma^2|w|^2 - \gamma^2|w - w_{opt}(\xi, \check{\xi}, y_d, \bar{\Sigma})|^2 \\ &\quad + 2(\theta - \check{\theta})'P_r(\check{\theta})\end{aligned}\tag{29}$$

where the worst-case disturbance with respect to the value function W is given by,⁴ $w_{opt} : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathbb{R} \times \mathcal{S}_{+(n+\sigma)} \rightarrow \mathbb{R}^q$,

$$w_{opt}(\xi, \check{\xi}, y_d, \bar{\Sigma}) = -\zeta^2 E' \begin{bmatrix} \bar{\mu}(\check{\theta}, \check{x}, y_d) \bar{C}_1 & C \end{bmatrix} (\xi - \check{\xi}) + \gamma^{-2} (I_q - \zeta^2 E' E) \bar{D}' \bar{\Sigma}^{-1} (\xi - \check{\xi})\tag{30}$$

which holds as long as $\Sigma > 0$ and $\check{\theta} \in \Theta_o$. Clearly, the closed-loop system is dissipative with storage function W and supply rate

$$-(Cx + \bar{C}_1\theta u + b_{p0}u - y_d)^2 + \gamma^2|w|^2\tag{31}$$

This completes the control design step. We will turn to present the main results in the next section.

5 Main Result

With the estimation and control design of the previous section, the state of the closed-loop system is given by

$$X := \begin{bmatrix} \theta' & x' & \bar{\Sigma}' & s_\Sigma & \check{\theta}' & \check{x}' & \bar{\Phi}' \end{bmatrix}'$$

which belongs to the open set

$$\mathcal{D} := \{X \mid \Sigma > 0, s_\Sigma > 0, \check{\theta} \in \Theta_o\}$$

The dynamics for X are

$$\dot{X} = F(X, y_d) + G(X, y_d)w = F(X, y_d) + G(X, y_d)\dot{M}\dot{w}; \quad X(0) = X_0\tag{32}$$

where F and G are smooth mappings of $\mathcal{D} \times \mathbb{R}$, respectively; and the initial condition

$$X_0 \in \mathcal{D}_0 := \{X_0 \in \mathcal{D} \mid \theta \in \Theta, \check{\theta}_0 \in \Theta, \Sigma(0) = \gamma^{-2}Q_0^{-1} > 0, s_\Sigma(0) = \gamma^2\text{Tr}(Q_0) \leq K_c\}$$

Since (29) holds, by Lemma 7 in Pan and Başar (2000), the value function W satisfies a Hamilton-Jacobi-Isaacs equation.

$$\begin{aligned}\frac{\partial W}{\partial X}(X)F(X, y_d) + \frac{1}{4\gamma^2} \frac{\partial W}{\partial X}(X)G(X, y_d) (G(X, y_d))' \left(\frac{\partial W}{\partial X}(X) \right)' \\ + Q(X, y_d) = 0; \quad \forall X \in \mathcal{D}, \forall y_d \in \mathbb{R}\end{aligned}\tag{33}$$

⁴See Appendix B for derivations.

where $Q : \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth and given by

$$Q(X, y_d) = |Cx + (b_{p0} + \bar{C}_1\theta)\mu(\check{\theta}, \check{x}, y_d) - y_d|^2 + \left| \xi - \check{\xi} \right|_{\bar{Q}(\Phi, \mu(\check{\theta}, \check{x}, y_d), s_\Sigma)}^2 - 2(\theta - \check{\theta})' P_r(\check{\theta}) \quad (34)$$

The closed-loop adaptive system possesses a strong robustness property, which will be stated precisely in the following theorem.

Theorem 1 *Consider the robust adaptive control problem formulated in Section 3 with Assumptions 1-7 holding. The robust adaptive controller μ defined by (28a), with the optimal choice (28b) for $\hat{\xi}$, achieves the following strong robustness properties for the closed-loop system.*

1. *Given $c_w \geq 0$, and $c_d \geq 0$, there exists a constant $c_c \geq 0$ and a compact set $\Theta_c \subset \Theta_o$, such that for any uncertainty $(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{\mathcal{W}}$ with*

$$|x_0| \leq c_w; \quad |y_d(t)| \leq c_d; \quad |\dot{w}(t)| \leq c_w; \quad \forall t \in [0, \infty)$$

all closed-loop state variables $x, \check{x}, \check{\theta}, \Sigma, s_\Sigma$, and Φ are bounded as follows, $\forall t \in [0, \infty)$,

$$|x(t)| \leq c_c; \quad |\check{x}(t)| \leq c_c; \quad \check{\theta}(t) \in \Theta_c; \quad |\overrightarrow{\Phi}(t)| \leq c_c$$

$$K_c^{-1} I_\sigma \leq \Sigma(t) \leq \gamma^{-2} Q_0^{-1}; \quad \gamma^2 \text{Tr}(Q_0) \leq s_\Sigma(t) \leq K_c$$

Therefore, there is a compact set $S \subseteq \mathcal{D}$ such that $X(t) \in S \forall t \in [0, \infty)$. Hence, there exists a constant $c_u \geq 0$ such that $|u(t)| \leq c_u$, $|\dot{\xi}(t)| \leq c_u$, $|\eta(t)| \leq c_u$, $|\lambda(t)| \leq c_u$, and $|\lambda_o(t)| \leq c_u$.

2. *The controller $\mu \in \mathcal{M}_u$ achieves disturbance attenuation level γ for any uncertainty $(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{\mathcal{W}}$.*
3. *For any uncertainty $(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{\mathcal{W}}$ with $\dot{w}_{[0, \infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $y_{d[0, \infty)} \in \mathcal{L}_\infty$, and $y_{d[0, \infty)}$ being uniformly continuous, the output of the system, $Cx + (\bar{C}_1\theta + b_{p0})u$, asymptotically tracks the reference trajectory, y_d , i.e.,*

$$\lim_{t \rightarrow +\infty} (Cx(t) + (\bar{C}_1\theta + b_{p0})u(t) - y_d(t)) = 0$$

Proof For the first statement, fix $c_w \geq 0$ and $c_d \geq 0$, consider any uncertainty $(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{\mathcal{W}}$ that satisfies

$$|x_0| \leq c_w; \quad |\dot{w}(t)| \leq c_w; \quad |y_d(t)| \leq c_d; \quad \forall t \in [0, \infty)$$

We define $[0, T_f)$ to be the maximal length interval on which the close-loop system (32) has a solution that lies in \mathcal{D} . By Lemma 1, we have Σ and s_Σ are upper and lower bounded as desired on $[0, T_f)$.

Introduce the vector of variables

$$X_e := \begin{bmatrix} \check{\theta}' & (\check{x} - \Phi\check{\theta})' \end{bmatrix}'$$

and two nonnegative and continuous functions defined on $\mathbb{R}^{n+\sigma}$

$$\begin{aligned} U_M(X_e) &:= K_c|\tilde{\theta}|^2 + \gamma^2|\tilde{x} - \Phi\tilde{\theta}|_{\Pi^{-1}}^2 \\ U_m(X_e) &:= \gamma^2|\tilde{\theta}|_{Q_0}^2 + \gamma^2|\tilde{x} - \Phi\tilde{\theta}|_{\Pi^{-1}}^2 \end{aligned}$$

then, we have, if we interpret W as $\bar{W}(t, X_e)$,

$$U_m(X_e) \leq \bar{W}(t, X_e) \leq U_M(X_e), \quad \forall (t, X_e) \in [0, T_f] \times \mathbb{R}^{n+\sigma}$$

Since $U_m(X_e)$ is continuous, nonnegative and radially unbounded, then $\forall \alpha \in \mathbb{R}$, the set $S_{1\alpha} := \{X_e \in \mathbb{R}^{n+\sigma} : U_m(X_e) \leq \alpha\}$ is compact or empty. Since $|\dot{w}(t)| \leq c_w, \forall t \in [0, \infty)$, we have the following inequality for the derivative of W :

$$\begin{aligned} \dot{W} &= -|Cx + (\bar{C}_1\theta + b_{p0})u - y_d|^2 - \gamma^4|x - \hat{x} - \Phi(\theta - \hat{\theta})|_{\Pi^{-1}\Delta\Pi^{-1}}^2 \\ &\quad - \epsilon(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|_{(u\bar{C}_1 + C\Phi)'(u\bar{C}_1 + C\Phi)}^2 + 2(\theta - \hat{\theta})'P_r(\hat{\theta}) + \gamma^2|w|^2 - \gamma^2|w - w_{opt}|^2 \\ &\leq -\gamma^4|x - \tilde{x} - \Phi(\theta - \tilde{\theta})|_{\Pi^{-1}\Delta\Pi^{-1}}^2 + 2(\theta - \tilde{\theta})'P_r(\tilde{\theta}) + \gamma^2\|\dot{M}\|_2^2 c_w^2 \end{aligned} \quad (35)$$

Since $-\gamma^4|x - \tilde{x} - \Phi(\theta - \tilde{\theta})|_{\Pi^{-1}\Delta\Pi^{-1}}^2 + 2(\theta - \tilde{\theta})'P_r(\tilde{\theta})$ will tend to $-\infty$ when X_e approaches the boundary of $\Theta_o \times \mathbb{R}^n$, then there exists a compact set $\Omega_1(c_w) \subset \Theta_o \times \mathbb{R}^n$, such that $\dot{W} < 0$ for $\forall X_e \in (\Theta_o \times \mathbb{R}^n) \setminus \Omega_1$. To apply the Lemma 9 in Pan and Başar (2000), we make the following substitutions,

$$\begin{aligned} U_m &\rightarrow W_1; \quad U_M \rightarrow W_2; \quad \bar{W} \rightarrow V; \quad S_{1\alpha} \rightarrow S_{1\alpha}; \quad \mathbb{R}^{n+\sigma} \rightarrow D; \quad c_1 \rightarrow \eta \\ S_{1c_1} &\rightarrow S_{1\eta}; \quad X_e \rightarrow \xi; \quad [0, T_f] \rightarrow [t_0, t_1]; \quad \Omega_1 \rightarrow K; \quad n + \sigma \rightarrow n \end{aligned}$$

Note that $X_e(t) \in \Theta_o \times \mathbb{R}^n, \forall t \in [0, T_f]$. Then, $\bar{W}(t, X_e(t)) \leq c_1$, and $X_e(t)$ belongs to a compact set $S_{1c_1} \subseteq \mathbb{R}^{n+\sigma}, \forall t \in [0, T_f]$, for some $c_1 \in \mathbb{R}$. It follows that the signal X_e is bounded, namely, $\tilde{\theta}$ and $\tilde{x} - \Phi\tilde{\theta}$ are bounded.

Based on the derivative of $\tilde{x} - \Phi\tilde{\theta}$,⁵ there is particular linear combination of the components of $\tilde{x} - \Phi\tilde{\theta}$, denoted by η_L ,

$$\begin{aligned} \frac{d(\tilde{x}(t) - \Phi(t)\tilde{\theta}(t))}{dt} &= A_f(\tilde{x}(t) - \Phi(t)\tilde{\theta}(t)) - \gamma^{-2}\Pi C'y(t) + \gamma^{-2}\Pi C'y_d(t) + (D + \gamma^{-2}\Pi C'E \\ &\quad - \zeta^2(\Pi C' + L)E)\dot{M}\dot{w}(t) \end{aligned} \quad (36a)$$

$$\eta_L = T_L(\tilde{x} - \Phi\tilde{\theta}) \quad (36b)$$

which is strictly minimum phase and has relative degree 1 with respect to y . Then by Lemma 8, the composite system of (1) and (36) has relative degree 1 from input u to output η_L ; and it may serve as a reference systems in the application of Lemma 11 in Pan and Başar (2000) in the following proof.

By Lemma 7, the system (36) with input y and output η_L may serve as reference system in the application of Lemma 11 of Pan and Başar (2000).

Based on the dynamics of η , the relative degree for each element of η is at least 1 with respect to the input y . Taking η_L as the output and y as the input of the reference system, we conclude η

⁵See Appendix B for detailed derivations.

is bounded by Lemma 11 of Pan and Başar (2000). Based on the dynamics of λ , the relative degree for each element of λ is at least 1 with respect to the input u . Taking η_L as the output and u as the input of the reference system, we conclude λ is also bounded.

