GENERALIZED MINIMUM PHASE PROPERTY FOR FINITE-DIMENSIONAL CONTINUOUS-TIME SISO LTI SYSTEMS WITH ADDITIVE DISTURBANCES. PART I: DEFINITION AND SOME PROPERTIES

ZIGANG PAN * AND TAMER BAŞAR †

January 29, 2013

Abstract. For finite-dimensional continuous-time single-input and single-output linear time-invariant systems, we introduce the concept of extended zero dynamics and generalize the concept of minimum phase, which accommodates the presence of disturbance inputs. By representing a SISO LTI system with a finite relative degree in its extended zero dynamics canonical form, we obtain its extended zero dynamics, which is simply its zero dynamics driven by the noiseless output of the system and the disturbance input. We then say a system is minimum phase if its extended zero dynamics is absent or satisfies that, for any bounded admissible initial condition, any bounded noiseless output, and any bounded admissible disturbance input waveform, the zero dynamics state trajectory is bounded. The system is minimum phase (according to this extended notion) if its zero dynamics is asymptotically stable. It is proved that the converse holds under the additional condition that the system be stabilizable from the control input. For a system to be minimum phase, it is necessary that the transfer function from the control input to the output has all zeros with negative real parts. The converse holds when the system is both controllable (from the control input) and observable. It is further shown that the generalized minimum phase property is necessary for the achievement of perfect tracking of any bounded reference trajectories with bounded derivatives up to certain order without any disturbances and the existence of bounded state trajectory for any admissible bounded initial condition, any admissible bounded disturbance input waveform, and any bounded reference trajectory with bounded derivatives up to certain order in model reference control of the system.

Key words. continuous-time systems, extended zero dynamics canonical form, minimum phase, extended zero dynamics.

AMS subject classifications. 34A30, 34H05

1. Introduction.

The minimum phase property for finite-dimensional continuous-time single-input and single-output (SISO) linear time-invariant (LTI) systems is of paramount importance in the theory of model reference control. The classical definition of the minimum phase property is in terms of the transfer function of the system. The system is said to be minimum phase if the numerator of the transfer function has roots all in the open left half of the complex plane ([3]).

In [4], the concept of zero dynamics for an affine finite-dimensional time-invariant nonlinear system is introduced, which is the dynamics of the system when the output of the system is kept to be identically zero. For controllable and observable finite-dimensional continuous-time SISO LTI systems, it turns out that the zero dynamics is asymptotically stable if and only if the zeros of its transfer function are in the open left half of the complex plane. The zero dynamics, as defined in [4], is an autonomous system without any inputs. This concept has its limitations in stability analysis due to the nonlinear nature of the problem. Recently, the minimum phase property for nonlinear systems has been revisited in [5], which is based on the concept of weakly uniform 0-detectability. The idea in [5] is in line of the development of input-output-to-state stability and output-to-state stability ([7]). It demonstrates the importance

*Present address: 4797 Bordeaux Lane, Mason, OH 45040, USA. Tel: 513-398-7825; Fax: 513-398-7825; Email: zigangpan2002@mac.com.

†Coordinated Science Laboratory, University of Illinois, 1308 West Main Street, Urbana, IL 61801, USA. Tel: 217-333-3607; Fax: 217-244-1653; Email: basar1@illinois.edu.

of concluding the boundedness of the states given that the system output and its derivatives up to certain order are bounded.

In model reference control for systems subject to exogeneous disturbance inputs, the objective is to make the system output to asymptotically track a desired reference trajectory while keeping all closed-loop states bounded when the disturbance input vanishes, and attenuate the effect of the disturbance input on the tracking error while keeping all closed-loop states bounded when the disturbance input is present. Then, the concept of minimum phase can be generalized to include systems with uncontrollable parts and/or unobservable parts and with exogeneous disturbance inputs. This is the main objective of this paper. For a finite-dimensional continuous-time SISO LTI system, when the relative degree r is well-defined, then it may be represented in the zero dynamics canonical form ([4]). When the system is also affected by disturbance inputs, the state transformation obtained while ignoring the disturbances leads to the extended zero dynamics canonical form. We define the extended zero dynamics of the system to be the dynamics of its zero dynamics states, which is the zero dynamics driven by the noiseless output of the system as well as the disturbance input. We show that the extended zero dynamics has a certain invariance property. We say that the system is minimum phase if the zero dynamics states are bounded for any admissible bounded initial condition, any bounded noiseless output, and any admissible bounded disturbance input waveform, or if the extended zero dynamics is absent. It is straightforward that if the extended zero dynamics is asymptotically stable or if it is absent, then the system is minimum phase (according to the generalized definition). The converse of this result holds if the system is stabilizable from the control input. For a system to be minimum phase, it is necessary that the transfer function from the control input to the output has all zeros with negative real parts. The converse holds when the system is both controllable (from the control input) and observable. An example is included to illustrate the generalized minimum phase concept that demonstrates that the extended zero dynamics does not need to be asymptotically stable or bounded input and bounded state stable in the usual sense for the full order system to be minimum phase. We also prove that the generalized minimum phase property is necessary for the achievement of perfect tracking of any bounded reference trajectories with bounded derivatives up to rth order without any disturbances and the existence of bounded state trajectory for any admissible bounded initial condition, any admissible bounded disturbance waveform, and any bounded reference trajectory with bounded derivatives up to rth order in model reference control of the system.

The balance of the paper is organized as follows. In the next section, we introduce the notations to be used in the paper. In § 3, we introduce the extended zero dynamics canonical form for finite-dimensional continuous-time SISO LTI systems, the generalized definition of minimum phase, and its relationship with classical definitions. In § 4, we prove the necessity of the generalized minimum phase property in model reference control of SISO LTI systems. The paper ends with some concluding remarks in § 5 and an Appendix which includes a number of technical lemmas and corollaries that are essential for the derivation in the main body of the paper.

2. Notations. We let $\mathbb{R}$ denote the real line; let $\mathbb{R}_e$ denote the extended real line, $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ to be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; and $\mathbb{C}$ to be the set of complex numbers. Unless specified, all signals, constants, and matrices are real. For a function f, we say that it belongs to $\mathcal{C}$ (or $\mathcal{C}_0$) if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously differentiable (Fréchet differentiability), which is equivalent to that all partial derivatives up to kth order are continuous when $\mathrm{dom}\,(f)$ is open, $k \in \mathbb{N}$. We say that a function is $\mathcal{L}_2$ if it is square integrable; and that it is $\mathcal{L}_\infty$ if it is bounded. We will write $\mathcal{C}_k(A,B)$ and $\mathcal{L}_p(A,B)$ to denote set of functions of A to B which are k-times continuously differentiable and set of functions of A to B which have a finite $\mathcal{L}_p$ norm, respectively. For any matrix A, A' denotes its transpose. For any $m, n \in \mathbb{Z}_+$ and any $m \times n$-dimensional matrix M, $\mathcal{R}\,(M)$ denotes the range space of M and $\mathcal{N}\,(M)$ denotes the null space of M. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, $|z|$ denotes $\sqrt{z'z}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, and any $n \times n$-dimensional symmetric matrix M, $|z|_M^2 := z'Mz$. For $n \times n$-dimensional symmetric matrices M_1 and M_2, where $n \in \mathbb{Z}_+$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{Z}_+$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. For $n \in \mathbb{Z}_+$, I_n denotes the $n \times n$-dimensional identity matrix. For $n \in \mathbb{Z}_+$ and $n \times n$-dimensional matrix A, we set $A^0 = I_n$. For any matrix M, $\|M\|_{p,p}$ denotes its p-induced norm, $1 \leq p \leq \infty$. For any $m, n \in \mathbb{Z}_+$, $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are zeros. For any $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$, $e_{n,k}$ denotes the kth n-dimensional unit vector, i.e., $\begin{bmatrix} \mathbf{0}_{1\times(k-1)} & 1 & \mathbf{0}_{1\times(n-k)} \end{bmatrix}'$. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f), \mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$ and $\|u_{[0,t_f)}\|_q = \left(\int_0^{t_f} |u(t)|^q \,\mathrm{d}t\right)^{1/q}$, $q \in [1,\infty)$. For a sufficiently smooth signal v, $v^{(i)}$ denotes the ith order derivative of v, $v^{[i]}$ denotes $\begin{bmatrix} v' & (v^{(1)})' & \cdots & (v^{(i)})' \end{bmatrix}'$, $i \in \mathbb{Z}_+$. For a $\mathcal{C}_\infty$ vector field f and a $\mathcal{C}_\infty$ function h, $L_f h$ denotes the derivative of h along f, which equals to $\frac{\partial h}{\partial x}(x) f(x)$ in local coordinates. $L_f^{k+1} h = L_f(L_f^k h)$, $k \in \mathbb{N}$; $L_f^0 h = h$. $\forall \lambda \in \mathbb{C}$, $\mathrm{Re}\,(\lambda)$ and $\mathrm{Im}\,(\lambda)$ denote the real part and the imaginary part of λ, respectively. We will denote constants or matrices of no specific interest or relevance to the analysis by $\star$. We will denote $m \times n$-dimensional matrices of no specific interest or relevance to the analysis by $\star_{m\times n}$.

3. Definition of minimum phase property. In this section, we will introduce a generalized definition of the minimum phase property that accommodates the presence of disturbance inputs, and show its relationship with respect to its classical definition. First, we will introduce the extended zero dynamics canonical form for SISO LTI systems.

LEMMA 3.1. *Consider a finite-dimensional continuous-time SISO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{3.1a}$$

$$y = Cx + Ew \tag{3.1b}$$

where x is the n-dimensional state, $n \in \mathbb{N}$; u is the scalar control input; y is the scalar output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; A, B, D, C, and E are constant matrices of appropriate dimensions. Let the system have relative degree $r \in \mathbb{N}$, $r < n$, from u to y, that is, $CB = \cdots = CA^{r-2}B = 0$ and $CA^{r-1}B \neq 0$. Then, there exists an invertible matrix T_o such that, in $\begin{bmatrix} x_z' & x_1 & \cdots & x_r \end{bmatrix}' = T_o^{-1} x$ coordinates, the system (3.1) admits the state space representation

$$\dot{x}_z = A_z x_z + A_{z1} x_1 + D_z w \tag{3.2a}$$

$$\dot{x}_i = a_{i1} x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \tag{3.2b}$$

$$\dot{x}_r = A_{rz} x_z + a_{r1} x_1 + b_0 u + D_r w \tag{3.2c}$$

$$y = x_1 + Ew \tag{3.2d}$$

where x_z is $(n-r)$-dimensional; x_i, $i = 1, \ldots, r$, are scalars; $b_0 = CA^{r-1}B \neq 0$ is the high-frequency gain of the system. The representation (3.2) is called the extended zero dynamics canonical form *of system (3.1).*

Furthermore, given any $\mathcal{C}_\infty$ diffeomorphism $\bar{x} = \begin{bmatrix} \bar{x}_z' & \bar{x}_1 & \cdots & \bar{x}_r \end{bmatrix}' = T^{-1}(x)$, where $T : \mathcal{D} \to \mathbb{R}^n$ and $\mathcal{D} \subseteq \mathbb{R}^n$ is open, that leads to a state space representation

$$\dot{\bar{x}}_z = f_z(\bar{x}_z, \bar{x}_1) + h_z(\bar{x})w \tag{3.3a}$$

$$\dot{\bar{x}}_i = f_i(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_i) + \bar{x}_{i+1} + h_i(\bar{x})w; \qquad i = 1, \ldots, r-1 \tag{3.3b}$$

$$\dot{\bar{x}}_r = f_r(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_r) + g_o(\bar{x})u + h_r(\bar{x})w \tag{3.3c}$$

$$y = \bar{x}_1 + Ew \tag{3.3d}$$

where $\bar{x}_z$ is $(n-r)$-dimensional; $\bar{x}_i$, $i = 1, \ldots, r$, are scalars, then, we have $\mathcal{D} = \mathcal{D}_z \times \mathbb{R}^r$ with $\mathcal{D}_z \subseteq \mathbb{R}^{n-r}$ being open; $\bar{x}_1 = x_1$, $g_o(\bar{x}) = b_0$, $\forall \bar{x} \in \mathcal{D}$; $\bar{x}_z = T_z^{-1}(x_z)$, where $T_z : \mathcal{D}_z \to \mathbb{R}^{n-r}$ is a $\mathcal{C}_\infty$ diffeomorphism, whose inverse is T_{Iz}; $f_z(\bar{x}_z, \bar{x}_1) = \bar{f}_z(\bar{x}_z) + f_{z1}(\bar{x}_z)\bar{x}_1$, $\bar{f}_z(\bar{x}_z) = \left(\frac{\partial T_{Iz}}{\partial x_z}(x_z) A_z x_z \right)\Big|_{x_z = T_z(\bar{x}_z)}$, $f_{z1}(\bar{x}_z) = \left(\frac{\partial T_{Iz}}{\partial x_z}(x_z) A_{z1} \right)\Big|_{x_z = T_z(\bar{x}_z)}$, $h_z(\bar{x}) = \left(\frac{\partial T_{Iz}}{\partial x_z}(x_z) D_z \right)\Big|_{x_z = T_z(\bar{x}_z)} = h_z(\bar{x}_z)$, $h_1(\bar{x}) = h_1$, and $h_i(\bar{x}) = h_i(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{i-1})$, $i = 2, \ldots, r$, when $r \geq 2$, $\forall \bar{x} \in \mathcal{D}$.

