

# GENERALIZED MINIMUM PHASE PROPERTY FOR FINITE-DIMENSIONAL CONTINUOUS-TIME SISO LTI SYSTEMS WITH ADDITIVE DISTURBANCES. PART I: DEFINITION AND SOME PROPERTIES \*

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**Abstract.** For finite-dimensional continuous-time single-input and single-output linear time-invariant systems, we introduce the concept of extended zero dynamics and generalize the concept of minimum phase, which accommodates the presence of disturbance inputs. By representing a SISO LTI system with a finite relative degree in its extended zero dynamics canonical form, we obtain its extended zero dynamics, which is simply its zero dynamics driven by the noiseless output of the system and the disturbance input. We then say a system is minimum phase if its extended zero dynamics is absent or satisfies that, for any bounded admissible initial condition, any bounded noiseless output, and any bounded admissible disturbance input waveform, the extended zero dynamics state trajectory is bounded. The system is minimum phase (according to this extended notion) if its zero dynamics is asymptotically stable. It is proved that the converse holds under the additional condition that the system be stabilizable from the control input. For a system to be minimum phase, it is necessary that the transfer function from the control input to the output has all zeros with negative real parts. The converse holds when the system is both controllable (from the control input) and observable. It is further shown that the generalized minimum phase property is necessary for the achievement of perfect tracking of any bounded reference trajectories with bounded derivatives up to certain order without any disturbances and the existence of bounded state trajectory for any admissible bounded initial condition, any admissible bounded disturbance input waveform, and any bounded reference trajectory with bounded derivatives up to certain order in model reference control of the system.

**Key words.** continuous-time systems, extended zero dynamics canonical form, minimum phase, extended zero dynamics.

**AMS subject classifications.** 34A30, 34H05

**1. Introduction.** The minimum phase property for finite-dimensional continuous-time single-input and single-output (SISO) linear time-invariant (LTI) systems is of paramount importance in the theory of model reference control. The classical definition of the minimum phase property is in terms of the transfer function of the system. The system is said to be minimum phase if the numerator of the transfer function has roots all in the open left half of the complex plane ([3]).

In [4], the concept of zero dynamics for an affine finite-dimensional time-invariant nonlinear system is introduced, which is the dynamics of the system when the output of the system is kept to be identically zero. For controllable and observable finite-dimensional continuous-time SISO LTI systems, it turns out that the zero dynamics is asymptotically stable if and only if the zeros of its transfer function are in the open left half of the complex plane. The zero dynamics, as defined in [4], is an autonomous system without any inputs. This concept has its limitations in stability analysis due to the nonlinear nature of the problem. Recently, the minimum phase property for nonlinear systems has been revisited in [5], which is based on the concept of weakly uniform 0-detectability. The idea in [5] is in line of the development of input-output-

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to-state stability and output-to-state stability ([8]). It demonstrates the importance of concluding the boundedness of the states given that the system output and its derivatives up to certain order are bounded.

In model reference control for systems subject to exogeneous disturbance inputs, the objective is to make the system output to asymptotically track a desired reference trajectory while keeping all closed-loop states bounded when the disturbance input vanishes, and attenuate the effect of the disturbance input on the tracking error while keeping all closed-loop states bounded when the disturbance input is present. Then, the concept of minimum phase can be generalized to include systems with uncontrollable parts and/or unobservable parts and with exogeneous disturbance inputs. This is the main objective of this paper. For a finite-dimensional continuous-time SISO LTI system, when the relative degree  $r$  is well-defined, then it may be represented in the zero dynamics canonical form ([4]). When the system is also affected by disturbance inputs, the state transformation obtained while ignoring the disturbances leads to the extended zero dynamics canonical form under additional state transformations. We define the extended zero dynamics of the system to be the dynamics of its zero dynamics states, which is the zero dynamics driven by the noiseless output of the system as well as the disturbance input. We show that the extended zero dynamics has a certain invariance property. We say that the system is minimum phase if the extended zero dynamics states are bounded for any admissible bounded initial condition, any bounded noiseless output, and any admissible bounded disturbance input waveform, or if the extended zero dynamics is absent. It is straightforward that if the extended zero dynamics is asymptotically stable or if it is absent, then the system is minimum phase (according to the generalized definition). The converse of this result holds if the system is stabilizable from the control input. For a system to be minimum phase, it is necessary that the transfer function from the control input to the output has all zeros with negative real parts. The converse holds when the system is both controllable (from the control input) and observable. An example is included to illustrate the generalized minimum phase concept that demonstrates that the extended zero dynamics does not need to be asymptotically stable or bounded input and bounded state stable in the usual sense for the full order system to be minimum phase. We also prove that the generalized minimum phase property is necessary for the achievement of perfect tracking of any bounded reference trajectories with bounded derivatives up to  $r$ th order without any disturbances and the existence of bounded state trajectory for any admissible bounded initial condition, any admissible bounded disturbance waveform, and any bounded reference trajectory with bounded derivatives up to  $r$ th order in model reference control of the system.

The balance of the paper is organized as follows. In the next section, we introduce the notations to be used in the paper. In § 3, we introduce the extended zero dynamics canonical form for finite-dimensional continuous-time SISO LTI systems, the generalized definition of minimum phase, and its relationship with classical definitions. In § 4, we prove the necessity of the generalized minimum phase property in model reference control of SISO LTI systems. The paper ends with some concluding remarks in § 5 and an Appendix which includes a number of technical lemmas and corollaries that are essential for the derivation in the main body of the paper.

**2. Notations.** We let  $\mathbb{R}$  denote the real line; let  $\mathbb{R}_e$  denote the extended real line,  $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ ;  $\mathbb{N}$  to be the set of natural numbers;  $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ ; and  $\mathbb{C}$  to be the set of complex numbers. Unless specified, all signals, constants, and matrices are real. For a function  $f$ , we say that it belongs to  $\mathcal{C}$  (or  $\mathcal{C}_0$ ) if it is

continuous; we say that it belongs to  $\mathcal{C}_k$  if it is  $k$ -times continuously differentiable (Fréchet differentiability), which is equivalent to that all partial derivatives up to  $k$ th order are continuous when  $\text{dom}(f)$  is open,  $k \in \mathbb{N}$ . We say that a function is  $L_2$  if it is square integrable; and that it is  $L_\infty$  if it is bounded. We will write  $\mathcal{C}_k(A, B)$  and  $L_p(A, B)$  to denote set of functions of  $A$  to  $B$  which are  $k$ -times continuously differentiable and set of functions of  $A$  to  $B$  which have a finite  $L_p$  norm, respectively. For any matrix  $A$ ,  $A'$  denotes its transpose. For any  $m, n \in \mathbb{Z}_+$  and any  $m \times n$ -dimensional matrix  $M$ ,  $\mathcal{R}(M)$  denotes the range space of  $M$  and  $\mathcal{N}(M)$  denotes the null space of  $M$ . For any vector  $z \in \mathbb{R}^n$ , where  $n \in \mathbb{Z}_+$ ,  $|z|$  denotes  $\sqrt{z'z}$ . For any vector  $z \in \mathbb{R}^n$ , where  $n \in \mathbb{Z}_+$ . For  $n \times n$ -dimensional symmetric matrices  $M_1$  and  $M_2$ , where  $n \in \mathbb{Z}_+$ , we write  $M_1 > M_2$  if  $M_1 - M_2$  is positive definite; we write  $M_1 \geq M_2$  if  $M_1 - M_2$  is positive semi-definite. For  $n \in \mathbb{Z}_+$ , the set of  $n \times n$ -dimensional positive definite matrices is denoted by  $\bar{\mathcal{S}}_{+n}$ . For  $n \in \mathbb{Z}_+$ ,  $I_n$  denotes the  $n \times n$ -dimensional identity matrix. For  $n \in \mathbb{Z}_+$  and  $n \times n$ -dimensional matrix  $A$ , we set  $A^0 = I_n$ . For any matrix  $M$ ,  $\|M\|_{p,p}$  denotes its  $p$ -induced norm,  $1 \leq p \leq \infty$ . For any  $m, n \in \mathbb{Z}_+$ ,  $\mathbf{0}_{m \times n}$  denotes the  $m \times n$ -dimensional matrix whose elements are zeros. For any  $n \in \mathbb{N}$  and  $k \in \{1, 2, \dots, n\}$ ,  $e_{n,k}$  denotes the  $k$ th  $n$ -dimensional unit vector, i. e.,  $\begin{bmatrix} \mathbf{0}_{1 \times (k-1)} & 1 & \mathbf{0}_{1 \times (n-k)} \end{bmatrix}'$ . For any waveform  $u_{[0,t_f]} \in \mathcal{C}([0, t_f], \mathbb{R}^p)$ , where  $t_f \in (0, \infty] \subset \mathbb{R}_e$  and  $p \in \mathbb{Z}_+$ ,  $\|u_{[0,t_f]}\|_\infty = \sup_{t \in [0, t_f]} |u(t)|$  and  $\|u_{[0,t_f]}\|_q = \left( \int_0^{t_f} |u(t)|^q dt \right)^{1/q}$ ,  $q \in [1, \infty)$ . For a sufficiently smooth signal  $v$ ,  $v^{(i)}$  denotes the  $i$ th order derivative of  $v$ ,  $v^{[i]}$  denotes  $\begin{bmatrix} v' & (v^{(1)})' & \dots & (v^{(i)})' \end{bmatrix}'$ ,  $i \in \mathbb{Z}_+$ . For a  $\mathcal{C}_\infty$  vector field  $f$  and a  $\mathcal{C}_\infty$  function  $h$ ,  $L_f h$  denotes the derivative of  $h$  along  $f$ , which equals to  $\frac{\partial h}{\partial x}(x)f(x)$  in local coordinates.  $L_f^{k+1}h = L_f(L_f^k h)$ ,  $k \in \mathbb{N}$ ;  $L_f^0 h = h$ .  $\forall \lambda \in \mathbb{C}$ ,  $\text{Re}(\lambda)$  and  $\text{Im}(\lambda)$  denote the real part and the imaginary part of  $\lambda$ , respectively. We will denote constants or matrices of no specific interest or relevance to the analysis by  $\star$ . We will denote  $m \times n$ -dimensional matrices of no specific interest or relevance to the analysis by  $\star_{m \times n}$ .

**3. Definition of minimum phase property.** In this section, we will introduce a generalized definition of the minimum phase property that accommodates the presence of disturbance inputs, and show its relationship with respect to its classical definition. First, we will introduce the extended zero dynamics canonical form for SISO LTI systems.

LEMMA 3.1. *Consider a finite-dimensional continuous-time SISO LTI system*

$$(3.1a) \quad \dot{x} = Ax + Bu + Dw$$

$$(3.1b) \quad y = Cx + Ew$$

where  $x$  is the  $n$ -dimensional state,  $n \in \mathbb{N}$ ;  $u$  is the scalar control input;  $y$  is the scalar output;  $w$  is the  $q$ -dimensional disturbance input,  $q \in \mathbb{Z}_+$ ;  $A$ ,  $B$ ,  $D$ ,  $C$ , and  $E$  are constant matrices of appropriate dimensions. Let the system have relative degree  $r \in \mathbb{N}$ ,  $r < n$ , from  $u$  to  $y$ , that is,  $CB = \dots = CA^{r-2}B = 0$  and  $CA^{r-1}B \neq 0$ . Then, there exists an invertible matrix  $T_o$  such that, in  $\begin{bmatrix} x'_z & x_1 & \dots & x_r \end{bmatrix}' = T_o^{-1}x$  coordinates, the system (3.1) admits the state space representation

$$(3.2a) \quad \dot{x}_z = A_z x_z + A_{z1} x_1 + D_z w$$

$$(3.2b) \quad \dot{x}_i = a_{i1} x_1 + x_{i+1} + D_i w; \quad i = 1, \dots, r-1$$

$$(3.2c) \quad \dot{x}_r = A_{rz} x_z + a_{r1} x_1 + b_0 u + D_r w$$

$$(3.2d) \quad y = x_1 + Ew$$

where  $x_z$  is  $(n-r)$ -dimensional;  $x_i$ ,  $i = 1, \dots, r$ , are scalars;  $b_0 = CA^{r-1}B \neq 0$  is the high-frequency gain of the system. The representation (3.2) is called the extended zero dynamics canonical form of system (3.1).

Furthermore, given any  $\mathcal{C}_\infty$  diffeomorphism  $\bar{x} = [\bar{x}'_z \ \bar{x}_1 \ \dots \ \bar{x}_r] = T^{-1}(x)$ , where  $T : \mathcal{D} \rightarrow \mathbb{R}^n$  and  $\mathcal{D} \subseteq \mathbb{R}^n$  is open, that leads to a state space representation

$$(3.3a) \quad \dot{\bar{x}}_z = f_z(\bar{x}_z, \bar{x}_1) + h_z(\bar{x})w$$

$$(3.3b) \quad \dot{\bar{x}}_i = f_i(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_i) + \bar{x}_{i+1} + h_i(\bar{x})w; \quad i = 1, \dots, r-1$$

$$(3.3c) \quad \dot{\bar{x}}_r = f_r(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_r) + g_o(\bar{x})u + h_r(\bar{x})w$$

$$(3.3d) \quad y = \bar{x}_1 + Ew$$

where  $\bar{x}_z$  is  $(n-r)$ -dimensional;  $\bar{x}_i$ ,  $i = 1, \dots, r$ , are scalars, then, we have  $\mathcal{D} = \mathcal{D}_z \times \mathbb{R}^r$  with  $\mathcal{D}_z \subseteq \mathbb{R}^{n-r}$  being open;  $\bar{x}_1 = x_1$ ,  $g_o(\bar{x}) = b_0$ ,  $\forall \bar{x} \in \mathcal{D}$ ;  $\bar{x}_z = T_z^{-1}(x_z)$ , where  $T_z : \mathcal{D}_z \rightarrow \mathbb{R}^{n-r}$  is a  $\mathcal{C}_\infty$  diffeomorphism, whose inverse is  $T_{Iz}$ ;  $f_z(\bar{x}_z, \bar{x}_1) = \bar{f}_z(\bar{x}_z) + f_{z1}(\bar{x}_z)\bar{x}_1$ ,  $\bar{f}_z(\bar{x}_z) = \left( \frac{\partial T_{Iz}}{\partial x_z}(x_z)A_z x_z \right) \Big|_{x_z=T_z(\bar{x}_z)}$ ,  $f_{z1}(\bar{x}_z) = \left( \frac{\partial T_{Iz}}{\partial x_z}(x_z)A_{z1} \right) \Big|_{x_z=T_z(\bar{x}_z)}$ ,  $h_z(\bar{x}) = \left( \frac{\partial T_{Iz}}{\partial x_z}(x_z)D_z \right) \Big|_{x_z=T_z(\bar{x}_z)} = h_z(\bar{x}_z)$ ,  $h_1(\bar{x}) = h_1$ , and  $h_i(\bar{x}) = h_i(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{i-1})$ ,  $i = 2, \dots, r$ , when  $r \geq 2$ ,  $\forall \bar{x} \in \mathcal{D}$ .

Hence, we observe that the dynamics (3.2a) is invariant (modulo  $\mathcal{C}_\infty$  diffeomorphisms) under  $\mathcal{C}_\infty$  diffeomorphism that brings system (3.1) into form (3.3). Henceforth, we will call (3.2a) the extended zero dynamics of (3.1).

*Proof.* We will apply the machinery of [4], and adapt it to our present problem.

Let  $V = [B \ \dots \ A^{r-1}B]_{n \times r}$  and  $U = \begin{bmatrix} C \\ \vdots \\ CA^{r-1} \end{bmatrix}_{r \times n}$ . Then, we have

$$UV = \begin{bmatrix} 0 & \dots & 0 & CA^{r-1}B \\ \vdots & \ddots & \ddots & \star \\ 0 & \ddots & \ddots & \vdots \\ CA^{r-1}B & \star & \dots & \star \end{bmatrix}_{r \times r}$$

which is clearly invertible. Hence,  $U$  and  $V$  are of rank  $r$ . Note that  $\text{rank}(V) = r$  implies that  $\dim(\mathcal{N}(V')) = n-r$ . Hence, there exists a real nonsingular  $n \times (n-r)$ -dimensional matrix  $K$  such that  $V'K = \mathbf{0}_{r \times (n-r)}$  and  $\text{rank}(K) = n-r$ . Define  $\bar{U} = \begin{bmatrix} K' \\ U \end{bmatrix}_{n \times n}$  and  $\bar{V} = [K \ V]_{n \times n}$ . Then, we have  $\bar{U}\bar{V} = \begin{bmatrix} K'K & \mathbf{0}_{(n-r) \times r} \\ UK & UV \end{bmatrix}$ . Since  $K$  is nonsingular, then  $K'K$  is invertible. Hence,  $\bar{U}\bar{V}$  is block lower triangular and is invertible. Then, we have  $\bar{U}$  and  $\bar{V}$  as invertible matrices.

Let  $x_z = K'x$  and  $z_i = CA^{i-1}x$ ,  $i = 1, \dots, r$ . Consider the coordinate transformation  $z := [x'_z \ z_1 \ \dots \ z_r]' = \bar{U}x$ . In  $z$  coordinates, system (3.1) admits the state space representation

$$\begin{aligned} \dot{z} &= \bar{U}\bar{A}\bar{U}^{-1}z + \bar{U}Bu + \bar{U}Dw =: \tilde{A}z + \tilde{B}u + \tilde{D}w \\ y &= C\bar{U}^{-1}z + Ew =: \tilde{C}z + Ew \end{aligned}$$

Note that

$$C = e'_{n,n-r+1}\bar{U} \Rightarrow \tilde{C} = e'_{n,n-r+1}; \quad \tilde{A} = \bar{U}A\bar{U}^{-1} =: \begin{bmatrix} \tilde{A}_z & \tilde{A}_{z1} & \cdots & \tilde{A}_{zr} \\ \tilde{A}_{1z} & \tilde{a}_{11} & \cdots & \tilde{a}_{1r} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{rz} & \tilde{a}_{r1} & \cdots & \tilde{a}_{rr} \end{bmatrix}$$

$$\tilde{B} = \bar{U}B = \begin{bmatrix} K' \\ U \end{bmatrix} B = \begin{bmatrix} \mathbf{0}_{(n-r) \times 1} \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-1) \times 1} \\ b_0 \end{bmatrix}$$

where  $\tilde{A}_z$  is  $(n-r) \times (n-r)$ -dimensional;  $\tilde{a}_{ij}$ ,  $i, j = 1, \dots, r$ , are scalar constants. Next, we calculate the matrix  $\tilde{A}$ , by the equality  $\bar{U}A = \tilde{A}\bar{U}$ . Thus, we have

$$\bar{U}A = \begin{bmatrix} K'A \\ CA \\ \vdots \\ CA^r \end{bmatrix} = \begin{bmatrix} \tilde{A}_z K' + \sum_{i=1}^r \tilde{A}_{zi} CA^{i-1} \\ \tilde{A}_{1z} K' + \sum_{i=1}^r \tilde{a}_{1i} CA^{i-1} \\ \vdots \\ \tilde{A}_{rz} K' + \sum_{i=1}^r \tilde{a}_{ri} CA^{i-1} \end{bmatrix} = \tilde{A}\bar{U}$$

Equating  $CA^j$  and  $\tilde{A}_{jz}K' + \sum_{i=1}^r \tilde{a}_{ji}CA^{i-1}$ ,  $j = 1, \dots, r-1$ , we have  $\tilde{A}_{jz} = \mathbf{0}_{1 \times (n-r)}$ ,  $\tilde{a}_{jj+1} = 1$ , and  $\tilde{a}_{ji} = 0$ , when  $j = 1, \dots, r-1$  and  $i = 1, \dots, r$ , and  $i \neq j+1$ .

Equating  $K'A$  and  $\tilde{A}_z K' + \sum_{i=1}^r \tilde{A}_{zi} CA^{i-1}$ , we have the following line of argument. Note that

$$K'AV = K' \begin{bmatrix} AB & \cdots & A^r B \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-r) \times 1} & \cdots & \mathbf{0}_{(n-r) \times 1} & K'A^r B \end{bmatrix}$$

$$= \frac{1}{b_0} K'A^r B C V$$

Therefore, we have  $(K'A - K'A^r B C / b_0)V = \mathbf{0}_{(n-r) \times r}$ . Then,  $V'(A'K - C'B'A'^r \cdot K/b_0) = \mathbf{0}_{r \times (n-r)}$ . Denote the column vectors of  $K$  by  $K_i$ ,  $i = 1, \dots, n-r$ . Then, the column vectors of  $A'K - C'B'A'^r K/b_0$ , that is  $A'K_i - C'B'A'^r K_i/b_0$ ,  $i = 1, \dots, n-r$ , are in the null space of  $V'$ , and therefore in the span of  $K$ . Hence, there exists  $(n-r) \times (n-r)$ -dimensional real matrix  $\tilde{\tilde{A}}_z$  such that  $A'K - C'B'A'^r K/b_0 = K\tilde{\tilde{A}}_z$ , which implies  $K'A = \tilde{\tilde{A}}_z' K' + K'A^r B C / b_0$ . Then, we have  $\tilde{A}_z = \tilde{\tilde{A}}_z'$ ,  $\tilde{A}_{z1} = K'A^r B / b_0$ , and  $\tilde{A}_{zj} = \mathbf{0}_{(n-r) \times 1}$ ,  $j = 2, \dots, r$ .