Based on the dynamics of λ_o (23c), we have λ_o is bounded. Based on the formula for Φ , (23d), we have Φ is bounded. Since $\tilde{x} - \Phi\tilde{\theta}$, Φ , and $\tilde{\theta}$ are bounded, we conclude that \tilde{x} is bounded. Define the following equations to separate x into two part:

$$\begin{aligned} x &= x_u + x_y \\ \dot{x}_u &= A_f x_u + \left(B + \bar{A}_{212}\theta - (\bar{C}_1\theta + b_{p0}) \left(\zeta^2 L + \Pi C' \left(\zeta^2 - \frac{1}{\gamma^2} \right) \right) \right) u \\ \dot{x}_y &= A_f x_y + \left(\bar{A}_{211}\theta + \zeta^2 L + \Pi C' \left(\zeta^2 - \frac{1}{\gamma^2} \right) \right) y + \left(D - \left(\zeta^2 L + \Pi C' \left(\zeta^2 - \frac{1}{\gamma^2} \right) \right) E \right) \dot{M}\dot{v} \end{aligned}$$

We observe that the relative degree for each element of x_u is at least 1 with respect to the input u , and the relative degree for each element of x_y is at least 1 with respect to the input y . Taking η_L as the output and u as the input of the reference system, we conclude x_u is bounded by Lemma 11 in Pan and Başar (2000). Similarly, Taking η_L as the output and y as the input of the reference system, we obtain that x_y is bounded. Then, we have x is bounded. Further, we have \tilde{x} is bounded, because x and \tilde{x} are bounded and $\tilde{x} = x - \tilde{x}$. Since \tilde{b}_0 is bounded, and bounded away from 0, we conclude that u is bounded.

Next, we need to prove the existence of a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c, \forall t \in [0, T_f]$. First introduce the function $\Upsilon : [0, T_f] \times (\Theta_o \times \mathbb{R}^n) \rightarrow \mathbb{R}$ by

$$\Upsilon(t, X_e) := \bar{W}(t, X_e) + (\rho_o - P(\check{\theta}))^{-1} P(\check{\theta})$$

We notice that, when $\check{\theta}$ approaches the boundary of Θ_o , $P(\check{\theta})$ approaches ρ_o . Then Υ approaches ∞ as X_e approaches the boundary of $\Theta_o \times \mathbb{R}^n$. We introduce two nonnegative and continuous functions defined on $\Theta_o \times \mathbb{R}^n$:

$$\begin{aligned} \Upsilon_M(X_e) &:= U_M(X_e) + (\rho_o - P(\check{\theta}))^{-1} P(\check{\theta}) \\ \Upsilon_m(X_e) &:= U_m(X_e) + (\rho_o - P(\check{\theta}))^{-1} P(\check{\theta}) \end{aligned}$$

Then, by the previous analysis, we have

$$\Upsilon_m(X_e) \leq \Upsilon(t, X_e) \leq \Upsilon_M(X_e), \quad \forall (t, X_e) \in [0, T_f] \times (\Theta_o \times \mathbb{R}^n)$$

Note that the set $S_{2\alpha} := \{X_e \in \Theta_o \times \mathbb{R}^n \mid \Upsilon_m(X_e) \leq \alpha\}$ is a compact set or an empty set, $\forall \alpha \in \mathbb{R}$. Then, we consider the derivative of Υ ,

$$\begin{aligned} \dot{\Upsilon} &= \dot{\bar{W}} + \rho_o (\rho_o - P(\check{\theta}))^{-2} \frac{\partial P}{\partial \theta}(\check{\theta}) \dot{\check{\theta}}(t) \\ &\leq -|Cx + (\bar{C}_1\theta + b_{p0})u - y_d|^2 - \gamma^4 |x - \hat{x} - \Phi(\theta - \hat{\theta})|_{\Pi^{-1}\Delta\Pi^{-1}}^2 + 2(\theta - \check{\theta})' P_r(\check{\theta}) \\ &\quad - \epsilon (\gamma^2 \zeta^2 - 1) |\theta - \hat{\theta}|_{(u\bar{C}_1 + C\Phi)'(u\bar{C}_1 + C\Phi)}^2 + \gamma^2 \|\dot{M}\|_2^2 c_w^2 + \rho_o (\rho_o - P(\check{\theta}))^{-2} \left(-\frac{\partial P}{\partial \theta}(\check{\theta}) \Sigma P_r(\check{\theta}) \right. \\ &\quad \left. - \frac{\partial P}{\partial \theta}(\check{\theta}) (u\Sigma\bar{C}'_1 + \Sigma\Phi'C') \left(y_d - C\tilde{x} - (\bar{C}_1\check{\theta} + b_{p0})u \right) + \frac{\partial P}{\partial \theta}(\check{\theta}) \gamma^2 \zeta^2 (u\Sigma\bar{C}'_1 + \Sigma\Phi'C') \right. \\ &\quad \left. \cdot \left(y - C\tilde{x} - (\bar{C}_1\check{\theta} + b_{p0})u \right) \right) \end{aligned}$$

which hold as long as $\Sigma > 0$ and $\check{\theta} \in \Theta_o$. We observe the following facts.

$$\begin{aligned} y_d - C\check{x} - u(\bar{C}_1\check{\theta} + b_{p0}) &= 0 \\ \frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2\zeta^2(u\Sigma\bar{C}'_1 + \Sigma\Phi'C')(y - C\check{x} - (\bar{C}_1\check{\theta} + b_{p0})u) \\ &\leq \left| \left(\frac{\partial P}{\partial \theta}(\check{\theta}) \right)' (\rho_o - P(\check{\theta}))^{-1} \right|^2 + \left| \frac{(\rho_o - P(\check{\theta}))}{2} \gamma^2 \zeta^2 (u\Sigma\bar{C}'_1 + \Sigma\Phi'C') (y - C\check{x} - (\bar{C}_1\check{\theta} + b_{p0})u) \right|^2 \end{aligned}$$

Then,

$$\begin{aligned} \dot{\Upsilon} &\leq -\gamma^4|x - \hat{x} - \Phi(\theta - \hat{\theta})|_{\Pi^{-1}\Delta\Pi^{-1}}^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) + \gamma^2\|\dot{M}\|_2^2c_w^2 \\ &\quad + \rho_o(\rho_o - P(\check{\theta}))^{-4} \left(-\frac{\partial P}{\partial \theta}(\check{\theta})\Sigma P_r(\check{\theta})(\rho_o - P(\check{\theta}))^2 + \left| \left(\frac{\partial P}{\partial \theta}(\check{\theta}) \right)' \right|^2 \right) \\ &\quad + \frac{\rho_o}{4} \left| \gamma^2\zeta^2(u\Sigma\bar{C}'_1 + \Sigma\Phi'C')(y - C\check{x} - (\bar{C}_1\check{\theta} + b_{p0})u) \right|^2 \\ &\leq -\gamma^4|x - \hat{x} - \Phi(\theta - \hat{\theta})|_{\Pi^{-1}\Delta\Pi^{-1}}^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) + \gamma^2\|\dot{M}\|_2^2c_w^2 \\ &\quad - \rho_o \left| \left(\frac{\partial P}{\partial \theta}(\check{\theta}) \right)' \right|^2 (\rho_o - P(\check{\theta}))^{-4} \left(K_c^{-1}p_r(\check{\theta})(\rho_o - P(\check{\theta}))^2 - 1 \right) + c \end{aligned}$$

where $c \geq 0$ is a constant, which holds as long as $\Sigma > 0$ and $\check{\theta} \in \Theta_o$. Since $\dot{\Upsilon}$ will tend to $-\infty$ when X_e approaches the boundary of $\Theta_o \times \mathbb{R}^n$, then there exists a compact set $\Omega_2(c_w) \subset \Theta_o \times \mathbb{R}^n$, such that $\forall X_e \in (\Theta_o \times \mathbb{R}^n) \setminus \Omega_2$, $\dot{\Upsilon} < 0$. To use the Lemma 9 in Pan and Başar (2000), we make the following substitutions,

$$\begin{aligned} \Upsilon_m &\rightarrow W_1; \quad \Upsilon_M \rightarrow W_2; \quad \Upsilon \rightarrow V; \quad S_{2\alpha} \rightarrow S_{1\alpha}; \quad \Theta_o \times \mathbb{R}^n \rightarrow D; \quad c_2 \rightarrow \eta \\ S_{2c_2} &\rightarrow S_{1\eta}; \quad X_e \rightarrow \xi; \quad [0, T_f] \rightarrow [t_0, t_1]; \quad \Omega_2 \rightarrow K; \quad n + \sigma \rightarrow n \end{aligned}$$

Then, $\Upsilon(t, X_e(t)) \leq c_2$, and $X_e(t)$ belongs to the compact set $S_{2c_2} \subset \Theta_o \times \mathbb{R}^n$, $\forall t \in [0, T_f]$, for some $c_2 \in \mathbb{R}$. Moreover, there exists a compact set $\Theta_c \subset \Theta_o$, such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f]$.

Then we can conclude that there exists a compact set $S \subset \mathcal{D}$, such that $X(t) \in S$, $\forall t \in [0, T_f]$. Therefore, it follows that $T_f = \infty$ and $X(t) \in S$, $\forall t \in [0, \infty)$.

For the second statement, we fix any uncertainty $(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{\mathcal{W}}$. Since y_d and \dot{w} are continuous functions, then, for any $t_f > 0$, there exist constants $c_d \geq 0$ and $c_w \geq 0$ such that $|y_d(t)| \leq c_d$ and $|\dot{w}(t)| \leq c_w$, $\forall t \in [0, t_f]$. By the first statement, there exists a solution $X : [0, t_f] \rightarrow \mathcal{D}$ for the closed-loop system (32). Then, by causality of system (32) and its smoothness, it implies that the closed-loop system (32) admits a unique solution on $[0, \infty)$. We set

$$\begin{aligned} l(t, \theta, x, y_{[0, t]}, y_{d[0, t]}) &:= \gamma^4|x - \hat{x}(t) - \Phi(t)(\theta - \hat{\theta}(t))|_{\Pi^{-1}\Delta\Pi^{-1}}^2 - 2(\theta - \check{\theta}(t))'P_r(\check{\theta}(t)) \\ &\quad + \epsilon(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}(t)|_{(u(t)\bar{C}_1 + C\Phi(t))'(u(t)\bar{C}_1 + C\Phi(t))}^2 \end{aligned}$$

Then, we have

$$\sup_{(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{\mathcal{W}}} \left\{ \int_0^{t_f} \left((Cx(\tau) + (\bar{C}_1\theta + b_{p0})u(\tau) - y_d(\tau))^2 + l(\tau, \theta, x(\tau), y_{[0, \tau]}, y_{d[0, \tau]}) \right) \right.$$

$$\begin{aligned}
& -\gamma^2|w(\tau)|^2) d\tau - \gamma^2 \left| \left[\begin{array}{cc} \theta' - \check{\theta}'_0 & x'_0 - \check{x}'_0 \end{array} \right]' \right|_{\bar{Q}_0}^2 \Big\} \\
& \leq \sup_{(x_0, \theta, \dot{w}_{[0, \infty)}, y_{d[0, \infty)}) \in \dot{W}} \left\{ \int_0^{t_f} \left((Cx(\tau) + (\bar{C}_1\theta + b_{p0})u(\tau) - y_d(\tau))^2 + l(\tau, \theta, x(\tau), y_{[0, \tau]}, y_{d[0, \tau]}) \right. \right. \\
& \quad \left. \left. - \gamma^2|w(\tau)|^2) d\tau - \gamma^2 \left| \left[\begin{array}{cc} \theta' - \check{\theta}'_0 & x'_0 - \check{x}'_0 \end{array} \right]' \right|_{\bar{Q}_0}^2 + \int_0^{t_f} \dot{W} d\tau - W(X(t_f)) + W(X(0)) \right\} \\
& \leq -W(X(t_f)) \leq 0
\end{aligned}$$

This establishes the second statement.