Hence, we observe that the dynamics (3.2a) is invariant (modulo $\mathcal{C}_\infty$ diffeomorphisms) under $\mathcal{C}_\infty$ diffeomorphism that brings system (3.1) into form (3.3). Henceforth, we will call (3.2a) the extended zero dynamics *of (3.1).*

Proof. We will apply the machinery of [4], and adapt it to our present problem. Let $V = \begin{bmatrix} B & \cdots & A^{r-1}B \end{bmatrix}_{n \times r}$ and $U = \begin{bmatrix} C \\ \vdots \\ CA^{r-1} \end{bmatrix}_{r \times n}$. Then, we have

$$UV = \begin{bmatrix} 0 & \cdots & 0 & CA^{r-1}B \\ \vdots & \iddots & \iddots & \star \\ 0 & \iddots & \iddots & \vdots \\ CA^{r-1}B & \star & \cdots & \star \end{bmatrix}_{r \times r}$$

which is clearly invertible. Hence, U and V are of rank r. Note that $\operatorname{rank}(V) = r$ implies that $\dim(\mathcal{N}(V')) = n - r$. Hence, there exists a real nonsingular $n \times (n-r)$-dimensional matrix K such that $V'K = \mathbf{0}_{r \times (n-r)}$ and $\operatorname{rank}(K) = n - r$. Define $\bar{U} = \begin{bmatrix} K' \\ U \end{bmatrix}_{n \times n}$ and $\bar{V} = \begin{bmatrix} K & V \end{bmatrix}_{n \times n}$. Then, we have $\bar{U}\bar{V} = \begin{bmatrix} K'K & \mathbf{0}_{(n-r) \times r} \\ UK & UV \end{bmatrix}$. Since K is nonsingular, then $K'K$ is invertible. Hence, $\bar{U}\bar{V}$ is block lower triangular and is invertible. Then, we have $\bar{U}$ and $\bar{V}$ as invertible matrices.

Let $x_z = K'x$ and $z_i = CA^{i-1}x$, $i = 1, \ldots, r$. Consider the coordinate transformation $z := \begin{bmatrix} x_z' & z_1 & \cdots & z_r \end{bmatrix}' = \bar{U}x$. In z coordinates, system (3.1) admits the state space representation

$$\dot{z} = \bar{U}A\bar{U}^{-1}z + \bar{U}Bu + \bar{U}Dw =: \tilde{A}z + \tilde{B}u + \tilde{D}w$$
$$y = C\bar{U}^{-1}z + Ew =: \tilde{C}z + Ew$$

Note that

$$C = e'_{n,n-r+1}\bar{U} \quad \Rightarrow \quad \tilde{C} = e'_{n,n-r+1}; \quad \tilde{A} = \bar{U}A\bar{U}^{-1} =: \begin{bmatrix} \tilde{A}_z & \tilde{A}_{z1} & \cdots & \tilde{A}_{zr} \\ \tilde{A}_{1z} & \tilde{a}_{11} & \cdots & \tilde{a}_{1r} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{rz} & \tilde{a}_{r1} & \cdots & \tilde{a}_{rr} \end{bmatrix}$$

$$\tilde{B} = \bar{U}B = \begin{bmatrix} K' \\ U \end{bmatrix} B = \begin{bmatrix} \mathbf{0}_{(n-r)\times 1} \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} \\ b_0 \end{bmatrix}$$

where $\tilde{A}_z$ is $(n-r)\times(n-r)$-dimensional; $\tilde{a}_{ij}$, $i,j = 1,\ldots,r$, are scalar constants. Next, we calculate the matrix $\tilde{A}$, by the equality $\bar{U}A = \tilde{A}\bar{U}$. Thus, we have

$$\bar{U}A = \begin{bmatrix} K'A \\ CA \\ \vdots \\ CA^r \end{bmatrix} = \begin{bmatrix} \tilde{A}_z K' + \sum_{i=1}^r \tilde{A}_{zi} CA^{i-1} \\ \tilde{A}_{1z}K' + \sum_{i=1}^r \tilde{a}_{1i} CA^{i-1} \\ \vdots \\ \tilde{A}_{rz}K' + \sum_{i=1}^r \tilde{a}_{ri} CA^{i-1} \end{bmatrix} = \tilde{A}\bar{U}$$

Equating CA^j and $\tilde{A}_{jz}K' + \sum_{i=1}^r \tilde{a}_{ji}CA^{i-1}$, $j = 1,\ldots,r-1$, we have $\tilde{A}_{jz} = \mathbf{0}_{1\times(n-r)}$, $\tilde{a}_{j\,j+1} = 1$, and $\tilde{a}_{ji} = 0$, when $j = 1,\ldots,r-1$ and $i = 1,\ldots,r$, and $i \neq j+1$.

Equating $K'A$ and $\tilde{A}_z K' + \sum_{i=1}^r \tilde{A}_{zi}CA^{i-1}$, we have the following line of argument. Note that

$$\begin{aligned} K'AV &= K' \begin{bmatrix} AB & \cdots & A^r B \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-r)\times 1} & \cdots & \mathbf{0}_{(n-r)\times 1} & K'A^r B \end{bmatrix} \\ &= \frac{1}{b_0} K'A^r BCV \end{aligned}$$

Therefore, we have $(K'A - K'A^r BC/b_0)V = \mathbf{0}_{(n-r)\times r}$. Then, $V'\,(A'K - C'B'A'^r \cdot K/b_0) = \mathbf{0}_{r\times(n-r)}$. Denote the column vectors of K by K_i, $i = 1,\ldots,n-r$. Then, the column vectors of $A'K - C'B'A'^r K/b_0$, that is $A'K_i - C'B'A'^r K_i/b_0$, $i = 1,\ldots,n-r$, are in the null space of V', and therefore in the span of K. Hence, there exists $(n-r)\times(n-r)$-dimensional real matrix $\tilde{\tilde{A}}_z$ such that $A'K - C'B'A'^r K/b_0 = K\tilde{\tilde{A}}_z$, which implies $K'A = \tilde{\tilde{A}}_z' K' + K'A^r BC/b_0$. Then, we have $\tilde{A}_z = \tilde{\tilde{A}}_z'$, $\tilde{A}_{z1} = K'A^r B/b_0$, and $\tilde{A}_{zj} = \mathbf{0}_{(n-r)\times 1}$, $j = 2,\ldots,r$.

Hence, in z coordinates, system (3.1) may be represented by

$$\begin{aligned} \dot{x}_z &= \tilde{A}_z x_z + \tilde{A}_{z1} z_1 + \tilde{D}_z w \\ \dot{z}_i &= z_{i+1} + \tilde{D}_i w; \qquad i = 1,\ldots,r-1 \\ \dot{z}_r &= \tilde{A}_{rz} x_z + \sum_{i=1}^r \tilde{a}_{ri} z_i + b_0 u + \tilde{D}_r w \\ y &= z_1 + Ew \end{aligned}$$

Let $z_f = \begin{bmatrix} z_1 & \cdots & z_r \end{bmatrix}'$. Then, the dynamics for z_f is

$$\text{(3.4a)}\; \dot{z}_f = \left[\begin{array}{c|ccc} \mathbf{0}_{(r-1)\times 1} & & I_{r-1} & \\ \hline \tilde{a}_{r1} & \tilde{a}_{r2} & \cdots & \tilde{a}_{rr} \end{array}\right] z_f + \begin{bmatrix} \mathbf{0}_{(r-1)\times 1} \\ b_0 \end{bmatrix} u$$

$$+\begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \tilde{A}_{rz} \end{bmatrix} x_z + \begin{bmatrix} \tilde{D}_1 \\ \vdots \\ \tilde{D}_r \end{bmatrix} w =: A_f z_f + B_f u + A_{fz} x_z + D_f w$$

(3.4b) $y = \begin{bmatrix} 1 & \mathbf{0}_{1\times(r-1)} \end{bmatrix} z_f + Ew =: C_f z_f + Ew$

It is clear that, in the dynamics (3.4), the transfer function from u to y is $H_f(s) = C_f\,(sI_r - A_f)^{-1} B_f = \dfrac{b_0}{s^r - \tilde{a}_{rr} s^{r-1} - \cdots - \tilde{a}_{r1}}$. Clearly, the dynamics (3.4) is observable. Then, there exists an invertible coordinate transformation $x_f = T_f^{-1} z_f$ that transforms (3.4) into the observer canonical form.

$$\begin{aligned} \dot{x}_f = {} & \left[\begin{array}{c|c} \tilde{a}_{rr} & \\ \vdots & I_{r-1} \\ \tilde{a}_{r2} & \\ \hline \tilde{a}_{r1} & \mathbf{0}_{1\times(r-1)} \end{array}\right] x_f + \begin{bmatrix} \mathbf{0}_{(r-1)\times 1} \\ b_0 \end{bmatrix} u + T_f^{-1} \begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \tilde{A}_{rz} \end{bmatrix} x_z \\ & + T_f^{-1} D_f w \\ y = {} & \begin{bmatrix} 1 & \mathbf{0}_{1\times(r-1)} \end{bmatrix} x_f + Ew \end{aligned}$$

Note that $T_f^{-1} \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & b_0 \end{bmatrix}' = \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & b_0 \end{bmatrix}'$ implies that the last column of T_f^{-1} is equal to $e_{r,r}$. Then, we have

$$T_f^{-1} \begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \tilde{A}_{rz} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \tilde{A}_{rz} \end{bmatrix}$$

Let the elements of x_f be $\begin{bmatrix} x_1 & \cdots & x_r \end{bmatrix}'$. Clearly, $y = z_1 + Ew = x_1 + Ew$, then, we have $z_1 = x_1$. Then, the system (3.1) admits the following state space representation, in $x := \begin{bmatrix} x_z' & x_1 & \cdots & x_r \end{bmatrix}'$ coordinates,

$$\begin{aligned} \dot{x}_z &= \tilde{A}_z x_z + \tilde{A}_{z1} x_1 + \tilde{D}_z w \\ \dot{x}_i &= \tilde{a}_{r\,r+1-i} x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \\ \dot{x}_r &= \tilde{A}_{rz} x_z + \tilde{a}_{r1} x_1 + b_0 u + D_r w \\ y &= x_1 + Ew \end{aligned}$$

where $\begin{bmatrix} D_1' & \cdots & D_r' \end{bmatrix}' = T_f^{-1} D_f$. Clearly, the above is in the form of (3.2). Hence, the desired matrix $T_o = \bar{U}^{-1} \begin{bmatrix} I_{n-r} & \mathbf{0} \\ \mathbf{0} & T_f \end{bmatrix}$.

Let $T : \mathcal{D} \to \mathbb{R}^n$ be any $\mathcal{C}_\infty$ diffeomorphism such that, in $\bar{x} = T^{-1}(x) = \begin{bmatrix} \bar{x}_z' & \bar{x}_1 & \cdots & \bar{x}_r \end{bmatrix}'$ coordinates, the system (3.1) admits the state space representation (3.3). Without loss of generality, assume that system (3.1) is given in the form of (3.2). Then,

$$A = \begin{bmatrix} A_z & A_{z1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & a_{11} & 1 & 0 & \cdots & 0 \\ \mathbf{0} & a_{21} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \mathbf{0} & a_{r-1\,1} & 0 & \cdots & 0 & 1 \\ A_{rz} & a_{r1} & 0 & \cdots & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} \\ b_0 \end{bmatrix}; \; D = \begin{bmatrix} D_z \\ D_1 \\ \vdots \\ D_r \end{bmatrix}$$

$$C = e_{n,n-r+1}'$$

$$\dot{\bar{x}} = f(\bar{x}) + g(\bar{x})u + h(\bar{x})w$$
$$y = \bar{x}_1 + Ew = C\bar{x} + Ew$$

$$f(\bar{x}) = \begin{bmatrix} f_z(\bar{x}_z, \bar{x}_1) \\ f_1(\bar{x}_z, \bar{x}_1) + \bar{x}_2 \\ \vdots \\ f_{r-1}(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{r-1}) + \bar{x}_r \\ f_r(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_r) \end{bmatrix}; \; g(\bar{x}) = \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} \\ g_o(\bar{x}) \end{bmatrix}; \; h(\bar{x}) = \begin{bmatrix} h_z(\bar{x}) \\ h_1(\bar{x}) \\ \vdots \\ h_r(\bar{x}) \end{bmatrix}$$

$Ax = \left(\frac{\partial T}{\partial \bar{x}}(\bar{x})f(\bar{x})\right)\big|_{\bar{x}=T^{-1}(x)}$; $B = \left(\frac{\partial T}{\partial \bar{x}}(\bar{x})g(\bar{x})\right)\big|_{\bar{x}=T^{-1}(x)}$; $D = \left(\frac{\partial T}{\partial \bar{x}}(\bar{x}) \cdot h(\bar{x})\right)\big|_{\bar{x}=T^{-1}(x)}$; $Cx = (C\bar{x})\big|_{\bar{x}=T^{-1}(x)}$, $\forall x \in \mathbb{R}^n$. Clearly, f, g, and h are $\mathcal{C}_\infty$ mappings of $\mathcal{D}$.