Hence, in  $z$  coordinates, system (3.1) may be represented by

$$\begin{aligned} \dot{x}_z &= \tilde{A}_z x_z + \tilde{A}_{z1} z_1 + \tilde{D}_z w \\ \dot{z}_i &= z_{i+1} + \tilde{D}_i w; \quad i = 1, \dots, r-1 \\ \dot{z}_r &= \tilde{A}_{rz} x_z + \sum_{i=1}^r \tilde{a}_{ri} z_i + b_0 u + \tilde{D}_r w \\ y &= z_1 + Ew \end{aligned}$$

Let  $z_f = [z_1 \ \cdots \ z_r]'$ . Then, the dynamics for  $z_f$  is

$$(3.4a) \dot{z}_f = \left[ \begin{array}{c|ccc} \mathbf{0}_{(r-1) \times 1} & & & \\ \hline \tilde{a}_{r1} & \tilde{a}_{r2} & \cdots & \tilde{a}_{rr} \end{array} \right] z_f + \begin{bmatrix} \mathbf{0}_{(r-1) \times 1} \\ b_0 \end{bmatrix} u$$

$$+ \begin{bmatrix} \mathbf{0}_{(r-1) \times (n-r)} \\ \tilde{A}_{rz} \end{bmatrix} x_z + \begin{bmatrix} \tilde{D}_1 \\ \vdots \\ \tilde{D}_r \end{bmatrix} w =: A_f z_f + B_f u + A_{fz} x_z + D_f w$$

$$(3.4b) \quad y = \begin{bmatrix} 1 & \mathbf{0}_{1 \times (r-1)} \end{bmatrix} z_f + Ew =: C_f z_f + Ew$$

It is clear that, in the dynamics (3.4), the transfer function from  $u$  to  $y$  is  $H_f(s) = C_f (sI_r - A_f)^{-1} B_f = \frac{b_0}{s^r - \tilde{a}_{rr}s^{r-1} - \dots - \tilde{a}_{r1}}$ . Clearly, the dynamics (3.4) is observable. Then, there exists an invertible coordinate transformation  $x_f = T_f^{-1} z_f$  that transforms (3.4) into the observer canonical form.

$$\begin{aligned} \dot{x}_f &= \left[ \begin{array}{c|c} \tilde{a}_{rr} & I_{r-1} \\ \vdots & \\ \hline \tilde{a}_{r2} & \\ \tilde{a}_{r1} & \mathbf{0}_{1 \times (r-1)} \end{array} \right] x_f + \begin{bmatrix} \mathbf{0}_{(r-1) \times 1} \\ b_0 \end{bmatrix} u + T_f^{-1} \begin{bmatrix} \mathbf{0}_{(r-1) \times (n-r)} \\ \tilde{A}_{rz} \end{bmatrix} x_z \\ &\quad + T_f^{-1} D_f w \\ y &= \begin{bmatrix} 1 & \mathbf{0}_{1 \times (r-1)} \end{bmatrix} x_f + Ew \end{aligned}$$

Note that  $T_f^{-1} \begin{bmatrix} \mathbf{0}_{1 \times (r-1)} & b_0 \end{bmatrix}' = \begin{bmatrix} \mathbf{0}_{1 \times (r-1)} & b_0 \end{bmatrix}'$  implies that the last column of  $T_f^{-1}$  is equal to  $e_{r,r}$ . Then, we have

$$T_f^{-1} \begin{bmatrix} \mathbf{0}_{(r-1) \times (n-r)} \\ \tilde{A}_{rz} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(r-1) \times (n-r)} \\ \tilde{A}_{rz} \end{bmatrix}$$

Let the elements of  $x_f$  be  $[x_1 \ \dots \ x_r]'$ . Clearly,  $y = z_1 + Ew = x_1 + Ew$ , then, we have  $z_1 = x_1$ . Then, the system (3.1) admits the following state space representation, in  $x := [x'_z \ x_1 \ \dots \ x_r]'$  coordinates,

$$\begin{aligned} \dot{x}_z &= \tilde{A}_z x_z + \tilde{A}_{z1} x_1 + \tilde{D}_z w \\ \dot{x}_i &= \tilde{a}_{r,r+1-i} x_1 + x_{i+1} + D_i w; \quad i = 1, \dots, r-1 \\ \dot{x}_r &= \tilde{A}_{rz} x_z + \tilde{a}_{r1} x_1 + b_0 u + D_r w \\ y &= x_1 + Ew \end{aligned}$$

where  $[D'_1 \ \dots \ D'_r] = T_f^{-1} D_f$ . Clearly, the above is in the form of (3.2). Hence, the desired matrix  $T_o = \bar{U}^{-1} \begin{bmatrix} I_{n-r} & \mathbf{0} \\ \mathbf{0} & T_f \end{bmatrix}$ .

Let  $T : \mathcal{D} \rightarrow \mathbb{R}^n$  be any  $\mathcal{C}_\infty$  diffeomorphism such that, in  $\bar{x} = T^{-1}(x) = [\bar{x}'_z \ \bar{x}_1 \ \dots \ \bar{x}_r]'$  coordinates, the system (3.1) admits the state space representation (3.3). Without loss of generality, assume that system (3.1) is given in the form of (3.2). Then,

$$A = \begin{bmatrix} A_z & A_{z1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & a_{11} & 1 & 0 & \dots & 0 \\ \mathbf{0} & a_{21} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \mathbf{0} & a_{r-1,1} & 0 & \dots & 0 & 1 \\ A_{rz} & a_{r1} & 0 & \dots & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} \mathbf{0}_{(n-1) \times 1} \\ b_0 \end{bmatrix}; \quad D = \begin{bmatrix} D_z \\ D_1 \\ \vdots \\ D_r \end{bmatrix}$$

$$C = e'_{n,n-r+1}$$

$$\begin{aligned}\dot{\bar{x}} &= f(\bar{x}) + g(\bar{x})u + h(\bar{x})w \\ y &= \bar{x}_1 + Ew = C\bar{x} + Ew\end{aligned}$$

$$f(\bar{x}) = \begin{bmatrix} f_z(\bar{x}_z, \bar{x}_1) \\ f_1(\bar{x}_z, \bar{x}_1) + \bar{x}_2 \\ \vdots \\ f_{r-1}(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{r-1}) + \bar{x}_r \\ f_r(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_r) \end{bmatrix}; \quad g(\bar{x}) = \begin{bmatrix} \mathbf{0}_{(n-1) \times 1} \\ g_o(\bar{x}) \end{bmatrix}; \quad h(\bar{x}) = \begin{bmatrix} h_z(\bar{x}) \\ h_1(\bar{x}) \\ \vdots \\ h_r(\bar{x}) \end{bmatrix}$$

$Ax = \left(\frac{\partial T}{\partial \bar{x}}(\bar{x})f(\bar{x})\right)|_{\bar{x}=T^{-1}(x)}$ ;  $B = \left(\frac{\partial T}{\partial \bar{x}}(\bar{x})g(\bar{x})\right)|_{\bar{x}=T^{-1}(x)}$ ;  $D = \left(\frac{\partial T}{\partial \bar{x}}(\bar{x}) \cdot h(\bar{x})\right)|_{\bar{x}=T^{-1}(x)}$ ;  $Cx = (C\bar{x})|_{\bar{x}=T^{-1}(x)}$ ,  $\forall x \in \mathbb{R}^n$ . Clearly,  $f$ ,  $g$ , and  $h$  are  $\mathcal{C}_\infty$  mappings of  $\mathcal{D}$ .

Let  $T^{-1}(x) := T_I(x) = [ (T_{I_z}(x))' \quad T_{I_1}(x) \quad \dots \quad T_{I_r}(x) ]'$  such that  $\bar{x}_z = T_{I_z}(x) \in \mathbb{R}^{n-r}$ ,  $\bar{x}_i = T_{I_i}(x) \in \mathbb{R}$ ,  $i = 1, \dots, r$ . Then,  $x_1 = Cx = (C\bar{x})|_{\bar{x}=T^{-1}(x)}$  implies that  $x_1 = \bar{x}_1|_{\bar{x}=T_I(x)} = T_{I_1}(x)$ ,  $\forall x \in \mathbb{R}^n$ .

CLAIM 3.1.1.  $\frac{\partial T_{I_z}}{\partial x_i}(x) = \mathbf{0}_{(n-r) \times 1}$ ,  $\forall x \in \mathbb{R}^n$ ,  $i = 1, \dots, r$ .

*Proof.* Note that  $\bar{x}_z = T_{I_z}(x)$  implies

$$\dot{\bar{x}}_z = f_z(\bar{x}_z, \bar{x}_1) + h_z(\bar{x})w = \left(\frac{\partial T_{I_z}}{\partial x}(x)(Ax + Bu + Dw)\right)|_{x=T(\bar{x})}$$

Hence, we have,  $\forall x \in \mathbb{R}^n$ ,

$$(3.5a) \quad f_z(T_{I_z}(x), x_1) = \frac{\partial T_{I_z}}{\partial x}(x)Ax$$

$$(3.5b) \quad \mathbf{0}_{(n-r) \times 1} = \frac{\partial T_{I_z}}{\partial x}(x)B$$

$$(3.5c) \quad h_z(T_I(x)) = \frac{\partial T_{I_z}}{\partial x}(x)D$$

By (3.5b), we have  $\frac{\partial T_{I_z}}{\partial x_r}(x)b_0 = \mathbf{0}_{(n-r) \times 1}$ ,  $\forall x \in \mathbb{R}^n$ , which implies that  $\frac{\partial T_{I_z}}{\partial x_r}(x) = \mathbf{0}_{(n-r) \times 1}$ ,  $\forall x \in \mathbb{R}^n$ .

We will prove the claim using mathematical induction on  $r + 1 - i$ .

1°  $r + 1 - i = 1 \Rightarrow i = r$ . We have shown that  $\frac{\partial T_{I_z}}{\partial x_r}(x) = \mathbf{0}_{(n-r) \times 1}$ ,  $\forall x \in \mathbb{R}^n$ . This completes the initialization step of the induction process. If  $r = 1$ , then the claim is proved. If  $r > 1$ , then we continue to the next step.

2° Assume the result holds for  $r + 1 - i \in \{1, \dots, k\}$ , with  $1 \leq k < r$ .

3° Consider the case  $r + 1 - i = k + 1$ . Then,  $i = r - k$ . By (3.5a), we have

$$f_z(T_{I_z}(x_z, x_1, \dots, x_r), x_1) = \frac{\partial T_{I_z}}{\partial x}(x_z, x_1, \dots, x_r)Ax, \quad \forall x \in \mathbb{R}^n$$

Then, by the inductive assumption,  $\forall x \in \mathbb{R}^n$ ,

$$\begin{aligned}f_z(T_{I_z}(x_z, x_1, \dots, x_r), x_1) &= \frac{\partial T_{I_z}}{\partial x_z}(x_z, x_1, \dots, x_r)(A_z x_z + A_{z1} x_1) \\ &\quad + \sum_{j=1}^{r-k} \frac{\partial T_{I_z}}{\partial x_j}(x_z, x_1, \dots, x_r)(a_{j1} x_1 + x_{j+1})\end{aligned}$$

Taking partial derivatives with respect to  $x_{r-k+1}$  on both sides of the above equation, noting that  $2 \leq r - k + 1 \leq r$ , we have,  $\forall x \in \mathbb{R}^n$ ,

$$\begin{aligned} & \frac{\partial f_z}{\partial \bar{x}_z}(T_{I_z}(x_z, x_1, \dots, x_r), x_1) \frac{\partial T_{I_z}}{\partial x_{r-k+1}}(x_z, x_1, \dots, x_r) \\ &= \frac{\partial^2 T_{I_z}}{\partial x_{r-k+1} \partial x_z}(x_z, x_1, \dots, x_r) (A_z x_z + A_{z1} x_1) \\ &+ \sum_{j=1}^{r-k} \frac{\partial^2 T_{I_z}}{\partial x_{r-k+1} \partial x_j}(x_z, x_1, \dots, x_r) (a_{j1} x_1 + x_{j+1}) + \frac{\partial T_{I_z}}{\partial x_{r-k}}(x_z, x_1, \dots, x_r) \end{aligned}$$

By the inductive assumption, then,  $\frac{\partial T_{I_z}}{\partial x_{r-k}}(x) = \mathbf{0}_{(n-r) \times 1}$ ,  $\forall x \in \mathbb{R}^n$ .

This completes the induction process.

This completes the proof of the claim.  $\square$

Therefore, we have shown that  $\frac{\partial T_{I_z}}{\partial x_i}(x) = \mathbf{0}_{(n-r) \times 1}$ ,  $\forall x \in \mathbb{R}^n$ ,  $i = 1, \dots, r$ . Then, by Mean Value Theorem,  $T_{I_z}(x) = T_{I_z}(x_z)$ ,  $\forall x \in \mathbb{R}^n$ .

**CLAIM 3.1.2.** *Let  $\mathcal{D}_i$  denotes the projection of  $\mathcal{D}$  onto the  $[\bar{x}'_z \ \bar{x}_1 \ \dots \ \bar{x}_i]'$  subspace,  $i = 0, \dots, r$ , which is an open subset of  $\mathbb{R}^{n-r+i}$  since  $\mathcal{D}$  is open in  $\mathbb{R}^n$ . We will denote  $\mathcal{D}_z := \mathcal{D}_0$ . By Claim 3.1.1, we have  $T_{I_z} : \mathbb{R}^{n-r} \rightarrow \mathcal{D}_z$  and  $T_{I_i} : \mathbb{R}^n \rightarrow \mathbb{R}$  are  $\mathcal{C}_\infty$ , and,  $\forall x \in \mathbb{R}^n$ ,  $[(T_{I_z}(x))' \ T_{I_1}(x) \ \dots \ T_{I_i}(x)]' \in \mathcal{D}_i$ ,  $\forall i = 1, \dots, r$ . Furthermore,  $\forall i \in \{1, \dots, r\}$ , we have*

$$\begin{aligned} L_{(Ax)}^{i-1}(Cx) &= CA^{i-1}x, \quad \forall x \in \mathbb{R}^n; \quad CA^{i-1} = \begin{bmatrix} \mathbf{0}_{1 \times (n-r)} & \star_{1 \times (i-1)} & 1 & \mathbf{0}_{1 \times (r-i)} \end{bmatrix} \\ L_{f(\bar{x})}^{i-1}(C\bar{x}) &= \bar{x}_i + \bar{l}_i(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{i-1}), \quad \forall \bar{x} \in \mathcal{D} \\ T_{I_i}(x) &= x_i + \bar{T}_{I_i}(x_z, x_1, \dots, x_{i-1}), \quad \forall x \in \mathbb{R}^n \end{aligned}$$

where  $\bar{l}_i : \mathcal{D}_{i-1} \rightarrow \mathbb{R}$  is  $\mathcal{C}_\infty$ ,  $\bar{T}_{I_1} : \mathbb{R}^{n-r} \rightarrow \mathbb{R}$  is the identically zero function, and  $\bar{T}_{I_i} : \mathbb{R}^{n-r+i-1} \rightarrow \mathbb{R}$  is  $\mathcal{C}_\infty$ , if  $i \geq 2$ .

*Proof.* The statements preceding the word ‘‘furthermore’’ are self evident. We will prove the rest of the claim using mathematical induction on  $i$ .

1°  $i = 1$ .  $L_{(Ax)}^0(Cx) = Cx$  and  $CA^0 = C = \begin{bmatrix} \mathbf{0}_{1 \times (n-r)} & 1 & \mathbf{0}_{1 \times (r-1)} \end{bmatrix}$ .  $L_{f(\bar{x})}^0(C\bar{x}) = C\bar{x} = \bar{x}_1$ ,  $\forall \bar{x} \in \mathcal{D}$ . We have shown that  $T_{I_1}(x) = x_1$ ,  $\forall x \in \mathbb{R}^n$ . This completes the initialization step. If  $r = 1$ , then the claim is proved. If  $r > 1$ , we continue to the next step.

2° Assume that the claim holds for  $i \in \{1, \dots, k\}$ , with  $1 \leq k < r$ .

3° Consider the case  $i = k + 1$ . We have

$$\begin{aligned} L_{(Ax)}^k(Cx) &= L_{(Ax)} L_{(Ax)}^{k-1}(Cx) = L_{(Ax)}(CA^{k-1}x) = CA^k x \\ CA^k &= (CA^{k-1})A = \begin{bmatrix} \mathbf{0}_{1 \times (n-r)} & \star_{1 \times k} & 1 & \mathbf{0}_{1 \times (r-k-1)} \end{bmatrix} \\ L_{f(\bar{x})}^k(C\bar{x}) &= L_{f(\bar{x})} L_{f(\bar{x})}^{k-1}(C\bar{x}) = L_{f(\bar{x})}(\bar{x}_k + \bar{l}_k(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{k-1})) \\ &= \bar{x}_{k+1} + f_k(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_k) + \frac{\partial \bar{l}_k}{\partial \bar{x}_z}(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{k-1}) f_z(\bar{x}_z, \bar{x}_1) \\ &\quad + \sum_{j=1}^{k-1} \frac{\partial \bar{l}_k}{\partial \bar{x}_j}(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{k-1}) (f_j(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_j) + \bar{x}_{j+1}) \\ &=: \bar{x}_{k+1} + \bar{l}_{k+1}(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_k) \end{aligned}$$



Clearly,  $\bar{l}_{k+1} : \mathcal{D}_k \rightarrow \mathbb{R}$  is  $\mathcal{C}_\infty$ . Note that  $C\bar{x}$  and  $f(\bar{x})$  are the representations of  $Cx$  and  $Ax$  in  $\bar{x}$  coordinates, respectively. Then, we have

$$L_{(Ax)}^k(Cx) = \left( L_{f(\bar{x})}^k(C\bar{x}) \right) \Big|_{\bar{x}=T_I(x)}, \quad \forall x \in \mathbb{R}^n$$

which implies that

$$\begin{aligned} CA^k x &= T_{I_{k+1}}(x) + \bar{l}_{k+1}(T_{I_z}(x_z), T_{I_1}(x), \dots, T_{I_k}(x)) \\ \Rightarrow \star x_1 + \dots + \star x_k + x_{k+1} &= T_{I_{k+1}}(x) + \bar{l}_{k+1}(T_{I_z}(x_z), T_{I_1}(x), \dots, T_{I_k}(x)) \\ \Rightarrow T_{I_{k+1}}(x) &= x_{k+1} + \bar{T}_{I_{k+1}}(x_z, x_1, \dots, x_k) \end{aligned}$$

where the last implication follows from the inductive assumption. Clearly,  $\bar{T}_{I_{k+1}} : \mathbb{R}^{n-r+k} \rightarrow \mathbb{R}$  is  $\mathcal{C}_\infty$ .

This completes the induction process.

Hence, the claim is proved.  $\square$

Note that  $L_B L_{(Ax)}^{r-1}(Cx) = CA^{r-1}B = b_0$ ,  $\forall x \in \mathbb{R}^n$ , and  $L_{g(\bar{x})} L_{f(\bar{x})}^{r-1}(C\bar{x}) = L_{g(\bar{x})}(\bar{x}_r + \bar{l}_r(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{r-1})) = g_o(\bar{x})$ ,  $\forall \bar{x} \in \mathcal{D}$ . Note that  $C\bar{x}$ ,  $f(\bar{x})$ ,  $g(\bar{x})$  are the representations of  $Cx$ ,  $Ax$ , and  $B$  in  $\bar{x}$  coordinates, respectively. Then, we have  $b_0 = g_o(\bar{x})|_{\bar{x}=T^{-1}(x)} \Rightarrow g_o(\bar{x}) = b_0$ ,  $\forall \bar{x} \in \mathcal{D}$ .

Now, we are in a position to show that  $\mathcal{D} = \mathcal{D}_z \times \mathbb{R}^r$ . Note that,  $\forall x \in \mathbb{R}^n$ , by Claims 3.1.1 and 3.1.2,

$$T^{-1}(x) = T_I(x) = \begin{bmatrix} T_{I_z}(x_z) \\ x_1 \\ x_2 + \bar{T}_{I_2}(x_z, x_1) \\ \vdots \\ x_r + \bar{T}_{I_r}(x_z, x_1, \dots, x_{r-1}) \end{bmatrix} \in \mathcal{D}$$

Since  $T : \mathcal{D} \rightarrow \mathbb{R}^n$  is a  $\mathcal{C}_\infty$  diffeomorphism, then  $T$  is bijective and  $\frac{\partial T_I}{\partial x}(x)$  is an invertible matrix,  $\forall x \in \mathbb{R}^n$ . Then,  $T_{I_z}$  must be surjective since  $T_I$  is surjective;  $\frac{\partial T_{I_z}}{\partial x_z}(x_z)$  must be an invertible matrix,  $\forall x_z \in \mathbb{R}^{n-r}$ , since  $\frac{\partial T_I}{\partial x}(x)$  has a block lower triangular structure. Furthermore,  $T_{I_z}$  must be injective since  $T_I$  is injective and  $\text{dom}(T_I) = \mathbb{R}^n$ . Hence,  $T_{I_z}$  is  $\mathcal{C}_\infty$ , bijective, and  $\frac{\partial T_{I_z}}{\partial x_z}(x_z)$  is an invertible matrix,  $\forall x_z \in \mathbb{R}^{n-r}$ . This implies that  $T_{I_z} : \mathbb{R}^{n-r} \rightarrow \mathcal{D}_z$  is a  $\mathcal{C}_\infty$  diffeomorphism by the Inverse Function Theorem [1], and  $T_{I_z}^{-1} := T_z : \mathcal{D}_z \rightarrow \mathbb{R}^{n-r}$ .  $\forall \bar{x} \in \mathcal{D}_z \times \mathbb{R}^r$ , by the block lower triangular structure of  $T_I$ ,  $\exists x \in \mathbb{R}^n$  such that  $\tilde{x} = T_I(x) \in \mathcal{D}$ . Then,  $\mathcal{D}_z \times \mathbb{R}^r \subseteq \mathcal{D} = T_I(\mathbb{R}^n)$ . By  $\mathcal{D}_z$  being the projection of  $\mathcal{D}$  to  $\bar{x}_z$  subspace, we have  $\mathcal{D} \subseteq \mathcal{D}_z \times \mathbb{R}^r$ . Hence,  $\mathcal{D} = \mathcal{D}_z \times \mathbb{R}^r$ .

It is a simple matter of algebra to realize that,  $\forall \bar{x} \in \mathcal{D}$ , we have

$$T(\bar{x}) = \begin{bmatrix} T_z(\bar{x}_z) \\ \bar{x}_1 \\ \bar{x}_2 + \bar{T}_2(\bar{x}_z, \bar{x}_1) \\ \vdots \\ \bar{x}_r + \bar{T}_r(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{r-1}) \end{bmatrix} \in \mathbb{R}^n$$

where  $\bar{T}_i : \mathcal{D}_i \rightarrow \mathbb{R}$  are  $\mathcal{C}_\infty$ ,  $i = 2, \dots, r$ .