For the last statement, we consider the following inequality,

$$\int_0^{t_f} \dot{W} d\tau \leq \int_0^{t_f} (|Cx(\tau) + (\bar{C}_1\theta + b_{p0})u(\tau) - y_d(\tau)|^2 + \gamma^2|\dot{M}\dot{w}(\tau)|^2) d\tau; \quad \forall t_f \geq 0$$

Then,

$$\int_0^\infty |Cx(\tau) + (\bar{C}_1\theta + b_{p0})u(\tau) - y_d(\tau)|^2 d\tau \leq \int_0^\infty (\gamma^2|\dot{M}\dot{w}(\tau)|^2) d\tau + W(X(0)) < +\infty$$

By Lemma 2, X is uniformly continuous. Since $y_d - C\check{x}$ is uniformly continuous and bounded, $\bar{C}_1\check{\theta} + b_{p0}$ is uniformly continuous and bounded away from 0, then u is uniformly continuous by Lemmas 3 and 4. Then, $Cx + (\bar{C}_1\theta + b_{p0})u - y_d$ is uniformly continuous. By Lemma 5, we have

$$\lim_{t \rightarrow +\infty} (Cx(t) + (\bar{C}_1\theta + b_{p0})u(t) - y_d(t)) = 0$$

This completes the proof of the theorem. \square

6 Example

In this section, we present one example to illustrate the main results of this paper. The design was carried out using MATLABTM symbolic computation tools, and the closed-loop system was simulated using SIMULINKTM.

Consider the following circuit problem in Figure 1, where v_i is the input voltage source; v_o is the measured output; v_e is an unknown sinusoidal voltage source; v_w is an unmeasured exogenous voltage source. The objective is to achieve asymptotic tracking of $v_o - v_w$ to the reference trajectory y_d .

The equations that describe the circuit are obtained as

$$y = v_e + b_0 u + \dot{w}$$

where $u = v_i$, $y = v_o$, $\dot{w} = v_w$, and b_0 is the ratio of v_t to v_i , which is assumed to be unknown and belong to the interval $[0.1, 1]$, whose true value is set to 1 for illustration purposes. v_e can be modeled as the output of a second-order linear system as the following,

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2; & \hat{x}_1(0) &= 1 \\
\dot{\hat{x}}_2 &= \bar{\theta}_2 \hat{x}_1; & \hat{x}_2(0) &= 1
\end{aligned}$$

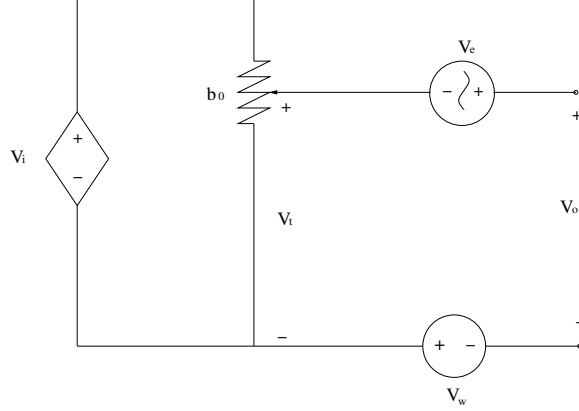


Figure 1: Diagram of the circuit.

where $\bar{\theta}_2$ is assumed to be unknown and belong to the interval $[-4, 0]$, whose true value is set to -4 for illustration purposes, and the initial conditions are set for illustration purposes. Note that the true system satisfies the Assumptions 1 and 2.

To normalize the parameters, we set $b_{p0} = 0.55$, and define $\theta = [\theta_1 \ \theta_2 \ \theta_3]'$, where $\theta_1 = (b_0 - b_{p0})/0.45$, $\theta_2 = (\bar{\theta}_2 + 2)/2$ and $\theta_3 = \theta_1\theta_2$. Then, the true value for parameter vector θ is $[1 \ -1 \ -1]'$, and θ_1 , θ_2 , and θ_3 all belong to the interval $[-1, 1]$.

Introduce the state transformation $x = \hat{x}$, and disturbance transformations, $w = \begin{bmatrix} 1 & \theta_2/3 \end{bmatrix}' \dot{w}$, we obtain the design model for the adaptive controller

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x + \left(y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1.1 & -0.9 \end{bmatrix} \right) \theta + \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0.55u + \begin{bmatrix} 0.45 & 0 & 0 \end{bmatrix} \theta u + \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{aligned}$$

For the adaptive control design, the ultimate lower bound for the achievable disturbance attenuation level is 1 with respect to w . We set the desired disturbance attenuation level $\gamma = 10$.

The convex function $P(\theta)$ is chosen as

$$P(\theta) = 0.8\theta_1^2 + 0.1(\theta_2^2 + \theta_3^2)$$

For other design and simulation parameters, we select

$$\begin{aligned} \tilde{x}_0 &= \begin{bmatrix} 0 & 0 \end{bmatrix}'; \quad \check{\theta}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}'; \quad Q_0 = 0.005I_3; \quad \Delta_1 = I_2; \quad p_n = \begin{bmatrix} 0 & 1 \end{bmatrix}' \\ \Phi_0 &= \mathbf{0}_{2 \times 3}; \quad K_c = 1.5; \quad \rho_o = 1.15; \quad \beta_\Delta = 0; \quad \epsilon = K_c^{-1} s_\Sigma; \quad \eta_0 = \mathbf{0}_{2 \times 1}; \quad \lambda_0 = \mathbf{0}_{2 \times 1} \end{aligned}$$

Then, we obtain

$$\Pi = \begin{bmatrix} 10.998 & 9.873 \\ 9.873 & 129.49 \end{bmatrix}$$

Set the reference trajectory, $y_d = \sqrt{|\sin(3t)|}$, which is uniformly continuous on $[0, \infty)$. We present two sets of simulation results for this example. The first set is to illustrate the regulatory

behaviour of the adaptive controller. We set $\dot{w}(t) \equiv 0$. The results are shown in Figure 2. We observe that the parameter estimates converge to their true values, and the tracking error converges to 0. The transient of the system is well-behaved, and the control magnitude is upper bounded by 2.2. The convergence rate is clearly exponential. When $t = 175$, the magnitude of the tracking error reduces to 10^{-14} , and the magnitude of the parameter estimation error reduces to 10^{-13} .

The second set of simulation results is to illustrate the robustness property of the adaptive controller. We set $\dot{w}(t) = 0.4 \sin(t)$. The simulation results are shown in Figure 3. We observe that the parameter estimates no longer converge to the true values, but the output tracking error satisfies the targeted attenuation level as desired according to Figure 3(f). The transient of the system is well-behaved which lasted about 5 seconds, and the control magnitude is upper bounded by 2.3.

7 Conclusions

In this paper, we studied the adaptive control design for tracking and disturbance attenuation for SISO linear systems with zero relative degree under noisy output measurements. We assume that the linear system has a known upper bound of the dynamic order, has a strictly minimum phase transfer function with known relative degree 0. We allow the system to be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov and those uncontrollable modes on the $j\omega$ -axis are uncontrollable from the disturbance input. Under these assumptions, the system may be transformed into the design model, which is linear in all of the unknown quantities. The objectives of the control design are to make the noiseless output of the system to asymptotically track the reference trajectory, and guarantee the boundedness of all closed-loop signals, while rejecting the uncertainties in the system, which comprises the initial state of the system, the true value of the unknown parameter vector, the waveform of the exogenous disturbance input, and the reference trajectory. We use H^∞ -optimal control formulation and game theoretic approach to derive the robust adaptive controller. We treat the unknown parameter vector as part of the expanded state vector, and formulate this adaptive control problem as a nonlinear H^∞ -optimal control problem with imperfect state measurements. For the design model, we assume that the measurement channel is noisy, such that the estimation step is a nonsingular optimization problem. We further assume that the unknown parameter vector belongs to a convex compact set characterized by a known smooth nonnegative radially unbounded and strictly convex function $P(\theta)$. Furthermore, for any parameter vector belonging to the set, the corresponding high frequency gain is never zero. Then, the *cost-to-come* function analysis is applied to derive the worst-case identifier and state estimator, which have a finite-dimensional structure. Using *a priori* information on the parameter vector, a smooth soft projection algorithm is applied in the estimation step, which relieves the persistency of excitation condition for the closed-loop system. Then, the closed-loop system is robust with or without the persistently exciting signals. After the estimation step is completed, the original problem becomes a nonlinear H^∞ -optimal control problem under full-information measurements. Then, the controller can be obtained directly from the cost function in one step. The controller then achieves the desired disturbance attenuation level, with the ultimate lower bound of the attenuation level being the noise intensity in the measurement channel. It guarantees the boundedness of all closed-loop signals and achieves asymptotic tracking of uniformly continuous bounded reference trajectories when the disturbance is of finite energy and bounded. Because of the assumptions we made on the unknown system, the adaptive controller can asymptotically cancel out the effect

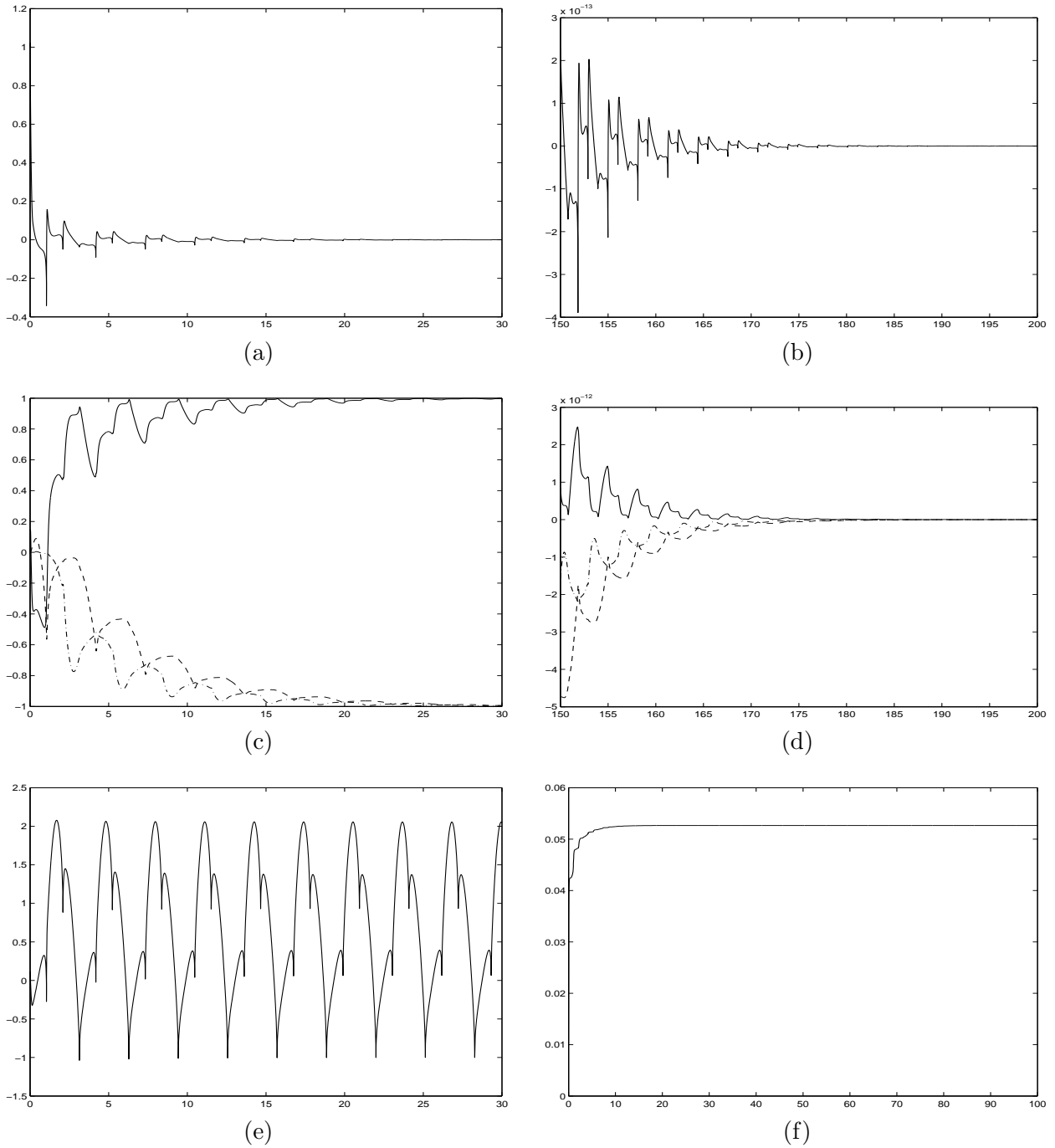


Figure 2: System response under reference trajectory $y_d(t) = \sqrt{|\sin(3t)|}$ and $\dot{w}(t) = 0$. (a) Tracking error; (b) Tracking error (long term); (c) Parameter estimates; solid line for $\hat{\theta}_1$; dash line for $\hat{\theta}_2$; and dash dot line for $\hat{\theta}_3$; (d) Parameter estimation error (long term); solid line for $\theta_1 - \hat{\theta}_1$; dash line for $\theta_2 - \hat{\theta}_2$; and dash dot line for $\theta_3 - \hat{\theta}_3$; (e) Control input; (f) A performance index $\int_0^t ((Cx + b_0u - y_d)^2 - \gamma^2|w|^2)d\tau$.