Let $T^{-1}(x) := T_I(x) = \begin{bmatrix} (T_{Iz}(x))' & T_{I1}(x) & \cdots & T_{Ir}(x) \end{bmatrix}'$ such that $\bar{x}_z = T_{Iz}(x) \in \mathbb{R}^{n-r}$, $\bar{x}_i = T_{Ii}(x) \in \mathbb{R}$, $i = 1, \ldots, r$. Then, $x_1 = Cx = (C\bar{x})\big|_{\bar{x}=T^{-1}(x)}$ implies that $x_1 = \bar{x}_1\big|_{\bar{x}=T_I(x)} = T_{I1}(x)$, $\forall x \in \mathbb{R}^n$.

Claim 3.1.1. $\frac{\partial T_{Iz}}{\partial x_i}(x) = \mathbf{0}_{(n-r)\times 1}$, $\forall x \in \mathbb{R}^n$, $i = 1, \ldots, r$.

Proof. Note that $\bar{x}_z = T_{Iz}(x)$ implies

$$\dot{\bar{x}}_z = f_z(\bar{x}_z, \bar{x}_1) + h_z(\bar{x})w = \left(\frac{\partial T_{Iz}}{\partial x}(x)\left(Ax + Bu + Dw\right)\right)\Big|_{x=T(\bar{x})}$$

Hence, we have, $\forall x \in \mathbb{R}^n$,

$$f_z(T_{Iz}(x), x_1) = \frac{\partial T_{Iz}}{\partial x}(x)Ax \tag{3.5a}$$

$$\mathbf{0}_{(n-r)\times 1} = \frac{\partial T_{Iz}}{\partial x}(x)B \tag{3.5b}$$

$$h_z(T_I(x)) = \frac{\partial T_{Iz}}{\partial x}(x)D \tag{3.5c}$$

By (3.5b), we have $\frac{\partial T_{Iz}}{\partial x_r}(x)b_0 = \mathbf{0}_{(n-r)\times 1}$, $\forall x \in \mathbb{R}^n$, which implies that $\frac{\partial T_{Iz}}{\partial x_r}(x) = \mathbf{0}_{(n-r)\times 1}$, $\forall x \in \mathbb{R}^n$.

We will prove the claim using mathematical induction on $r + 1 - i$.

1° $r + 1 - i = 1 \Rightarrow i = r$. We have shown that $\frac{\partial T_{Iz}}{\partial x_r}(x) = \mathbf{0}_{(n-r)\times 1}$, $\forall x \in \mathbb{R}^n$. This completes the initialization step of the induction process. If $r = 1$, then the claim is proved. If $r > 1$, then we continue to the next step.

2° Assume the result holds for $r + 1 - i \in \{1, \ldots, k\}$, with $1 \le k < r$.

3° Consider the case $r + 1 - i = k + 1$. Then, $i = r - k$. By (3.5a), we have

$$f_z(T_{Iz}(x_z, x_1, \ldots, x_r), x_1) = \frac{\partial T_{Iz}}{\partial x}(x_z, x_1, \ldots, x_r)Ax, \qquad \forall x \in \mathbb{R}^n$$

Then, by the inductive assumption, $\forall x \in \mathbb{R}^n$,

$$\begin{aligned} f_z(T_{Iz}(x_z, x_1, \ldots, x_r), x_1) = &\frac{\partial T_{Iz}}{\partial x_z}(x_z, x_1, \ldots, x_r)\left(A_z x_z + A_{z1}x_1\right) \\ &+ \sum_{j=1}^{r-k} \frac{\partial T_{Iz}}{\partial x_j}(x_z, x_1, \ldots, x_r)\left(a_{j1}x_1 + x_{j+1}\right) \end{aligned}$$

Taking partial derivatives with respect to x_{r-k+1} on both sides of the above equation, noting that $2 \le r-k+1 \le r$, we have, $\forall x \in \mathbb{R}^n$,

$$\begin{aligned}
&\frac{\partial f_z}{\partial \bar{x}_z}(T_{Iz}(x_z, x_1, \ldots, x_r), x_1) \frac{\partial T_{Iz}}{\partial x_{r-k+1}}(x_z, x_1, \ldots, x_r) \\
&= \frac{\partial^2 T_{Iz}}{\partial x_{r-k+1} \partial x_z}(x_z, x_1, \ldots, x_r)\,(A_z x_z + A_{z1} x_1) \\
&\quad + \sum_{j=1}^{r-k} \frac{\partial^2 T_{Iz}}{\partial x_{r-k+1} \partial x_j}(x_z, x_1, \ldots, x_r)\,(a_{j1} x_1 + x_{j+1}) + \frac{\partial T_{Iz}}{\partial x_{r-k}}(x_z, x_1, \ldots, x_r)
\end{aligned}$$

By the inductive assumption, then, $\frac{\partial T_{Iz}}{\partial x_{r-k}}(x) = \mathbf{0}_{(n-r)\times 1}$, $\forall x \in \mathbb{R}^n$.

This completes the induction process.

This completes the proof of the claim. $\square$

Therefore, we have shown that $\frac{\partial T_{Iz}}{\partial x_i}(x) = \mathbf{0}_{(n-r)\times 1}$, $\forall x \in \mathbb{R}^n$, $i = 1, \ldots, r$. Then, by Mean Value Theorem, $T_{Iz}(x) = T_{Iz}(x_z)$, $\forall x \in \mathbb{R}^n$.

CLAIM 3.1.2. *Let $\mathcal{D}_i$ denotes the projection of $\mathcal{D}$ onto the $\begin{bmatrix} \bar{x}_z' & \bar{x}_1 & \cdots & \bar{x}_i \end{bmatrix}'$ subspace, $i = 0, \ldots, r$, which is an open subset of $\mathbb{R}^{n-r+i}$ since $\mathcal{D}$ is open in $\mathbb{R}^n$. We will denote $\mathcal{D}_z := \mathcal{D}_0$. By Claim 3.1.1, we have $T_{Iz} : \mathbb{R}^{n-r} \to \mathcal{D}_z$ and $T_{Ii} : \mathbb{R}^n \to \mathbb{R}$ are $\mathcal{C}_\infty$, and, $\forall x \in \mathbb{R}^n$, $\begin{bmatrix} (T_{Iz}(x))' & T_{I1}(x) & \cdots & T_{Ii}(x) \end{bmatrix}' \in \mathcal{D}_i$, $\forall i = 1, \ldots, r$. Furthermore, $\forall i \in \{1, \ldots, r\}$, we have*

$$\begin{aligned}
L_{(Ax)}^{i-1}(Cx) &= CA^{i-1}x, \ \forall x \in \mathbb{R}^n; \quad CA^{i-1} = \begin{bmatrix} \mathbf{0}_{1\times(n-r)} & \star_{1\times(i-1)} & 1 & \mathbf{0}_{1\times(r-i)} \end{bmatrix} \\
L_{f(\bar{x})}^{i-1}(C\bar{x}) &= \bar{x}_i + \bar{l}_i(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{i-1}), \quad \forall \bar{x} \in \mathcal{D} \\
T_{Ii}(x) &= x_i + \bar{T}_{Ii}(x_z, x_1, \ldots, x_{i-1}), \quad \forall x \in \mathbb{R}^n
\end{aligned}$$

where $\bar{l}_i : \mathcal{D}_{i-1} \to \mathbb{R}$ is $\mathcal{C}_\infty$, $\bar{T}_{I1} : \mathbb{R}^{n-r} \to \mathbb{R}$ is the identically zero function, and $\bar{T}_{Ii} : \mathbb{R}^{n-r+i-1} \to \mathbb{R}$ is $\mathcal{C}_\infty$, if $i \ge 2$.

Proof. The statements preceding the word "furthermore" are self evident. We will prove the rest of the claim using mathematical induction on i.

1° $i = 1$. $L_{(Ax)}^0(Cx) = Cx$ and $CA^0 = C = \begin{bmatrix} \mathbf{0}_{1\times(n-r)} & 1 & \mathbf{0}_{1\times(r-1)} \end{bmatrix}$. $L_{f(\bar{x})}^0(C\bar{x}) = C\bar{x} = \bar{x}_1$, $\forall \bar{x} \in \mathcal{D}$. We have shown that $T_{I1}(x) = x_1$, $\forall x \in \mathbb{R}^n$. This completes the initialization step. If $r = 1$, then the claim is proved. If $r > 1$, we continue to the next step.

2° Assume that the claim holds for $i \in \{1, \ldots, k\}$, with $1 \le k < r$.

3° Consider the case $i = k+1$. We have

$$\begin{aligned}
L_{(Ax)}^k(Cx) &= L_{(Ax)} L_{(Ax)}^{k-1}(Cx) = L_{(Ax)}(CA^{k-1}x) = CA^k x \\
CA^k &= (CA^{k-1})A = \begin{bmatrix} \mathbf{0}_{1\times(n-r)} & \star_{1\times k} & 1 & \mathbf{0}_{1\times(r-k-1)} \end{bmatrix} \\
L_{f(\bar{x})}^k(C\bar{x}) &= L_{f(\bar{x})} L_{f(\bar{x})}^{k-1}(C\bar{x}) = L_{f(\bar{x})}(\bar{x}_k + \bar{l}_k(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{k-1})) \\
&= \bar{x}_{k+1} + f_k(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_k) + \frac{\partial \bar{l}_k}{\partial \bar{x}_z}(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{k-1}) f_z(\bar{x}_z, \bar{x}_1) \\
&\quad + \sum_{j=1}^{k-1} \frac{\partial \bar{l}_k}{\partial \bar{x}_j}(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{k-1})\,(f_j(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_j) + \bar{x}_{j+1}) \\
&=: \bar{x}_{k+1} + \bar{l}_{k+1}(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_k)
\end{aligned}$$

Clearly, $\bar{l}_{k+1} : \mathcal{D}_k \to \mathbb{R}$ is $\mathcal{C}_\infty$. Note that $C\bar{x}$ and $f(\bar{x})$ are the representations of Cx and Ax in $\bar{x}$ coordinates, respectively. Then, we have

$$L^k_{(Ax)}(Cx) = \left(L^k_{f(\bar{x})}(C\bar{x}) \right)\Big|_{\bar{x}=T_I(x)}, \qquad \forall x \in \mathbb{R}^n$$

which implies that

$$\begin{aligned} & CA^k x = T_{I\,k+1}(x) + \bar{l}_{k+1}(T_{Iz}(x_z), T_{I1}(x), \ldots, T_{Ik}(x)) \\ \Rightarrow\; & \star x_1 + \cdots + \star x_k + x_{k+1} = T_{I\,k+1}(x) + \bar{l}_{k+1}(T_{Iz}(x_z), T_{I1}(x), \ldots, T_{Ik}(x)) \\ \Rightarrow\; & T_{I\,k+1}(x) = x_{k+1} + \bar{T}_{I\,k+1}(x_z, x_1, \ldots, x_k) \end{aligned}$$

where the last implication follows from the inductive assumption. Clearly, $\bar{T}_{I\,k+1} : \mathbb{R}^{n-r+k} \to \mathbb{R}$ is $\mathcal{C}_\infty$.

This completes the induction process.

Hence, the claim is proved. $\square$

Note that $L_B L^{r-1}_{(Ax)}(Cx) = CA^{r-1}B = b_0$, $\forall x \in \mathbb{R}^n$, and $L_{g(\bar{x})} L^{r-1}_{f(\bar{x})}(C\bar{x}) = L_{g(\bar{x})}(\bar{x}_r + \bar{l}_r(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{r-1})) = g_o(\bar{x})$, $\forall \bar{x} \in \mathcal{D}$. Note that $C\bar{x}$, $f(\bar{x})$, $g(\bar{x})$ are the representations of Cx, Ax, and B in $\bar{x}$ coordinates, respectively. Then, we have $b_0 = g_o(\bar{x})|_{\bar{x}=T^{-1}(x)} \Rightarrow g_o(\bar{x}) = b_0$, $\forall \bar{x} \in \mathcal{D}$.