Then,  $\forall \bar{x} \in \mathcal{D}$ ,

$$\dot{\bar{x}}_z = \left( \frac{\partial T_{I_z}}{\partial x_z}(x_z) (A_z x_z + A_{z1} x_1 + D_z w) \right) \Big|_{x_z=T_z(\bar{x}_z), x_1=\bar{x}_1} = f_z(\bar{x}_z, \bar{x}_1) + h_z(\bar{x})w$$

Therefore  $f_z(\bar{x}_z, \bar{x}_1) = \bar{f}_z(\bar{x}_z) + f_{z1}(\bar{x}_z)\bar{x}_1$  with  $\bar{f}_z(\bar{x}_z) = \left( \frac{\partial T_{Iz}}{\partial x_z}(x_z)A_z x_z \right) \Big|_{x_z=T_z(\bar{x}_z)}$  and  $f_{z1}(\bar{x}_z) = \left( \frac{\partial T_{Iz}}{\partial x_z}(x_z)A_{z1} \right) \Big|_{x_z=T_z(\bar{x}_z)}$ ,  $h_z(\bar{x}) = \left( \frac{\partial T_{Iz}}{\partial x_z}(x_z)D_z \right) \Big|_{x_z=T_z(\bar{x}_z)} = h_z(\bar{x}_z)$ ,  $\forall \bar{x} \in \mathcal{D}$ .

To obtain the formula for  $h_i$ 's, we will distinguish between 2 exhaustive and mutually exclusive cases: Case 1:  $r = 1$ ; Case 2:  $r \geq 2$ .

Case 1:  $r = 1$ .  $\dot{x}_1 = A_{1z}x_z + a_{11}x_1 + b_0u + D_1w = f_1(\bar{x}_z, \bar{x}_1) + g_o(\bar{x})u + h_1(\bar{x})w$ . Hence, we have  $h_1(\bar{x}) = D_1$ ,  $\forall \bar{x} \in \mathcal{D}$ .

Case 2:  $r \geq 2$ .  $\dot{x}_1 = a_{11}x_1 + x_2 + D_1w = f_1(\bar{x}_z, \bar{x}_1) + \bar{x}_2 + h_1(\bar{x})w$ . Hence, we have  $h_1(\bar{x}) = D_1$ ,  $\forall \bar{x} \in \mathcal{D}$ . For  $i = 2, \dots, r-1$ , by Claim 3.1.2, we have

$$\begin{aligned} \dot{\bar{x}}_i &= f_i(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_i) + \bar{x}_{i+1} + h_i(\bar{x})w \\ &= a_{i1}x_1 + x_{i+1} + D_iw + \frac{\partial \bar{T}_{Ii}}{\partial x_z}(x_z, x_1, \dots, x_{i-1})(A_z x_z + A_{z1}x_1 + D_z w) \\ &\quad + \sum_{j=1}^{i-1} \frac{\partial \bar{T}_{Ii}}{\partial x_j}(x_z, x_1, \dots, x_{i-1})(a_{j1}x_1 + x_{j+1} + D_j w) \end{aligned}$$

Hence,  $\forall \bar{x} \in \mathcal{D}$ ,

$$\begin{aligned} h_i(\bar{x}) &= \left( D_i + \frac{\partial \bar{T}_{Ii}}{\partial x_z}(x_z, x_1, \dots, x_{i-1})D_z + \sum_{j=1}^{i-1} \frac{\partial \bar{T}_{Ii}}{\partial x_j}(x_z, x_1, \dots, x_{i-1})D_j \right) \Big|_{x=T(\bar{x})} \\ &=: h_i(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{i-1}); \quad i = 2, \dots, r-1 \end{aligned}$$

Furthermore,

$$\begin{aligned} \dot{\bar{x}}_r &= f_r(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_r) + g_o(\bar{x})u + h_r(\bar{x})w = A_{rz}x_z + a_{r1}x_1 + b_0u + D_r w \\ &\quad + \frac{\partial \bar{T}_{Ir}}{\partial x_z}(x_z, x_1, \dots, x_{r-1})(A_z x_z + A_{z1}x_1 + D_z w) + \sum_{j=1}^{r-1} \frac{\partial \bar{T}_{Ir}}{\partial x_j}(x_z, x_1, \dots, x_{r-1}) \\ &\quad \cdot (a_{j1}x_1 + x_{j+1} + D_j w) \end{aligned}$$

Hence,  $\forall \bar{x} \in \mathcal{D}$ ,

$$\begin{aligned} h_r(\bar{x}) &= \left( D_r + \frac{\partial \bar{T}_{Ir}}{\partial x_z}(x_z, x_1, \dots, x_{r-1})D_z + \sum_{j=1}^{r-1} \frac{\partial \bar{T}_{Ir}}{\partial x_j}(x_z, x_1, \dots, x_{r-1})D_j \right) \Big|_{x=T(\bar{x})} \\ &=: h_r(\bar{x}_z, \bar{x}_1, \dots, \bar{x}_{r-1}) \end{aligned}$$

This completes the proof of the lemma.  $\square$

REMARK 3.1. We observe in the previous lemma that the zero dynamics of the system (3.1) according to [4] is exactly  $\dot{x}_z = A_z x_z$ . The extended zero dynamics is simply the zero dynamics together with driving terms which include the noiseless output of the system and the disturbance input.

LEMMA 3.2. Consider a finite-dimensional continuous-time SISO LTI system

$$(3.6a) \quad \dot{x} = Ax + Bu + Dw$$

$$(3.6b) \quad y = Cx + Ew$$

where  $x$  is the  $n$ -dimensional state,  $n \in \mathbb{N}$ ;  $u$  is the scalar control input;  $y$  is the scalar output;  $w$  is the  $q$ -dimensional disturbance input,  $q \in \mathbb{Z}_+$ ;  $A$ ,  $B$ ,  $D$ ,  $C$ , and  $E$

are constant matrices of appropriate dimensions. Let the system have relative degree  $r = n$ , from  $u$  to  $y$ , that is,  $CB = \dots = CA^{n-2}B = 0$  and  $CA^{n-1}B \neq 0$ . Then, there exists an invertible matrix  $T_o$  such that, in  $[x_1 \ \dots \ x_r]'$  =  $T_o^{-1}x$  coordinates, the system (3.6) admits the state space representation

$$(3.7a) \quad \dot{x}_i = a_{i1}x_1 + x_{i+1} + D_i w; \quad i = 1, \dots, r-1$$

$$(3.7b) \quad \dot{x}_r = a_{r1}x_1 + b_0 u + D_r w$$

$$(3.7c) \quad y = x_1 + Ew$$

where  $x_i$ ,  $i = 1, \dots, r$ , are scalars;  $b_0 = CA^{r-1}B \neq 0$  is the high-frequency gain of the system. The representation (3.7) is called the extended zero dynamics canonical form of system (3.6) (which is also the observer canonical form). The extended zero dynamics for the system is clearly absent.

*Proof.* Define  $V = [B \ \dots \ A^{n-1}B]_{n \times n}$  and  $U = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n \times n}$ . Then, we

have

$$UV = \begin{bmatrix} 0 & \dots & 0 & CA^{r-1}B \\ \vdots & \ddots & \ddots & \star \\ 0 & \ddots & \ddots & \vdots \\ CA^{r-1}B & \star & \dots & \star \end{bmatrix}_{r \times r}$$

which is clearly invertible. Hence,  $U$  and  $V$  are invertible. Then, the system (3.6) is observable, and is controllable from  $u$ . Then, there exists a real invertible transformation  $[x_1 \ \dots \ x_n]'$  =  $T_o^{-1}x$  that transforms the system into observer canonical form.

$$\dot{x}_i = a_{i1}x_1 + x_{i+1} + b_i u + D_i w; \quad i = 1, \dots, r-1$$

$$\dot{x}_r = a_{r1}x_1 + b_r u + D_r w$$

$$y = x_1 + Ew$$

Because the relative degree from  $u$  to  $y$  is  $r = n$ , then, we have  $b_1 = \dots = b_{n-1} = 0$ . It is straightforward to obtain that  $b_n = CA^{n-1}B = b_0 \neq 0$ . This completes the proof of the lemma.  $\square$

LEMMA 3.3. Consider a finite-dimensional continuous-time SISO LTI system

$$(3.8a) \quad \dot{x} = Ax + Bu + Dw$$

$$(3.8b) \quad y = Cx + b_0 u + Ew$$

where  $x$  is the  $n$ -dimensional state,  $n \in \mathbb{N}$ ;  $u$  is the scalar control input;  $y$  is the scalar output;  $w$  is the  $q$ -dimensional disturbance input,  $q \in \mathbb{Z}_+$ ; and  $A$ ,  $B$ ,  $D$ ,  $C$ ,  $b_0$ , and  $E$  are constant matrices of appropriate dimensions. Let  $b_0 \neq 0$ , i. e., that the system admits relative degree 0 from  $u$  to  $y$ , and  $b_0$  is the high-frequency gain of the system. Then, the system (3.8) admits the following representation:

$$(3.9a) \quad \dot{x} = (A - \frac{1}{b_0}BC)x + \frac{1}{b_0}B(y - Ew) + Dw =: \hat{A}x + \hat{B}(y - Ew) + Dw$$

$$(3.9b) \quad y = Cx + b_0 u + Ew$$

The representation (3.9) is called the zero dynamics canonical form of (3.8).

Furthermore, for any  $\mathcal{C}_1$  diffeomorphism  $\bar{x} = T^{-1}(x)$ ,  $T : \mathcal{D} \rightarrow \mathbb{R}^n$ , where  $\mathcal{D} \subseteq \mathbb{R}^n$  is open, that leads to a representation

$$(3.10a) \quad \dot{\bar{x}} = \hat{f}(\bar{x}) + \hat{g}(\bar{x})(y - Ew) + \hat{h}(\bar{x})w$$

$$(3.10b) \quad y = \hat{c}(\bar{x}) + b_0u + Ew$$

we must have  $\hat{f}(\bar{x}) = \left( \frac{\partial T_I}{\partial x}(x) \hat{A}x \right) \Big|_{x=T(\bar{x})}$ ,  $\hat{g}(\bar{x}) = \left( \frac{\partial T_I}{\partial x}(x) \hat{B} \right) \Big|_{x=T(\bar{x})}$ ,  $\hat{h}(\bar{x}) = \left( \frac{\partial T_I}{\partial x}(x) D \right) \Big|_{x=T(\bar{x})}$ ,  $\hat{c}(\bar{x}) = (Cx) \Big|_{x=T(\bar{x})}$ ,  $\forall \bar{x} \in \mathcal{D}$ , where  $T_I : \mathbb{R}^n \rightarrow \mathcal{D}$  is the inverse function of  $T$ .

Hence, we observe that the dynamics (3.9a) is invariant (modulo  $\mathcal{C}_1$  diffeomorphisms) under  $\mathcal{C}_1$  diffeomorphism that bring the system (3.8) into the form of (3.10). Then, the dynamics (3.9a) is called the extended zero dynamics of (3.8).

*Proof.* Note that  $u = \frac{1}{b_0}(y - Cx - Ew)$ . Substitution of this equality into (3.8a) immediately leads to (3.9a). Hence, (3.9) is a representation of (3.8).

For any  $\mathcal{C}_1$  diffeomorphism  $\bar{x} = T^{-1}(x)$ ,  $T : \mathcal{D} \rightarrow \mathbb{R}^n$ , where  $\mathcal{D} \subseteq \mathbb{R}^n$  is open, the system (3.8) admits the state space representation

$$\begin{aligned} \dot{\bar{x}} &= f(\bar{x}) + g(\bar{x})u + h(\bar{x})w \\ y &= c(\bar{x}) + b_0u + Ew \end{aligned}$$

where we have  $f(\bar{x}) = \left( \frac{\partial T_I}{\partial x}(x) Ax \right) \Big|_{x=T(\bar{x})}$ ,  $g(\bar{x}) = \left( \frac{\partial T_I}{\partial x}(x) B \right) \Big|_{x=T(\bar{x})}$ ,  $h(\bar{x}) = \left( \frac{\partial T_I}{\partial x}(x) D \right) \Big|_{x=T(\bar{x})}$ ,  $c(\bar{x}) = (Cx) \Big|_{x=T(\bar{x})}$ ,  $\forall \bar{x} \in \mathcal{D}$ , and  $T_I : \mathbb{R}^n \rightarrow \mathcal{D}$  is the inverse function of  $T$ . Then, we have that  $u = \frac{1}{b_0}(y - c(\bar{x}) - Ew)$ . Substitution of this relationship into the above, we have

$$\begin{aligned} \dot{\bar{x}} &= f(\bar{x}) + \frac{1}{b_0}g(\bar{x})(y - Ew) - \frac{1}{b_0}g(\bar{x})c(\bar{x}) + h(\bar{x})w = \hat{f}(\bar{x}) + \hat{g}(\bar{x})(y - Ew) + \hat{h}(\bar{x})w \\ y &= \hat{c}(\bar{x}) + b_0u + Ew \end{aligned}$$

This completes the proof of the lemma.  $\square$

REMARK 3.2. We observe that, in the previous lemma, the zero dynamics of the system (3.8) is exactly  $\dot{x} = \hat{A}x$ . The extended zero dynamics is simply the zero dynamics together with the driving terms which include the noiseless output of the system and the disturbance input.

DEFINITION 3.4. Consider a finite-dimensional continuous-time SISO LTI system

$$(3.11) \quad y = b_0u + Ew$$

where  $u$  is the scalar control input;  $y$  is the scalar output;  $w$  is the  $q$ -dimensional disturbance input,  $q \in \mathbb{Z}_+$ ; and  $b_0$  and  $E$  are constant matrices of appropriate dimensions. Let  $b_0 \neq 0$ , which implies that the system admits relative degree 0 from  $u$  to  $y$ , and  $b_0$  is the high-frequency gain of the system. Then, (3.11) is called the extended zero dynamics canonical form. Clearly, the extended zero dynamics is absent in this case.

This completes our presentation of the definition of extended zero dynamics and the concept of extended zero dynamics canonical form for finite-dimensional

continuous-time SISO LTI systems with a finite relative degree. Next, we present a definition of the admissible class of disturbance waveforms.

**DEFINITION 3.5.**  $\mathcal{W}_d \subseteq \mathcal{C}([0, \infty), \mathbb{R}^q)$ ,  $q \in \mathbb{Z}_+$ , is said to be of class  $\mathcal{B}_q$  if it is nonempty and  $\forall w_{[0, \infty)} \in \mathcal{W}_d$ ,  $\forall t_f \in [0, \infty)$ ,  $\exists \bar{w}_{[0, \infty)} \in \mathcal{W}_d$ , such that  $w_{[0, t_f]} = \bar{w}_{[0, t_f]}$  and  $\|\bar{w}_{[0, \infty)}\|_\infty < \infty$ .

**EXAMPLE 3.1.**  $\mathcal{C}([0, \infty), \mathbb{R}^q)$  and  $\mathcal{C}_k([0, \infty), \mathbb{R}^q)$  are of class  $\mathcal{B}_q$ ,  $\forall k \in \mathbb{N} \cup \{\infty\}$  and  $\forall q \in \mathbb{Z}_+$ . The singleton set  $\{w_{[0, \infty)} \in \mathcal{C}([0, \infty), \mathbb{R}^q) \mid w(t) = \mathbf{0}_{q \times 1}, \forall t \in [0, \infty)\}$  is of class  $\mathcal{B}_q$ .

This definition allows estimation of the smooth part of an arbitrary varying signal using an observer.

**LEMMA 3.6.** Let  $\mathcal{W}_{d1}$  and  $\mathcal{W}_{d2}$  be of class  $\mathcal{B}_{q_1}$  and class  $\mathcal{B}_{q_2}$ , respectively,  $q_1, q_2 \in \mathbb{Z}_+$ . Then,  $\mathcal{W}_{d1} \times \mathcal{W}_{d2}$  is of class  $\mathcal{B}_{q_1+q_2}$ . Furthermore, if  $q_1 = q_2 = q$ , then  $\mathcal{W}_{d1} \cup \mathcal{W}_{d2}$  is of class  $\mathcal{B}_q$ .

*Proof.* Clearly, we have  $\mathcal{W}_{d1} \times \mathcal{W}_{d2} \subseteq \mathcal{C}([0, \infty), \mathbb{R}^{q_1+q_2})$  and is nonempty. For any  $(w_{1[0, \infty)}, w_{2[0, \infty)}) \in \mathcal{W}_{d1} \times \mathcal{W}_{d2}$  and  $\forall t_f \in [0, \infty)$ . Since  $\mathcal{W}_{d1}$  is of class  $\mathcal{B}_{q_1}$ , then,  $\exists \bar{w}_{1[0, \infty)} \in \mathcal{W}_{d1}$ , such that  $w_{1[0, t_f]} = \bar{w}_{1[0, t_f]}$  and  $\|\bar{w}_{1[0, \infty)}\|_\infty < +\infty$ . Since  $\mathcal{W}_{d2}$  is of class  $\mathcal{B}_{q_2}$ , then,  $\exists \bar{w}_{2[0, \infty)} \in \mathcal{W}_{d2}$ , such that  $w_{2[0, t_f]} = \bar{w}_{2[0, t_f]}$  and  $\|\bar{w}_{2[0, \infty)}\|_\infty < +\infty$ . Then, we have  $(\bar{w}_{1[0, \infty)}, \bar{w}_{2[0, \infty)}) \in \mathcal{W}_{d1} \times \mathcal{W}_{d2}$ ,  $(w_{1[0, t_f]}, w_{2[0, t_f]}) = (\bar{w}_{1[0, t_f]}, \bar{w}_{2[0, t_f]})$ , and  $\|(\bar{w}_{1[0, \infty)}, \bar{w}_{2[0, \infty)})\|_\infty < +\infty$ . Hence,  $\mathcal{W}_{d1} \times \mathcal{W}_{d2}$  is of class  $\mathcal{B}_{q_1+q_2}$ .

Let  $q_1 = q_2 = q$ . Clearly,  $\mathcal{W}_{d1} \cup \mathcal{W}_{d2} \subseteq \mathcal{C}([0, \infty), \mathbb{R}^q)$  and is nonempty.  $\forall w_{[0, \infty)} \in \mathcal{W}_{d1} \cup \mathcal{W}_{d2}$ ,  $\exists i \in \{1, 2\}$ , such that  $w_{[0, \infty)} \in \mathcal{W}_{di}$ .  $\forall t_f \in [0, \infty)$ , Since  $\mathcal{W}_{di}$  is of class  $\mathcal{B}_q$ , then,  $\exists \bar{w}_{[0, \infty)} \in \mathcal{W}_{di} \subseteq \mathcal{W}_{d1} \cup \mathcal{W}_{d2}$ , such that  $w_{[0, t_f]} = \bar{w}_{[0, t_f]}$  and  $\|\bar{w}_{[0, \infty)}\|_\infty < \infty$ . Hence, we have  $\mathcal{W}_{d1} \cup \mathcal{W}_{d2}$  is of class  $\mathcal{B}_q$ .  $\square$

Now, we are in a position to introduce a generalized definition of minimum phase that also accommodates the presence of disturbances.

**DEFINITION 3.7.** Consider a finite-dimensional continuous-time SISO LTI system

$$(3.12a) \quad \dot{x} = Ax + Bu + Dw; \quad x(0) = x_0$$

$$(3.12b) \quad y = Cx + Ku + Ew$$

where  $x$  is the  $n$ -dimensional state vector,  $n \in \mathbb{Z}_+$ ;  $x_0 \in \mathcal{D}_0 \neq \emptyset$ ,  $\mathcal{D}_0 \subseteq \mathbb{R}^n$  is the set of admissible initial conditions;  $u$  is the scalar control input;  $y$  is the scalar output;  $w$  is the  $q$ -dimensional disturbance input,  $q \in \mathbb{Z}_+$ ;  $w_{[0, \infty)} \in \mathcal{W}_d$  of class  $\mathcal{B}_q$ ,  $\mathcal{W}_d$  is the set of admissible disturbance waveforms;  $A$ ,  $B$ ,  $D$ ,  $C$ ,  $K$ , and  $E$  are constant matrices of appropriate dimensions. Let the relative degree from  $u$  to  $y$  be  $r \in \mathbb{Z}_+$ .

1. If  $r = 0$  and  $n = 0$ , then, the system (3.12) admits the extended zero dynamics canonical form of (3.11), where the extended zero dynamics is absent. We will henceforth say that the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

2. If  $r \in \mathbb{N}$  and  $n = r$ , then, the system admits the extended zero dynamics canonical form (3.7), where the extended zero dynamics is absent. We will henceforth say that the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

3. If  $r = 0$  and  $n \in \mathbb{N}$ , then, the system (3.12) admits the extended zero dynamics canonical form (3.9). We will say that the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if its extended zero dynamics (3.9a) satisfies that  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ,  $\forall x_0 \in \mathcal{D}_0$  with  $|x_0| \leq c_w$ ,  $\forall w_{[0, \infty)} \in \mathcal{W}_d$  with  $\|w_{[0, \infty)}\|_\infty \leq c_w$ ,  $\forall y_{[0, \infty)} \in \mathcal{C}$  with  $\|y_{[0, \infty)}\|_\infty \leq c_w$ , we have  $\|x_{[0, \infty)}\|_\infty \leq c_c$ .

4. If  $r \in \mathbb{N}$  and  $r < n$ , then, the system (3.12) admits the extended zero dynamics canonical form (3.2). We will say that the system (3.12) is minimum phase

with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if its extended zero dynamics (3.2a) with the coordinate transformation  $x_z = T_z x$  satisfies that  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ,  $\forall x_{z0} \in T_z(\mathcal{D}_0)$  with  $|x_{z0}| \leq c_w$ ,  $\forall w_{[0, \infty)} \in \mathcal{W}_d$  with  $\|w_{[0, \infty)}\|_\infty \leq c_w$ ,  $\forall x_{1[0, \infty)} \in \mathcal{C}_r$  with  $\|x_{1[0, \infty)}^{[r]}\|_\infty \leq c_w$ , we have  $\|x_{z[0, \infty)}\|_\infty \leq c_c$ .