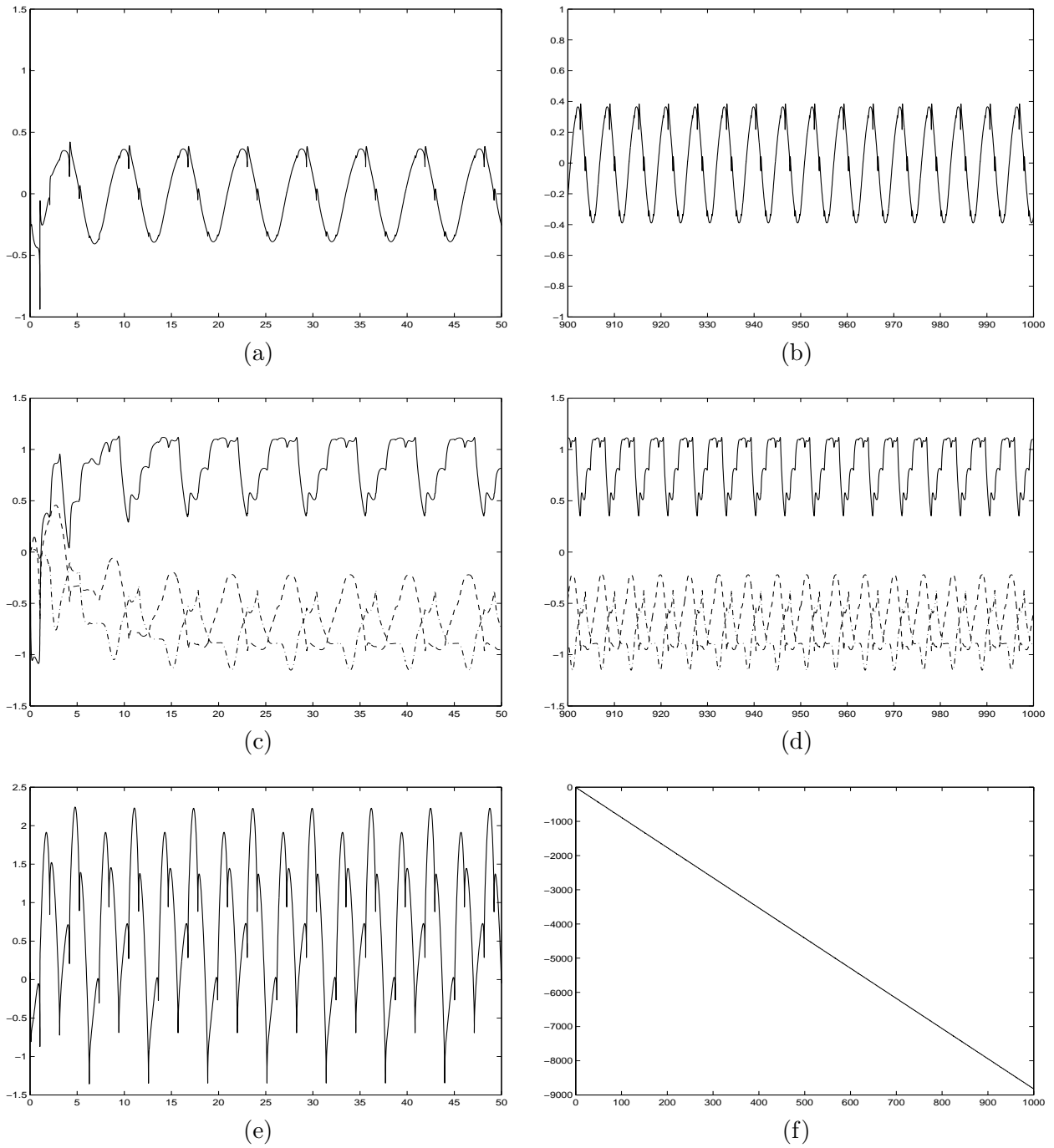


Figure 3: System response under reference trajectory $y_d(t) = \sqrt{|\sin(3t)|}$ and $\dot{w}(t) = 0.4\sin(t)$. (a) Tracking error; (b) Tracking error (long term); (c) Parameter estimates; solid line for $\hat{\theta}_1$; dash line for $\hat{\theta}_2$; and dash dot line for $\hat{\theta}_3$; (d) Parameter estimates (long term); solid line for $\check{\theta}_1$; dash line for $\check{\theta}_2$; and dash dot line for $\check{\theta}_3$; (e) Control input; (f) A performance index $\int_0^t ((Cx + b_0u - y_d)^2 - \gamma^2|w|^2)d\tau$.

of exogenous sinusoidal inputs with unknown magnitudes, phases, and frequencies, as long as we extend our system model to incorporate the knowledge of the existence of such sinusoidal inputs. This property of our adaptive controller has significant impact on active noise cancellation applications. This feature is illustrated by a numerical example, which corroborates all of our theoretical findings.

Future research directions that are of interest are described as follows. One direction lies in the generalization of the results to nonlinear systems. Another fruitful direction lies in the extension of the results to multiple-input and multiple-output systems. The class of MIMO systems under study involves two subsystems, \mathbf{S}_1 , and \mathbf{S}_2 , interconnected to each other, where the connection is serial. Preliminary results have been obtained for this class of MIMO systems.

A Seven Lemmas

Lemma 2 *Assume $f : [0, \infty) \rightarrow \mathbb{R}$ is differentiable in t , and satisfies*

$$|f^{(1)}(t)| \leq M < \infty; \quad \forall t \in [0, \infty)$$

for some $M \in \mathbb{R}$, $M \geq 0$. Then, f is uniformly continuous on $[0, \infty)$.

Proof Given a $\epsilon > 0$, we take $\delta(\epsilon) = \frac{\epsilon}{M+1}$. Then, if $x, y \in [0, \infty)$ and $|x - y| \leq \delta(\epsilon)$, from the Mean Value Theorem, we have

$$|f(x) - f(y)| = |f^{(1)}(\xi)| \cdot |x - y| \leq M \cdot \delta(\epsilon) = M \cdot \frac{\epsilon}{M+1} < \epsilon$$

This completes the proof of this Lemma. □

Lemma 3 *Assume $g : [0, \infty) \rightarrow \mathbb{R}$ and $h : [0, \infty) \rightarrow \mathbb{R}$ are uniformly continuous. If $\exists M > 0$, such that $|g(t)| \leq M$ and $|h(t)| \leq M$, $\forall t \in [0, \infty)$, then $g \cdot h$ is uniformly continuous on $[0, \infty)$.*

Proof Since g is uniformly continuous, $\forall \epsilon > 0$, $\exists \delta_g(\epsilon) > 0$ such that $|g(x) - g(y)| < \epsilon$, $\forall x, y \in [0, \infty)$ with $|x - y| < \delta_g(\epsilon)$; since h is uniformly continuous, $\forall \epsilon > 0$, $\exists \delta_h(\epsilon) > 0$ such that $|h(x) - h(y)| < \epsilon$, $\forall x, y \in [0, \infty)$ with $|x - y| < \delta_h(\epsilon)$. Then, given an $\epsilon > 0$, we take $\delta(\epsilon) = \min\{\delta_h(\frac{\epsilon}{2M}), \delta_g(\frac{\epsilon}{2M})\}$, and we have

$$\begin{aligned} |g(x) \cdot h(x) - g(y) \cdot h(y)| &= |g(x) \cdot h(x) - g(x) \cdot h(y) + g(x) \cdot h(y) - g(y) \cdot h(y)| \\ &\leq |g(x)| \cdot |h(x) - h(y)| + |h(y)| \cdot |g(x) - g(y)| \\ &\leq M|h(x) - h(y)| + M|g(x) - g(y)| \\ &< \epsilon; \quad \forall x, y \in [0, \infty) \text{ with } |x - y| < \delta(\epsilon) \end{aligned}$$

This completes the proof of this Lemma. □

Lemma 4 *Assume $h : [0, \infty) \rightarrow \mathbb{R}$ is uniformly continuous. If $\exists M > 0$, such that $|h(t)| \geq M$, $\forall t \in [0, \infty)$, then $\frac{1}{h(t)}$ is uniformly continuous on $[0, \infty)$.*

Proof Since h is uniformly continuous, $\forall \epsilon > 0, \exists \delta_h(\epsilon) > 0$ such that $|h(x) - h(y)| < \epsilon, \forall x, y \in [0, \infty)$ with $|x - y| < \delta_h(\epsilon)$. Then, given an $\epsilon > 0$, we take $\delta(\epsilon) = \delta_h(\frac{\epsilon}{M^2})$, and we have

$$\begin{aligned} \left| \frac{1}{h(x)} - \frac{1}{h(y)} \right| &= \left| \frac{h(x) - h(y)}{h(x) \cdot h(y)} \right| \\ &\leq M^2 |h(x) - h(y)| \\ &< \epsilon; \quad \forall x, y \in [0, \infty) \text{ with } |x - y| < \delta(\epsilon) \end{aligned}$$

This completes the proof of this Lemma. \square

Lemma 5 *If $f : [0, \infty) \rightarrow \mathbb{R}$ is uniformly continuous in t , and satisfies:*

$$\int_0^\infty |f(\tau)|^p d\tau < \infty; \quad p \in (0, \infty)$$

then $\lim_{t \rightarrow \infty} f(t) = 0$.

Proof We prove it by contradiction. Suppose $\lim_{t \rightarrow \infty} f(t) \neq 0$, then $\exists \epsilon_0 > 0, \forall T > 0$ exists $t_0 > T$ such that $|f(t_0)| > \epsilon_0$.

From the definition of uniformly continuous, take $\epsilon = \frac{\epsilon_0}{2} > 0$, there exists $\delta > 0$ such that $\forall t_1, t_2 \in [0, \infty)$ with $|t_1 - t_2| < \delta$, we have

$$|f(t_1) - f(t_2)| \leq \epsilon$$

For the above t_0 , it follows $|f(t)| \geq \frac{\epsilon_0}{2}$ when $t \in [t_0 - \delta, t_0 + \delta] \cap [0, \infty)$.

First we take $T_1 = \delta$, then we may obtain $t_1 > T_1$ and a time interval $[t_1 - \delta, t_1 + \delta]$, such that $|f(t)| \geq \frac{\epsilon_0}{2}, \forall t \in [t_1 - \delta, t_1 + \delta]$. Next, take $T_2 = t_1 + 2\delta$, then we obtain a $t_2 > T_2$ and a time interval $[t_2 - \delta, t_2 + \delta]$, such that $|f(t)| \geq \frac{\epsilon_0}{2}, \forall t \in [t_2 - \delta, t_2 + \delta]$. Continue such procedure, we obtain a sequence $\{t_n\}_{n=1}^\infty$ and corresponding disjoint time intervals $[t_n - \delta, t_n + \delta], n = 1, 2, \dots$. Then

$$\begin{aligned} \int_0^\infty |f(\tau)|^p d\tau &\geq \int_{t_1 - \delta}^{t_1 + \delta} |f(\tau)|^p d\tau + \int_{t_2 - \delta}^{t_2 + \delta} |f(\tau)|^p d\tau + \dots + \int_{t_n - \delta}^{t_n + \delta} |f(\tau)|^p d\tau + \dots \\ &\geq \int_{t_1 - \delta}^{t_1 + \delta} \left| \frac{1}{2} \epsilon_0 \right|^p d\tau + \int_{t_2 - \delta}^{t_2 + \delta} \left| \frac{1}{2} \epsilon_0 \right|^p d\tau + \dots + \int_{t_n - \delta}^{t_n + \delta} \left| \frac{1}{2} \epsilon_0 \right|^p d\tau + \dots \\ &\geq \left(\frac{1}{2} \epsilon_0 \right)^p 2\delta N \quad \forall N \in \mathbb{N} \end{aligned}$$

which contradicts $\int_0^\infty |f(\tau)|^p d\tau < \infty$.