Now, we are in a position to show that $\mathcal{D} = \mathcal{D}_z \times \mathbb{R}^r$. Note that, $\forall x \in \mathbb{R}^n$, by Claims 3.1.1 and 3.1.2,

$$T^{-1}(x) = T_I(x) = \begin{bmatrix} T_{Iz}(x_z) \\ x_1 \\ x_2 + \bar{T}_{I2}(x_z, x_1) \\ \vdots \\ x_r + \bar{T}_{Ir}(x_z, x_1, \ldots, x_{r-1}) \end{bmatrix} \in \mathcal{D}$$

Since $T : \mathcal{D} \to \mathbb{R}^n$ is a $\mathcal{C}_\infty$ diffeomorphism, then T is bijective and $\frac{\partial T_I}{\partial x}(x)$ is an invertible matrix, $\forall x \in \mathbb{R}^n$. Then, T_{Iz} must be surjective since T_I is surjective; $\frac{\partial T_{Iz}}{\partial x_z}(x_z)$ must be an invertible matrix, $\forall x_z \in \mathbb{R}^{n-r}$, since $\frac{\partial T_I}{\partial x}(x)$ has a block lower triangular structure. Furthermore, T_{Iz} must be injective since T_I is injective and $\mathrm{dom}\,(T_I) = \mathbb{R}^n$. Hence, T_{Iz} is $\mathcal{C}_\infty$, bijective, and $\frac{\partial T_{Iz}}{\partial x_z}(x_z)$ is an invertible matrix, $\forall x_z \in \mathbb{R}^{n-r}$. This implies that $T_{Iz} : \mathbb{R}^{n-r} \to \mathcal{D}_z$ is a $\mathcal{C}_\infty$ diffeomorphism by the Inverse Function Theorem [1], and $T_{Iz}^{-1} := T_z : \mathcal{D}_z \to \mathbb{R}^{n-r}$. $\forall \tilde{x} \in \mathcal{D}_z \times \mathbb{R}^r$, by the block lower triangular structure of T_I, $\exists x \in \mathbb{R}^n$ such that $\tilde{x} = T_I(x) \in \mathcal{D}$. Then, $\mathcal{D}_z \times \mathbb{R}^r \subseteq \mathcal{D} = T_I(\mathbb{R}^n)$. By $\mathcal{D}_z$ being the projection of $\mathcal{D}$ to $\bar{x}_z$ subspace, we have $\mathcal{D} \subseteq \mathcal{D}_z \times \mathbb{R}^r$. Hence, $\mathcal{D} = \mathcal{D}_z \times \mathbb{R}^r$.

It is a simple matter of algebra to realize that, $\forall \bar{x} \in \mathcal{D}$, we have

$$T(\bar{x}) = \begin{bmatrix} T_z(\bar{x}_z) \\ \bar{x}_1 \\ \bar{x}_2 + \bar{T}_2(\bar{x}_z, \bar{x}_1) \\ \vdots \\ \bar{x}_r + \bar{T}_r(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{r-1}) \end{bmatrix} \in \mathbb{R}^n$$

where $\bar{T}_i : \mathcal{D}_i \to \mathbb{R}$ are $\mathcal{C}_\infty$, $i = 2, \ldots, r$.

Then, $\forall \bar{x} \in \mathcal{D}$,

$$\dot{\bar{x}}_z = \left(\frac{\partial T_{Iz}}{\partial x_z}(x_z)\,(A_z x_z + A_{z1} x_1 + D_z w) \right)\Big|_{x_z = T_z(\bar{x}_z), x_1 = \bar{x}_1} = f_z(\bar{x}_z, \bar{x}_1) + h_z(\bar{x}) w$$

Therefore $f_z(\bar{x}_z, \bar{x}_1) = \bar{f}_z(\bar{x}_z) + f_{z1}(\bar{x}_z)\bar{x}_1$ with $\bar{f}_z(\bar{x}_z) = \left(\frac{\partial T_{Iz}}{\partial x_z}(x_z)A_z x_z\right)\Big|_{x_z=T_z(\bar{x}_z)}$ and $f_{z1}(\bar{x}_z) = \left(\frac{\partial T_{Iz}}{\partial x_z}(x_z)A_{z1}\right)\Big|_{x_z=T_z(\bar{x}_z)}$, $h_z(\bar{x}) = \left(\frac{\partial T_{Iz}}{\partial x_z}(x_z)D_z\right)\Big|_{x_z=T_z(\bar{x}_z)} = h_z(\bar{x}_z)$, $\forall \bar{x} \in \mathcal{D}$.

To obtain the formula for h_i's, we will distinguish between 2 exhaustive and mutually exclusive cases: Case 1: $r = 1$; Case 2: $r \geq 2$.

Case 1: $r = 1$. $\dot{x}_1 = A_{1z}x_z + a_{11}x_1 + b_0 u + D_1 w = f_1(\bar{x}_z, \bar{x}_1) + g_o(\bar{x})u + h_1(\bar{x})w$. Hence, we have $h_1(\bar{x}) = D_1$, $\forall \bar{x} \in \mathcal{D}$.

Case 2: $r \geq 2$. $\dot{x}_1 = a_{11}x_1 + x_2 + D_1 w = f_1(\bar{x}_z, \bar{x}_1) + \bar{x}_2 + h_1(\bar{x})w$. Hence, we have $h_1(\bar{x}) = D_1$, $\forall \bar{x} \in \mathcal{D}$. For $i = 2, \ldots, r-1$, by Claim 3.1.2, we have

$$\begin{aligned}
\dot{\bar{x}}_i &= f_i(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_i) + \bar{x}_{i+1} + h_i(\bar{x})w \\
&= a_{i1}x_1 + x_{i+1} + D_i w + \frac{\partial \bar{T}_{Ii}}{\partial x_z}(x_z, x_1, \ldots, x_{i-1})\left(A_z x_z + A_{z1}x_1 + D_z w\right) \\
&\quad + \sum_{j=1}^{i-1} \frac{\partial \bar{T}_{Ii}}{\partial x_j}(x_z, x_1, \ldots, x_{i-1})\left(a_{j1}x_1 + x_{j+1} + D_j w\right)
\end{aligned}$$

Hence, $\forall \bar{x} \in \mathcal{D}$,

$$\begin{aligned}
h_i(\bar{x}) &= \left(D_i + \frac{\partial \bar{T}_{Ii}}{\partial x_z}(x_z, x_1, \ldots, x_{i-1})D_z + \sum_{j=1}^{i-1} \frac{\partial \bar{T}_{Ii}}{\partial x_j}(x_z, x_1, \ldots, x_{i-1})D_j\right)\Big|_{x=T(\bar{x})} \\
&=: h_i(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{i-1}); \qquad i = 2, \ldots, r-1
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\dot{\bar{x}}_r &= f_r(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_r) + g_o(\bar{x})u + h_r(\bar{x})w = A_{rz}x_z + a_{r1}x_1 + b_0 u + D_r w \\
&\quad + \frac{\partial \bar{T}_{Ir}}{\partial x_z}(x_z, x_1, \ldots, x_{r-1})\left(A_z x_z + A_{z1}x_1 + D_z w\right) + \sum_{j=1}^{r-1} \frac{\partial \bar{T}_{Ir}}{\partial x_j}(x_z, x_1, \ldots, x_{r-1}) \\
&\quad \cdot\left(a_{j1}x_1 + x_{j+1} + D_j w\right)
\end{aligned}$$

Hence, $\forall \bar{x} \in \mathcal{D}$,

$$\begin{aligned}
h_r(\bar{x}) &= \left(D_r + \frac{\partial \bar{T}_{Ir}}{\partial x_z}(x_z, x_1, \ldots, x_{r-1})D_z + \sum_{j=1}^{r-1} \frac{\partial \bar{T}_{Ir}}{\partial x_j}(x_z, x_1, \ldots, x_{r-1})D_j\right)\Big|_{x=T(\bar{x})} \\
&=: h_r(\bar{x}_z, \bar{x}_1, \ldots, \bar{x}_{r-1})
\end{aligned}$$

This completes the proof of the lemma. □

Remark 3.1. *We observe in the previous lemma that the zero dynamics of the system (3.1) according to [4] is exactly $\dot{x}_z = A_z x_z$. The extended zero dynamics is simply the zero dynamics together with driving terms which include the noiseless output of the system and the disturbance input.*

Lemma 3.2. *Consider a finite-dimensional continuous-time SISO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{3.6a}$$

$$y = Cx + Ew \tag{3.6b}$$

where x is the n-dimensional state, $n \in \mathbb{N}$; u is the scalar control input; y is the scalar output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; A, B, D, C, and E

are constant matrices of appropriate dimensions. Let the system have relative degree $r = n$, from u to y, that is, $CB = \cdots = CA^{n-2}B = 0$ and $CA^{n-1}B \neq 0$. Then, there exists an invertible matrix T_o such that, in $\begin{bmatrix} x_1 & \cdots & x_r \end{bmatrix}' = T_o^{-1}x$ coordinates, the system (3.6) admits the state space representation

$$\dot{x}_i = a_{i1}x_1 + x_{i+1} + D_iw; \qquad i = 1, \dots, r-1 \tag{3.7a}$$
$$\dot{x}_r = a_{r1}x_1 + b_0u + D_rw \tag{3.7b}$$
$$y = x_1 + Ew \tag{3.7c}$$

where x_i, $i = 1, \dots, r$, are scalars; $b_0 = CA^{r-1}B \neq 0$ is the high-frequency gain of the system. The representation (3.7) is called the extended zero dynamics canonical form *of system (3.6) (which is also the observer canonical form). The* extended zero dynamics *for the system is clearly absent.*

Proof. Define $V = \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}_{n\times n}$ and $U = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n\times n}$. Then, we have

$$UV = \begin{bmatrix} 0 & \cdots & 0 & CA^{r-1}B \\ \vdots & \iddots & \iddots & \star \\ 0 & \iddots & \iddots & \vdots \\ CA^{r-1}B & \star & \cdots & \star \end{bmatrix}_{r\times r}$$

which is clearly invertible. Hence, U and V are invertible. Then, the system (3.6) is observable, and is controllable from u. Then, there exists a real invertible transformation $\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}' = T_o^{-1}x$ that transforms the system into observer canonical form.

$$\begin{aligned} \dot{x}_i &= a_{i1}x_1 + x_{i+1} + b_iu + D_iw; \qquad i = 1, \dots, r-1 \\ \dot{x}_r &= a_{r1}x_1 + b_ru + D_rw \\ y &= x_1 + Ew \end{aligned}$$

Because the relative degree from u to y is $r = n$, then, we have $b_1 = \cdots = b_{n-1} = 0$. It is straightforward to obtain that $b_n = CA^{n-1}B = b_0 \neq 0$. This completes the proof of the lemma. □

LEMMA 3.3. *Consider a finite-dimensional continuous-time SISO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{3.8a}$$
$$y = Cx + b_0u + Ew \tag{3.8b}$$

where x is the n-dimensional state, $n \in \mathbb{N}$; u is the scalar control input; y is the scalar output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; and A, B, D, C, b_0, and E are constant matrices of appropriate dimensions. Let $b_0 \neq 0$, i. e., that the system admits relative degree 0 from u to y, and b_0 is the high-frequency gain of the system. Then, the system (3.8) admits the following representation:

$$\dot{x} = (A - \frac{1}{b_0}BC)x + \frac{1}{b_0}B(y - Ew) + Dw =: \hat{A}x + \hat{B}(y - Ew) + Dw \tag{3.9a}$$
$$y = Cx + b_0u + Ew \tag{3.9b}$$

The representation (3.9) is called the zero dynamics canonical form *of (3.8).*

Furthermore, for any $\mathcal{C}_1$ diffeomorphism $\bar{x} = T^{-1}(x)$, $T : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is open, that leads to a representation

$$\dot{\bar{x}} = \hat{f}(\bar{x}) + \hat{g}(\bar{x})(y - Ew) + \hat{h}(\bar{x})w \tag{3.10a}$$

$$y = \hat{c}(\bar{x}) + b_0 u + Ew \tag{3.10b}$$

we must have $\hat{f}(\bar{x}) = \left(\frac{\partial T_I}{\partial x}(x)\hat{A}x\right)\Big|_{x=T(\bar{x})}$, $\hat{g}(\bar{x}) = \left(\frac{\partial T_I}{\partial x}(x)\hat{B}\right)\Big|_{x=T(\bar{x})}$, $\hat{h}(\bar{x}) = \left(\frac{\partial T_I}{\partial x}(x)D\right)\Big|_{x=T(\bar{x})}$, $\hat{c}(\bar{x}) = (Cx)|_{x=T(\bar{x})}$, $\forall \bar{x} \in \mathcal{D}$, where $T_I : \mathbb{R}^n \to \mathcal{D}$ is the inverse function of T.

Hence, we observe that the dynamics (3.9a) is invariant (modulo $\mathcal{C}_1$ diffeomorphisms) under $\mathcal{C}_1$ diffeomorphism that bring the system (3.8) into the form of (3.10). Then, the dynamics (3.9a) is called the extended zero dynamics *of (3.8).*

Proof. Note that $u = \frac{1}{b_0}(y - Cx - Ew)$. Substitution of this equality into (3.8a) immediately leads to (3.9a). Hence, (3.9) is a representation of (3.8).