Note that, in the last item of the above definition,  $x_1$  is set to be any desired trajectory that is  $\mathcal{C}_r$  and  $x_1$  together with its derivatives up to order  $r$  are bounded. This generalized notion of minimum phase has important significance in the context of model reference control (see § 4). In the following, we prove a necessary and sufficient condition for the last item of the above definition where  $x_1$  is set to be any bounded signal in  $\mathcal{C}$ . First, we have the following preliminary result, which is useful in its own right.

LEMMA 3.8. *Consider the system (3.12) of Definition 3.7. Let  $r < n$ . Then, the following statements hold.*

1. *When  $r > 0$ . (3.12) admits the extended zero dynamics canonical form (3.2). Then, (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  implies that the system*

$$(3.13) \quad \dot{z} = A_z z + A_{z1} v; \quad z(0) = \mathbf{0}_{(n-r) \times 1}$$

*is bounded input and bounded state stable.*

2. *When  $r = 0$ . (3.12) admits the extended zero dynamics canonical form (3.9). Then, (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  implies that the system*

$$(3.14) \quad \dot{z} = \hat{A} z + \hat{B} v; \quad z(0) = \mathbf{0}_{n \times 1}$$

*is bounded input and bounded state stable.*

*Proof.* For the first statement, we will prove the lemma using an argument of contradiction. Suppose (3.13) is not bounded input and bounded state stable. By Corollary A.10,  $\exists v_{[0, \infty)} \in \mathcal{C}_r$ , such that  $\|v_{[0, \infty)}^{[r]}\|_\infty =: \bar{c}_w \in (0, \infty) \subset \mathbb{R}$  and  $\|z_{[0, \infty)}\|_\infty = +\infty$ .

Fix  $x_{z0} \in T_z(\mathcal{D}_0)$ , where  $x_z = T_z x$  is the coordinate transformation for the extended zero dynamics. Then,  $\exists c_w \in [\bar{c}_w, \infty) \subset \mathbb{R}$ , such that  $|x_{z0}| \leq c_w$  and  $\exists w_{[0, \infty)} \in \mathcal{W}_d$  with  $\|w_{[0, \infty)}\|_\infty \leq c_w$ . Set  $x_{1[0, \infty)} = \rho v_{[0, \infty)} \in \mathcal{C}_r$  with  $\rho \in (0, 1)$ . Then, clearly,  $\|x_{1[0, \infty)}^{[r]}\|_\infty = \rho \bar{c}_w \leq c_w$ . Let  $x_{z[0, \infty)}$  be the solution to (3.2a) with initial condition  $x_z(0) = x_{z0}$  and inputs  $x_{1[0, \infty)}$  and  $w_{[0, \infty)}$ . Then,  $x_{z[0, \infty)}$  may be generated by

$$\begin{aligned} \dot{z} &= A_z z + A_{z1} v; & z(0) &= \mathbf{0}_{(n-r) \times 1} \\ \dot{\zeta} &= A_z \zeta + D_z w; & \zeta(0) &= x_{z0} \\ x_{z[0, \infty)} &= \zeta_{[0, \infty)} + \rho z_{[0, \infty)} \end{aligned}$$

Then,  $\|x_{z[0, \infty)}\|_\infty = +\infty$  for some  $\rho \in (0, 1)$ . This contradicts with the assumption that (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

Hence, (3.13) is bounded input and bounded state stable.

The second statement is a direct consequence of Lemma A.11. This completes the proof of the lemma.  $\square$

PROPOSITION 3.9. *Consider the system (3.12) of Definition 3.7. Let  $r \in \mathbb{N}$  and  $r < n$ . It admits the extended zero dynamics canonical form (3.2). Then, (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if, and only if, its extended zero dynamics*

(3.2a) with coordinate transformation  $x_z = T_z x$  satisfies:  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ,  $\forall x_{z0} \in T_z(\mathcal{D}_0)$  with  $|x_{z0}| \leq c_w$ ,  $\forall w_{[0, \infty)} \in \mathcal{W}_d$  with  $\|w_{[0, \infty)}\|_\infty \leq c_w$ ,  $\forall x_{1[0, \infty)} \in \mathcal{C}$  with  $\|x_{1[0, \infty)}\|_\infty \leq c_w$ , we have  $\|x_{z[0, \infty)}\|_\infty \leq c_c$ .

*Proof.* “If”. This is straightforward.

“Only if”.  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\forall x_{z0} \in T_z(\mathcal{D}_0)$  with  $|x_{z0}| \leq c_w$ ,  $\forall w_{[0, \infty)} \in \mathcal{W}_d$  with  $\|w_{[0, \infty)}\|_\infty \leq c_w$ ,  $\forall x_{1[0, \infty)} \in \mathcal{C}$  with  $\|x_{1[0, \infty)}\|_\infty \leq c_w$ . The trajectory  $x_{z[0, \infty)}$  may be generated by

$$\begin{aligned} \dot{z} &= A_z z + A_{z1} x_1; & z(0) &= \mathbf{0}_{(n-r) \times 1} \\ \dot{\zeta} &= A_z \zeta + D_z w; & \zeta(0) &= x_{z0} \\ x_{z[0, \infty)} &= \zeta_{[0, \infty)} + z_{[0, \infty)} \end{aligned}$$

By Lemmas 3.8 and A.3,  $\exists c_{c2} \in [0, \infty) \subset \mathbb{R}$ , which depends only on the matrices  $A_z$  and  $A_{z1}$ , such that  $\|z_{[0, \infty)}\|_\infty \leq c_{c2} c_w$ . Since (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ , and the identically zero function  $\mathbf{0}_{[0, \infty)} \in \mathcal{C}_r$  with  $\|\mathbf{0}_{[0, \infty)}^{[r]}\|_\infty = 0$ , then,  $\exists c_{c1} \in [0, \infty) \subset \mathbb{R}$ , which depends only on  $c_w$ , such that  $\|\zeta_{[0, \infty)}\|_\infty \leq c_{c1}$ . Hence,  $\|x_{z[0, \infty)}\|_\infty \leq c_{c1} + c_{c2} c_w$ .

This completes the proof of the proposition.  $\square$

Next, we present a result that links the generalized minimum phase property to the asymptotic stability property of the extended zero dynamics.

**PROPOSITION 3.10.** *Consider the system (3.12) of Definition 3.7. Let  $r \in \mathbb{Z}_+$ . Then, the following statements hold.*

1. *Let  $r = n = 0$ . The system is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  since the extended zero dynamics is absent.*
2. *Let  $1 \leq r = n$ . The system is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  since the extended zero dynamics is absent.*
3. *Let  $0 = r < n$ . The system is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if its extended zero dynamics (3.9a) is such that the matrix  $\hat{A}$  is Hurwitz. On the other hand, if the system is stabilizable from  $u$  and is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ , then the matrix  $\hat{A}$  is Hurwitz.*
4. *Let  $1 \leq r < n$ . The system is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if its extended zero dynamics (3.2a) is such that the matrix  $A_z$  is Hurwitz. On the other hand, if the system is stabilizable from  $u$  and is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ , then the matrix  $A_z$  is Hurwitz.*

*Proof.* The statements 1 and 2 are immediate from the definition of minimum phase property.

For statement 3, let  $0 = r < n$ . By Lemma 3.3, the extended zero dynamics of the system is (3.9a). If the matrix  $\hat{A}$  is Hurwitz, by Lemmas A.1 and A.2, the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

On the other hand, if the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  and is stabilizable from  $u$ , we will show that the matrix  $\hat{A}$  is Hurwitz. By the stabilizability of the pair  $(A, B)$ , we have the pair  $(\hat{A}, \hat{B})$  is stabilizable. Since (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ , by Lemma 3.8, the following system

$$\dot{x}_u = \hat{A} x_u + \hat{B} v; \quad x_u(0) = \mathbf{0}_{n \times 1}$$

is bounded input and bounded state stable. Since the triple  $(\hat{A}, \hat{B}, I_n)$  is stabilizable and detectable, then, by Corollary A.8,  $\hat{A}$  is Hurwitz.



For statement 4, without loss of generality, we assume that the system (3.12) is given in the extended zero dynamics canonical form (3.2), by Lemma 3.1. The extended zero dynamics is (3.2a) and  $x_z = T_z x$ . If the matrix  $A_z$  is Hurwitz, by Lemmas A.1 and A.2, system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

On the other hand, if the system (3.12) is stabilizable from  $u$  and is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . Then, the matrix

$$(3.15) \quad [ \lambda I - A \mid B ] = \left[ \begin{array}{cccc|c} \lambda I_{n-r} - A_z & -A_{z1} & & & \\ & \lambda - a_{11} & -1 & & \\ & -a_{21} & \lambda & \ddots & \\ & \vdots & & \ddots & -1 \\ -A_{rz} & -a_{r1} & & & \lambda \end{array} \middle| b_0 \right]$$

has rank  $n$ ,  $\forall \lambda \in \mathbb{C}$  with  $\text{Re}(\lambda) \geq 0$ , where all empty spaces in the above matrix are zeros. Observe that the sub-block matrix of (3.15), consisting the last  $r$  rows and the last  $r$  columns, is invertible. Then, the matrix  $[ \lambda I_{n-r} - A_z \quad -A_{z1} ]$  has rank  $n-r$ ,  $\forall \lambda \in \mathbb{C}$  with  $\text{Re}(\lambda) \geq 0$ . Hence, the pair  $(A_z, A_{z1})$  is stabilizable. Since (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ , by Lemma 3.8, we have the following system

$$(3.16) \quad \dot{x}_{zu} = A_z x_{zu} + A_{z1} v; \quad x_{zu}(0) = \mathbf{0}_{(n-r) \times 1}$$

is bounded input and bounded state stable. Since the triple  $(A_z, A_{z1}, I_{n-r})$  is stabilizable and detectable, then, by Corollary A.8, the matrix  $A_z$  is Hurwitz.

This completes the proof of the proposition.  $\square$

Next, we present a result that states that a controllable and observable system is minimum phase if, and only if, its transfer function has all zeros with negative real parts. This demonstrates that Definition 3.7 is truly a generalization of the minimum phase concept.

**PROPOSITION 3.11.** *Consider the system (3.12) of Definition 3.7. Assume that the system admits relative degree  $r \in \mathbb{Z}_+$  from  $u$  to  $y$ ; and it is controllable from  $u$  and is observable. Then, the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if, and only if, the transfer function  $H(s) = C(sI_n - A)^{-1}B + K$  has all zeros with negative real parts.*

*Proof.* We will consider separately four exhaustive and mutually exclusive cases:

Case 1:  $r = n = 0$ ; Case 2:  $1 \leq r = n$ ; Case 3:  $0 = r < n$ ; Case 4:  $1 \leq r < n$ .

Case 1:  $r = n = 0$ . Then, the system (3.12) is given by  $y = Ku + Ew$ , where  $K \neq 0$ . Clearly, the extended zero dynamics of (3.12) is absent. The transfer function from  $u$  to  $y$  is  $H(s) = K$ . Clearly,  $H(s)$  does not have any zeros. Thus, (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  (according to Definition 3.7) if, and only if,  $H(s)$  has all zeros with negative real parts. This case is proved.

Case 2:  $1 \leq r = n$ . By Lemma 3.2, the system (3.12) admits the state space representation (3.7). Then, the extended zero dynamics for (3.12) is absent. Hence, the system is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . The transfer function  $H(s)$  from  $u$  to  $y$  is  $\frac{b_0}{s^r - a_{11}s^{r-1} - \dots - a_{r1}}$ , which does not admit any zeros. Hence,  $H(s)$  has all zeros with negative real parts. Thus, the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if, and only if,  $H(s)$  has all zeros with negative real parts. This case is proved.



Case 3:  $0 = r < n$ . Let the transfer function from  $u$  to  $y$  be

$$H(s) = \frac{b_0 s^n + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = b_0 + \frac{(b_1 - b_0 a_1) s^{n-1} + \cdots + (b_n - b_0 a_n)}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

Without loss of generality, assume (3.12) is given in observer canonical form. Then,  $K = b_0 \neq 0$ ,

$$A = \left[ \begin{array}{c|c} -a_1 & I_{n-1} \\ \vdots & \\ -a_{n-1} & \\ \hline -a_n & \mathbf{0}_{1 \times (n-1)} \end{array} \right]; \quad B = \begin{bmatrix} b_1 - b_0 a_1 \\ \vdots \\ b_n - b_0 a_n \end{bmatrix}; \quad C = [1 \quad \mathbf{0}_{1 \times (n-1)}]$$

By Lemma 3.3, the extended zero dynamics of (3.12) is given by

$$\dot{x} = \left( A - \frac{1}{b_0} BC \right) x + \frac{1}{b_0} B(y - Ew) + Dw =: \hat{A}x + \hat{B}y + \hat{D}w$$

where

$$\hat{A} = \left[ \begin{array}{c|c} -b_1/b_0 & I_{n-1} \\ \vdots & \\ -b_{n-1}/b_0 & \\ \hline -b_n/b_0 & \mathbf{0}_{1 \times (n-1)} \end{array} \right]$$

The characteristic function of  $\hat{A}$  is  $s^n + \frac{b_1}{b_0} s^{n-1} + \cdots + \frac{b_n}{b_0}$ . Hence,  $\hat{A}$  is Hurwitz if, and only if, all zeros of  $H(s)$  have negative real parts.

When the matrix  $\hat{A}$  is Hurwitz, by Proposition 3.10, the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

On the other hand, when the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ , again by Proposition 3.10, the matrix  $\hat{A}$  is Hurwitz, since the system is controllable from  $u$ .

Thus, we have shown that the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if, and only if, the matrix  $\hat{A}$  is Hurwitz, which holds if, and only if, the transfer function  $H(s)$  have all zeros with negative real parts. Hence, this case is proved.

Case 4:  $1 \leq r < n$ . Let the transfer function from  $u$  to  $y$  be given by  $H(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$  with  $m = n - r \in \mathbb{N}$  and  $b_0 \neq 0$ . By Lemma A.5, there exists a real invertible coordinate transformation  $[x'_f \quad x'_z]' = T^{-1}x = [T'_f \quad T'_z]'x$  such that, in  $[x'_f \quad x'_z]'$  coordinates, the system (3.12) admits the state space representation:

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_z \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_f \\ x_z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x_f \\ x_z \end{bmatrix} + Ew$$

where  $x_z$  is  $m$ -dimensional;  $x_f = [x_1 \quad \cdots \quad x_r]'$  with  $x_i, i = 1, \dots, r$ , being scalars; the partitioning of the system matrices are compatible with the partitioning of the

state vector; and

$$A_{11} = \left[ \begin{array}{c|c} \bar{a}_1 & I_{r-1} \\ \vdots & \\ \hline \bar{a}_{r-1} & \\ \bar{a}_r & \mathbf{0}_{1 \times (r-1)} \end{array} \right]; \quad A_{12} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}; \quad A_{21} = \left[ \begin{array}{c|c} \bar{a}_{r+1} & \\ \vdots & \\ \hline \bar{a}_n & \mathbf{0}_{m \times (r-1)} \end{array} \right]$$

$$A_{22} = \left[ \begin{array}{c|c} -b_1/b_0 & I_{m-1} \\ \vdots & \\ \hline -b_{m-1}/b_0 & \\ -b_m/b_0 & \mathbf{0}_{1 \times (m-1)} \end{array} \right]; \quad B_1 = \begin{bmatrix} \mathbf{0}_{(r-1) \times 1} \\ b_0 \end{bmatrix}; \quad B_2 = \mathbf{0}_{m \times 1}$$

$$C_1 = \begin{bmatrix} 1 & \mathbf{0}_{1 \times (r-1)} \end{bmatrix}; \quad C_2 = \mathbf{0}_{1 \times m}$$

In  $[x'_z \quad x'_f]'$  coordinates, the system (3.12) is represented by

$$(3.17a) \quad \dot{x}_z = A_{22}x_z + \begin{bmatrix} \bar{a}_{r+1} \\ \vdots \\ \bar{a}_n \end{bmatrix} x_1 + D_2w =: A_{22}x_z + A_{21,1}x_1 + D_2w$$

$$(3.17b) \quad \dot{x}_f = A_{12}x_z + A_{11}x_f + B_1u + D_1w$$

$$(3.17c) \quad y = x_1 + Ew$$

which is clearly in the form of (3.2). Hence, (3.17a) is the extended zero dynamics of (3.12). The characteristic function of  $A_{22}$  is  $s^m + \frac{b_1}{b_0}s^{m-1} + \dots + \frac{b_m}{b_0}$ . The matrix  $A_{22}$  is Hurwitz if, and only if, the transfer function  $H(s)$  has all zeros with negative real parts.

When the matrix  $A_{22}$  is Hurwitz, by Proposition 3.10, the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

On the other hand, when the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ , by Proposition 3.10, the matrix  $A_{22}$  is Hurwitz, since the system is controllable from  $u$ .

Thus, we have shown that the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  if, and only if, the matrix  $A_{22}$  is Hurwitz, which holds if, and only if, the transfer function  $H(s)$  have all zeros with negative real parts. Hence, this case is proved.

This completes the proof of the proposition.  $\square$

**PROPOSITION 3.12.** *Consider the system (3.12) of Definition 3.7 with  $r \in \mathbb{Z}_+$ . Assume that it is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . Then, the transfer function from  $u$  to  $y$ ,  $H(s) = K + C(sI_n - A)^{-1}B$ , has all zeros with negative real parts.*

*Proof.* We will distinguish four exhaustive and mutually exclusive cases: Case 1:  $r = n = 0$ ; Case 2:  $0 < r = n$ ; Case 3:  $0 = r < n$ ; Case 4:  $0 < r < n$ .

Case 1:  $r = n = 0$ . Then, (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  and  $H(s) = K \neq 0$  (since  $r = 0$ ). The result holds.

Case 2:  $0 < r = n$ . Then, (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  and admits extended zero dynamics canonical form (3.7). This implies that  $H(s) = \frac{b_0}{s^n - a_{11}s^{n-1} - \dots - a_{n1}}$ , which does not have any zeros. The results holds in this case.

Case 3:  $0 = r < n$ . Then, (3.12) admits extended zero dynamics canonical form (3.9). Using Laplace transform, we may calculate the transfer function from  $u$  to  $y$  of

(3.9), which should equal to  $H(s)$ . Then, we have

$$\frac{b_0}{1 - C(sI_n - \hat{A})^{-1}\hat{B}} = b_0 + C(sI_n - A)^{-1}B = H(s)$$

This yields

$$C(sI_n - \hat{A})^{-1}\hat{B} = \frac{H(s) - b_0}{H(s)}$$

By Lemma 3.8, the system  $\dot{z} = \hat{A}z + \hat{B}v$ ,  $z(0) = \mathbf{0}_{n \times 1}$  is bounded input and bounded state. By Lemma A.3, there exists a  $k \in [0, \infty) \subset \mathbb{R}$  and  $\lambda \in (0, \infty) \subset \mathbb{R}$  such that  $\|e^{\hat{A}t}\hat{B}\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty) \subset \mathbb{R}$ . This implies that  $C(sI_n - \hat{A})^{-1}\hat{B}$  has all poles with negative real parts. By the above equation, the numerator of  $H(s)$  is equal to the denominator of  $C(sI_n - \hat{A})^{-1}\hat{B}$ . Hence,  $H(s)$  has all zeros with negative real parts.

Case 4:  $0 < r < n$ . Then, (3.12) admits extended zero dynamics canonical form (3.2). Using Laplace transform, we may calculate the transfer function from  $u$  to  $y$  of (3.2), which should equal to  $H(s)$ . Then, we have

$$\begin{aligned} sX_z(s) &= A_z X_z(s) + A_{z1}Y(s) \\ sX_i(s) &= a_{i1}Y(s) + X_{i+1}(s); \quad i = 1, \dots, r-1 \\ sX_r(s) &= A_{rz}X_z(s) + a_{r1}Y(s) + b_0U(s) \end{aligned}$$

which yields

$$H(s) = C(sI_n - A)^{-1}B = \frac{b_0}{s^r - a_{11}s^{r-1} - \dots - a_{r1} - A_{rz}(sI_{n-r} - A_z)^{-1}A_{z1}}$$

By Lemma 3.8, the system  $\dot{z} = A_z z + A_{z1}v$ ,  $z(0) = \mathbf{0}_{(n-r) \times 1}$  is bounded input and bounded state. By Lemma A.3, there exists a  $k \in [0, \infty) \subset \mathbb{R}$  and  $\lambda \in (0, \infty) \subset \mathbb{R}$  such that  $\|e^{A_z t}A_{z1}\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty) \subset \mathbb{R}$ . This implies that  $A_{rz}(sI_{n-r} - A_z)^{-1}A_{z1}$  has all poles with negative real parts. By the above equation, the numerator of  $H(s)$  is equal to the denominator of  $A_{rz}(sI_{n-r} - A_z)^{-1}A_{z1}$  or a factor of it. Hence,  $H(s)$  has all zeros with negative real parts.

This completes the proof of the proposition.  $\square$

**REMARK 3.3.** *We conclude, based on the previous three lemmas, that if a finite-dimensional continuous-time SISO LTI system with  $0 < r < n$  is minimum phase according to [4], then it is also minimum phase according to the generalized definition; on the other hand, if it is minimum phase according to the generalized definition, and it is stabilizable from  $u$ , then it is minimum phase according to [4].*

*Under controllability and observability assumptions, a finite-dimensional continuous-time SISO LTI system with relative degree  $r \in \mathbb{Z}_+$  is minimum phase according to the generalized definition if, and only if, it is minimum phase according to the classical definition, that is, the transfer function from  $u$  to  $y$  have all zeros with negative real parts. When the system is not necessarily controllable and observable, it is minimum phase according to the generalized definition implies that its transfer function has all zeros with negative real parts.*

We will present an example that illustrates the generalized definition of minimum phase.