Therefore $\lim_{t \rightarrow \infty} f(t) = 0$. \square

Lemma 6 *Consider the system (1) under Assumption 2. Assume $\dot{n} < n$. Then, we can always add additional $(n - \dot{n})$ -dimensional dynamics, such that the expanded system is of order n and satisfies Assumption 2. Furthermore, the expanded system admits the same mapping from $(\dot{x}_0, u_{[0, \infty)}, \dot{w}_{[0, \infty)})$ to $y_{[0, \infty)}$ as system (1).*

Proof We will discuss two exhaustive and mutually exclusive cases: Case 1: $\dot{n} = 0$; Case 2: $\dot{n} > 0$. First, consider Case 1: $\dot{n} = 0$. We expand the system as following,

$$\dot{\tilde{x}} = \tilde{A}_{22} \tilde{x}; \quad \tilde{x}(0) = 0 \tag{37a}$$

$$y = \tilde{C} \tilde{x} + b_0 u + \dot{E} \dot{w} \tag{37b}$$

where \tilde{x} is the n -dimensional state vector; \tilde{A}_{22} is Hurwitz, and $(\tilde{A}_{22}, \tilde{C})$ is observable. We notice that, since (37a) is a stable system with zero initial conditions, the trajectory \tilde{x} will always be zeros. Then, the trajectory y will remain the same for the expanded system (37) and the original system (1) when the trajectories for u and \dot{w} remain unchanged. Clearly, (37) satisfies Assumption 2.

Next, we consider Case 2: $\dot{n} > 0$. Since system (1) satisfies Assumption 2, we assume it is given in observer canonical form. Then, we expand the original system to n dimensional by adding the \tilde{x} dynamics, which is $(n - \dot{n})$ -dimensional,

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} \dot{A} & \tilde{A}_{12} \\ \mathbf{0} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} \dot{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \dot{D} \\ \mathbf{0} \end{bmatrix} \dot{w}; \quad \begin{bmatrix} \tilde{x}(0) \\ \tilde{x}(0) \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ \mathbf{0} \end{bmatrix} \quad (38a)$$

$$y = \begin{bmatrix} \dot{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{x} \end{bmatrix} + b_0 u + \dot{E} \dot{w} \quad (38b)$$

where $\tilde{A}_{12} = \begin{bmatrix} \mathbf{0} & -\tilde{C}' \end{bmatrix}'$, \tilde{C} is a row vector, \tilde{A}_{22} is Hurwitz, and the pair $(\tilde{A}_{22}, \tilde{C})$ is observable.

We observe that the trajectories \tilde{x} and y in (38) are as same as those in (1), when \tilde{x}_0 , u and \dot{w} remain unchanged.

To check the observability of (38), fix any $\lambda \in \mathbf{C}$, we observe

$$\begin{bmatrix} \dot{C} & \mathbf{0} \\ \lambda I - \dot{A} & -\tilde{A}_{12} \\ \mathbf{0} & \lambda I - \tilde{A}_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & \cdots & 0 \\ * & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & -1 \\ \hline & & * & \\ \hline & & & \tilde{C} \\ & & & \lambda I - \tilde{A}_{22} \end{array} \right] \quad (39)$$

where $*$ stands for arbitrary scalars or matrices. Since the pair $(\tilde{A}_{22}, \tilde{C})$ is observable, the matrix (39) is of full column rank $\forall \lambda \in \mathbf{C}$. Then, system (38) is observable.

Since the expanded dynamics, \tilde{x} , is uncontrollable from u , the transfer function of (38) is the same one of (1), and it is strictly minimum phase with relative degree 0.

To check others conditions of Assumption 2, we follow the derivation below,

First, we transform the system (1) into Jordan canonical form representation. We choose an invertible complex matrix T_J . The transformation $\dot{x} = T_J z$ will transform the dynamics (1) into

$$\dot{z} = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} z + \begin{bmatrix} B_{J1} \\ \vdots \\ B_{Jk} \end{bmatrix} u + \begin{bmatrix} D_{J1} \\ \vdots \\ D_{Jk} \end{bmatrix} \dot{w} \quad (40a)$$

$$y = \begin{bmatrix} C_{J1} & \cdots & C_{Jk} \end{bmatrix} z + b_0 u + \dot{E} \dot{w} \quad (40b)$$

where J_i is a Jordan block associated with eigenvalue λ_i , $i = 1, \dots, k, k \in \mathbf{N}$; in the column vectors of T_J , complex conjugate pairs appear together. Since a single-output Jordan canonical form state space representation is observable only if there is only one Jordan block associated with each distinct eigenvalue, then $i \neq j \Rightarrow \lambda_i \neq \lambda_j, \forall i, j \in \{1, \dots, k\}$. Partition $T_J = \begin{bmatrix} T_{J1} & \cdots & T_{Jk} \end{bmatrix}$

accordingly. For any $i, j \in \{1, \dots, k\}$, λ_i is the complex conjugate of λ_j , implies that T_{J_i} is the complex conjugate of T_{J_j} . By Assumption 2, if λ_i is any uncontrollable mode from u , then the last row of B_i is zero; in addition, if $\text{Re}(\lambda_i) = 0$, then the last row of D_i is zero. When $\text{Re}(\lambda_i) = 0$ and λ_i is an uncontrollable mode, the second to last row of B_{J_i} must be nonzero, otherwise, the uncontrollable part of (1) with respect to u will contain a Jordan block associated with λ_i of order at least 2, which is not stable in the sense of Lyapunov. Next, we separate the above Jordan Canonical form into controllable and uncontrollable parts with respect to u , and we denote them as z_c and $z_{\bar{c}}$, respectively. Then, we separate the uncontrollable part $z_{\bar{c}}$ into $\text{Re}(\lambda) < 0$ and $\text{Re}(\lambda) = 0$ parts, and we denote them as z_3 and z_4 , respectively. This corresponds to the existence of a permutation matrix T_c , and the transformation $z = T_c[z'_c \ z'_3 \ z'_4]'$ will transform (40) into

$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \left[\begin{array}{c|c} \bar{A}_c & \bar{A}_a \\ \hline \mathbf{0}_{(n-n_1) \times n_1} & \begin{array}{cc} A_{33} & \mathbf{0} \\ \mathbf{0} & \bar{A}_{44} \end{array} \end{array} \right] \begin{bmatrix} z_c \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \bar{D}_c \\ D_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (41a)$$

$$y = \left[\bar{C}_c \mid \bar{C}_3 \ \bar{C}_4 \right] \begin{bmatrix} z_c \\ z_3 \\ z_4 \end{bmatrix} + b_0 u + \dot{E} \dot{w} \quad (41b)$$

where z_c is the n_1 -dimensional controllable part; z_3 and z_4 are n_2 - and n_3 -dimensional, respectively, and compose the uncontrollable part; $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, 3$. The matrix T_c can be taken such

that $\bar{A}_c = \begin{bmatrix} \bar{J}_{c1} & & \\ & \ddots & \\ & & \bar{J}_{ck_1} \end{bmatrix}$, $\bar{A}_{33} = \begin{bmatrix} \bar{J}_{31} & & \\ & \ddots & \\ & & \bar{J}_{3k_2} \end{bmatrix}$, and $\bar{A}_{44} = \begin{bmatrix} \bar{J}_{41} & & \\ & \ddots & \\ & & \bar{J}_{4k_3} \end{bmatrix}$, where

$0 \leq k_1, k_2, k_3 \leq k$; J_{ci} , $i = 1, \dots, k_1$, J_{3j} , $j = 1, \dots, k_2$, J_{4l} , $l = 1, \dots, k_3$, are Jordan blocks associated with eigenvalues λ_{ci} , $i = 1, \dots, k_1$, λ_{3j} , $j = 1, \dots, k_2$, λ_{4l} , $l = 1, \dots, k_3$, respectively. $\lambda_{ci} \neq \lambda_{cj}$, if $i \neq j$; $\lambda_{3i} \neq \lambda_{3j}$, if $i \neq j$; and $\lambda_{4i} \neq \lambda_{4j}$, if $i \neq j$. Furthermore, $\forall i \in \{1, \dots, k_3\}$, \bar{J}_{4i} is of order 1, and λ_{4i} has zero real part; $\forall i \in \{1, \dots, k_2\}$, λ_{3i} has negative real part. Complex conjugate pairs appear together in each of the set $\{\lambda_{c1}, \dots, \lambda_{ck_1}\}$, $\{\lambda_{31}, \dots, \lambda_{3k_2}\}$, $\{\lambda_{41}, \dots, \lambda_{4k_3}\}$. Partition $T_J T_c = \left[T_{Jc1} \ \dots \ T_{Jck_1} \mid T_{J31} \ \dots \ T_{J3k_2} \mid T_{J41} \ \dots \ T_{J4k_3} \right]$ accordingly. For any $i, j \in \{1, \dots, k_1\}$, λ_{ci} is the complex conjugate of λ_{cj} , implies T_{Jci} is the complex conjugate of T_{Jcj} . Similar statements can be made for λ_{3i} and λ_{3j} , and also for λ_{4i} and λ_{4j} .

It is easy to see that, we can choose three complex invertible matrices $T_{\bar{J}_c}, T_{\bar{J}_3}, T_{\bar{J}_4}$, such that $T_{\bar{J}} = \text{block diagonal}(T_{\bar{J}_c}, T_{\bar{J}_3}, T_{\bar{J}_4})$, whose partitioning is the compatible with that of $[z'_c \ z'_3 \ z'_4]'$, and $T_1 = T_J T_c T_{\bar{J}}$ is a real invertible matrix. Then, the coordinate transformation $\dot{x} = T_1[x'_c \ \tilde{x}'_3 \ \tilde{x}'_4]'$ will transform (1) into

$$\begin{bmatrix} \dot{x}_c \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{bmatrix} = \left[\begin{array}{c|c} A_c & A_{c,34} \\ \hline \mathbf{0}_{(n-n_1) \times n_1} & \begin{array}{cc} \dot{A}_{33} & \mathbf{0} \\ \mathbf{0} & \dot{A}_{44} \end{array} \end{array} \right] \begin{bmatrix} x_c \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} B_c \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} D_c \\ \dot{D}_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (42a)$$

$$y = \left[C_c \mid \dot{C}_3 \ \dot{C}_4 \right] \begin{bmatrix} x_c \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + b_0 u + \dot{E} \dot{w} \quad (42b)$$

where all matrices are real, the triple (A_c, B_c, C_c) is controllable and observable, \dot{A}_{33} is Hurwitz, \dot{A}_{44} has all eigenvalues lying on the $j\omega$ -axis and all Jordan blocks being of order 1.