For any $\mathcal{C}_1$ diffeomorphism $\bar{x} = T^{-1}(x)$, $T : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is open, the system (3.8) admits the state space representation

$$\begin{aligned}\dot{\bar{x}} &= f(\bar{x}) + g(\bar{x})u + h(\bar{x})w \\ y &= c(\bar{x}) + b_0 u + Ew\end{aligned}$$

where we have $f(\bar{x}) = \left(\frac{\partial T_I}{\partial x}(x)Ax\right)\big|_{x=T(\bar{x})}$, $g(\bar{x}) = \left(\frac{\partial T_I}{\partial x}(x)B\right)\big|_{x=T(\bar{x})}$, $h(\bar{x}) = \left(\frac{\partial T_I}{\partial x}(x)D\right)\big|_{x=T(\bar{x})}$, $c(\bar{x}) = (Cx)|_{x=T(\bar{x})}$, $\forall \bar{x} \in \mathcal{D}$, and $T_I : \mathbb{R}^n \to \mathcal{D}$ is the inverse function of T. Then, we have that $u = \frac{1}{b_0}(y - c(\bar{x}) - Ew)$. Substitution of this relationship into the above, we have

$$\begin{aligned}&\dot{\bar{x}} = f(\bar{x}) + \frac{1}{b_0}g(\bar{x})(y - Ew) - \frac{1}{b_0}g(\bar{x})c(\bar{x}) + h(\bar{x})w = \hat{f}(\bar{x}) + \hat{g}(\bar{x})(y - Ew) + \hat{h}(\bar{x})w \\ &y = \hat{c}(\bar{x}) + b_0 u + Ew\end{aligned}$$

This completes the proof of the lemma. □

REMARK 3.2. *We observe that, in the previous lemma, the zero dynamics of the system (3.8) is exactly $\dot{x} = \hat{A}x$. The extended zero dynamics is simply the zero dynamics together with the driving terms which include the noiseless output of the system and the disturbance input.*

DEFINITION 3.4. *Consider a finite-dimensional continuous-time SISO LTI system*

$$y = b_0 u + Ew \tag{3.11}$$

where u is the scalar control input; y is the scalar output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; and b_0 and E are constant matrices of appropriate dimensions. Let $b_0 \neq 0$, which implies that the system admits relative degree 0 from u to y, and b_0 is the high-frequency gain of the system. Then, (3.11) is called the extended zero dynamics canonical form. *Clearly, the* extended zero dynamics *is absent in this case.*

This completes our presentation of the definition of extended zero dynamics and the concept of extended zero dynamics canonical form for finite-dimensional continuous-time SISO LTI systems with a finite relative degree. Next, we present a definition of the admissible class of disturbance waveforms.

DEFINITION 3.5. *$\mathcal{W}_d \subseteq \mathcal{C}([0,\infty),\mathbb{R}^q)$, $q \in \mathbb{Z}_+$, is said to be of* class $\mathcal{B}_q$ *if it is nonempty and $\forall w_{[0,\infty)} \in \mathcal{W}_d$, $\forall t_f \in [0,\infty)$, $\exists \bar{w}_{[0,\infty)} \in \mathcal{W}_d$, such that $w_{[0,t_f]} = \bar{w}_{[0,t_f]}$ and $\left\| \bar{w}_{[0,\infty)} \right\|_\infty < \infty$.*

EXAMPLE 3.1. *$\mathcal{C}([0,\infty),\mathbb{R}^q)$ and $\mathcal{C}_k([0,\infty),\mathbb{R}^q)$ are of class $\mathcal{B}_q$, $\forall k \in \mathbb{N} \cup \{\infty\}$ and $\forall q \in \mathbb{Z}_+$. The singleton set $\{ w_{[0,\infty)} \in \mathcal{C}([0,\infty),\mathbb{R}^q) \mid w(t) = \mathbf{0}_{q\times 1},\ \forall t \in [0,\infty) \}$ is of class $\mathcal{B}_q$.*

This definition allows estimation of the smooth part of an arbitrary varying signal using an observer.

LEMMA 3.6. *Let $\mathcal{W}_{d1}$ and $\mathcal{W}_{d2}$ be of class $\mathcal{B}_{q_1}$ and class $\mathcal{B}_{q_2}$, respectively, $q_1, q_2 \in \mathbb{Z}_+$. Then, $\mathcal{W}_{d1} \times \mathcal{W}_{d2}$ is of class $\mathcal{B}_{q_1+q_2}$. Furthermore, if $q_1 = q_2 = q$, then $\mathcal{W}_{d1} \cup \mathcal{W}_{d2}$ is of class $\mathcal{B}_q$.*

Proof. Clearly, we have $\mathcal{W}_{d1} \times \mathcal{W}_{d2} \subseteq \mathcal{C}([0,\infty),\mathbb{R}^{q_1+q_2})$ and is nonempty. For any $(w_{1[0,\infty)}, w_{2[0,\infty)}) \in \mathcal{W}_{d1} \times \mathcal{W}_{d2}$ and $\forall t_f \in [0,\infty)$. Since $\mathcal{W}_{d1}$ is of class $\mathcal{B}_{q_1}$, then, $\exists \bar{w}_{1[0,\infty)} \in \mathcal{W}_{d1}$, such that $w_{1[0,t_f]} = \bar{w}_{1[0,t_f]}$ and $\left\| \bar{w}_{1[0,\infty)} \right\|_\infty < +\infty$. Since $\mathcal{W}_{d2}$ is of class $\mathcal{B}_{q_2}$, then, $\exists \bar{w}_{2[0,\infty)} \in \mathcal{W}_{d2}$, such that $w_{2[0,t_f]} = \bar{w}_{2[0,t_f]}$ and $\left\| \bar{w}_{2[0,\infty)} \right\|_\infty < +\infty$. Then, we have $(\bar{w}_{1[0,\infty)}, \bar{w}_{2[0,\infty)}) \in \mathcal{W}_{d1} \times \mathcal{W}_{d2}$, $(w_{1[0,t_f]}, w_{2[0,t_f]}) = (\bar{w}_{1[0,t_f]}, \bar{w}_{2[0,t_f]})$, and $\left\| (\bar{w}_{1[0,\infty)}, \bar{w}_{2[0,\infty)}) \right\|_\infty < +\infty$. Hence, $\mathcal{W}_{d1} \times \mathcal{W}_{d2}$ is of class $\mathcal{B}_{q_1+q_2}$.

Let $q_1 = q_2 = q$. Clearly, $\mathcal{W}_{d1} \cup \mathcal{W}_{d2} \subseteq \mathcal{C}([0,\infty),\mathbb{R}^q)$ and is nonempty. $\forall w_{[0,\infty)} \in \mathcal{W}_{d1} \cup \mathcal{W}_{d2}$, $\exists i \in \{1,2\}$, such that $w_{[0,\infty)} \in \mathcal{W}_{di}$. $\forall t_f \in [0,\infty)$, Since $\mathcal{W}_{di}$ is of class $\mathcal{B}_q$, then, $\exists \bar{w}_{[0,\infty)} \in \mathcal{W}_{di} \subseteq \mathcal{W}_{d1} \cup \mathcal{W}_{d2}$, such that $w_{[0,t_f]} = \bar{w}_{[0,t_f]}$ and $\left\| \bar{w}_{[0,\infty)} \right\|_\infty < \infty$. Hence, we have $\mathcal{W}_{d1} \cup \mathcal{W}_{d2}$ is of class $\mathcal{B}_q$. ∎

Now, we are in a position to introduce a generalized definition of minimum phase that also accommodates the presence of disturbances.

DEFINITION 3.7. *Consider a finite-dimensional continuous-time SISO LTI system*

$$\dot{x} = Ax + Bu + Dw; \qquad x(0) = x_0 \tag{3.12a}$$

$$y = Cx + Ku + Ew \tag{3.12b}$$

where x is the n-dimensional state vector, $n \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is the set of admissible initial conditions; u is the scalar control input; y is the scalar output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$, $\mathcal{W}_d$ is the set of admissible disturbance waveforms; A, B, D, C, K, and E are constant matrices of appropriate dimensions. Let the relative degree from u to y be $r \in \mathbb{Z}_+$.

1. *If $r = 0$ and $n = 0$, then, the system (3.12) admits the extended zero dynamics canonical form of (3.11), where the extended zero dynamics is absent. We will henceforth say that the system (3.12) is* minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.
2. *If $r \in \mathbb{N}$ and $n = r$, then, the system admits the extended zero dynamics canonical form (3.7), where the extended zero dynamics is absent. We will henceforth say that the system (3.12) is* minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.
3. *If $r = 0$ and $n \in \mathbb{N}$, then, the system (3.12) admits the extended zero dynamics canonical form (3.9). We will say that the system (3.12) is* minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ *if its extended zero dynamics (3.9a) satisfies that $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\left\| y_{[0,\infty)} \right\|_\infty \leq c_w$, we have $\left\| x_{[0,\infty)} \right\|_\infty \leq c_c$.*
4. *If $r \in \mathbb{N}$ and $r < n$, then, the system (3.12) admits the extended zero dynamics canonical form (3.2). We will say that the system (3.12) is* minimum phase

with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ *if its extended zero dynamics (3.2a) with the coordinate transformation* $x_z = T_z x$ *satisfies that* $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_{z0} \in T_z(\mathcal{D}_0)$ *with* $|x_{z0}| \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ *with* $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall x_{1[0,\infty)} \in \mathcal{C}_r$ *with* $\left\| x^{[r]}_{1[0,\infty)} \right\|_\infty \le c_w$, *we have* $\left\| x_{z[0,\infty)} \right\|_\infty \le c_c$.

Note that, in the last item of the above definition, x_1 is set to be any desired trajectory that is $\mathcal{C}_r$ and x_1 together with its derivatives up to order r are bounded. This generalized notion of minimum phase has important significance in the context of model reference control (see § 4). In the following, we prove a necessary and sufficient condition for the last item of the above definition where x_1 is set to be any bounded signal in $\mathcal{C}$. First, we have the following preliminary result, which is useful in its own right.

LEMMA 3.8. *Consider the system (3.12) of Definition 3.7. Let* $r < n$. *Then, the following statements hold.*

1. *When* $r > 0$. *(3.12) admits the extended zero dynamics canonical form (3.2). Then, (3.12) is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ *implies that the system*

$$\dot z = A_z z + A_{z1} v; \qquad z(0) = \mathbf{0}_{(n-r)\times 1} \tag{3.13}$$

is bounded input and bounded state stable.

2. *When* $r = 0$. *(3.12) admits the extended zero dynamics canonical form (3.9). Then, (3.12) is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ *implies that the system*

$$\dot z = \hat A z + \hat B v; \qquad z(0) = \mathbf{0}_{n\times 1} \tag{3.14}$$

is bounded input and bounded state stable.

Proof. For the first statement, we will prove the lemma using an argument of contradiction. Suppose (3.13) is not bounded input and bounded state stable. By Corollary A.10, $\exists v_{[0,\infty)} \in \mathcal{C}_r$, such that $\left\| v^{[r]}_{[0,\infty)} \right\|_\infty =: \bar c_w \in (0,\infty) \subset \mathbb{R}$ and $\left\| z_{[0,\infty)} \right\|_\infty = +\infty$.

Fix $x_{z0} \in T_z(\mathcal{D}_0)$, where $x_z = T_z x$ is the coordinate transformation for the extended zero dynamics. Then, $\exists c_w \in [\bar c_w, \infty) \subset \mathbb{R}$, such that $|x_{z0}| \le c_w$ and $\exists w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$. Set $x_{1[0,\infty)} = \rho v_{[0,\infty)} \in \mathcal{C}_r$ with $\rho \in (0,1)$. Then, clearly, $\left\| x^{[r]}_{1[0,\infty)} \right\|_\infty = \rho \bar c_w \le c_w$. Let $x_{z[0,\infty)}$ be the solution to (3.2a) with initial condition $x_z(0) = x_{z0}$ and inputs $x_{1[0,\infty)}$ and $w_{[0,\infty)}$. Then, $x_{z[0,\infty)}$ may be generated by

$$\begin{aligned} \dot z &= A_z z + A_{z1} v; \qquad z(0) = \mathbf{0}_{(n-r)\times 1} \\ \dot\zeta &= A_z \zeta + D_z w; \qquad \zeta(0) = x_{z0} \\ x_{z[0,\infty)} &= \zeta_{[0,\infty)} + \rho z_{[0,\infty)} \end{aligned}$$

Then, $\left\| x_{z[0,\infty)} \right\|_\infty = +\infty$ for some $\rho \in (0,1)$. This contradicts with the assumption that (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

Hence, (3.13) is bounded input and bounded state stable.

The second statement is a direct consequence of Lemma A.11. This completes the proof of the lemma. □

LEMMA 3.9. *Consider the system (3.12) of Definition 3.7. Let* $r \in \mathbb{N}$ *and* $r < n$. *It admits the extended zero dynamics canonical form (3.2). Then, (3.12) is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ *if, and only if, its extended zero dynamics (3.2a) with* *coordinate transformation* $x_z = T_z x$ *satisfies:* $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_{z0} \in T_z(\mathcal{D}_0)$ *with* $|x_{z0}| \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ *with* $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall x_{1[0,\infty)} \in \mathcal{C}$ *with* $\left\| x_{1[0,\infty)} \right\|_\infty \le c_w$, *we have* $\left\| x_{z[0,\infty)} \right\|_\infty \le c_c$.