EXAMPLE 3.2. Consider the following 4th order system

$$(3.18a) \quad \dot{x}_z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_z + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} w; \quad x_z(0) = \begin{bmatrix} x_{z1,0} \\ 0 \end{bmatrix}$$

$$(3.18b) \quad \dot{x}_1 = x_2; \quad x_1(0) = x_{1,0}$$

$$(3.18c) \quad \dot{x}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_z + u + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} w; \quad x_2(0) = x_{2,0}$$

$$(3.18d) \quad y = x_1$$

with  $r = 2$ ,  $x_0 = [x_{z1,0} \ 0 \ x_{1,0} \ x_{2,0}]'$ . Clearly, (3.18) is in the extended zero dynamics canonical form (3.2). Define  $\mathcal{D}_0 = \{x_0 \in \mathbb{R}^4 \mid x_{z1,0}, x_{1,0}, x_{2,0} \in \mathbb{R}\}$ . Let the elements of  $w$  be  $[w_1 \ w_2 \ w_3]'$ .  $w_{1[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}) =: \mathcal{W}_{d1}$ , which is of class  $\mathcal{B}_1$ .  $w_{2[0,\infty)} \in \mathcal{W}_{d2}$ , which is defined by

$$\mathcal{W}_{d2} = \{w_{2[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}) \mid \|w_{2[0,\infty)}\|_\infty < \infty, \phi(t) = \int_0^t w_2(\tau) d\tau, \\ \forall t \in [0,\infty), \|\phi_{[0,\infty)}\|_\infty \leq c_2\}$$

where  $c_2 \in [0,\infty) \subset \mathbb{R}$  is a fixed constant. Clearly,  $\mathcal{W}_{d2}$  is of class  $\mathcal{B}_1$ .  $w_{3[0,\infty)} \in \mathcal{W}_{d3}$ , which is defined by

$$\mathcal{W}_{d3} = \{w_{3[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}) \mid \|w_{3[0,\infty)}\|_\infty < \infty, \psi_2(t) = \int_0^t w_3(\tau) d\tau, \\ \forall t \in [0,\infty), \|\psi_{2[0,\infty)}\|_\infty \leq c_{3,2}, \psi_1(t) = \int_0^t \psi_2(\tau) d\tau, \forall t \in [0,\infty), \\ \|\psi_{1[0,\infty)}\|_\infty \leq c_{3,1}\}$$

where  $c_{3,2}, c_{3,1} \in [0,\infty) \subset \mathbb{R}$  are fixed constants. Clearly,  $\mathcal{W}_{d3}$  is of class  $\mathcal{B}_1$ . Then,  $w_{[0,\infty)} \in \mathcal{W}_{d1} \times \mathcal{W}_{d2} \times \mathcal{W}_{d3} =: \mathcal{W}_d$ , which is of class  $\mathcal{B}_3$  by Lemma 3.6.

$\forall c_w \in [0,\infty) \subset \mathbb{R}$ . Let  $c_c = \sqrt{(c_w + c_2 + c_{3,1})^2 + c_{3,2}^2} \in [0,\infty) \subset \mathbb{R}$ .  $\forall x_{z0} = [x_{z1,0} \ 0]'$   $\in T_z(\mathcal{D}_0)$  with  $|x_{z0}| \leq c_w$ , we have  $|x_{z1,0}| \leq c_w$ .  $\forall w_{[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{[0,\infty)}\|_\infty \leq c_w$ .  $\forall x_{1[0,\infty)} \in \mathcal{C}_2$  with  $\|x_{1[0,\infty)}^{[2]}\|_\infty \leq c_w$ . The state trajectory,  $x_{z[0,\infty)}$ , for the extended zero dynamics (3.18a) may be generated by

$$\begin{aligned} \dot{x}_{zo} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{zo}; \quad x_{zo}(0) = \begin{bmatrix} x_{z1,0} \\ 0 \end{bmatrix} \\ \dot{x}_{zu} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{zu} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_2; \quad x_{zu}(0) = \mathbf{0}_{2 \times 1} \\ \dot{x}_{zv} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{zv} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_3; \quad x_{zv}(0) = \mathbf{0}_{2 \times 1} \\ x_{z[0,\infty)} &= x_{zo[0,\infty)} + x_{zu[0,\infty)} + x_{zv[0,\infty)} \end{aligned}$$

Let  $x_z = [x_{z1} \ x_{z2}]'$ ,  $x_{zo} = [x_{zo1} \ x_{zo2}]'$ ,  $x_{zu} = [x_{zu1} \ x_{zu2}]'$ , and  $x_{zv} = [x_{zv1} \ x_{zv2}]'$ . Then,  $\forall t \in [0,\infty)$ , we have  $x_{zo1}(t) = x_{z1,0}$ ,  $x_{zo2}(t) = 0$ ,  $x_{zu1}(t) = \int_0^t w_2(\tau) d\tau$ ,  $x_{zu2}(t) = 0$ ,  $x_{zv2}(t) = \int_0^t w_3(\tau) d\tau$ , and  $x_{zv1}(t) = \int_0^t x_{zv2}(\tau) d\tau$ . Hence, we have  $\|x_{zo1[0,\infty)}\|_\infty \leq c_w$ ,  $\|x_{zo2[0,\infty)}\|_\infty = 0$ ,  $\|x_{zu1[0,\infty)}\|_\infty \leq c_2$ ,  $\|x_{zu2[0,\infty)}\|_\infty = 0$ ,  $\|x_{zv2[0,\infty)}\|_\infty \leq c_{3,2}$ , and  $\|x_{zv1[0,\infty)}\|_\infty \leq c_{3,1}$ . Then,  $\|x_{z1[0,\infty)}\|_\infty = \|x_{zo1[0,\infty)} +$

$x_{zu1[0,\infty)} + x_{zv1[0,\infty)} \Big\|_\infty \leq c_w + c_2 + c_{3,1}$  and  $\|x_{z2[0,\infty)}\|_\infty = \|x_{zo2[0,\infty)} + x_{zu2[0,\infty)} + x_{zv2[0,\infty)}\|_\infty \leq c_{3,2}$ . Therefore, we have  $\|x_z[0,\infty)\|_\infty \leq c_c$ . Hence, the system (3.18) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . Note that the extended zero dynamics (3.18a) is neither asymptotically stable nor bounded input and bounded state stable with respect to inputs  $x_1$  and  $w$  in the usual sense.

#### 4. Necessity of minimum phase property in model reference control.

We state and prove the following result on the necessity of the generalized minimum phase condition in model reference control of finite-dimensional continuous-time SISO LTI systems.

**THEOREM 4.1.** *Consider the system (3.12) of Definition 3.7. Let (3.12) admit the relative degree  $r \in \{0, \dots, n\}$  with respect to  $u$ . Let  $y_{d[0,\infty)} \in \mathcal{C}_r([0, \infty), \mathbb{R})$  be the reference trajectory and further denote  $Y_d := \begin{bmatrix} y_d & y_d^{(1)} & \dots & y_d^{(r)} \end{bmatrix}'$ , and  $Y_{d0} := \begin{bmatrix} y_d(0) & y_d^{(1)}(0) & \dots & y_d^{(r-1)}(0) \end{bmatrix}' \in \mathbb{R}^r$ . Assume that*

- (i) *the identically  $\mathbf{0}_{q \times 1}$  function belongs to  $\mathcal{W}_d$ ;*
- (ii) *the set  $\mathcal{D}_0$  satisfies, when  $1 \leq r < n$ , by Lemma 3.1, system  $S_P$  is in the extended zero dynamics canonical form (3.2), and  $\mathcal{D}_0 = \mathcal{D}_{z0} \times \mathbb{R}^r$ , where  $\mathcal{D}_{z0} \subseteq \mathbb{R}^{n-r}$  is nonempty.*

Assume that there exists a finite-dimensional model reference controller,  $S_C$

$$(4.1a) \quad \dot{\xi} = f(\xi, y, Y_d); \quad \xi(0) = \xi_0(\check{x}_0, Y_{d0})$$

$$(4.1b) \quad u = h(\xi, y, Y_d)$$

where  $\xi$  is  $\bar{n}$ -dimensional,  $\bar{n} \in \mathbb{Z}_+$ ; and  $f$  and  $h$  are locally Lipschitz on  $\mathcal{D}_\xi \times \mathbb{R} \times \mathbb{R}^{r+1}$ ;  $\mathcal{D}_\xi \subseteq \mathbb{R}^{\bar{n}}$  is nonempty;  $\check{x}_0 \in \mathcal{D}_0$  is an estimate of  $x_0$ ;  $\xi_0 : \mathcal{D}_0 \times \mathbb{R}^r \rightarrow \mathcal{D}_\xi$ ; such that  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ;  $\forall Y_{d0} \in \mathbb{R}^r$  with  $|Y_{d0}| \leq c_w$ ;  $\forall y_{d[0,\infty)} \in \mathcal{C}$  with  $\|Y_{d[0,\infty)}\|_\infty \leq c_w$ , we have

1.  $\forall x_0 \in \mathcal{D}_0$  with  $|x_0| \leq c_w$ ;  $\forall \check{x}_0 \in \mathcal{D}_0$  with  $|\check{x}_0| \leq c_w$ ;  $\forall w_{[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{[0,\infty)}\|_\infty \leq c_w$ ; there exists a unique solution  $X_{[0,\infty)} := (x_{[0,\infty)}, \xi_{[0,\infty)})$  to the closed-loop system  $S$  such that  $\|x_{[0,\infty)}\|_\infty \leq c_c$ ,  $\xi(t) \in \mathcal{D}_\xi$ ,  $\forall t \in [0, \infty)$ , such that  $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$ ;

2. when  $r = 0$  and  $n \in \mathbb{N}$ ,  $\forall x_0 \in \mathcal{D}_0$ , set  $\check{x}_0 = x_0 \in \mathcal{D}_0$  and  $w_{[0,\infty)}$  equals to the identically  $\mathbf{0}_{q \times 1}$  function in  $\mathcal{W}_d$ , then  $Cx(t) + Ku(t) = y_d(t)$ ,  $\forall t \in [0, \infty) \subset \mathbb{R}$ ;

3. when  $1 \leq r < n$ ,  $\forall x_{z0} \in \mathcal{D}_{z0}$ ,  $\exists x_0 \in \mathcal{D}_0$  whose first  $n - r$  coordinates equal to  $x_{z0}$ , set  $\check{x}_0 = x_0 \in \mathcal{D}_0$  and  $w_{[0,\infty)}$  equals to the identically  $\mathbf{0}_{q \times 1}$  function in  $\mathcal{W}_d$ , then  $y(t) = y_d(t)$ ,  $\forall t \in [0, \infty) \subset \mathbb{R}$ .

Then, the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ .

*Proof.* We will consider separately four exhaustive and mutually exclusive cases:

Case 1:  $0 = r = n$ ; Case 2:  $1 \leq r = n$ ; Case 3:  $0 = r < n$ ; Case 4:  $1 \leq r < n$ .

Case 1:  $0 = r = n$ . Clearly, the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  since its extended zero dynamics is absent. This completes the proof for this case.

Case 2:  $1 \leq r = n$ . Clearly, the system (3.12) is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  since its extended zero dynamics is absent. This completes the proof for this case.

Case 3:  $0 = r < n$ . By Lemma 3.3, the system  $S_P$  admits the following extended zero dynamics canonical form representation,  $K \neq 0$ ,

$$(4.2a) \quad \dot{x} = \hat{A}x + \hat{B}(y - Ew) + Dw =: \hat{A}x + \hat{B}y + \hat{D}w; \quad x(0) = x_0$$

$$(4.2b) \quad y = Cx + Ku + Ew$$

We need the following result.

CLAIM 4.1.1. *The system  $\dot{z} = \hat{A}z + \hat{B}v$ ;  $z(0) = \mathbf{0}_{n \times 1}$  is bounded input and bounded state stable.*

*Proof.* Suppose that the claim is false. By Corollary A.10,  $\exists v_{[0,\infty)} \in \mathcal{C}$  such that  $\|v_{[0,\infty)}\|_\infty =: c_v \in (0, \infty) \subset \mathbb{R}$  and  $\|z_{[0,\infty)}\|_\infty = +\infty$ . Choose  $w_{[0,\infty)}$  to be the identically  $\mathbf{0}_{q \times 1}$  function in  $\mathcal{W}_d$ . Fix  $x_0 \in \mathcal{D}_0$  and  $\tilde{x}_0 = x_0$ .  $\forall \rho \in (0, 1) \subset \mathbb{R}$ , let  $y_{d[0,\infty)} = \rho v_{[0,\infty)}$  and  $c_w = \max\{c_v, |x_0|\} \in \mathbb{R}$ . By 1.,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ , where  $c_c$  depends only on  $c_w$  and  $S_C$ , such that the closed-loop system  $S$  admits a unique solution  $X_{[0,\infty)}$  with  $\|x_{[0,\infty)}\|_\infty \leq c_c$ , and  $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$ . By 2., we have  $Cx_{[0,\infty)} + Ku_{[0,\infty)} = y_{d[0,\infty)} = \rho v_{[0,\infty)}$ . This further implies that  $y_{[0,\infty)} = y_{d[0,\infty)} = \rho v_{[0,\infty)}$ . Note that  $x_{[0,\infty)}$  is the unique solution to  $\dot{x} = \hat{A}x + \hat{B}y_d$ ,  $x(0) = x_0$ . By linearity, we have  $x_{[0,\infty)} = \rho z_{[0,\infty)} + x_{I[0,\infty)}$ , where  $x_{I[0,\infty)}$  satisfies  $\dot{x}_I = \hat{A}_I x_I$ ,  $x_I(0) = x_0$ . Hence, we have  $\|x_{[0,\infty)}\|_\infty = +\infty$  for some  $\rho \in (0, 1)$ . This is a contradiction. Hence, the claim is proved.  $\square$

Back to proof of the lemma. Let  $\bar{P} := [\hat{B} \quad \hat{A}\hat{B} \quad \dots \quad \hat{A}^{n-1}\hat{B}]$  and  $n_c := \text{rank}(\bar{P})$ . We will further distinguish 3 exhaustive and mutually exclusive cases: Case 3.1:  $n_c = 0$ ; Case 3.2:  $1 \leq n_c < n$ ; Case 3.3:  $n_c = n$ .

Case 3.1:  $n_c = 0$ . Then, we have  $\hat{B} = \mathbf{0}_{n \times 1}$ . Then, the extended zero dynamics is given by

$$(4.3) \quad \dot{x} = \hat{A}x + Dw; \quad x(0) = x_0$$

By the assumption,  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ,  $\forall x_0 \in \mathcal{D}_0$  with  $|x_0| \leq c_w$ , let  $\tilde{x}_0 = x_0$ ,  $\forall w_{[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{[0,\infty)}\|_\infty \leq c_w$ , let  $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)} \in \mathcal{C}$ , then, there exists a unique solution  $X_{[0,\infty)}$  to the closed-loop system with  $\|x_{[0,\infty)}\|_\infty \leq c_c$ . Clearly,  $x_{[0,\infty)}$  is the unique solution to (4.3). Then, the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . This completes the proof for this sub-case.

Case 3.2:  $1 \leq n_c < n$ . Without loss of generality, assume that (4.2a) is partitioned into controllable and uncontrollable parts (with respect to  $y$ ). We have  $x = [x'_c \quad x'_{\bar{c}}]'$

$$(4.4) \quad \begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \hat{A}_c & \hat{A}_{c\bar{c}} \\ \mathbf{0} & \hat{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \hat{B}_c \\ \mathbf{0} \end{bmatrix} y + \begin{bmatrix} \hat{D}_c \\ \hat{D}_{\bar{c}} \end{bmatrix} w$$

where  $x_c$  is  $n_c$ -dimensional;  $x_{\bar{c}}$  is  $n_{\bar{c}} := n - n_c$  dimensional; and the pair  $(\hat{A}_c, \hat{B}_c)$  is controllable.

By Claim 4.1.1 and Lemma A.3,  $\exists k \in [0, \infty) \subset \mathbb{R}$ ,  $\exists \lambda \in (0, \infty) \subset \mathbb{R}$ , such that  $\|e^{\hat{A}t} \hat{B}\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty)$ . Note that

$$\|e^{\hat{A}t} \hat{B}\|_{2,2} = \left\| \begin{bmatrix} e^{\hat{A}_c t} & \star \\ \mathbf{0} & e^{\hat{A}_{\bar{c}} t} \end{bmatrix} \begin{bmatrix} \hat{B}_c \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} = \|e^{\hat{A}_c t} \hat{B}_c\|_{2,2} \leq ke^{-\lambda t}; \quad \forall t \in [0, \infty)$$

Hence, we have the system  $\dot{z}_c = \hat{A}_c z_c + \hat{B}_c v$ ,  $z_c(0) = \mathbf{0}_{n_c \times 1}$  is bounded input and bounded state stable by Lemma A.3. By the controllability of the pair  $(\hat{A}_c, \hat{B}_c)$  and Lemma A.7, we have that  $\hat{A}_c$  is Hurwitz.

We will show that the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  by an argument of contradiction. Suppose  $S_P$  is not minimum phase.  $\exists c_w \in [0, \infty) \subset \mathbb{R}$ ,

$\forall i \in \mathbb{N}$ ,  $\exists x_{0(i)} \in \mathcal{D}_0$  with  $|x_{0(i)}| \leq c_w$ ,  $\exists w_{(i)[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{(i)[0,\infty)}\|_\infty \leq c_w$ ,  $\exists y_{(i)[0,\infty)} \in \mathcal{C}$  with  $\|y_{(i)[0,\infty)}\|_\infty \leq c_w$ , such that  $\|x_{(i)[0,\infty)}\|_\infty > i$ , where  $x_{(i)[0,\infty)}$  is the solution to (4.2a) with initial condition  $x_{0(i)}$  and inputs  $y_{(i)[0,\infty)}$  and  $w_{(i)[0,\infty)}$ . Partition  $x_{(i)}$  as  $\begin{bmatrix} x'_{c(i)} & x'_{\bar{c}(i)} \end{bmatrix}'$ . Then, we have  $\limsup_{i \rightarrow \infty} \|x_{\bar{c}(i)[0,\infty)}\|_\infty = +\infty$  since  $\hat{A}_c$  is Hurwitz ( $\|x_{\bar{c}(i)[0,\infty)}\|_\infty =: \bar{c}_{wi}$ , then,  $\|x_{c(i)[0,\infty)}\|_\infty \leq k_1 c_w + k_2 \sqrt{\bar{c}_{wi}^2 + 2c_w^2}$  for some  $k_1, k_2 \in [0, \infty) \subset \mathbb{R}$  that is independent of  $i \in \mathbb{N}$ ). Hence,  $\forall i \in \mathbb{N}$ , let  $x_0 = x_{0(i)}$ ,  $w_{[0,\infty)} = w_{(i)[0,\infty)}$ ,  $\check{x}_0 = x_0$ , and  $y_{d[0,\infty)}$  to be the identically  $\mathbf{0}_{q \times 1}$  function. By 1.,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ , where  $c_c$  depends only on  $c_w$  and  $S_C$ , such that the closed-loop system admits unique solution  $X_{\{i\}[0,\infty)}$  with continuous signals  $u_{\{i\}[0,\infty)}$  and  $y_{\{i\}[0,\infty)}$  and  $\|x_{\{i\}[0,\infty)}\|_\infty \leq c_c$ . Note that  $x_{\{i\}[0,\infty)}$  is the unique solution to (4.4) with initial condition  $x_{0(i)}$  and inputs  $y_{\{i\}[0,\infty)}$  and  $w_{(i)[0,\infty)}$ . By the uniqueness of solution to  $x_{\bar{c}}$  dynamics, we have  $x_{\bar{c}\{i\}[0,\infty)} = x_{\bar{c}(i)[0,\infty)}$ . Hence,  $\limsup_{i \rightarrow \infty} \|x_{\{i\}[0,\infty)}\|_\infty \geq \limsup_{i \rightarrow \infty} \|x_{\bar{c}\{i\}[0,\infty)}\|_\infty = \limsup_{i \rightarrow \infty} \|x_{\bar{c}(i)[0,\infty)}\|_\infty = +\infty$ . This is a contradiction. Hence, the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . This completes the proof for this sub-case.

Case 3.3:  $n_c = n$ . Then, the pair  $(\hat{A}, \hat{B})$  is controllable. By Claim 4.1.1 and Lemma A.7, we have that the matrix  $\hat{A}$  is Hurwitz. Then, by Proposition 3.10, the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . This completes the proof for this sub-case.

This completes the proof for Case 3.

Case 4:  $1 \leq r < n$ . By Lemma 3.1, the system  $S_P$  admits the extended zero dynamics canonical form representation (3.2). We need the following result.

CLAIM 4.1.2. *The system  $\dot{z} = A_z z + A_{z1} v$ ;  $z(0) = \mathbf{0}_{n_z \times 1}$  is bounded input and bounded state stable, where  $n_z := n - r$ .*

*Proof.* Suppose that the claim is false. By Corollary A.10,  $\exists v_{[0,\infty)} \in \mathcal{C}_r$  such that  $\|v_{[0,\infty)}^{[r]}\|_\infty =: c_v \in (0, \infty) \subset \mathbb{R}$  and  $\|z_{[0,\infty)}\|_\infty = +\infty$ . Choose  $w_{[0,\infty)}$  to be the identically  $\mathbf{0}_{q \times 1}$  function in  $\mathcal{W}_d$  and  $x_{z0} \in \mathcal{D}_{z0}$ . Let  $y_{d[0,\infty)} = \rho v_{[0,\infty)}$ . Clearly, we have  $|Y_{d0}| \leq c_v$  and  $y_{d[0,\infty)}^{(r)} \in \mathcal{C}$  with  $\|Y_{d[0,\infty)}\|_\infty \leq c_v$ . By 3.,  $\exists x_0 \in \mathcal{D}_0$  whose first  $n_z$  coordinates equal to  $x_{z0}$ , and let  $\check{x}_0 = x_0 \in \mathcal{D}_0$  such that, under the model reference controller  $S_C$ , the closed-loop system admits a solution  $X_{[0,\infty)}$  with  $x_{1[0,\infty)} = y_{[0,\infty)} = y_{d[0,\infty)} = \rho v_{[0,\infty)}$ . Let  $c_w = \max\{c_v, |x_0|\} \in (0, \infty) \subset \mathbb{R}$ . By 1.,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ , where  $c_c$  depends only on  $c_w$  and  $S_C$ , such that the unique solution  $X_{[0,\infty)}$  to the closed-loop system satisfies  $\|x_{[0,\infty)}\|_\infty \leq c_c$ , and  $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$ . Note that  $x_{z[0,\infty)}$  is the unique solution to  $\dot{x}_z = A_z x_z + A_{z1} y_d$ ,  $x_z(0) = x_{z0}$ . By linearity, we have  $x_{z[0,\infty)} = \rho z_{[0,\infty)} + x_{zI[0,\infty)}$ , where  $x_{zI[0,\infty)}$  satisfies  $\dot{x}_{zI} = A_z x_{zI}$ ,  $x_{zI}(0) = x_{z0}$ . Hence, we have  $\|x_{[0,\infty)}\|_\infty \geq \|x_{z[0,\infty)}\|_\infty = +\infty$  for some  $\rho \in (0, 1)$ . This is a contradiction. This completes the proof of the claim.  $\square$

Back to proof of the lemma. Let  $P_z := \begin{bmatrix} A_{z1} & A_z A_{z1} & \cdots & A_z^{n_z-1} A_{z1} \end{bmatrix}$  and  $n_{zc} := \text{rank}(P_z)$ . We will further distinguish 3 exhaustive and mutually exclusive cases: Case 4.1:  $n_{zc} = 0$ ; Case 4.2:  $1 \leq n_{zc} < n_z$ ; Case 4.3:  $n_{zc} = n_z$ .