Then, (38) admits the following state space representation.

$$\begin{bmatrix} \dot{x}_c \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A_c & \dot{A}_{c,34} & \dot{\tilde{A}}_c \\ \mathbf{0}_{(n-n_1) \times n_1} & \dot{A}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dot{A}_{44} \\ \mathbf{0} & \mathbf{0} & \dot{\tilde{A}} \end{bmatrix} \begin{bmatrix} x_c \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B_c \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} D_c \\ \dot{D}_3 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \dot{w}$$

$$y = \begin{bmatrix} C_c & \dot{C}_3 & \dot{C}_4 & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_c \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x} \end{bmatrix} + b_0 u + \dot{E} \dot{w}$$

We define a coordinate transformation $\tilde{x}_4 = \tilde{T}\tilde{x} + \tilde{x}_4$, such that $\tilde{T}\dot{\tilde{A}} - \dot{A}_{44}\tilde{T} + \dot{\tilde{A}}_4 = 0$. Then, the system (38) admits the following state space representation,

$$\begin{bmatrix} \dot{x}_c \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} A_c & \dot{\tilde{A}}_{c,34} \\ \mathbf{0} & \dot{A}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dot{A}_{44} \end{bmatrix} \begin{bmatrix} x_c \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} B_c \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} D_c \\ \dot{D}_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (43a)$$

$$y = \begin{bmatrix} C_c & \dot{\tilde{C}}_3 & \dot{C}_4 \end{bmatrix} \begin{bmatrix} x_c \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + b_0 u + \dot{E} \dot{w} \quad (43b)$$

where $\tilde{x}_3 = \begin{bmatrix} \tilde{x}_3 \\ \tilde{x} \end{bmatrix}$, and $\dot{\tilde{A}}_{33} = \begin{bmatrix} \dot{A}_{33} & \dot{\tilde{A}}_3 \\ \mathbf{0} & \dot{\tilde{A}} \end{bmatrix}$. Since \dot{A}_{33} and $\dot{\tilde{A}}$ are Hurwitz, then $\dot{\tilde{A}}_{33}$ is Hurwitz.

Based on (43), clearly the expanded system satisfies Assumption 2.

This complete the proof of the lemma. \square

Lemma 7 Consider a linear time-invariant system with the following state space representation

$$\dot{x} = Ax + Bu + Dw \quad (44a)$$

$$y = Cx + Ew \quad (44b)$$

where x is the n -dimensional state vector, $n \in \mathbb{N}$; w is q -dimensional disturbance input, $q \in \mathbb{N}$; u is the scalar input, and y is the scalar output; A is Hurwitz; and the transfer function from u to y is $H(s) = C(sI_n - A)^{-1}B$, which is strictly minimum phase and has relative degree $r \in \mathbb{N}$; Then the system (44) admits state space representation (70) in Pan and Başar (2000), which satisfies Assumption 10 of Pan and Başar (2000), under some real invertible coordinate transformation, with index $r_1 = r$.

Proof First, we introduce a linear state diffeomorphism for (44) to decompose the system into observable and unobservable part. We choose a real invertible matrix T_o , then the equivalence transformation $x = T_o[z'_o \ z'_\bar{o}]'$ will transform the dynamics of x into

$$\begin{bmatrix} \dot{z}_o \\ \dot{z}_\bar{o} \end{bmatrix} = \begin{bmatrix} A_o & \mathbf{0} \\ A_{o\bar{o}} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} z_o \\ z_\bar{o} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u + \begin{bmatrix} D_o \\ D_{\bar{o}} \end{bmatrix} w \quad (45a)$$

$$y = \begin{bmatrix} C_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} z_o \\ z_\bar{o} \end{bmatrix} + Ew \quad (45b)$$

where z_o is an \bar{n} -dimensional vector, $\bar{n} \in \mathbb{N}$ and $A_{\bar{o}}$ is Hurwitz.

Second, we introduce a linear state diffeomorphism for the z_o subsystem of (45) to decompose it into controllable and uncontrollable parts. We choose an invertible real matrix T_c , then the transformation $z_o = T_c[z'_{co} \ z'_{\bar{c}o}]'$ will transform the dynamics of z_o into

$$\begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{\bar{c}o} \end{bmatrix} = \begin{bmatrix} A_{co} & A_{c\bar{c}o} \\ \mathbf{0} & A_{\bar{c}o} \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{\bar{c}o} \end{bmatrix} + \begin{bmatrix} B_{co} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} D_{co} \\ D_{\bar{c}o} \end{bmatrix} w \quad (46a)$$

$$y = \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{\bar{c}o} \end{bmatrix} + Ew \quad (46b)$$

where z_{co} is n_1 -dimensional state vector, $n_1 \in \mathbb{N}$, and the matrix $A_{\bar{c}o}$ is Hurwitz. We note that all matrices are real, and the triple (A_{co}, B_{co}, C_{co}) is controllable and observable, and the transfer function of (44) from u to y is

$$H(s) = C_{co} (sI_{n_1} - A_{co})^{-1} B_{co} =: \frac{b_0 s^{n_1-r} + b_1 s^{n_1-r-1} + \dots + b_{n_1-r}}{s^{n_1} + a_1 s^{n_1-1} + \dots + a_{n_1}}$$

with $b_0 \neq 0$.

By Lemma 12 in Pan and Başar (2000), there exists a real invertible matrix T , such that

$$\begin{bmatrix} T^{-1} & \mathbf{0}_{n_1 \times 1} \\ \mathbf{0}_{1 \times n_1} & 1 \end{bmatrix} \begin{bmatrix} A_{co} & B_{co} \\ C_{co} & 0 \end{bmatrix} \begin{bmatrix} T & \mathbf{0}_{n_1 \times 1} \\ \mathbf{0}_{1 \times n_1} & 1 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \bar{A}_{21} & \bar{A}_{22} & \mathbf{0} \\ \bar{C}_1 & \mathbf{0} & 0 \end{bmatrix}$$

where

$$\bar{A}_{11} = \begin{bmatrix} \bar{a}_1 & & \\ & \vdots & \\ & & I_{r-1} \\ \hline \bar{a}_{r-1} & & \\ \bar{a}_r & & \mathbf{0}_{1 \times (r-1)} \end{bmatrix}; \quad \bar{A}_{12} = \begin{cases} \begin{bmatrix} \mathbf{0} \\ 1 \\ \vdots \end{bmatrix} & n_1 > r \\ \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} & n_1 = r \end{cases} \quad (47a)$$

$$\bar{A}_{21} = \begin{bmatrix} \bar{a}_{r+1} & & \\ & \vdots & \\ & & \mathbf{0}_{(n_1-r) \times (r-1)} \\ \hline \bar{a}_{n_1} & & \end{bmatrix}; \quad \bar{A}_{22} = \begin{cases} \begin{bmatrix} -b_1/b_0 & & \\ & \vdots & \\ & & I_{n_1-r-1} \\ \hline -b_{n_1-r-1}/b_0 & & \\ -b_{n_1-r}/b_0 & & \mathbf{0}_{1 \times (n_1-r-1)} \end{bmatrix} & n_1 > r \\ \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} & n_1 = r \end{cases} \quad (47b)$$

$$\bar{B}_1 = \begin{bmatrix} \mathbf{0}_{1 \times (r-1)} & b_0 \end{bmatrix}'; \quad \bar{C}_1 = \begin{bmatrix} 1 & \mathbf{0}_{1 \times (r-1)} \end{bmatrix} \quad (47c)$$

Since $H(s)$ is strictly minimum phase, \bar{A}_{22} is Hurwitz.

Hence the desired state transformation matrix is

$$\bar{T} = T_o \begin{bmatrix} T_c & \mathbf{0} \\ \mathbf{0} & I_{n-\bar{n}} \end{bmatrix} \begin{bmatrix} T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\bar{n}-n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-\bar{n}} \end{bmatrix}$$

Then, the coordinate transformation $x = \bar{T}[z'_1 \ z'_2 \ z'_{\bar{c}o} \ z'_o]'$ will transform system (44) into

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_{\bar{c}o} \\ \dot{z}_o \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{12,3} & \mathbf{0} \\ \bar{A}_{21} & \bar{A}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{\bar{c}o} & \mathbf{0} & \mathbf{0} \\ A_{\bar{o},12} & A_{\bar{o},3} & A_{\bar{o}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_{\bar{c}o} \\ z_o \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \mathbf{0} \\ \mathbf{0} \\ B_{\bar{o}} \end{bmatrix} u + \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \\ D_{\bar{c}o} \\ D_{\bar{o}} \end{bmatrix} w \quad (48a)$$

$$y = \begin{bmatrix} \bar{C}_1 & \mathbf{0} & C_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_{\bar{c}o} \\ z_o \end{bmatrix} + Ew \quad (48b)$$

Let $\bar{x}_1 = z_1$, $\bar{x}_2 = z_2$, $\bar{x}_3 = z_{\bar{c}o}$ and $\bar{x}_5 = z_o$. Then, in the $[\bar{x}'_1 \ \bar{x}'_2 \ \bar{x}'_3 \ \bar{x}'_5]'$ coordinate, the system (48) admits the state space representation (70) as in Pan and Başar (2000), and satisfies Assumption 10 in Pan and Başar (2000) with $r_1 = r$.

This complete the proof of the lemma. \square

The next lemma establishes a result that system (1) composed with another linear system may still satisfy Assumption 10 of (Pan and Başar 2000).

Lemma 8 *Consider system (1) under Assumption 2 and a second linear system, which admits the following state space representation*

$$\dot{\eta} = A_2\eta + B_2y + D_2\dot{w} \quad (49a)$$

$$\eta_L = C_2\eta + E_2\dot{w} \quad (49b)$$

where η is the n_2 -dimensional state vector, $n_2 \in \mathbb{N}$; η_L is the scalar output; y and \dot{w} are the same signals as in (1); A_2 is Hurwitz; and the transfer function of (49) from y to η_L is $H_2(s) = C_2(sI_{n_2} - A_2)^{-1}B_2$, which is strictly minimum phase and has relative degree $r_2 \in \mathbb{N}$. By Remark 1, W.L.O.G., assume $\dot{n} = n \in \mathbb{N}$.

Then the composite system of (1) and (49), which is given by the state space representation

$$\begin{bmatrix} \dot{\eta} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A_2 & B_2\dot{C} \\ \mathbf{0}_{n \times n_2} & \dot{A} \end{bmatrix} \begin{bmatrix} \eta \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_2b_0 \\ \dot{B} \end{bmatrix} u + \begin{bmatrix} B_2\dot{E} + D_2 \\ \dot{D} \end{bmatrix} \dot{w} \quad (50a)$$

$$\eta_L = \begin{bmatrix} C_2 & \mathbf{0}_{1 \times n} \end{bmatrix} \begin{bmatrix} \eta \\ \hat{x} \end{bmatrix} + E_2\dot{w} \quad (50b)$$

admits state space representation (70) as in Pan and Başar (2000), which satisfies Assumption 10 of Pan and Başar (2000) with $r_1 = r_2$, under some real invertible coordinate transformation.

Proof By analysis that is similar to that in the Case 2 in the proof of Lemma 6, there exists a real invertible matrix T_1 such that system (1) admits the state space representation (42) in the coordinates of $T_1^{-1}\hat{x} = [x'_c \ \bar{x}'_3 \ \bar{x}'_4]'$, where x_c is n_1 -dimensional, $n_1 \in \mathbb{N} \cup \{0\}$; the triple (A_c, B_c, C_c) is controllable and observable; the matrix \dot{A}_{33} is Hurwitz; all of the eigenvalue of the matrix \dot{A}_{44} are on the $j\omega$ -axis and all Jordan blocks of \dot{A}_{44} are of order 1.

Now, we will discuss 2 exhaustive and mutually exclusive cases: Case 1: $n_1 > 0$; Case 2: $n_1 = 0$. First, consider Case 1: $n_1 > 0$. The transfer function of (42) from u to y is

$$H(s) = C_c(sI_{n_1} - A_c)^{-1}B_c + b_0 = \frac{b_1s^{n_1-1} + \dots + b_{n_1}}{s^{n_1} + a_1s^{n_1-1} + \dots + a_{n_1}} + b_0$$

where $a_i, i = 1, \dots, n_1, b_j, j = 0, \dots, n_1,$ are some constants and $b_0 \neq 0$. Then, there exists a $r_0 \in \{1, \dots, n_1\}$, such that $b_{r_0} \neq 0$, and $b_j = 0, 1 \leq j \leq r_0 - 1$.