Proof. "If". This is straightforward.

"Only if". $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall x_{z0} \in T_z(\mathcal{D}_0)$ with $|x_{z0}| \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall x_{1[0,\infty)} \in \mathcal{C}$ with $\left\| x_{1[0,\infty)} \right\|_\infty \le c_w$. The trajectory $x_{z[0,\infty)}$ may be generated by

$$\begin{aligned} \dot z &= A_z z + A_{z1} x_1; \qquad z(0) = \mathbf{0}_{(n-r)\times 1} \\ \dot\zeta &= A_z \zeta + D_z w; \qquad \zeta(0) = x_{z0} \\ x_{z[0,\infty)} &= \zeta_{[0,\infty)} + z_{[0,\infty)} \end{aligned}$$

By Lemmas 3.8 and A.3, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only on the matrices A_z and A_{z1}, such that $\left\| z_{[0,\infty)} \right\|_\infty \le c_{c2} c_w$. Since (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, and the identically zero function $\mathbf{0}_{[0,\infty)} \in \mathcal{C}_r$ with $\left\| \mathbf{0}^{[r]}_{[0,\infty)} \right\|_\infty = 0$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\| \zeta_{[0,\infty)} \right\|_\infty \le c_{c1}$. Hence, $\left\| x_{z[0,\infty)} \right\|_\infty \le c_{c1} + c_{c2} c_w$.

This completes the proof of the lemma. ∎

Next, we present a result that links the generalized minimum phase property to the asymptotic stability property of the extended zero dynamics.

LEMMA 3.10. *Consider the system (3.12) of Definition 3.7. Let* $r \in \mathbb{Z}_+$. *Then, the following statements hold.*

1. *Let* $r = n = 0$. *The system is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ *since the extended zero dynamics is absent.*
2. *Let* $1 \le r = n$. *The system is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ *since the extended zero dynamics is absent.*
3. *Let* $0 = r < n$. *The system is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ *if its extended zero dynamics (3.9a) is such that the matrix* $\hat A$ *is Hurwitz. On the other hand, if the system is stabilizable from* u *and is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$, *then the matrix* $\hat A$ *is Hurwitz.*
4. *Let* $1 \le r < n$. *The system is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ *if its extended zero dynamics (3.2a) is such that the matrix* A_z *is Hurwitz. On the other hand, if the system is stabilizable from* u *and is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$, *then the matrix* A_z *is Hurwitz.*

Proof. The statements 1 and 2 are immediate from the definition of minimum phase property.

For statement 3, let $0 = r < n$. By Lemma 3.3, the extended zero dynamics of the system is (3.9a). If the matrix $\hat A$ is Hurwitz, by Lemmas A.1 and A.2, the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

On the other hand, if the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and is stabilizable from u, we will show that the matrix $\hat A$ is Hurwitz. By the stabilizability of the pair (A, B), we have the pair $(\hat A, \hat B)$ is stabilizable. Since (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, by Lemma 3.8, the following system

$$\dot x_u = \hat A x_u + \hat B v; \qquad x_u(0) = \mathbf{0}_{n\times 1}$$

is bounded input and bounded state stable. Since the triple $(\hat A, \hat B, I_n)$ is stabilizable and detectable, then, by Corollary A.8, $\hat A$ is Hurwitz.

For statement 4, without loss of generality, we assume that the system (3.12) is given in the extended zero dynamics canonical form (3.2), by Lemma 3.1. The extended zero dynamics is (3.2a) and $x_z = T_z x$. If the matrix A_z is Hurwitz, by Lemmas A.1 and A.2, system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

On the other hand, if the system (3.12) is stabilizable from u and is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. Then, the matrix

$$\left[\begin{array}{c|c} \lambda I - A & B \end{array}\right] = \left[\begin{array}{ccccc|c} \lambda I_{n-r} - A_z & -A_{z1} & & & & \\ & \lambda - a_{11} & -1 & & & \\ & -a_{21} & \lambda & \ddots & & \\ & \vdots & & \ddots & -1 & \\ -A_{rz} & -a_{r1} & & & \lambda & b_0 \end{array}\right] \tag{3.15}$$

has rank n, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) \geq 0$, where all empty spaces in the above matrix are zeros. Observe that the sub-block matrix of (3.15), consisting the last r rows and the last r columns, is invertible. Then, the matrix $\left[\begin{array}{cc} \lambda I_{n-r} - A_z & -A_{z1} \end{array}\right]$ has rank $n-r$, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) \geq 0$. Hence, the pair (A_z, A_{z1}) is stabilizable. Since (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, by Lemma 3.8, we have the following system

$$\dot{x}_{zu} = A_z x_{zu} + A_{z1} v; \qquad x_{zu}(0) = \mathbf{0}_{(n-r)\times 1} \tag{3.16}$$

is bounded input and bounded state stable. Since the triple (A_z, A_{z1}, I_{n-r}) is stabilizable and detectable, then, by Corollary A.8, the matrix A_z is Hurwitz.

This completes the proof of the lemma. □

Next, we present a lemma that states that a controllable and observable system is minimum phase if, and only if, its transfer function has all zeros with negative real parts. This demonstrates that Definition 3.7 is truly a generalization of the minimum phase concept.

Lemma 3.11. *Consider the system (3.12) of Definition 3.7. Assume that the system admits relative degree $r \in \mathbb{Z}_+$ from u to y; and it is controllable from u and is observable. Then, the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ if, and only if, the transfer function $H(s) = C\,(sI_n - A)^{-1}B + K$ has all zeros with negative real parts.*

Proof. We will consider separately four exhaustive and mutually exclusive cases: Case 1: $r = n = 0$; Case 2: $1 \leq r = n$; Case 3: $0 = r < n$; Case 4: $1 \leq r < n$.

Case 1: $r = n = 0$. Then, the system (3.12) is given by $y = Ku + Ew$, where $K \neq 0$. Clearly, the extended zero dynamics of (3.12) is absent. The transfer function from u to y is $H(s) = K$. Clearly, $H(s)$ does not have any zeros. Thus, (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ (according to Definition 3.7) if, and only if, $H(s)$ has all zeros with negative real parts. This case is proved.

Case 2: $1 \leq r = n$. By Lemma 3.2, the system (3.12) admits the state space representation (3.7). Then, the extended zero dynamics for (3.12) is absent. Hence, the system is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. The transfer function $H(s)$ from u to y is $\frac{b_0}{s^r - a_{11}s^{r-1} - \cdots - a_{r1}}$, which does not admit any zeros. Hence, $H(s)$ has all zeros with negative real parts. Thus, the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ if, and only if, $H(s)$ has all zeros with negative real parts. This case is proved.

Case 3: $0 = r < n$. Let the transfer function from u to y be

$$H(s) = \frac{b_0 s^n + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = b_0 + \frac{(b_1 - b_0 a_1)s^{n-1} + \cdots + (b_n - b_0 a_n)}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

Without loss of generality, assume (3.12) is given in observer canonical form. Then, $K = b_0 \neq 0$,

$$A = \left[\begin{array}{c|c} -a_1 & \\ \vdots & I_{n-1} \\ -a_{n-1} & \\ \hline -a_n & \mathbf{0}_{1\times(n-1)} \end{array}\right]; \qquad B = \begin{bmatrix} b_1 - b_0 a_1 \\ \vdots \\ b_n - b_0 a_n \end{bmatrix}; \qquad C = \begin{bmatrix} 1 & \mathbf{0}_{1\times(n-1)} \end{bmatrix}$$

By Lemma 3.3, the extended zero dynamics of (3.12) is given by

$$\dot{x} = (A - \frac{1}{b_0}BC)x + \frac{1}{b_0}B(y - Ew) + Dw =: \hat{A}x + \hat{B}y + \hat{D}w$$

where

$$\hat{A} = \left[\begin{array}{c|c} -b_1/b_0 & \\ \vdots & I_{n-1} \\ -b_{n-1}/b_0 & \\ \hline -b_n/b_0 & \mathbf{0}_{1\times(n-1)} \end{array}\right]$$

The characteristic function of $\hat{A}$ is $s^n + \frac{b_1}{b_0}s^{n-1} + \cdots + \frac{b_n}{b_0}$. Hence, $\hat{A}$ is Hurwitz if, and only if, all zeros of $H(s)$ have negative real parts.

When the matrix $\hat{A}$ is Hurwitz, by Lemma 3.10, the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

On the other hand, when the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, again by Lemma 3.10, the matrix $\hat{A}$ is Hurwitz, since the system is controllable from u.

Thus, we have shown that the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ if, and only if, the matrix $\hat{A}$ is Hurwitz, which holds if, and only if, the transfer function $H(s)$ have all zeros with negative real parts. Hence, this case is proved.

Case 4: $1 \leq r < n$. Let the transfer function from u to y be given by $H(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$ with $m = n - r \in \mathbb{N}$ and $b_0 \neq 0$. By Lemma A.5, there exists a real invertible coordinate transformation $\begin{bmatrix} x_f' & x_z' \end{bmatrix}' = T^{-1}x = \begin{bmatrix} T_f' & T_z' \end{bmatrix}' x$ such that, in $\begin{bmatrix} x_f' & x_z' \end{bmatrix}'$ coordinates, the system (3.12) admits the state space representation:

$$\begin{aligned} \begin{bmatrix} \dot{x}_f \\ \dot{x}_z \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_f \\ x_z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w \\ y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_f \\ x_z \end{bmatrix} + Ew \end{aligned}$$

where x_z is m-dimensional; $x_f = \begin{bmatrix} x_1 & \cdots & x_r \end{bmatrix}'$ with x_i, $i = 1, \ldots, r$, being scalars; the partitioning of the system matrices are compatible with the partitioning of the state vector; and

$$A_{11} = \left[\begin{array}{c|c} \bar{a}_1 & \\ \vdots & I_{r-1} \\ \bar{a}_{r-1} & \\ \hline \bar{a}_r & \mathbf{0}_{1\times(r-1)} \end{array}\right];\ A_{12} = \left[\begin{array}{cc} & \mathbf{0} \\ 1 & \end{array}\right];\ A_{21} = \left[\begin{array}{c|c} \bar{a}_{r+1} & \\ \vdots & \mathbf{0}_{m\times(r-1)} \\ \bar{a}_n & \end{array}\right]$$

$$A_{22} = \left[\begin{array}{c|c} -b_1/b_0 & \\ \vdots & I_{m-1} \\ -b_{m-1}/b_0 & \\ \hline -b_m/b_0 & \mathbf{0}_{1\times(m-1)} \end{array}\right];\quad \begin{array}{c} B_1 = \begin{bmatrix} \mathbf{0}_{(r-1)\times 1} \\ b_0 \end{bmatrix};\ B_2 = \mathbf{0}_{m\times 1} \\ C_1 = \begin{bmatrix} 1 & \mathbf{0}_{1\times(r-1)} \end{bmatrix};\ C_2 = \mathbf{0}_{1\times m} \end{array}$$

In $\begin{bmatrix} x_z' & x_f' \end{bmatrix}'$ coordinates, the system (3.12) is represented by

$$\dot{x}_z = A_{22}x_z + \begin{bmatrix} \bar{a}_{r+1} \\ \vdots \\ \bar{a}_n \end{bmatrix} x_1 + D_2 w =: A_{22}x_z + A_{21,1}x_1 + D_2 w \tag{3.17a}$$

$$\dot{x}_f = A_{12}x_z + A_{11}x_f + B_1 u + D_1 w \tag{3.17b}$$

$$y = x_1 + Ew \tag{3.17c}$$

which is clearly in the form of (3.2). Hence, (3.17a) is the extended zero dynamics of (3.12). The characteristic function of A_{22} is $s^m + \frac{b_1}{b_0}s^{m-1} + \cdots + \frac{b_m}{b_0}$. The matrix A_{22} is Hurwitz if, and only if, the transfer function $H(s)$ has all zeros with negative real parts.

When the matrix A_{22} is Hurwitz, by Lemma 3.10, the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

On the other hand, when the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, by Lemma 3.10, the matrix A_{22} is Hurwitz, since the system is controllable from u.

Thus, we have shown that the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ if, and only if, the matrix A_{22} is Hurwitz, which holds if, and only if, the transfer function $H(s)$ have all zeros with negative real parts. Hence, this case is proved.

This completes the proof of the lemma. ∎

LEMMA 3.12. *Consider the system (3.12) of Definition 3.7 with* $r \in \mathbb{Z}_+$. *Assume that it is minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$. *Then, the transfer function from* u *to* y, $H(s) = K + C(sI_n - A)^{-1}B$, *has all zeros with negative real parts.*

Proof. We will distinguish four exhaustive and mutually exclusive cases: Case 1: $r = n = 0$; Case 2: $0 < r = n$; Case 3: $0 = r < n$; Case 4: $0 < r < n$.