Case 4.1:  $n_{zc} = 0$ . Then, we have  $A_{z1} = \mathbf{0}_{n_z \times 1}$ . Then, the extended zero dynamics is given by

$$(4.5) \quad \dot{x}_z = A_z x_z + D_z w; \quad x_z(0) = x_{z0}$$

By the assumption,  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ,  $\forall x_{z0} \in \mathcal{D}_{z0}$  with  $|x_{z0}| \leq c_w$ , let  $x_0 = \begin{bmatrix} x'_{z0} & \mathbf{0}_{1 \times r} \end{bmatrix}' \in \mathcal{D}_{z0} \times \mathbb{R}^r = \mathcal{D}_0$  and  $\check{x}_0 = x_0$ ,  $\forall w_{[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{[0,\infty)}\|_\infty \leq c_w$ , let  $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)} \in \mathcal{C}_r$  with  $\|Y_{d[0,\infty)}\|_\infty = 0$  and  $|Y_{d0}| = 0$ , then,

there exists a unique solution  $X_{[0,\infty)}$  to the closed-loop system with  $\|x_{[0,\infty)}\|_\infty \leq c_c$ . This implies that  $\|x_{z[0,\infty)}\|_\infty \leq \|x_{[0,\infty)}\|_\infty \leq c_c$  and  $x_{z[0,\infty)}$  is the unique solution to (4.5). Since the extended zero dynamics (4.5) is independent of  $x_{1[0,\infty)}$ , then the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . This completes the proof for this sub-case.

Case 4.2:  $1 \leq n_{zc} < n_z$ . Without loss of generality, assume that (3.2a) is partitioned into controllable and uncontrollable parts (with respect to  $x_1$ ). We have  $x_z = [x'_{zc} \quad x'_{z\bar{c}}]'$

$$(4.6) \quad \begin{bmatrix} \dot{x}_{zc} \\ \dot{x}_{z\bar{c}} \end{bmatrix} = \begin{bmatrix} A_{zc} & A_{zc\bar{c}} \\ \mathbf{0} & A_{z\bar{c}} \end{bmatrix} \begin{bmatrix} x_{zc} \\ x_{z\bar{c}} \end{bmatrix} + \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} x_1 + \begin{bmatrix} D_{zc} \\ D_{z\bar{c}} \end{bmatrix} w$$

where  $x_{zc}$  is  $n_{zc}$ -dimensional;  $x_{z\bar{c}}$  is  $n_{z\bar{c}} := n_z - n_{zc}$  dimensional; and the pair  $(A_{zc}, A_{zc1})$  is controllable.

By Claim 4.1.2 and Lemma A.3,  $\exists k \in [0, \infty) \subset \mathbb{R}$ ,  $\exists \lambda \in (0, \infty) \subset \mathbb{R}$ , such that  $\|e^{A_z t} A_{z1}\|_{2,2} \leq k e^{-\lambda t}$ ,  $\forall t \in [0, \infty)$ . Note that,  $\forall t \in [0, \infty)$ ,

$$\|e^{A_z t} A_{z1}\|_{2,2} = \left\| \begin{bmatrix} e^{A_{zc} t} & \star \\ \mathbf{0} & e^{A_{z\bar{c}} t} \end{bmatrix} \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} = \|e^{A_{zc} t} A_{zc1}\|_{2,2} \leq k e^{-\lambda t}$$

Hence, we have that the system  $\dot{z}_c = A_{zc} z_c + A_{zc1} v$ ,  $z_c(0) = \mathbf{0}_{n_{zc} \times 1}$  is bounded input and bounded state stable by Lemma A.3. By the controllability of the pair  $(A_{zc}, A_{zc1})$  and Lemma A.7, we have that  $A_{zc}$  is Hurwitz.

We will show that the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$  by an argument of contradiction. Suppose  $S_P$  is not minimum phase.  $\exists c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\forall i \in \mathbb{N}$ ,  $\exists x_{z0(i)} \in \mathcal{D}_{z0}$  with  $|x_{z0(i)}| \leq c_w$ ,  $\exists w_{(i)[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{(i)[0,\infty)}\|_\infty \leq c_w$ ,  $\exists x_{1(i)[0,\infty)} \in \mathcal{C}_r$  with  $\|x_{1(i)[0,\infty)}^{[r]}\|_\infty \leq c_w$ , such that  $\|x_{z(i)[0,\infty)}\|_\infty > i$ , where  $x_{z(i)[0,\infty)}$  is the solution to (4.6) with initial condition specified by  $x_{z0(i)}$  and inputs  $x_{1(i)[0,\infty)}$  and  $w_{(i)[0,\infty)}$ . Partition  $x_{z(i)}$  as  $[x'_{zc(i)} \quad x'_{z\bar{c}(i)}]'$ . Then, we have  $\limsup_{i \rightarrow \infty} \|x_{z\bar{c}(i)[0,\infty)}\|_\infty = +\infty$  since  $A_{zc}$  is Hurwitz ( $\|x_{z\bar{c}(i)[0,\infty)}\|_\infty =: \bar{c}_{wi}$ , then,  $\|x_{zc(i)[0,\infty)}\|_\infty \leq k_1 c_w + k_2 \sqrt{\bar{c}_{wi}^2 + 2c_w^2}$ , for some  $k_1, k_2 \in [0, \infty) \subset \mathbb{R}$  that is independent of  $i \in \mathbb{N}$ ). Let  $x_0 = x_{0(i)} := [x'_{z0(i)} \quad \mathbf{0}_{1 \times r}]' \in \mathcal{D}_0$ ,  $w_{[0,\infty)} = w_{(i)[0,\infty)}$ ,  $\check{x}_0 = x_0$ , and  $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)}$  with  $\|Y_{d[0,\infty)}\|_\infty = 0$  and  $|Y_{d0}| = 0$ . By 1.,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ , where  $c_c$  depends only on  $c_w$  and  $S_C$ , such that the closed-loop system admits unique solution  $X_{\{i\}[0,\infty)}$  with continuous signals  $u_{\{i\}[0,\infty)}$  and  $y_{\{i\}[0,\infty)}$  and  $\|x_{\{i\}[0,\infty)}\|_\infty \leq c_c$ . Note that  $x_{\{i\}[0,\infty)}$  is the unique solution to (3.2) with initial condition  $x_{0(i)}$  and inputs  $u_{\{i\}[0,\infty)}$  and  $w_{(i)[0,\infty)}$ . Then, the component  $x_{z\bar{c}\{i\}[0,\infty)}$  of  $x_{\{i\}[0,\infty)}$  is the unique solution to  $x_{z\bar{c}}$  dynamics in (4.6), which is independent of  $x_{1\{i\}[0,\infty)}$ . We have  $x_{z\bar{c}\{i\}[0,\infty)} = x_{z\bar{c}(i)[0,\infty)}$ . Hence,  $\limsup_{i \rightarrow \infty} \|x_{\{i\}[0,\infty)}\|_\infty \geq \limsup_{i \rightarrow \infty} \|x_{z\bar{c}\{i\}[0,\infty)}\|_\infty = \limsup_{i \rightarrow \infty} \|x_{z\bar{c}(i)[0,\infty)}\|_\infty = +\infty$ . This is a contradiction. Hence, the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . This completes the proof for this sub-case.

Case 4.3:  $n_{zc} = n_z$ . Then, the pair  $(A_z, A_{z1})$  is controllable. By Claim 4.1.2 and Lemma A.7, we have that the matrix  $A_z$  is Hurwitz. Then, by Proposition 3.10, the system  $S_P$  is minimum phase with respect to  $\mathcal{D}_0$  and  $\mathcal{W}_d$ . This completes the proof for this sub-case.

This completes the proof of the lemma.  $\square$



**5. Conclusions.** In this paper, we have obtained the extended zero dynamics canonical form for finite-dimensional continuous-time SISO LTI systems with a finite relative degree  $r$ . Based on the canonical form, we introduced the definition of extended zero dynamics for the system, which is the zero dynamics defined in [4] together with driving terms involving the noiseless output of the system and the disturbance input. Then, we generalized the concept of minimum phase for such systems that accommodates the disturbance input and subsumes the classical definition of minimum phase. A system is said to be minimum phase with respect to the set of admissible initial conditions and the set of admissible disturbance input waveforms if the extended zero dynamics satisfies that, for any bounded admissible initial condition, any bounded noiseless output, and any bounded admissible disturbance input waveform, the extended zero dynamics state trajectory is bounded or if the extended zero dynamics is absent. If the extended zero dynamics is asymptotically stable, then the system is clearly minimum phase. The converse holds under the additional assumption that the system is stabilizable from the control input. For a system to be minimum phase, it is necessary that the transfer function from the control input to the output has all zeros with negative real parts. The converse holds when the system is both controllable (from the control input) and observable. An example is presented to illustrate the generalized concept of minimum phase that the extended zero dynamics does not need to be asymptotically stable or bounded input and bounded state stable in the usual sense. It is further shown that the generalized minimum phase property is necessary for the achievement of perfect tracking of any bounded reference trajectories with bounded derivatives up to  $r$ th order without any disturbances and the existence of bounded state trajectory for any admissible bounded initial condition, any admissible bounded disturbance waveform, and any bounded reference trajectory with bounded derivatives up to  $r$ th order in model reference control of the system.

We envision that this generalized definition of minimum phase will lead to broader results for model reference adaptive control. Future research on this topic is under way to investigate the minimum phase properties for feedback interconnected systems and to investigate the boundedness of the inverse of a minimum phase system [7]. It is fruitful to extend the generalized definition of minimum phase to finite-dimensional continuous-time square MIMO LTI systems that admits a dynamic extension to a system with vector relative degree. Thus, this generalized minimum phase condition might be proved to be necessary and sufficient for model reference robust adaptive control design.

**Appendix A. A number of useful results.** We first present two lemmas that are standard in linear system theory [2].

LEMMA A.1. *Consider finite-dimensional continuous-time LTI system:*

$$(A.1a) \quad \dot{x} = Ax + Bu; \quad x(0) = x_0$$

$$(A.1b) \quad y = Cx + Ku$$

where  $x$  is the  $n$ -dimensional state,  $n \in \mathbb{Z}_+$ ;  $x_0 \in \mathbb{R}^n$ ;  $u$  is the  $p$ -dimensional input,  $p \in \mathbb{Z}_+$ ; and  $y$  is the  $m$ -dimensional output,  $m \in \mathbb{Z}_+$ ;  $A$ ,  $B$ ,  $C$ , and  $K$  are constant matrices of appropriate dimensions. Let  $u_{[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}^p)$ . Then,  $y_{[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}^m)$  and  $x_{[0,\infty)} \in \mathcal{C}_1([0,\infty), \mathbb{R}^n)$  are given by,  $\forall t \in [0,\infty) \subset \mathbb{R}$ ,

$$(A.2a) \quad x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau) d\tau$$

$$(A.2b) \quad y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau) d\tau + Ku(t)$$

LEMMA A.2. *Let  $n \in \mathbb{Z}_+$  and  $A$  be an  $n \times n$ -dimensional Hurwitz matrix. Then,  $\exists k \in [0, \infty) \subset \mathbb{R}$  and  $\lambda \in (0, \infty) \subset \mathbb{R}$  such that  $\|e^{At}\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty) \subset \mathbb{R}$ .*

We present here a lemma on the bounded input and bounded output stability of LTI systems.

LEMMA A.3. *Consider finite-dimensional continuous-time LTI system:*

$$(A.3) \quad S: \quad \begin{cases} \dot{x} = Ax + Bu; & x(0) = x_0 = \mathbf{0}_{n \times 1} \\ y = Cx + Du \end{cases}$$

where  $x$  is the  $n$ -dimensional state,  $n \in \mathbb{Z}_+$ ;  $u$  is the  $p$ -dimensional input,  $p \in \mathbb{Z}_+$ ; and  $y$  is the  $m$ -dimensional output,  $m \in \mathbb{Z}_+$ ;  $A$ ,  $B$ ,  $C$ , and  $D$  are constant matrices of appropriate dimensions. Then, the following statements are equivalent.

1. *There exist  $k \in [0, \infty) \subset \mathbb{R}$  and  $\lambda \in (0, \infty) \subset \mathbb{R}$  such that  $\|Ce^{At}B\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty)$ .*

2. *Viewed as a linear operator  $S : L_q([0, \infty), \mathbb{R}^p) \rightarrow L_q([0, \infty), \mathbb{R}^m)$ , where  $q \in [1, \infty] \subset \mathbb{R}_e$ , the system  $S$  is bounded.*

3.  *$\forall c_u \in [0, \infty) \subset \mathbb{R}$ ,  $\forall t_f \in (0, \infty) \subset \mathbb{R}_e$ ,  $\forall u_{[0,t_f]} \in \mathcal{C}$  with  $\|u_{[0,t_f]}\|_\infty \leq c_u$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ , such that  $\|y_{[0,t_f]}\|_\infty \leq c_c$ .*

4.  *$\forall c_u \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ,  $\forall t_f \in (0, \infty) \subset \mathbb{R}_e$ ,  $\forall u_{[0,t_f]} \in \mathcal{C}$  with  $\|u_{[0,t_f]}\|_\infty \leq c_u$ , we have  $\|y_{[0,t_f]}\|_\infty \leq c_c$ .*

*Proof.* The proof of the lemma is standard and follows the path  $1 \Rightarrow 2 \Rightarrow 4 \Rightarrow 3 \Rightarrow 1$ . The proof is omitted here.  $\square$

DEFINITION A.4. *For the finite-dimensional continuous-time LTI system (A.3) under the assumptions of Lemma A.3, we say that  $S$  is bounded input and bounded output stable (BIBO) if any of 1 – 4 holds. If, in addition,  $C = I_n$  and  $D = \mathbf{0}_{m \times p}$ , then we say that  $S$  is bounded input and bounded state stable (BIBS).*

Next, we present a canonical form for a finite-dimensional SISO LTI system that is both controllable and observable.

LEMMA A.5. *Consider a finite-dimensional continuous-time SISO LTI system*

$$(A.4a) \quad \dot{x} = Ax + Bu$$

$$(A.4b) \quad y = Cx$$

where  $x$  is the  $n$ -dimensional state vector;  $u$  is a scalar control input;  $y$  is a scalar output; and  $n \in \mathbb{N}$ . Assume the triple  $(A, B, C)$  is controllable and observable. Let the transfer function from  $u$  to  $y$  be given by,  $b_0 \neq 0$ ,

$$(A.5) H(s) = C(sI_n - A)^{-1}B = \frac{b_0s^m + b_1s^{m-1} + \dots + b_m}{s^n + a_1s^{n-1} + \dots + a_n} = \frac{\sum_{i=0}^m b_i s^{m-i}}{s^n + \sum_{i=1}^n a_i s^{n-i}}$$

which admits relative degree  $r = n - m \in \{1, \dots, n\}$ . Then, there exists a unique coordinate transformation  $\bar{x} = Tx$ , within the class of  $\mathcal{C}_1$  diffeomorphisms, where  $T$  is invertible, such that, in the  $\bar{x}$  coordinate, the system (A.4) admits the state space representation

$$(A.6a) \quad \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$(A.6b) \quad y = \bar{C}\bar{x}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}; \quad \bar{A}_{11} = \left[ \begin{array}{c|c} \bar{a}_1 & I_{r-1} \\ \vdots & \\ \bar{a}_{r-1} & \\ \hline \bar{a}_r & \mathbf{0}_{1 \times (r-1)} \end{array} \right]$$

$$\bar{A}_{12} = \begin{cases} \left[ \begin{array}{c|c} \mathbf{0} & \\ \hline 1 & \end{array} \right] & m \geq 1 \\ \left[ \begin{array}{c|c} \mathbf{0} & \\ \hline 1 & \end{array} \right] & m = 0 \end{cases} \quad \bar{A}_{21} = \left[ \begin{array}{c|c} \bar{a}_{r+1} & \\ \vdots & \\ \bar{a}_n & \end{array} \right] \mathbf{0}_{(n-r) \times (r-1)}$$

$$\bar{A}_{22} = \begin{cases} \left[ \begin{array}{c|c} -b_1/b_0 & I_{m-1} \\ \vdots & \\ -b_{m-1}/b_0 & \\ \hline -b_m/b_0 & \mathbf{0}_{1 \times (m-1)} \end{array} \right] & m \geq 1 \\ \left[ \begin{array}{c|c} \mathbf{0} & \\ \hline -b_m/b_0 & \mathbf{0}_{1 \times (m-1)} \end{array} \right] & m = 0 \end{cases} \quad \bar{B} = [ \mathbf{0}_{1 \times (r-1)} \quad b_0 \quad \mathbf{0}_{1 \times m} ]'$$

$$\bar{C} = [ 1 \quad \mathbf{0}_{1 \times (n-1)} ]$$

and the constants  $\bar{a}_1, \dots, \bar{a}_n$  are uniquely determined by

$$(A.7) \quad s^n + \sum_{i=1}^n a_i s^{n-i} = \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right) \left( \sum_{i=0}^m b_i s^{m-i} \right) / b_0 - \sum_{i=r+1}^n \bar{a}_i s^{n-i}$$

*Proof.* We will first show the existence of a real transformation  $\bar{x} = Tx$  that transforms the system into the canonical form (A.6). Since the pair  $(A, C)$  is observable, there exists a real coordinate transformation  $z = T_0x$ , such that the system (A.4) admits the observer canonical form representation in the  $z$  coordinates,

$$\dot{z} = \left[ \begin{array}{c|c} -a_1 & I_{n-1} \\ \vdots & \\ -a_{n-1} & \\ \hline -a_n & \mathbf{0}_{1 \times (n-1)} \end{array} \right] z + \left[ \begin{array}{c} \mathbf{0}_{(r-1) \times 1} \\ b_0 \\ \vdots \\ b_m \end{array} \right] u =: A_0z + B_0u$$

$$y = \bar{C}z$$

Now, we will distinguish 2 exhaustive and mutually exclusive cases: Case 1:  $r = n$ ; Case 2:  $r < n$ .

Case 1:  $r = n$ . In this case, the system representation in  $z$  coordinate is in the canonical form (A.6). Then, the desired  $T$  is  $T_0$  for this case.

Case 2:  $r < n$ . In this case, we will introduce a sequence of  $r$  state transformations that will transform the system into the representation (A.6).

By the first transformation, we will transform  $B_0$  into  $\bar{B}$ . Let  $z_1 = T_1z$ , where  $T_1$  is the same as  $I_n$  except the  $r$ th column, which equals to

$$\left[ \mathbf{0}_{1 \times (r-1)} \quad 1 \quad -b_1/b_0 \quad \cdots \quad -b_m/b_0 \right]'$$

Then, its inverse  $T_1^{-1}$  is also the same as  $I_n$  except the  $r$ th column, which equals to  $\left[ \mathbf{0}_{1 \times (r-1)} \quad 1 \quad b_1/b_0 \quad \cdots \quad b_m/b_0 \right]'$ . This transforms the system (A.4) into

$$\dot{z}_1 = A_1z_1 + \bar{B}u$$

$$y = \bar{C}z_1$$

where the matrix  $A_1 = T_1 A_0 T_1^{-1}$ . Now, we will further distinguish 2 exhaustive and mutually exclusive sub-cases: Case 2a:  $r = 1$ ; Case 2b:  $r > 1$ .

Case 2a:  $r = 1$ . In this case,  $A_1$  is in the same form as  $\bar{A}$ . Hence, the desired  $T = T_1 T_0$  for this sub-case.

Case 2b:  $r > 1$ . In this case,  $A_1$  is the same as the matrix  $\bar{A}$  except the 1st and  $r$ th columns, which are  $[a_{11} \ a_{21} \ \cdots \ a_{n1}]'$  and  $[\mathbf{0}_{1 \times (r-2)} \ 1 \ d_{11} \ \cdots \ d_{m+11}]'$ , respectively, and  $a_{j1}$ ,  $j = 1, \dots, n$  and  $d_{j1}$ ,  $j = 1, \dots, m+1$ , are some constants in  $\mathbb{R}$ . We will continue in this case to introduce additional  $r-1$  transformations.

By the  $i$ th ( $2 \leq i \leq r-1$ ) transformation, we will transform the  $(r-i+2)$ nd column of the matrix  $A_i$  into the  $(r-i+1)$ st unit vector. Let  $z_i = T_i z_{i-1}$ , where  $T_i$  is the same as  $I_n$  except the  $(r-i+1)$ st column, which equals to

$$[\mathbf{0}_{1 \times (r-i)} \ 1 \ -d_{1i-1} \ \cdots \ -d_{m+i-1i-1}]'$$

Then, its inverse  $T_i^{-1}$  is also the same as  $I_n$  except the  $(r-i+1)$ st column, which equals to  $[\mathbf{0}_{1 \times (r-i)} \ 1 \ d_{1i-1} \ \cdots \ d_{m+i-1i-1}]'$ . This transforms the system (A.4) into

$$\begin{aligned} \dot{z}_i &= A_i z_i + \bar{B}u \\ y &= \bar{C}z_i \end{aligned}$$

where the matrix  $A_i = T_i A_{i-1} T_i^{-1}$  is the same as the matrix  $\bar{A}$  except the 1st and  $(r-i+1)$ st columns, which are

$$[a_{1i} \ a_{2i} \ \cdots \ a_{ni}]' \text{ and } [\mathbf{0}_{1 \times (r-i-1)} \ 1 \ d_{1i} \ \cdots \ d_{m+i i}]'$$

respectively, and  $a_{ji}$ ,  $j = 1, \dots, n$  and  $d_{ji}$ ,  $j = 1, \dots, m+i$ , are some constants in  $\mathbb{R}$ .