By Lemma 12 of Pan and Başar (2000), there exists a real invertible matrix T , which transforms the triple (A_c, B_c, C_c) into

$$\left[\begin{array}{c|c} T^{-1} & \mathbf{0}_{n_1 \times 1} \\ \hline \mathbf{0}_{1 \times n_1} & 1 \end{array} \right] \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array} \right] \left[\begin{array}{c|c} T & \mathbf{0}_{n_1 \times 1} \\ \hline \mathbf{0}_{1 \times n_1} & 1 \end{array} \right] = \left[\begin{array}{c|c|c} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \hline \bar{A}_{21} & \bar{A}_{22} & \mathbf{0} \\ \hline \bar{C}_1 & \mathbf{0} & 0 \end{array} \right]$$

where

$$\begin{aligned} \bar{A}_{11} &= \left[\begin{array}{c|c} \bar{a}_1 & \\ \vdots & I_{r_0-1} \\ \hline \bar{a}_{r_0-1} & \\ \hline \bar{a}_{r_0} & \mathbf{0}_{1 \times (r_0-1)} \end{array} \right]; & \bar{A}_{12} &= \begin{cases} \left[\begin{array}{c} \mathbf{0} \\ 1 \end{array} \right] & n_1 > r_0 \\ \left[\begin{array}{c} \mathbf{0} \\ 1 \end{array} \right] & n_1 = r_0 \end{cases} \\ \bar{A}_{21} &= \left[\begin{array}{c|c} \bar{a}_{r_0+1} & \\ \vdots & \mathbf{0}_{(n_1-r_0) \times (r_0-1)} \\ \hline \bar{a}_{n_1} & \end{array} \right]; & \bar{A}_{22} &= \begin{cases} \left[\begin{array}{c|c} -b_{r_0+1}/b_{r_0} & \\ \vdots & I_{n_1-r_0-1} \\ \hline -b_{n_1-1}/b_{r_0} & \\ \hline -b_{n_1}/b_{r_0} & \mathbf{0}_{1 \times (n_1-r_0-1)} \end{array} \right] & n_1 > r_0 \\ \left[\begin{array}{c} \mathbf{0} \end{array} \right] & n_1 = r_0 \end{cases} \\ \bar{B}_1 &= \left[\begin{array}{c|c} \mathbf{0}_{1 \times (r_0-1)} & b_{r_0} \end{array} \right]'; & \bar{C}_1 &= \left[\begin{array}{c|c} 1 & \mathbf{0}_{1 \times (r_0-1)} \end{array} \right] \end{aligned}$$

Then, the coordinate transformation $T_3^{-1}T_4^{-1}[\eta' \ \dot{x}']' = [\eta' \ \bar{x}'_1 \ \bar{x}'_2 \ \bar{x}'_3 \ \bar{x}'_4]'$, where $T_3 =$ block diagonal (I_{n_2}, T, I_{n-n_1}) , $T_4 =$ block diagonal (I_{n_2}, T_1) , will transform system (50) into

$$\begin{bmatrix} \dot{\eta} \\ \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \bar{C}_1 & \mathbf{0} & B_2 \bar{C}_3 & B_2 \bar{C}_4 \\ \mathbf{0} & \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\ \mathbf{0} & \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{A}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{A}_{44} \end{bmatrix} \begin{bmatrix} \eta \\ \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} B_2 b_0 \\ \bar{B}_1 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} B_2 \dot{E} + D_2 \\ \bar{D}_1 \\ \bar{D}_2 \\ \bar{D}_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (51a)$$

$$\eta_L = \left[\begin{array}{c|c|c|c} C_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{bmatrix} \eta \\ \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + E_2 \dot{w} \quad (51b)$$

We will focus on the $[\eta' \ \bar{x}'_1 \ \bar{x}'_2]'$ dynamics. First, we claim that any uncontrollable mode of the following pair

$$\left(\left(\begin{array}{c|c|c} A_2 & B_2 \bar{C}_1 & \mathbf{0} \\ \hline \mathbf{0} & \bar{A}_{11} & \bar{A}_{12} \\ \hline \mathbf{0} & \bar{A}_{21} & \bar{A}_{22} \end{array} \right), \begin{bmatrix} b_0 B_2 \\ \bar{B}_1 \\ \mathbf{0} \end{bmatrix} \right)$$

is an eigenvalue of the matrix A_2 , which has negative real part. We show it as follows. Let $\lambda \in \mathbb{C}$ be a uncontrollable mode of the above pair, then the following matrix

$$\left[\begin{array}{ccc|c} \lambda I - A_2 & -B_2\bar{C}_1 & \mathbf{0} & b_0B_2 \\ \mathbf{0} & \lambda I - \bar{A}_{11} & -\bar{A}_{12} & \bar{B}_1 \\ \mathbf{0} & -\bar{A}_{21} & \lambda I - \bar{A}_{22} & \mathbf{0} \end{array} \right]$$

is not full row rank. Since $\left[\begin{array}{cc|c} \lambda I - \bar{A}_{11} & -\bar{A}_{12} & \bar{B}_1 \\ -\bar{A}_{21} & \lambda I - \bar{A}_{22} & \mathbf{0} \end{array} \right]$ has full row rank for any λ , it implies that the matrix $\lambda I - A_2$ is singular, i.e. λ is an eigenvalue of A_2 .

Next, we claim that any unobservable mode of the following pair

$$\left(\left[\begin{array}{ccc|c} A_2 & B_2\bar{C}_1 & \mathbf{0} & \\ \mathbf{0} & \bar{A}_{11} & \bar{A}_{12} & \\ \mathbf{0} & \bar{A}_{21} & \bar{A}_{22} & \end{array} \right], \left[C_2 \mid \mathbf{0} \ \mathbf{0} \right] \right)$$

is an eigenvalue of A_2 or is a zero of $H_2(s)$, which then has negative real part. We will prove it as follows. Let $\lambda \in \mathbb{C}$ be a unobservable mode, then the following matrix

$$\left[\begin{array}{ccc|c} C_2 & \mathbf{0} & \mathbf{0} & \\ \lambda I - A_2 & -B_2\bar{C}_1 & \mathbf{0} & \\ \mathbf{0} & \lambda I - \bar{A}_{11} & -\bar{A}_{12} & \\ \mathbf{0} & -\bar{A}_{21} & \lambda I - \bar{A}_{22} & \end{array} \right] \quad (52)$$

is not full column rank. Since the submatrix of (52), consisting of the last n_1 rows and the last $n_1 - 1$ columns of (52), has full column rank, and the submatrix of (52), consisting of the first $n_2 + 1$ rows and the last $n_1 - 1$ columns of (52), is zero matrix, then the submatrix $\left[\begin{array}{cc|c} C_2 & \mathbf{0} & \\ \lambda I - A_2 & -B_2 & \end{array} \right]$ must be singular, i.e.,

$$\det \left(\left[\begin{array}{cc|c} C_2 & \mathbf{0} & \\ \lambda I - A_2 & -B_2 & \end{array} \right] \right) = 0 = \det \left(\left[\begin{array}{cc|c} \mathbf{0} & C_2 & \\ -B_2 & \lambda I - A_2 & \end{array} \right] \right)$$

There are two exhaustive and mutually exclusive cases. Case 1a: the matrix $\lambda I - A_2$ is singular, it implies the unobservable mode is an eigenvalue of A_2 . Case 1b: the matrix $\lambda I - A_2$ is invertible. Then the above determinant equality implies

$$\det \left(\left[\begin{array}{cc|c} C_2(\lambda I - A_2)^{-1}B_2 & \mathbf{0} & \\ \mathbf{0} & \lambda I - A_2 & \end{array} \right] \right) = H_2(\lambda) \det(\lambda I - A_2) = 0$$

This further implies that λ is a zero of $H_2(\lambda)$. Therefore, this claim is proved.

Next, we decompose the $[\eta' \ \bar{x}'_1 \ \bar{x}'_2]'$ dynamics into observable and unobservable parts, and then decompose the observable part into controllable and uncontrollable part. The real invertible transformation $[x'_{co} \ x'_{\bar{co}} \ x'_{\bar{o}}]' = T_{co}^{-1}[\eta' \ \bar{x}'_1 \ \bar{x}'_2]'$ will transform (51) into

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{\bar{co}} \\ \dot{x}_{\bar{o}} \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} A_{co} & * & \mathbf{0} & * & * \\ \mathbf{0} & A_{\bar{co}} & \mathbf{0} & * & * \\ * & * & A_{\bar{o}} & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{A}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{A}_{44} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{\bar{co}} \\ x_{\bar{o}} \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} B_{co} \\ \mathbf{0} \\ B_{\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} D_{co} \\ D_{\bar{co}} \\ D_{\bar{o}} \\ \dot{D}_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (53a)$$

$$\eta_L = \begin{bmatrix} C_{co} & C_{\bar{c}o} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{\bar{c}o} \\ x_{\bar{o}} \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + E_2 \dot{w} \quad (53b)$$

where x_{co} is the controllable and observable part, and is \bar{n}_3 -dimensional, $\bar{n}_3 \in \mathbb{N}$; $x_{\bar{c}o}$ is the uncontrollable and observable part; and $x_{\bar{o}}$ is the unobservable part; *'s denote arbitrary constant matrices. Since the uncontrollable and unobservable modes of $[\eta' \ \bar{x}'_1 \ \bar{x}'_2]'$ have negative real parts, then $A_{\bar{c}o}$ and $A_{\bar{o}}$ are Hurwitz. We note that the triple (A_{co}, B_{co}, C_{co}) is controllable and observable, and the transfer function for (53) from u to η_L is

$$C_{co}(sI_{\bar{n}_3} - A_{co})^{-1}B_{co} = H_2(s)H(s)$$

which is strictly minimum phase and its relative degree is r_2 .

By Lemma 12 in Pan and Başar (2000), there exists a real invertible matrix T_5 , and the coordinate transformation $x_{co} = T_5[\bar{x}'_1 \ \bar{x}'_2]'$ will transform system (53) into

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{o}} \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} A_{111} & A_{112} & * & \mathbf{0} & * & * \\ A_{121} & A_{122} & * & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & A_{\bar{c}o} & \mathbf{0} & * & * \\ * & * & * & A_{\bar{o}} & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{A}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{A}_{44} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ x_{\bar{c}o} \\ x_{\bar{o}} \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} B_{11} \\ \mathbf{0} \\ \mathbf{0} \\ B_{\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} D_{11} \\ D_{12} \\ D_{\bar{c}o} \\ D_{\bar{o}} \\ \dot{D}_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (54a)$$

$$\eta_L = \begin{bmatrix} C_{11} & \mathbf{0} & C_{\bar{c}o} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ x_{\bar{c}o} \\ x_{\bar{o}} \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + E_2 \dot{w} \quad (54b)$$

where *'s stand for some arbitrary constant matrices;

$$A_{111} = \left[\begin{array}{c|c} \tilde{a}_1 & \\ \vdots & I_{r_2-1} \\ \hline \tilde{a}_{r_2-1} & \\ \tilde{a}_{r_2} & \mathbf{0}_{1 \times (r_2-1)} \end{array} \right]; \quad A_{112} = \begin{cases} \begin{bmatrix} \mathbf{0} \\ 1 \\ \vdots \end{bmatrix} & \bar{n}_3 > r_2 \\ \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} & \bar{n}_3 = r_2 \end{cases}$$

$$A_{121} = \left[\begin{array}{c|c} \tilde{a}_{r_2+1} & \\ \vdots & \mathbf{0}_{(\bar{n}_3-r_2) \times (r_2-1)} \\ \tilde{a}_{\bar{n}_3} & \end{array} \right]; \quad A_{122} = \begin{cases} \left[\begin{array}{c|c} -\tilde{b}_{r_2+1}/\tilde{b}_{r_2} & \\ \vdots & I_{\bar{n}_3-r_2-1} \\ \hline -\tilde{b}_{\bar{n}_3-1}/\tilde{b}_{r_2} & \\ -\tilde{b}_{\bar{n}_3}/\tilde{b}_{r_2} & \mathbf{0}_{1 \times (\bar{n}_3-r_2-1)} \end{array} \right] & \bar{n}_3 > r_2 \\ \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} & \bar{n}_3 = r_2 \end{cases}$$

$$B_{11} = \begin{bmatrix} \mathbf{0}_{1 \times (r_2-1)} & \tilde{b}_{r_2} \end{bmatrix}'; \quad C_{11} = \begin{bmatrix} 1 & \mathbf{0}_{1 \times (r_2-1)} \end{bmatrix}$$

where $\tilde{a}_i, i = 1, \dots, \bar{n}_3, \tilde{b}_i, i = r_2, \dots, \bar{n}_3$, are some constants; $\tilde{b}_{r_2} \neq 0$; the matrix A_{122} is Hurwitz, and the dimension of \bar{x}_1 is r_2 . We choose $\bar{x}_3 = [x'_{\bar{c}o} \ \bar{x}'_3]'$, $\bar{x}_4 = \bar{x}_4$, and $\bar{x}_5 = x_{\bar{o}}$. Then the composite

system (54), in the coordinate of $[\bar{x}'_1 \ \bar{x}'_2 \ \bar{x}'_3 \ \bar{x}'_4 \ \bar{x}'_5]'$, admits the state space representation (70) as in Pan and Başar (2000) and satisfies Assumption 10 of Pan and Başar (2000) with $r_1 = r_2$.