Case 1: $r = n = 0$. Then, (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and $H(s) = K \neq 0$ (since $r = 0$). The result holds.

Case 2: $0 < r = n$. Then, (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits extended zero dynamics canonical form (3.7). This implies that $H(s) = \frac{b_0}{s^n - a_{11}s^{n-1} - \cdots - a_{n1}}$, which does not have any zeros. The results holds in this case.

Case 3: $0 = r < n$. Then, (3.12) admits extended zero dynamics canonical form (3.9). Using Laplace transform, we may calculate the transfer function from u to y of (3.9), which should equal to $H(s)$. Then, we have

$$\frac{b_0}{1 - C(sI_n - \hat{A})^{-1}\hat{B}} = b_0 + C(sI_n - A)^{-1}B = H(s)$$

This yields

$$C(sI_n - \hat{A})^{-1}\hat{B} = \frac{H(s) - b_0}{H(s)}$$

By Lemma 3.8, the system $\dot{z} = \hat{A}z + \hat{B}v$, $z(0) = \mathbf{0}_{n\times 1}$ is bounded input and bounded state. By Lemma A.3, there exists a $k \in [0,\infty) \subset \mathbb{R}$ and $\lambda \in (0,\infty) \subset \mathbb{R}$ such that $\left\| \mathrm{e}^{\hat{A}t}\hat{B} \right\|_{2,2} \leq k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty) \subset \mathbb{R}$. This implies that $C(sI_n - \hat{A})^{-1}\hat{B}$ has all poles with negative real parts. By the above equation, the numerator of $H(s)$ is equal to the denominator of $C(sI_n - \hat{A})^{-1}\hat{B}$. Hence, $H(s)$ has all zeros with negative real parts.

Case 4: $0 < r < n$. Then, (3.12) admits extended zero dynamics canonical form (3.2). Using Laplace transform, we may calculate the transfer function from u to y of (3.2), which should equal to $H(s)$. Then, we have

$$\begin{aligned} sX_z(s) &= A_z X_z(s) + A_{z1}Y(s) \\ sX_i(s) &= a_{i1}Y(s) + X_{i+1}(s); \qquad i = 1,\ldots,r-1 \\ sX_r(s) &= A_{rz}X_z(s) + a_{r1}Y(s) + b_0U(s) \end{aligned}$$

which yields

$$H(s) = C(sI_n - A)^{-1}B = \frac{b_0}{s^r - a_{11}s^{r-1} - \cdots - a_{r1} - A_{rz}(sI_{n-r} - A_z)^{-1}A_{z1}}$$

By Lemma 3.8, the system $\dot{z} = A_z z + A_{z1}v$, $z(0) = \mathbf{0}_{(n-r)\times 1}$ is bounded input and bounded state. By Lemma A.3, there exists a $k \in [0,\infty) \subset \mathbb{R}$ and $\lambda \in (0,\infty) \subset \mathbb{R}$ such that $\left\| \mathrm{e}^{A_z t}A_{z1} \right\|_{2,2} \leq k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty) \subset \mathbb{R}$. This implies that $A_{rz}(sI_{n-r} - A_z)^{-1}A_{z1}$ has all poles with negative real parts. By the above equation, the numerator of $H(s)$ is equal to the denominator of $A_{rz}(sI_{n-r} - A_z)^{-1}A_{z1}$ or a factor of it. Hence, $H(s)$ has all zeros with negative real parts.

This completes the proof of the lemma. □

REMARK 3.3. *We conclude, based on the previous three lemmas, that if a finite-dimensional continuous-time SISO LTI system with $0 < r < n$ is minimum phase according to [4], then it is also minimum phase according to the generalized definition; on the other hand, if it is minimum phase according to the generalized definition, and it is stabilizable from u, then it is minimum phase according to [4].*

Under controllability and observability assumptions, a finite-dimensional continuous-time SISO LTI system with relative degree $r \in \mathbb{Z}_+$ is minimum phase according to the generalized definition if, and only if, it is minimum phase according to the classical definition, that is, the transfer function from u to y have all zeros with negative real parts. When the system is not necessarily controllable and observable, it is minimum phase according to the generalized definition implies that its transfer function has all zeros with negative real parts.

We will present an example that illustrates the generalized definition of minimum phase.

EXAMPLE 3.2. Consider the following 4th order system

$$\dot{x}_z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_z + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} w; \qquad x_z(0) = \begin{bmatrix} x_{z1,0} \\ 0 \end{bmatrix} \tag{3.18a}$$

$$\dot{x}_1 = x_2; \qquad x_1(0) = x_{1,0} \tag{3.18b}$$

$$\dot{x}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_z + u + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} w; \qquad x_2(0) = x_{2,0} \tag{3.18c}$$

$$y = x_1 \tag{3.18d}$$

with $r = 2$, $x_0 = \begin{bmatrix} x_{z1,0} & 0 & x_{1,0} & x_{2,0} \end{bmatrix}'$. Clearly, (3.18) is in the zero dynamics canonical form (3.2). Define $\mathcal{D}_0 = \{x_0 \in \mathbb{R}^4 \mid x_{z1,0}, x_{1,0}, x_{2,0} \in \mathbb{R}\}$. Let the elements of w be $\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}'$. $w_{1[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}) =: \mathcal{W}_{d1}$, which is of class $\mathcal{B}_1$. $w_{2[0,\infty)} \in \mathcal{W}_{d2}$, which is defined by

$$\mathcal{W}_{d2} = \{w_{2[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}) \mid \left\| w_{2[0,\infty)} \right\|_\infty < \infty,\ \phi(t) = \int_0^t w_2(\tau)\,\mathrm{d}\tau,$$
$$\forall t \in [0,\infty),\ \left\| \phi_{[0,\infty)} \right\|_\infty \le c_2\}$$

where $c_2 \in [0,\infty) \subset \mathbb{R}$ is a fixed constant. Clearly, $\mathcal{W}_{d2}$ is of class $\mathcal{B}_1$. $w_{3[0,\infty)} \in \mathcal{W}_{d3}$, which is defined by

$$\mathcal{W}_{d3} = \{w_{3[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}) \mid \left\| w_{3[0,\infty)} \right\|_\infty < \infty,\ \psi_2(t) = \int_0^t w_3(\tau)\,\mathrm{d}\tau,$$
$$\forall t \in [0,\infty),\ \left\| \psi_{2[0,\infty)} \right\|_\infty \le c_{3,2},\ \psi_1(t) = \int_0^t \psi_2(\tau)\,\mathrm{d}\tau,\ \forall t \in [0,\infty),$$
$$\left\| \psi_{1[0,\infty)} \right\|_\infty \le c_{3,1}\}$$

where $c_{3,2}, c_{3,1} \in [0,\infty) \subset \mathbb{R}$ are fixed constants. Clearly, $\mathcal{W}_{d3}$ is of class $\mathcal{B}_1$. Then, $w_{[0,\infty)} \in \mathcal{W}_{d1} \times \mathcal{W}_{d2} \times \mathcal{W}_{d3} =: \mathcal{W}_d$, which is of class $\mathcal{B}_3$ by Lemma 3.6.

$\forall c_w \in [0,\infty) \subset \mathbb{R}$. Let $c_c = \sqrt{(c_w + c_2 + c_{3,1})^2 + c_{3,2}^2} \in [0,\infty) \subset \mathbb{R}$. $\forall x_{z0} = \begin{bmatrix} x_{z1,0} & 0 \end{bmatrix}' \in T_z(\mathcal{D}_0)$ with $|x_{z0}| \le c_w$, we have $|x_{z1,0}| \le c_w$. $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$. $\forall x_{1[0,\infty)} \in \mathcal{C}_2$ with $\left\| x_{1[0,\infty)}^{[2]} \right\|_\infty \le c_w$. The state trajectory, $x_{z[0,\infty)}$, for the extended zero dynamics (3.18a) may be generated by

$$\dot{x}_{zo} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{zo}; \qquad x_{zo}(0) = \begin{bmatrix} x_{z1,0} \\ 0 \end{bmatrix}$$
$$\dot{x}_{zu} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{zu} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_2; \qquad x_{zu}(0) = \mathbf{0}_{2\times 1}$$
$$\dot{x}_{zv} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{zv} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_3; \qquad x_{zv}(0) = \mathbf{0}_{2\times 1}$$
$$x_{z[0,\infty)} = x_{zo[0,\infty)} + x_{zu[0,\infty)} + x_{zv[0,\infty)}$$

Let $x_z = \begin{bmatrix} x_{z1} & x_{z2} \end{bmatrix}'$, $x_{zo} = \begin{bmatrix} x_{zo1} & x_{zo2} \end{bmatrix}'$, $x_{zu} = \begin{bmatrix} x_{zu1} & x_{zu2} \end{bmatrix}'$, and $x_{zv} = \begin{bmatrix} x_{zv1} & x_{zv2} \end{bmatrix}'$. Then, $\forall t \in [0,\infty)$, we have $x_{zo1}(t) = x_{z1,0}$, $x_{zo2}(t) = 0$, $x_{zu1}(t) = \int_0^t w_2(\tau)\,\mathrm{d}\tau$, $x_{zu2}(t) = 0$, $x_{zv2}(t) = \int_0^t w_3(\tau)\,\mathrm{d}\tau$, and $x_{zv1}(t) = \int_0^t x_{zv2}(\tau)\,\mathrm{d}\tau$. Hence, we have $\left\| x_{zo1[0,\infty)} \right\|_\infty \le c_w$, $\left\| x_{zo2[0,\infty)} \right\|_\infty = 0$, $\left\| x_{zu1[0,\infty)} \right\|_\infty \le c_2$, $\left\| x_{zu2[0,\infty)} \right\|_\infty = 0$, $\left\| x_{zv2[0,\infty)} \right\|_\infty \le c_{3,2}$, and $\left\| x_{zv1[0,\infty)} \right\|_\infty \le c_{3,1}$. Then, $\left\| x_{z1[0,\infty)} \right\|_\infty = \left\| x_{zo1[0,\infty)} + x_{zu1[0,\infty)} + x_{zv1[0,\infty)} \right\|_\infty \le c_w + c_2 + c_{3,1}$ and $\left\| x_{z2[0,\infty)} \right\|_\infty = \left\| x_{zo2[0,\infty)} + x_{zu2[0,\infty)} + x_{zv2[0,\infty)} \right\|_\infty \le c_{3,2}$. Therefore, we have $\left\| x_{z[0,\infty)} \right\|_\infty \le c_c$. Hence, the system (3.18) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. Note that the extended zero dynamics (3.18a) is neither asymptotically stable nor bounded input and bounded state stable with respect to inputs x_1 and w in the usual sense.

4. Necessity of minimum phase property in model reference control. We state and prove the following result on the necessity of the generalized minimum phase condition in model reference control of finite-dimensional continuous-time SISO LTI systems.

PROPOSITION 4.1. *Consider the system (3.12) of Definition 3.7. Let (3.12) admit the relative degree $r \in \{0, \ldots, n\}$ with respect to u. Assume that*

(i) the identically $\mathbf{0}_{q\times 1}$ function belongs to $\mathcal{W}_d$;

(ii) the set $\mathcal{D}_0$ satisfies, when $1 \le r < n$, by Lemma 3.1, system S_P admits the zero dynamics canonical form (3.2), without loss of generality, assume (3.12) is in zero dynamics canonical form (3.2) and $\mathcal{D}_0 = \mathcal{D}_{z0} \times \mathbb{R}^r$, where $\mathcal{D}_{z0} \subseteq \mathbb{R}^{n-r}$ is nonempty.