By the  $r$ th transformation, which is the last one, we will transform the 2nd column of the matrix  $A_{r-1}$  into the 1st unit vector. Let  $z_r = T_r z_{r-1}$ , where  $T_r$  is the same as  $I_n$  except the 1st column, which equals to  $[1 \ -d_{1r-1} \ \cdots \ -d_{n-1r-1}]'$ . Then, its inverse  $T_r^{-1}$  is also the same as  $I_n$  except the 1st column, which equals to  $[1 \ d_{1r-1} \ \cdots \ d_{n-1r-1}]'$ . This transforms the system (A.4) into

$$\begin{aligned} \dot{z}_r &= A_r z_r + \bar{B}u \\ y &= \bar{C}z_r \end{aligned}$$

where the matrix  $A_r = T_r A_{r-1} T_r^{-1}$  is the same as the matrix  $\bar{A}$ . Hence, the desired state transformation is given by  $\bar{x} = z_r = T_r \cdots T_1 T_0 x =: Tx$  for this sub-case.

Thus, in all cases, we have shown the existence of a real transformation  $\bar{x} = Tx$  that transforms the system into the canonical form (A.6).

Next, we show that the constants  $\bar{a}_1, \dots, \bar{a}_n$  are uniquely determined by (A.7). Note that

$$H(s) = \bar{C}(sI_n - \bar{A})^{-1} \bar{B} = b_0 M_{1r}(s)$$

where  $M_{1r}$  is the  $1r$ -element of  $(sI_n - \bar{A})^{-1}$ . By, Cramer's rule, the following holds,

$$H(s) = \frac{b_0 (-1)^{r+1} \det(\bar{\Delta}_{r1})}{\det(sI_n - \bar{A})}$$

where  $\bar{\Delta}_{r1}$  is the matrix obtained from  $sI_n - \bar{A}$  by removing the  $r$ th row and the 1st column. It is clear that

$$\det(\bar{\Delta}_{r1}) = \begin{cases} (-1)^{r-1} \det(\bar{\Delta}) & m \geq 1 \\ (-1)^{r-1} & m = 0 \end{cases} = (-1)^{r-1} / b_0 \sum_{i=0}^m b_i s^{m-i}$$

where  $\bar{\Delta} = sI_m - \bar{A}_{22}$ .

Let  $\Delta_n = sI_n - \bar{A}$ . We will distinguish 3 exhaustive and mutually exclusive cases:

Case A:  $m = 0$ ; Case B:  $m = 1$ ; Case C:  $m \geq 2$ .

Case A:  $m = 0$ . In this case, we have

$$\det(\Delta_n) = s^n - \sum_{i=1}^n \bar{a}_i s^{n-i} = \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right) \left( \sum_{i=0}^m b_i s^{m-i} \right) / b_0 - \sum_{i=r+1}^n \bar{a}_i s^{n-i}$$

Case B:  $m = 1$ . In this case, by expanding the last row of  $\Delta_n$  using the Laplace Theorem, we have

$$\begin{aligned} \det(\Delta_n) &= (-\bar{a}_n) (-1)^{n+1} \det(\bar{\Delta}_{n,n1}) + (s + b_1/b_0) \det(sI_r - \bar{A}_{11}) \\ &= \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right) \left( \sum_{i=0}^m b_i s^{m-i} \right) / b_0 - \sum_{i=r+1}^n \bar{a}_i s^{n-i} \end{aligned}$$

where  $\bar{\Delta}_{n,n1}$  is a matrix obtained from  $\Delta_n$  by removing the  $n$ th row and the 1st column.

Case C:  $m \geq 2$ . By expanding the last row of  $\Delta_n$  using the Laplace Theorem, we have

$$\begin{aligned} \det(\Delta_n) &= (-\bar{a}_n) (-1)^{n+1} \det(\bar{\Delta}_{n,n1}) + (b_m/b_0) (-1)^{n+r+1} \det(\bar{\Delta}_{n,nr+1}) \\ &\quad + s \det(\bar{\Delta}_{n,nn}) \end{aligned}$$

where  $\bar{\Delta}_{n,ij}$  denotes the matrix obtained from  $\Delta_n$  by removing the  $i$ th row and the  $j$ th column,  $i, j = 1, \dots, n$ . It is clear that

$$\begin{aligned} \det(\bar{\Delta}_{n,n1}) &= (-1)^{n-1} \\ \det(\bar{\Delta}_{n,nr+1}) &= (-1)^{m-1} \det(sI_r - \bar{A}_{11}) = (-1)^{m-1} \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right) \end{aligned}$$

Let  $\Delta_{n-1} = \bar{\Delta}_{n,nn}$ , which is a sub-matrix of  $\Delta_n$  consisting of the first  $n-1$  rows and first  $n-1$  columns. Then, we have

$$\det(\Delta_n) = -\bar{a}_n + (b_m/b_0) \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right) + s \det(\Delta_{n-1})$$

Repeated application of the Laplace Theorem by expanding the last row of  $\Delta_i$ ,  $i = n-1, \dots, r+2$ , we have

$$\det(\Delta_i) = -\bar{a}_i + (b_{i-r}/b_0) (s^r - \bar{a}_1 s^{r-1} - \dots - \bar{a}_r) + s \det(\Delta_{i-1})$$

where  $\Delta_{i-1}$  is the matrix obtained from  $\Delta_i$  by removing the last row and the last column. The matrix  $\Delta_i$  is a sub-matrix of  $\Delta_n$  consisting of the first  $i$  rows and first  $i$  columns.

Apply the Laplace Theorem to  $\det(\Delta_{r+1})$  by expanding its last row, we have

$$\det(\Delta_{r+1}) = (-\bar{a}_{r+1}) (-1)^{r+2} \det(\bar{\Delta}_{r+1,r+11}) + (s + b_1/b_0) \det(sI_r - \bar{A}_{11})$$

where  $\bar{\Delta}_{r+1,r+11}$  is a matrix obtained from  $\Delta_{r+1}$  by removing the  $(r+1)$ st row and the 1st column. It is clear  $\det(\bar{\Delta}_{r+1,r+11}) = (-1)^r$ . Then,

$$\det(\Delta_{r+1}) = -\bar{a}_{r+1} + (s + b_1/b_0) \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right)$$

Based on the recursive formula for  $\det(\Delta_i)$ , we conclude

$$\det(\Delta_n) = \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right) \left( \sum_{i=0}^m b_i s^{m-i} \right) / b_0 - \sum_{i=r+1}^n \bar{a}_i s^{n-i}$$

In all three cases, we obtain  $\det(\Delta_n) = \left( s^r - \sum_{i=1}^r \bar{a}_i s^{r-i} \right) \left( \sum_{i=0}^m b_i s^{m-i} \right) / b_0 - \sum_{i=r+1}^n \bar{a}_i s^{n-i}$ . Then,

$$H(s) = \frac{\sum_{i=0}^m b_i s^{m-i}}{\det(\Delta_n)}$$

Compare the above formula with (A.5), we obtain (A.7). Clearly, the constants  $\bar{a}_1, \dots, \bar{a}_n$  are uniquely determined by (A.7) since  $s^r - \sum_{i=1}^r \bar{a}_i s^{r-i}$  and  $-\sum_{i=r+1}^n \bar{a}_i s^{n-i}$  are the quotient and remainder of  $s^n + \sum_{i=1}^n a_i s^{n-i}$  divided by  $\left( \sum_{i=0}^m b_i s^{m-i} \right) / b_0$ , respectively.

Finally, we prove the uniqueness of the transformation  $\bar{x} = Tx$  within the class of all  $\mathcal{C}_1$  diffeomorphisms. Let  $\bar{x} = T(x)$  be any  $\mathcal{C}_1$  diffeomorphism that leads to the state space representation (A.6). Denote the elements of  $\bar{x}$  by  $\bar{x} = [\bar{x}_1 \ \cdots \ \bar{x}_n]'$ . Then, we have  $\bar{x}_i = T_i(x)$ ,  $i = 1, 2, \dots, n$ . Note that  $y = \bar{C}\bar{x} = \bar{x}_1 = Cx$ , then  $T_1(x) = Cx =: M_1x$ . We will obtain a recursive formula for  $T_i(x)$ ,  $i = 1, \dots, r$ . When  $i = 1$ , we have  $\bar{x}_1 = T_1(x) = M_1x$ . Assume that we have shown that  $\bar{x}_i = T_i(x) = M_ix$ ,  $i = 1, \dots, k$ , where  $1 \leq k < r$ . Then, take derivative of  $\bar{x}_k$ , we have

$$\dot{\bar{x}}_k = \bar{a}_k \bar{x}_1 + \bar{x}_{k+1} = M_k(Ax + Bu)$$

This implies that

$$\bar{x}_{k+1} = T_{k+1}(x) = M_k Ax - \bar{a}_k M_1 x =: M_{k+1}x$$

and  $M_k B = 0$ . The above induction argument shows that  $\bar{x}_k = T_k(x) = M_k x$ ,  $k = 1, \dots, r$ , and  $M_k$  satisfies the recursive formula

$$M_1 = C; \quad M_{k+1} = M_k A - \bar{a}_k M_1, \quad k = 1, \dots, r-1$$

Now, we will distinguish 4 exhaustive and mutually exclusive cases: Case I:  $n = 1$ ; Case II:  $r = n \geq 2$ ; Case III:  $r = n - 1 \geq 1$ ; Case IV:  $1 \leq r \leq n - 2$ .

Case I:  $n = 1$ . Clearly,  $T(x) = T_1(x) = M_1x$  is unique. Case II:  $r = n \geq 2$ . We have  $T_i(x) = M_ix$ ,  $i = 1, \dots, n$ , such that  $M_1 = C$ ;  $M_{k+1} = M_k A - \bar{a}_k M_1$ ,  $k = 1, \dots, n-1$ . This shows that  $T(x)$  is uniquely determined in this case.

Case III:  $1 \leq r = n - 1$ . Take the derivative of  $\bar{x}_r$  to yield,

$$\dot{\bar{x}}_{n-1} = \bar{a}_{n-1} \bar{x}_1 + \bar{x}_n + b_0 u = M_{n-1}(Ax + Bu)$$

This implies that

$$\bar{x}_n = T_n(x) = M_{n-1}Ax - \bar{a}_{n-1}M_1x =: M_nx$$

and  $M_n B = b_0$ . Hence,  $\bar{x}_k = T_k(x) = M_k x$ ,  $k = 1, \dots, n$ , and  $M_k$  satisfies

$$M_1 = C; \quad M_{k+1} = M_k A - \bar{a}_k M_1, \quad k = 1, \dots, n-1$$

This shows that  $T(x)$  is unique in this case.









constant matrices of appropriate dimensions. Assume that the system is bounded input and bounded output stable from  $w$  to  $z$ ; and it is controllable from  $w$  and observable from  $z$ . Then, the matrix  $A$  is Hurwitz.

*Proof.* We will show this by an argument of contradiction. Suppose  $A$  is not Hurwitz. Then,  $A$  admits an eigenvalue  $\lambda \in \mathbb{C}$  with nonnegative real part. We will distinguish two exhaustive and mutually exclusive cases: Case 1:  $\lambda \in \mathbb{R}$ ; Case 2:  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

Case 1:  $\lambda \in \mathbb{R}$ . Without loss of generality, assume that the matrix  $A$  is given in the real Jordan canonical form (A.8) with  $\lambda$  being the eigenvalue of the first Jordan block; that is  $\lambda = \lambda_1$ . By the controllability of the system, there exists  $w_{[0,1]} \in \mathcal{C}$  such that  $w(0) = \mathbf{0}_{q \times 1}$ ,  $w(1) = \mathbf{0}_{q \times 1}$ , and transfer the state from  $x(0) = \mathbf{0}_{n \times 1}$  to  $x(1) = \begin{bmatrix} 1 & \mathbf{0}_{1 \times (n-1)} \end{bmatrix}'$ . Since the pair  $(A, C)$  is observable, then  $C = \begin{bmatrix} c_1 & \star_{m \times (n-1)} \end{bmatrix}$  with  $c_1 \neq \mathbf{0}_{m \times 1}$ . Set  $w_{[1,\infty)}$  to be the identically  $\mathbf{0}_{q \times 1}$  function. Then, on the interval  $[1, \infty)$ ,  $z_{[1,\infty)}$  is generated by

$$\begin{aligned} \dot{x}_1 &= \lambda x_1; & x_1(1) &= 1 \\ z &= c_1 x_1 \end{aligned}$$

Since  $\lambda \in [0, \infty) \subset \mathbb{R}$ , then  $\|z_{[1,\infty)}\|_2 = \infty$ . Hence, we have  $w_{[0,\infty)} \in \mathcal{C}$ ,  $\|w_{[0,\infty)}\|_2 = \|w_{[0,1]}\|_2 < +\infty$ , and  $\|z_{[0,\infty)}\|_2 = +\infty$ . Thus, the system (A.10) viewed as a linear operator  $S : L_2([0, \infty), \mathbb{R}^q) \rightarrow L_2([0, \infty), \mathbb{R}^m)$  is not bounded. This contradicts the assumption that (A.10) is bounded input and bounded output stable. Hence, we have arrived at a contradiction in this case.

Case 2:  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Without loss of generality, assume that  $A$  is given in the real Jordan canonical form (A.8) with  $\lambda$  being the eigenvalue of the  $B_1$ , that is  $\lambda = \mu_1$ . By the controllability of the system, there exists  $w_{[0,1]} \in \mathcal{C}$  such that  $w(0) = \mathbf{0}_{q \times 1}$ ,  $w(1) = \mathbf{0}_{q \times 1}$ , and transfer the state from  $x(0) = \mathbf{0}_{n \times 1}$  to  $x(1) = \begin{bmatrix} \mathbf{0}_{1 \times (\sum_{i=1}^l n_i)} & 1 & \mathbf{0}_{1 \times (n - \sum_{i=1}^l n_i - 1)} \end{bmatrix}'$ . Since the pair  $(A, C)$  is observable, then  $C = \begin{bmatrix} \star_{m \times (\sum_{i=1}^l n_i)} & c_1 & \star_{m \times (m_1 - 1)} & c_2 & \star_{m \times (n - \sum_{i=1}^l n_i - m_1 - 1)} \end{bmatrix}$  with  $c_1$  and  $c_2$  not both being zero vectors. Set  $w_{[1,\infty)}$  to be the identically  $\mathbf{0}_{q \times 1}$  function. Then, on the interval  $[1, \infty)$ ,  $z_{[1,\infty)}$  is generated by

$$\begin{aligned} \dot{x}_1 &= \operatorname{Re}(\lambda) x_1 + \operatorname{Im}(\lambda) x_2; & x_1(1) &= 1 \\ \dot{x}_2 &= -\operatorname{Im}(\lambda) x_1 + \operatorname{Re}(\lambda) x_2; & x_2(1) &= 0 \\ z &= c_1 x_1 + c_2 x_2 \end{aligned}$$

Let  $a = \operatorname{Re}(\lambda) \geq 0$  and  $b = \operatorname{Im}(\lambda) \neq 0$ . Then  $x_1(t) = e^{a(t-1)} \cos(b(t-1))$  and  $x_2(t) = -e^{a(t-1)} \sin(b(t-1))$ ,  $\forall t \in [1, \infty)$ . Thus,  $\|z_{[1,\infty)}\|_2 = +\infty$ . Hence, we have  $w_{[0,\infty)} \in \mathcal{C}$ ,  $\|w_{[0,\infty)}\|_2 = \|w_{[0,1]}\|_2 < \infty$ , and  $\|z_{[0,\infty)}\|_2 = +\infty$ . Thus, the system (A.10) viewed as a linear operator  $S : L_2([0, \infty), \mathbb{R}^q) \rightarrow L_2([0, \infty), \mathbb{R}^m)$  is not bounded. This contradicts the assumption that (A.10) is bounded input and bounded output stable. Hence, we have arrived at a contradiction in this case.

In both cases, we have arrived at a contradiction. Hence, the hypothesis does not hold. Then,  $A$  is Hurwitz.

This completes the proof of the lemma.  $\square$

**COROLLARY A.8.** *Consider the system (A.10). Assume that the system is bounded input and bounded output stable from  $w$  to  $z$ ; and it is stabilizable from  $w$  and detectable from  $z$ . Then, the matrix  $A$  is Hurwitz.*

*Proof.* Let  $Q = [ C' \ A'C' \ \dots \ A'^{n-1}C' ]'$  and  $\text{rank}(Q) = n_o \in \{0, 1, \dots, n\}$ . We will distinguish 3 exhaustive and mutually exclusive cases: Case 1:  $n_o = 0$ ; Case 2:  $1 \leq n_o < n$ ; Case 3:  $n_o = n$ .

Case 1:  $n_o = 0$ . Then,  $C = \mathbf{0}_{m \times n}$ . Since the pair  $(A, C)$  is detectable, then, the matrix

$$\begin{bmatrix} C \\ \lambda I_n - A \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m \times n} \\ \lambda I_n - A \end{bmatrix}$$

has rank  $n$ ,  $\forall \lambda \in \mathbb{C}$  with  $\text{Re}(\lambda) \geq 0$ . Thus, all eigenvalues of  $A$  has negative real parts. Hence,  $A$  is Hurwitz. This case is proved.

Case 2:  $1 \leq n_o < n$ . We will partition the system (A.10) into observable and unobservable parts. There exists a real invertible matrix  $T$  such that, in  $[ x'_o \ x'_{\bar{o}} ]' = T^{-1}x$  coordinate, the system (A.10) admits the state space representation:

$$(A.11a) \quad \dot{x}_o = A_o x_o + D_o w$$

$$(A.11b) \quad \dot{x}_{\bar{o}} = A_{\bar{o}o} x_o + A_{\bar{o}} x_{\bar{o}} + D_{\bar{o}} w$$

$$(A.11c) \quad z = C_o x_o$$

where  $x_o$  is  $n_o$ -dimensional;  $x_{\bar{o}}$  is  $n_{\bar{o}}$ -dimensional,  $n_{\bar{o}} = n - n_o \in \mathbb{N}$ ; and the pair  $(A_o, C_o)$  is observable. By the detectability of the pair  $(A, C)$ , the matrix  $A_{\bar{o}}$  is Hurwitz. Since the pair  $(A, D)$  is stabilizable, then, the matrix

$$\left[ \begin{array}{cc|c} \lambda I_{n_o} - A_o & \mathbf{0} & D_o \\ -A_{\bar{o}o} & \lambda I_{n_{\bar{o}}} - A_{\bar{o}} & D_{\bar{o}} \end{array} \right]$$

has rank  $n$ ,  $\forall \lambda \in \mathbb{C}$  with  $\text{Re}(\lambda) \geq 0$ . Therefore, the matrix  $[ \lambda I_{n_o} - A_o \ D_o ]$  has rank  $n_o$ ,  $\forall \lambda \in \mathbb{C}$  with  $\text{Re}(\lambda) \geq 0$ . Hence, the pair  $(A_o, D_o)$  is stabilizable. Let  $P_o = [ D_o \ A_o D_o \ \dots \ A_o^{n_o-1} D_o ]$  and  $\text{rank}(P_o) = n_{co} \in \{0, \dots, n_o\}$ . We will further distinguish three exhaustive and mutually exclusive cases: Case 2a:  $n_{co} = 0$ ; Case 2b:  $1 \leq n_{co} < n_o$ ; Case 2c:  $n_{co} = n_o$ .

Case 2a:  $n_{co} = 0$ . Then,  $D_o = \mathbf{0}_{n_o \times q}$ . Since the pair  $(A_o, D_o)$  is stabilizable, then, the matrix  $[ \lambda I_{n_o} - A_o \ \mathbf{0}_{n_o \times q} ]$  has rank  $n_o$ ,  $\forall \lambda \in \mathbb{C}$  with  $\text{Re}(\lambda) \geq 0$ . Hence, the matrix  $A_o$  is Hurwitz. Then, we have  $A = T \begin{bmatrix} A_o & \mathbf{0} \\ A_{\bar{o}o} & A_{\bar{o}} \end{bmatrix} T^{-1}$  is Hurwitz. This sub-case is proved.