Case 2: $n_1 = 0$. Then, system (1) admits transfer function $H(s) = b_0$ from u to y .

The composite system (50) admits the following state space representation in the coordinates of $[\eta' \ \bar{x}'_3 \ \bar{x}'_4]'$,

$$\begin{bmatrix} \dot{\eta} \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} A_2 & B_2\dot{C}_3 & B_2\dot{C}_4 \\ \mathbf{0} & \bar{A}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{A}_{44} \end{bmatrix} \begin{bmatrix} \eta \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} B_2b_0 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} B_2\dot{E} + D_2 \\ \bar{D}_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (55a)$$

$$\eta_L = \begin{bmatrix} C_2 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + E_2\dot{w} \quad (55b)$$

By an argument that is similar to that used in the proof of Lemma 7, there exists a real invertible matrix \bar{T} , such that the coordinate transformation $\eta = \bar{T}[z'_1 \ z'_2 \ z'_{\bar{c}o} \ z'_{\bar{o}}]'$ will transform (49) into the form of (48). Then, the system (55) admits the following state space representation

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{\bar{o}} \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & * & \mathbf{0} & * & * \\ \bar{A}_{21} & \bar{A}_{22} & * & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & A_{\bar{c}o} & \mathbf{0} & * & * \\ * & * & * & A_{\bar{o}} & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{A}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{A}_{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_{\bar{c}o} \\ z_{\bar{o}} \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \mathbf{0} \\ \mathbf{0} \\ B_{\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \\ D_{\bar{c}o} \\ D_{\bar{o}} \\ \bar{D}_3 \\ \mathbf{0} \end{bmatrix} \dot{w} \quad (56a)$$

$$\eta_L = \begin{bmatrix} \bar{C}_1 & \mathbf{0} & C_{\bar{c}o} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_{\bar{c}o} \\ z_{\bar{o}} \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + E_2\dot{w} \quad (56b)$$

where $*$'s stand for some arbitrary constant matrices; the dimension of $[z'_1 \ z'_2]'$ is $n_4 \in \mathbb{N}$; and \bar{A}_{11} , \bar{A}_{12} , \bar{A}_{21} , \bar{A}_{22} , \bar{B}_1 , and \bar{C}_1 admit structures as in (47) with $n_1 = n_4$ and $r = r_2$. the matrices \bar{A}_{22} , $A_{\bar{c}o}$, and $A_{\bar{o}}$ are Hurwitz. We choose $\bar{x}_1 = z_1$, $\bar{x}_2 = z_2$, $\bar{x}_3 = [z'_{\bar{c}o} \ \bar{x}'_3]'$, $\bar{x}_4 = \bar{x}_4$, and $\bar{x}_5 = z_{\bar{o}}$. Then the composite system (56), in the coordinates of $[\bar{x}'_1 \ \bar{x}'_2 \ \bar{x}'_3 \ \bar{x}'_4 \ \bar{x}'_5]'$, admits the state space representation (70) as in Pan and Başar (2000), which satisfies Assumption 10 of Pan and Başar (2000) with index $r_1 = r_2$.

This completes the proof of this Lemma. \square

B Some Derivations

$$\begin{aligned} & \frac{d(\bar{x} - \Phi\tilde{\theta})}{dt} \\ &= \dot{x} - \dot{\bar{x}} - \dot{\Phi}\tilde{\theta} + \Phi\dot{\tilde{\theta}} \\ &= Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Bu + Dw - \left(-\Phi\Sigma P_r(\tilde{\theta}) + A\bar{x} + Bu - (u\Phi\Sigma\bar{C}'_1 \right. \end{aligned}$$

$$\begin{aligned}
& +(\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\tilde{x} - u(\bar{C}_1\tilde{\theta} + b_{p0})) + (\gamma^2u\Phi\Sigma\bar{C}'_1 + \Pi C' + \gamma^2\Phi\Sigma\Phi'C' + L) \\
& \zeta^2(y - C\tilde{x} - u(\bar{C}_1\tilde{\theta} + b_{p0})) - \left[\Phi\Sigma \quad \gamma^{-2}\Pi + \Phi\Sigma\Phi' \right] \bar{Q}\xi_c + (y\bar{A}_{211} + u\bar{A}_{212})\tilde{\theta}) \\
& - \left(A_f\Phi + y\bar{A}_{211} + (\bar{A}_{212} + (\gamma^{-2} - \zeta^2)\Pi C'\bar{C}_1 - \zeta^2L\bar{C}_1)u \right)\tilde{\theta} + \Phi \left(-\Sigma P_r(\tilde{\theta}) - (u\Sigma\bar{C}'_1 \right. \\
& \left. + \Sigma\Phi'C')(y_d - C\tilde{x} - u(\bar{C}_1\tilde{\theta} + b_{p0})) - \left[\Sigma \quad \Sigma\Phi' \right] \bar{Q}\xi_c + \gamma^2\zeta^2(u\Sigma\bar{C}'_1 + \Sigma\Phi'C')(y - C\tilde{x} \right. \\
& \left. - u(\bar{C}_1\tilde{\theta} + b_{p0})) \right) \\
= & A\tilde{x} - A_f\Phi\tilde{\theta} - ((\gamma^{-2} - \zeta^2)\Phi C'\bar{C}_1 - \zeta^2L\bar{C}_1)u\tilde{\theta} + Dw + \left[\mathbf{0} \quad \gamma^{-2}\Pi \right] \bar{Q}\xi_c + \gamma^{-2}\Pi C'y_d \\
& + \gamma^{-2}\Pi C'(-C\tilde{x} - u(\bar{C}_1\tilde{\theta} + b_{p0})) - \zeta^2(\Pi C' + L)y + \zeta^2(\Pi C' + L)(C\tilde{x} + u(\bar{C}_1\tilde{\theta} + b_{p0})) \\
= & A\tilde{x} - A_f\Phi\tilde{\theta} + Dw + \left[\mathbf{0} \quad \gamma^{-2}\Pi \right] \bar{Q}\xi_c - ((\gamma^{-2} - \zeta^2)\Pi C'\bar{C}_1\tilde{\theta} - \zeta^2L\bar{C}_1\tilde{\theta} \\
& + \gamma^{-2}\Pi C'(\bar{C}_1\tilde{\theta} + b_{p0}) - \zeta^2(\Pi C' + L)(\bar{C}_1\tilde{\theta} + b_{p0}))u + \left(-\gamma^{-2}\Pi C'C + \zeta^2(\Pi C' + L)C \right)\tilde{x} \\
& + \gamma^{-2}\Pi C'y_d - \zeta^2(\Pi C' + L)y + \gamma^{-2}\Pi C'y - \gamma^{-2}\Pi C'y \\
= & A\tilde{x} - A_f\Phi\tilde{\theta} + \left(D - \zeta^2(\Pi C' + L)E + \gamma^{-2}\Pi C'E \right)w + \left[\mathbf{0} \quad \gamma^{-2}\Pi \right] \bar{Q}\xi_c + \gamma^{-2}\Pi C'(y_d - y) \\
& - \left((\gamma^{-2} - \zeta^2)\Pi C'\bar{C}_1\tilde{\theta} - \zeta^2L\bar{C}_1\tilde{\theta} + \gamma^{-2}\Pi C'(\bar{C}_1\tilde{\theta} + b_{p0}) - \zeta^2(\Pi C' + L)(\bar{C}_1\tilde{\theta} + b_{p0}) \right. \\
& \left. + \zeta^2(\Pi C' + L)(\bar{C}_1\tilde{\theta} + b_{p0}) - \gamma^{-2}\Pi C'(\bar{C}_1\tilde{\theta} + b_{p0}) \right)u - (\gamma^{-2}\Pi C'C - \zeta^2(\Pi C' + L)C)\tilde{x} \\
= & (A + \gamma^{-2}\Pi C'C - \zeta^2(\Pi C' + L)C)\tilde{x} - A_f\Phi\tilde{\theta} + \left(D - \zeta^2(\Pi C' + L)E + \gamma^{-2}\Pi C'E \right)w \\
& + \left[\mathbf{0} \quad \gamma^{-2}\Pi \right] \bar{Q}\xi_c + \gamma^{-2}\Pi C'y_d - \gamma^{-2}\Pi C'y - \left((\gamma^{-2} - \zeta^2)\Pi C'\bar{C}_1\tilde{\theta} - \zeta^2L\bar{C}_1\tilde{\theta} \right. \\
& \left. - \gamma^{-2}\Pi C'\bar{C}_1\tilde{\theta} + \zeta^2(\Pi C' + L)\bar{C}_1\tilde{\theta} \right)u \\
= & A_f(\tilde{x} - \Phi\tilde{\theta}) + \gamma^{-2}\Pi C'y_d - \gamma^{-2}\Pi C'y + \left(D - \zeta^2(\Pi C' + L)E + \gamma^{-2}\Pi C'E \right)w + \left[\mathbf{0} \quad \gamma^{-2}\Pi \right] \bar{Q}\xi_c
\end{aligned}$$

Next we give the derivation of w_{opt} . First, we notice that

$$\begin{aligned}
v &= \zeta(y - C\tilde{x} - (b_{p0} + \bar{C}_1\tilde{\theta})u) \\
&= \zeta(\bar{C}\xi + b_{p0}u + Ew - \bar{C}\check{\xi} - b_{p0}u) \\
&= \zeta\bar{C}(\xi - \check{\xi}) + \zeta Ew \\
w_* &= \zeta^2 E'(y - b_{p0}u - \bar{C}\xi) + \gamma^{-2}(I_q - \zeta^2 E'E)\bar{D}'\bar{\Sigma}^{-1}(\xi - \check{\xi}) \\
&= \zeta^2 E'Ew + \Delta
\end{aligned}$$

where $\Delta = \gamma^{-2}(I_q - \zeta^2 E'E)\bar{D}'\bar{\Sigma}^{-1}(\xi - \check{\xi})$. Then, we have $E\Delta = 0$.

$$\begin{aligned}
v^2 &= \zeta^2(\bar{C}(\xi - \check{\xi}) + Ew)'(\bar{C}(\xi - \check{\xi}) + Ew) \\
&= \zeta^2|\bar{C}(\xi - \check{\xi})|^2 + \zeta^2|Ew|^2 + 2\zeta^2(\bar{C}(\xi - \check{\xi}))'Ew \\
|w - w_*|^2 &= |w|^2 + |\zeta^2 E'Ew + \Delta|^2 - 2(\zeta^2 E'Ew + \Delta)'w \\
&= |w|^2 + |\zeta^2 E'Ew|^2 + |\Delta|^2 - 2\Delta'w - 2\zeta^2|Ew|^2 \\
v^2 + |w - w_*|^2 &= |w|^2 - 2(\Delta - \zeta^2 E'\bar{C}(\xi - \check{\xi}))'w + |\Delta|^2 + \zeta^2|\bar{C}(\xi - \check{\xi})|^2 \\
&= |w - (\Delta - \zeta^2 E'\bar{C}(\xi - \check{\xi}))|^2 - \zeta^4(\bar{C}(\xi - \check{\xi}))'EE'\bar{C}(\xi - \check{\xi}) + \zeta^2|\bar{C}(\xi - \check{\xi})|^2 \\
&= |w - (\Delta - \zeta^2 E'\bar{C}(\xi - \check{\xi}))|^2 \\
&= |w - w_{opt}|^2
\end{aligned}$$

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