Case 2b:  $1 \leq n_{co} < n_o$ . We will further partition the observable part (A.11a) into controllable and uncontrollable parts. There exists a real invertible matrix  $T_1$  such that, in  $[ x'_{co} \ x'_{\bar{c}o} ]' = T_1^{-1}x_o$  coordinates, the system (A.11a) and (A.11c) admits the state space representation

$$\dot{x}_{co} = A_{co} x_{co} + A_{c\bar{c}o} x_{\bar{c}o} + D_{co} w$$

$$\dot{x}_{\bar{c}o} = A_{\bar{c}o} x_{\bar{c}o}$$

$$z = C_{co} x_{co} + C_{\bar{c}o} x_{\bar{c}o}$$

where  $x_{co}$  is  $n_{co}$ -dimensional;  $x_{\bar{c}o}$  is  $n_{\bar{c}o}$ -dimensional,  $n_{\bar{c}o} = n_o - n_{co} \in \mathbb{N}$ ; and the pair  $(A_{co}, D_{co})$  is controllable. By the stabilizability of the pair  $(A_o, D_o)$ , the matrix  $A_{\bar{c}o}$  is Hurwitz. By the observability of the pair  $(A_o, C_o)$ , the matrix

$$\begin{bmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n_o-1} \end{bmatrix} T_1 = \begin{bmatrix} C_{co} & C_{\bar{c}o} \\ C_{co} A_{co} & \star_{m \times n_{\bar{c}o}} \\ \vdots & \vdots \\ C_{co} A_{co}^{n_o-1} & \star_{m \times n_{\bar{c}o}} \end{bmatrix}$$

has rank  $n_o$ . Therefore, the pair  $(A_{co}, C_{co})$  is observable. In  $[x'_{co} \ x'_{\bar{co}} \ x'_{\bar{o}}]'$  =  $\bar{T}^{-1}x$  coordinates, the system (A.10) admits the state space representation

$$\begin{aligned} \begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{\bar{co}} \\ \dot{x}_{\bar{o}} \end{bmatrix} &= \begin{bmatrix} A_{co} & A_{c\bar{co}} & \mathbf{0} \\ \mathbf{0} & A_{\bar{co}} & \mathbf{0} \\ A_{\bar{o}co} & A_{\bar{o}\bar{co}} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{\bar{co}} \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} D_{co} \\ \mathbf{0} \\ D_{\bar{o}} \end{bmatrix} w \\ z &= [C_{co} \ C_{\bar{co}} \ \mathbf{0}] \begin{bmatrix} x_{co} \\ x_{\bar{co}} \\ x_{\bar{o}} \end{bmatrix} \end{aligned}$$

Since (A.10) is bounded input and bounded output stable, then, by Lemma A.3,  $\exists k \in [0, \infty) \subset \mathbb{R}$  and  $\exists \lambda \in (0, \infty) \subset \mathbb{R}$  such that  $\|Ce^{At}D\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty)$ . Note that,  $\forall t \in [0, \infty)$ ,

$$\begin{aligned} \|Ce^{At}D\|_{2,2} &= \left\| [C_{co} \ C_{\bar{co}} \ \mathbf{0}] \begin{bmatrix} e^{A_{co}t} & \star & \mathbf{0} \\ \mathbf{0} & e^{A_{\bar{co}}t} & \mathbf{0} \\ \star & \star & e^{A_{\bar{o}}t} \end{bmatrix} \begin{bmatrix} D_{co} \\ \mathbf{0} \\ D_{\bar{o}} \end{bmatrix} \right\|_{2,2} \\ &= \|C_{co}e^{A_{co}t}D_{co}\|_{2,2} \leq ke^{-\lambda t} \end{aligned}$$

Hence, by Lemma A.3, the system

$$\begin{aligned} \dot{\bar{x}} &= A_{co}\bar{x} + D_{co}w; & \bar{x}(0) &= \mathbf{0}_{n_{co} \times 1} \\ y &= C_{co}\bar{x} \end{aligned}$$

is bounded input and bounded output stable. This coupled with the controllability and observability of the triple  $(A_{co}, D_{co}, C_{co})$ , by Lemma A.7, then the matrix  $A_{co}$  is Hurwitz. Hence, the matrix  $A = \bar{T} \begin{bmatrix} A_{co} & A_{c\bar{co}} & \mathbf{0} \\ \mathbf{0} & A_{\bar{co}} & \mathbf{0} \\ A_{\bar{o}co} & A_{\bar{o}\bar{co}} & A_{\bar{o}} \end{bmatrix} \bar{T}^{-1}$  is Hurwitz. This sub-case is proved.

Case 2c:  $n_{co} = n_o$ . Then, it means the pair  $(A_o, D_o)$  is controllable. Hence, the triple  $(A_o, D_o, C_o)$  is controllable and observable. Since the system (A.10) is bounded input and bounded output stable, then, by Lemma A.3,  $\exists k \in [0, \infty) \subset \mathbb{R}$  and  $\exists \lambda \in (0, \infty) \subset \mathbb{R}$  such that  $\|Ce^{At}D\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty)$ . Note that,  $\forall t \in [0, \infty)$ ,

$$\|Ce^{At}D\|_{2,2} = \left\| [C_o \ \mathbf{0}] \begin{bmatrix} e^{A_o t} & \mathbf{0} \\ \star & e^{A_{\bar{o}} t} \end{bmatrix} \begin{bmatrix} D_o \\ D_{\bar{o}} \end{bmatrix} \right\|_{2,2} = \|C_o e^{A_o t} D_o\|_{2,2} \leq ke^{-\lambda t}$$

Hence, by Lemma A.3, the system

$$\begin{aligned} \dot{\bar{x}} &= A_o\bar{x} + D_o w; & \bar{x}(0) &= \mathbf{0}_{n_o \times 1} \\ y &= C_o\bar{x} \end{aligned}$$

is bounded input and bounded output stable. This coupled with the controllability and observability of the triple  $(A_o, D_o, C_o)$ , by Lemma A.7, then the matrix  $A_o$  is Hurwitz. Hence, the matrix  $A = T \begin{bmatrix} A_o & \mathbf{0} \\ A_{\bar{o}o} & A_{\bar{o}} \end{bmatrix} T^{-1}$  is Hurwitz. This sub-case is proved.

This completes the proof for Case 2.

Case 3:  $n_o = n$ . Then, the pair  $(A, C)$  is observable. Let  $\text{rank}(P) = n_c$ , where  $P = \begin{bmatrix} D & AD & \cdots & A^{n-1}D \end{bmatrix}$ . We will further distinguish 3 exhaustive and mutually exclusive cases: Case 3a:  $n_c = 0$ ; Case 3b:  $1 \leq n_c < n$ ; Case 3c:  $n_c = n$ .

Case 3a:  $n_c = 0$ . Then,  $D = \mathbf{0}_{n \times q}$ . Since the pair  $(A, D)$  is stabilizable, then the matrix  $\begin{bmatrix} \lambda I_n - A & D \end{bmatrix}$  has rank  $n$ ,  $\forall \lambda \in \mathbb{C}$  with  $\text{Re}(\lambda) \geq 0$ . Hence the matrix  $A$  has all eigenvalues with negative real parts. Hence,  $A$  is Hurwitz. This sub-case is proved.

Case 3b:  $1 \leq n_c < n$ . We will partition (A.10) into controllable and uncontrollable parts. There exists a real invertible matrix  $T$  such that, in  $\begin{bmatrix} x'_c & x'_{\bar{c}} \end{bmatrix}' = T^{-1}x$  coordinates, the system (A.10) admits the state space representation:

$$\begin{aligned} \dot{x}_c &= A_c x_c + A_{c\bar{c}} x_{\bar{c}} + D_c w \\ \dot{x}_{\bar{c}} &= A_{\bar{c}} x_{\bar{c}} \\ z &= C_c x_c + C_{\bar{c}} x_{\bar{c}} \end{aligned}$$

where  $x_c$  is  $n_c$ -dimensional and  $x_{\bar{c}}$  is  $n_{\bar{c}}$ -dimensional,  $n_{\bar{c}} = n - n_c \in \mathbb{N}$ ; the pair  $(A_c, D_c)$  is controllable. By the stabilizability of the pair  $(A, D)$ , the matrix  $A_{\bar{c}}$  is Hurwitz. By the observability of the pair  $(A, C)$ , we have

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T = \begin{bmatrix} C_c & C_{\bar{c}} \\ C_c A_c & \star_{m \times n_{\bar{c}}} \\ \vdots & \vdots \\ C_c A_c^{n-1} & \star_{m \times n_{\bar{c}}} \end{bmatrix}$$

has rank  $n$ . Hence, the pair  $(A_c, C_c)$  is observable. Hence, the triple  $(A_c, D_c, C_c)$  is controllable and observable.

Since (A.10) is bounded input and bounded output stable, then, by Lemma A.3,  $\exists k \in [0, \infty) \subset \mathbb{R}$  and  $\exists \lambda \in (0, \infty) \subset \mathbb{R}$  such that  $\|Ce^{At}D\|_{2,2} \leq ke^{-\lambda t}$ ,  $\forall t \in [0, \infty)$ . Note that,  $\forall t \in [0, \infty)$ ,

$$\|Ce^{At}D\|_{2,2} = \left\| \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} e^{A_c t} & \star \\ \mathbf{0} & e^{A_{\bar{c}} t} \end{bmatrix} \begin{bmatrix} D_c \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} = \|C_c e^{A_c t} D_c\|_{2,2} \leq ke^{-\lambda t}$$

Hence, by Lemma A.3, the system

$$\begin{aligned} \dot{\bar{x}} &= A_{\bar{c}} \bar{x} + D_c w; & \bar{x}(0) &= \mathbf{0}_{n_{\bar{c}} \times 1} \\ y &= C_{\bar{c}} \bar{x} \end{aligned}$$

is bounded input and bounded output stable. This coupled with the controllability and observability of the triple  $(A_c, D_c, C_c)$ , by Lemma A.7, then the matrix  $A_c$  is Hurwitz. Hence, the matrix  $A = T \begin{bmatrix} A_c & A_{c\bar{c}} \\ \mathbf{0} & A_{\bar{c}} \end{bmatrix} T^{-1}$  is Hurwitz. This sub-case is proved.

Case 3c:  $n_c = n$ . Hence, the system (A.10) is controllable and observable. By Lemma A.7, the matrix  $A$  is Hurwitz. This sub-case is proved.

This completes the proof of the corollary.  $\square$

Next, we present a technical lemma and a technical corollary that are essential for the proof of Proposition 3.9.

LEMMA A.9. *Consider a finite-dimensional single-input LTI system:*

$$(A.12) \quad \dot{z} = Az + Bv; \quad z(0) = \mathbf{0}_{n \times 1}$$

where  $z$  is the  $n$ -dimensional state,  $n \in \mathbb{Z}_+$ ;  $v$  is the scalar control input; and  $A$  and  $B$  are constant matrices of appropriate dimensions. Assume that (A.12) is controllable but is not bounded input and bounded state stable. Let  $r \in \mathbb{Z}_+$ . Then,  $\exists v_{[0,\infty)} \in C_r$ , such that  $\|v_{[0,\infty)}^{[r]}\|_\infty \in (0, \infty) \subset \mathbb{R}$  and  $\|z_{[0,\infty)}\|_\infty = +\infty$ .

*Proof.* Without loss of generality, assume that  $A$  is given in the real Jordan canonical form (A.8). Partition  $B$  accordingly. Clearly, the matrix  $A$  is not Hurwitz. Then,  $A$  admits an eigenvalue  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) \geq 0$ . We will distinguish 2 exhaustive and mutually exclusive cases: Case 1:  $\lambda \in \mathbb{R}$ ; Case 2:  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

Case 1:  $\lambda \in \mathbb{R}$ . Without loss of generality, assume  $\lambda$  is the eigenvalue of the first Jordan block; that is  $\lambda = \lambda_1$ . Then, the element of  $z$ ,  $z_a$ , corresponding to the last row of the first Jordan block, satisfies the dynamics:  $\dot{z}_a = \lambda z_a + b_a v$ ,  $z_a(0) = 0$ , where  $b_a \neq 0$  since the pair  $(A, B)$  is controllable. Let  $v_{[0,\infty)}$  be given by a constant function with value 1. Then,  $\|v_{[0,\infty)}^{[r]}\|_\infty = 1$  and  $z_a(t) = \int_0^t e^{\lambda(t-\tau)} b_a d\tau = \begin{cases} \frac{b_a}{\lambda} e^{\lambda t} \Big|_{\tau=0}^t = \frac{b_a}{\lambda} (e^{\lambda t} - 1) & \lambda > 0 \\ b_a t & \lambda = 0 \end{cases}$ ,  $\forall t \in [0, \infty)$ . Hence,  $\|z_{[0,\infty)}\|_\infty \geq \|z_{a[0,\infty)}\|_\infty = \infty$ . This completes the proof for this case.

Case 2:  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Without loss of generality, assume  $\lambda = a + bi$  is the eigenvalue associated with the first complex Jordan block; that is  $\lambda = \mu_1$ . Note that  $a \geq 0$  and  $b \neq 0$ . Then, the element of  $z$ ,  $z_b$ , corresponding to the last row of the first complex Jordan block, satisfies the dynamics

$$\begin{aligned} \dot{z}_b &= a z_b + b z_c + b_b v; & z_b(0) &= 0 \\ \dot{z}_c &= -b z_b + a z_c + b_c v; & z_c(0) &= 0 \end{aligned}$$

where  $b_b$  and  $b_c$  are not both 0s, since the pair  $(A, B)$  is controllable. The state transition matrix for the above 2nd order system is

$$\Phi(t) = \begin{bmatrix} e^{at} \cos(bt) & e^{at} \sin(bt) \\ -e^{at} \sin(bt) & e^{at} \cos(bt) \end{bmatrix}$$

Then, we have

$$\begin{aligned} z_b(t) &= \int_0^t e^{a(t-\tau)} (\cos(b(t-\tau)) b_b + \sin(b(t-\tau)) b_c) v(\tau) d\tau \\ &= \sqrt{b_b^2 + b_c^2} \int_0^t e^{a(t-\tau)} \sin(b(t-\tau) + \phi) v(\tau) d\tau; \quad \forall t \in [0, \infty) \end{aligned}$$

where  $\sin(\phi) = \frac{b_b}{\sqrt{b_b^2 + b_c^2}}$  and  $\cos(\phi) = \frac{b_c}{\sqrt{b_b^2 + b_c^2}}$ . Choose  $v_{[0,\infty)}$  to be  $v(t) = \sin(-bt + \phi)$ ,  $\forall t \in [0, \infty)$ . Then,  $\|v_{[0,\infty)}^{[r]}\|_\infty \leq 1 + |b| + \dots + |b|^r$ . Furthermore, we have, when  $a = 0$ ,

$$z_b(2k\pi/|b|) = \sqrt{b_b^2 + b_c^2} \int_0^{2k\pi/|b|} (\sin(-b\tau + \phi))^2 d\tau = \sqrt{b_b^2 + b_c^2} k\pi/|b|; \quad k = 1, 2, \dots$$

when  $a > 0$ ,

$$z_b(2k\pi/|b|) = \sqrt{b_b^2 + b_c^2} e^{2k\pi a/|b|} \int_0^{2k\pi/|b|} e^{-a\tau} (\sin(-b\tau + \phi))^2 d\tau$$

$$\begin{aligned}
&= \sqrt{b_b^2 + b_c^2} e^{2k\pi a/|b|} \left( -\frac{1}{2a} e^{-a\tau} + \frac{e^{-a\tau}}{2(a^2 + 4b^2)} (a \cos(-2b\tau + 2\phi) \right. \\
&\quad \left. + 2b \sin(-2b\tau + 2\phi)) \right) \Big|_{\tau=0}^{2k\pi/|b|} \\
&= \sqrt{b_b^2 + b_c^2} (e^{2k\pi a/|b|} - 1) \left( \frac{1}{2a} - \frac{1}{2(a^2 + 4b^2)} (a \cos(2\phi) + 2b \sin(2\phi)) \right) \\
&= \sqrt{b_b^2 + b_c^2} (e^{2k\pi a/|b|} - 1) \left( \frac{1}{2a} - \frac{1}{2\sqrt{a^2 + 4b^2}} \sin(2\phi + \theta) \right); \quad k = 1, 2, \dots
\end{aligned}$$

where  $\sin(\theta) = \frac{a}{\sqrt{a^2 + 4b^2}}$  and  $\cos(\theta) = \frac{2b}{\sqrt{a^2 + 4b^2}}$ . Hence, we have  $\|z_{[0,\infty)}\|_\infty \geq \|z_{b[0,\infty)}\|_\infty = +\infty$ . This completes the proof for this case.

This completes the proof for the lemma.  $\square$

COROLLARY A.10. Consider a finite-dimensional continuous-time LTI system:

$$(A.13) \quad \dot{z} = Az + Bv; \quad z(0) = \mathbf{0}_{n \times 1}$$

where  $z$  is the  $n$ -dimensional state,  $n \in \mathbb{Z}_+$ ;  $v$  is the  $p$ -dimensional control input,  $p \in \mathbb{Z}_+$ ; and  $A$  and  $B$  are constant matrices of appropriate dimensions. Assume that (A.13) is not bounded input and bounded state stable. Let  $r \in \mathbb{Z}_+$ . Then,  $\exists v_{[0,\infty)} \in \mathcal{C}_r$ , such that  $\|v_{[0,\infty)}^{[r]}\|_\infty \in (0, \infty) \subset \mathbb{R}$  and  $\|z_{[0,\infty)}\|_\infty = +\infty$ .

*Proof.* Consider the special case  $p = 1$ . Let  $P = [B \ \cdots \ A^{n-1}B]$  and  $\text{rank}(P) = n_c$ . Clearly,  $0 < n_c \leq n$  since the system (A.13) is not bounded input and bounded state stable. When  $n_c = n$ , then, the system (A.13) is controllable. The result holds by Lemma A.9.

When  $0 < n_c < n$ . Without loss of generality, assume that the system (A.13) is partitioned into controllable and uncontrollable parts:

$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} A_c & A_{c\bar{c}} \\ \mathbf{0} & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_c \\ z_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ \mathbf{0} \end{bmatrix} v; \quad \begin{bmatrix} z_c(0) \\ z_{\bar{c}}(0) \end{bmatrix} = \mathbf{0}_{n \times 1}$$

where  $z_c$  is  $n_c$ -dimensional and the pair  $(A_c, B_c)$  is controllable. Clearly,  $z_{\bar{c}}(t) = \mathbf{0}_{(n-n_c) \times 1}$ ,  $\forall t \in [0, \infty)$ . Then, we have

$$(A.14) \quad \dot{z}_c = A_c z_c + B_c v; \quad z_c(0) = \mathbf{0}_{n_c \times 1}$$

By the fact that (A.13) is not bounded input and bounded state stable, then (A.14) is not bounded input and bounded state stable. Then, by Lemma A.9,  $\exists v_{[0,\infty)} \in \mathcal{C}_r$ , such that  $\|v_{[0,\infty)}^{[r]}\|_\infty \in (0, \infty) \subset \mathbb{R}$ , and  $\|z_{c[0,\infty)}\|_\infty = +\infty$ . Hence,  $\|z_{[0,\infty)}\|_\infty = +\infty$ . This completes the proof for the special case.

Consider the general case  $p \in \mathbb{Z}_+$ . Let column vectors of  $B$  be  $[B_1 \ \cdots \ B_p]$  and the elements of  $v$  be  $[v_1 \ \cdots \ v_p]$ . Since (A.13) is not bounded input and bounded state stable, by Lemma A.3, then,  $\exists i_0 \in \{1, \dots, p\}$ , such that the system  $\dot{\xi} = A\xi + B_{i_0}v_{i_0}$ ,  $\xi(0) = \mathbf{0}_{n \times 1}$ , is not bounded input and bounded state stable. Without loss of generality, assume  $i_0 = 1$ . By Corollary A.10,  $\exists v_{1[0,\infty)} \in \mathcal{C}_r$ , such that  $\|v_{1[0,\infty)}^{[r]}\|_\infty \in (0, \infty) \subset \mathbb{R}$  and  $\|\xi_{[0,\infty)}\|_\infty = +\infty$ . Set  $v_i$ ,  $i = 2, \dots, p$ , to constantly zero functions. Then, we have  $v_{[0,\infty)} \in \mathcal{C}_r$ , such that  $\|v_{[0,\infty)}^{[r]}\|_\infty = \|v_{1[0,\infty)}^{[r]}\|_\infty \in (0, \infty) \subset \mathbb{R}$ , and  $\|z_{[0,\infty)}\|_\infty = \|\xi_{[0,\infty)}\|_\infty = +\infty$ .

This completes the proof of this lemma.  $\square$

Finally, we present a technical lemma that is needed for the proof of Proposition 3.10.

LEMMA A.11. *Consider a finite-dimensional LTI system*

$$(A.15) \quad \dot{x} = Ax + Bu + Dw; \quad x(0) = x_0$$

where  $x$  is the  $n$ -dimensional state,  $n \in \mathbb{Z}_+$ ;  $u$  is the  $p$ -dimensional control input,  $p \in \mathbb{Z}_+$ ; and  $w$  is the  $q$ -dimensional disturbance input,  $q \in \mathbb{Z}_+$ ; and  $A$ ,  $B$ , and  $D$  are constant matrices of appropriate dimensions;  $x_0 \in \mathbb{R}^n$  is fixed; and  $w_{[0,\infty)} \in \mathcal{W}_d$  of class  $\mathcal{B}_q$ . Assume that  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$ ,  $\forall w_{[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{[0,\infty)}\|_\infty \leq c_w$ ,  $\forall u_{[0,\infty)} \in \mathcal{C}$  with  $\|u_{[0,\infty)}\|_\infty \leq c_w$ , then  $\|x_{[0,\infty)}\|_\infty \leq c_c$ . Then, we conclude that the following system

$$(A.16) \quad \dot{z} = Az + Bv; \quad z(0) = \mathbf{0}_{n \times 1}$$

is bounded input and bounded state stable.

*Proof.* We will first show that the system (A.16) satisfies  $\forall c_w \in [0, \infty) \subset \mathbb{R}$ ,  $\forall v_{[0,\infty)} \in \mathcal{C}$  with  $\|v_{[0,\infty)}\|_\infty \leq c_w$ ,  $\exists c_c \in [0, \infty) \subset \mathbb{R}$  such that  $\|z_{[0,\infty)}\|_\infty \leq c_c$ .

We will prove this by using an argument of contradiction. Suppose the result is not true. Then,  $\exists c_y \in (0, \infty) \subset \mathbb{R}$ ,  $\exists \bar{v}_{[0,\infty)} \in \mathcal{C}$  with  $\|\bar{v}_{[0,\infty)}\|_\infty \leq c_y$  such that  $\|\bar{z}_{[0,\infty)}\|_\infty = \infty$ , where  $\bar{z}_{[0,\infty)}$  is the solution to (A.16) with  $\bar{v}_{[0,\infty)}$  as the input.

Then,  $\exists c_w \in [c_y, \infty) \subset \mathbb{R}$ ,  $\exists w_{[0,\infty)} \in \mathcal{W}_d$  with  $\|w_{[0,\infty)}\|_\infty \leq c_w$ . Let  $\xi_{[0,\infty)}$  be the solution to the following system

$$\dot{\xi} = A\xi + Dw; \quad \xi(0) = x_0$$

Choose  $u_{[0,\infty)} = \rho \bar{v}_{[0,\infty)}$  with  $\rho \in (0, 1)$ . Then, the solution to (A.15) is given by  $x_{[0,\infty)} = \xi_{[0,\infty)} + \rho \bar{z}_{[0,\infty)}$ . Clearly,  $\|x_{[0,\infty)}\|_\infty = \infty$  for some  $\rho \in (0, 1)$ . This contradicts with the assumption of the lemma. Hence, the result holds.

Then, the system (A.16) is bounded input and bounded state stable, by the causality of the system and Lemma A.3.

This completes the proof of the lemma.  $\square$

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