Let $y_{d[0,\infty)} \in \mathcal{C}_r([0,\infty), \mathbb{R})$ be the reference trajectory and further denote $Y_d := \begin{bmatrix} y_d & y_d^{(1)} & \cdots & y_d^{(r)} \end{bmatrix}'$, and $Y_{d0} := \begin{bmatrix} y_d(0) & y_d^{(1)}(0) & \cdots & y_d^{(r-1)}(0) \end{bmatrix}' \in \mathbb{R}^r$. Assume that there exists a finite-dimensional model reference controller, S_C

$$\dot{\xi} = f(\xi, y, Y_d); \qquad \xi(0) = \xi_0(\check{x}_0, Y_{d0}) \tag{4.1a}$$

$$u = h(\xi, y, Y_d) \tag{4.1b}$$

where ξ is $\bar{n}$-dimensional, $\bar{n} \in \mathbb{Z}_+$; and f and h are locally Lipschitz on $\mathcal{D}_\xi \times \mathbb{R} \times \mathbb{R}^{r+1}$; $\mathcal{D}_\xi \subseteq \mathbb{R}^{\bar{n}}$ is nonempty; $\check{x}_0 \in \mathcal{D}_0$ is an estimate of x_0; $\xi_0 : \mathcal{D}_0 \times \mathbb{R}^r \to \mathcal{D}_\xi$; such that $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$; $\forall Y_{d0} \in \mathbb{R}^r$ with $|Y_{d0}| \le c_w$; $\forall y^{(r)}_{d[0,\infty)} \in \mathcal{C}$ with $\left\| Y_{d[0,\infty)} \right\|_\infty \le c_w$, we have

1. $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$; $\forall \check{x}_0 \in \mathcal{D}_0$ with $|\check{x}_0| \le c_w$; $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$; there exists a unique solution $X_{[0,\infty)} := (x_{[0,\infty)}, \xi_{[0,\infty)})$ to the closed-loop system S such that $\left\| x_{[0,\infty)} \right\|_\infty \le c_c$, $\xi(t) \in \mathcal{D}_\xi$, $\forall t \in [0,\infty)$, such that $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$;

2. when $r = 0$ and $n \in \mathbb{N}$, $\forall x_0 \in \mathcal{D}_0$, set $\check{x}_0 = x_0 \in \mathcal{D}_0$ and $w_{[0,\infty)}$ equals to the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$, then $Cx(t) + Ku(t) = y_d(t)$, $\forall t \in [0,\infty) \subset \mathbb{R}$;

3. when $1 \le r < n$, $\forall x_{z0} \in \mathcal{D}_{z0}$, $\exists x_0 \in \mathcal{D}_0$ whose first $n - r$ coordinates equal to x_{z0}, set $\check{x}_0 = x_0 \in \mathcal{D}_0$ and $w_{[0,\infty)}$ equals to the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$, then $y(t) = y_d(t)$, $\forall t \in [0,\infty) \subset \mathbb{R}$.

Then, the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

Proof. We will consider separately four exhaustive and mutually exclusive cases: Case 1: $0 = r = n$; Case 2: $1 \le r = n$; Case 3: $0 = r < n$; Case 4: $1 \le r < n$.

Case 1: $0 = r = n$. Clearly, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof for this case.

Case 2: $1 \le r = n$. Clearly, the system (3.12) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof for this case.

Case 3: $0 = r < n$. By Lemma 3.3, the system S_P admits the following extended zero dynamics canonical form representation, $K \ne 0$,

$$\dot{x} = \hat{A}x + \hat{B}(y - Ew) + Dw =: \hat{A}x + \hat{B}y + \hat{D}w; \qquad x(0) = x_0 \tag{4.2a}$$

$$y = Cx + Ku + Ew \tag{4.2b}$$

We need the following result.

CLAIM 4.1.1. *The system $\dot{z} = \hat{A}z + \hat{B}v$; $z(0) = \mathbf{0}_{n\times 1}$ is bounded input and bounded state stable.*

Proof. Suppose that the claim is false. By Corollary A.10, $\exists v_{[0,\infty)} \in \mathcal{C}$ such that $\left\| v_{[0,\infty)} \right\|_\infty =: c_v \in (0,\infty) \subset \mathbb{R}$ and $\left\| z_{[0,\infty)} \right\|_\infty = +\infty$. Choose $w_{[0,\infty)}$ to be the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$. Fix $x_0 \in \mathcal{D}_0$ and $\check{x}_0 = x_0$. $\forall \rho \in (0,1) \subset \mathbb{R}$, let $y_{d[0,\infty)} = \rho v_{[0,\infty)}$ and $c_w = \max\{c_v, |x_0|\} \in \mathbb{R}$. By 1., $\exists c_c \in [0,\infty) \subset \mathbb{R}$, where c_c depends only on c_w and S_C, such that the closed-loop system S admits a unique solution $X_{[0,\infty)}$ with $\left\| x_{[0,\infty)} \right\|_\infty \leq c_c$, and $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$. By 2., we have $Cx_{[0,\infty)} + Ku_{[0,\infty)} = y_{d[0,\infty)} = \rho v_{[0,\infty)}$. This further implies that $y_{[0,\infty)} = y_{d[0,\infty)} = \rho v_{[0,\infty)}$. Note that $x_{[0,\infty)}$ is the unique solution to $\dot{x} = \hat{A}x + \hat{B}y_d$, $x(0) = x_0$. By linearity, we have $x_{[0,\infty)} = \rho z_{[0,\infty)} + x_{I[0,\infty)}$, where $x_{I[0,\infty)}$ satisfies $\dot{x}_I = \hat{A}x_I$, $x_I(0) = x_0$. Hence, we have $\left\| x_{[0,\infty)} \right\|_\infty = +\infty$ for some $\rho \in (0,1)$. This is a contradiction. Hence, the claim is proved. $\square$

Back to proof of the lemma. Let $\bar{P} := \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \cdots & \hat{A}^{n-1}\hat{B} \end{bmatrix}$ and $n_c := \text{rank}(\bar{P})$. We will further distinguish 3 exhaustive and mutually exclusive cases: Case 3.1: $n_c = 0$; Case 3.2: $1 \leq n_c < n$; Case 3.3: $n_c = n$.

Case 3.1: $n_c = 0$. Then, we have $\hat{B} = \mathbf{0}_{n\times 1}$. Then, the extended zero dynamics is given by

$$\dot{x} = \hat{A}x + Dw; \qquad x(0) = x_0 \tag{4.3}$$

By the assumption, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, let $\check{x}_0 = x_0$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, let $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)} \in \mathcal{C}$, then, there exists a unique solution $X_{[0,\infty)}$ to the closed-loop system with $\left\| x_{[0,\infty)} \right\|_\infty \leq c_c$. Clearly, $x_{[0,\infty)}$ is the unique solution to (4.3). Then, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

Case 3.2: $1 \leq n_c < n$. Without loss of generality, assume that (4.2a) is partitioned into controllable and uncontrollable parts (with respect to y). We have $x = \begin{bmatrix} x_c' & x_{\bar{c}}' \end{bmatrix}'$

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \hat{A}_c & \hat{A}_{c\bar{c}} \\ \mathbf{0} & \hat{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \hat{B}_c \\ \mathbf{0} \end{bmatrix} y + \begin{bmatrix} \hat{D}_c \\ \hat{D}_{\bar{c}} \end{bmatrix} w \tag{4.4}$$

where x_c is n_c-dimensional; $x_{\bar{c}}$ is $n_{\bar{c}} := n - n_c$ dimensional; and the pair $(\hat{A}_c, \hat{B}_c)$ is controllable.

By Claim 4.1.1 and Lemma A.3, $\exists k \in [0,\infty) \subset \mathbb{R}$, $\exists \lambda \in (0,\infty) \subset \mathbb{R}$, such that $\|\mathrm{e}^{\hat{A}t}\hat{B}\|_{2,2} \leq k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty)$. Note that

$$\|\mathrm{e}^{\hat{A}t}\hat{B}\|_{2,2} = \left\| \begin{bmatrix} \mathrm{e}^{\hat{A}_c t} & \star \\ \mathbf{0} & \mathrm{e}^{\hat{A}_{\bar{c}} t} \end{bmatrix} \begin{bmatrix} \hat{B}_c \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} = \|\mathrm{e}^{\hat{A}_c t}\hat{B}_c\|_{2,2} \leq k\mathrm{e}^{-\lambda t}; \qquad \forall t \in [0,\infty)$$

Hence, we have the system $\dot{z}_c = \hat{A}_c z_c + \hat{B}_c v$, $z_c(0) = \mathbf{0}_{n_c\times 1}$ is bounded input and bounded state stable by Lemma A.3. By the controllability of the pair $(\hat{A}_c, \hat{B}_c)$ and Lemma A.7, we have that $\hat{A}_c$ is Hurwitz.

We will show that the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ by an argument of contradiction. Suppose S_P is not minimum phase. $\exists c_w \in [0,\infty) \subset \mathbb{R}$, $\forall i \in \mathbb{N}$, $\exists x_{0(i)} \in \mathcal{D}_0$ with $|x_{0(i)}| \leq c_w$, $\exists w_{(i)[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{(i)[0,\infty)} \right\|_\infty \leq c_w$, $\exists y_{(i)[0,\infty)} \in \mathcal{C}$ with $\left\| y_{(i)[0,\infty)} \right\|_\infty \leq c_w$, such that $\left\| x_{(i)[0,\infty)} \right\|_\infty > i$, where $x_{(i)[0,\infty)}$ is the solution to (4.2a) with initial condition $x_{0(i)}$ and inputs $y_{(i)[0,\infty)}$ and $w_{(i)[0,\infty)}$. Partition $x_{(i)}$ as $\begin{bmatrix} x_{c(i)}' & x_{\bar{c}(i)}' \end{bmatrix}'$. Then, we have $\limsup_{i\to\infty} \left\| x_{\bar{c}(i)[0,\infty)} \right\|_\infty = +\infty$ since

$\hat{A}_c$ is Hurwitz ($\left\| x_{\bar{c}(i)[0,\infty)} \right\|_\infty =: \bar{c}_{wi}$, then, $\left\| x_{c(i)[0,\infty)} \right\|_\infty \le k_1 c_w + k_2 \sqrt{\bar{c}_{wi}^2 + 2c_w^2}$ for some $k_1, k_2 \in [0,\infty) \subset \mathbb{R}$ that is independent of $i \in \mathbb{N}$). Hence, $\forall i \in \mathbb{N}$, let $x_0 = x_{0(i)}$, $w_{[0,\infty)} = w_{(i)[0,\infty)}$, $\check{x}_0 = x_0$, and $y_{d[0,\infty)}$ to be the identically $\mathbf{0}_{q\times 1}$ function. By 1., $\exists c_c \in [0,\infty) \subset \mathbb{R}$, where c_c depends only on c_w and S_C, such that the closed-loop system admits unique solution $X_{\{i\}[0,\infty)}$ with continuous signals $u_{\{i\}[0,\infty)}$ and $y_{\{i\}[0,\infty)}$ and $\left\| x_{\{i\}[0,\infty)} \right\|_\infty \le c_c$. Note that $x_{\{i\}[0,\infty)}$ is the unique solution to (4.4) with initial condition $x_{0(i)}$ and inputs $y_{\{i\}[0,\infty)}$ and $w_{(i)[0,\infty)}$. By the uniqueness of solution to $x_{\bar{c}}$ dynamics, we have $x_{\bar{c}\{i\}[0,\infty)} = x_{\bar{c}(i)[0,\infty)}$. Hence, $\limsup_{i\to\infty} \left\| x_{\{i\}[0,\infty)} \right\|_\infty \ge \limsup_{i\to\infty} \left\| x_{\bar{c}\{i\}[0,\infty)} \right\|_\infty = \limsup_{i\to\infty} \left\| x_{\bar{c}(i)[0,\infty)} \right\|_\infty = +\infty$. This is a contradiction. Hence, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

Case 3.3: $n_c = n$. Then, the pair $(\hat{A}, \hat{B})$ is controllable. By Claim 4.1.1 and Lemma A.7, we have that the matrix $\hat{A}$ is Hurwitz. Then, by Lemma 3.10, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

This completes the proof for Case 3.

Case 4: $1 \le r < n$. By Lemma 3.1, the system S_P admits the extended zero dynamics canonical form representation (3.2). We need the following result.

CLAIM 4.1.2. *The system $\dot{z} = A_z z + A_{z1} v$; $z(0) = \mathbf{0}_{n_z\times 1}$ is bounded input and bounded state stable, where $n_z := n - r$.*

Proof. Suppose that the claim is false. By Corollary A.10, $\exists v_{[0,\infty)} \in \mathcal{C}_r$ such that $\left\| v^{[r]}_{[0,\infty)} \right\|_\infty =: c_v \in (0,\infty) \subset \mathbb{R}$ and $\left\| z_{[0,\infty)} \right\|_\infty = +\infty$. Choose $w_{[0,\infty)}$ to be the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$ and $x_{z0} \in \mathcal{D}_{z0}$. Let $y_{d[0,\infty)} = \rho v_{[0,\infty)}$. Clearly, we have $|Y_{d0}| \le c_v$ and $y^{(r)}_{d[0,\infty)} \in \mathcal{C}$ with $\left\| Y_{d[0,\infty)} \right\|_\infty \le c_v$. By 3., $\exists x_0 \in \mathcal{D}_0$ whose first n_z coordinates equal to x_{z0}, and let $\check{x}_0 = x_0 \in \mathcal{D}_0$ such that, under the model reference controller S_C, the closed-loop system admits a solution $X_{[0,\infty)}$ with $x_{1[0,\infty)} = y_{[0,\infty)} = y_{d[0,\infty)} = \rho v_{[0,\infty)}$. Let $c_w = \max\{c_v, |x_0|\} \in (0,\infty) \subset \mathbb{R}$. By 1., $\exists c_c \in [0,\infty) \subset \mathbb{R}$, where c_c depends only on c_w and S_C, such that the unique solution $X_{[0,\infty)}$ to the closed-loop system satisfies $\left\| x_{[0,\infty)} \right\|_\infty \le c_c$, and $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$. Note that $x_{z[0,\infty)}$ is the unique solution to $\dot{x}_z = A_z x_z + A_{z1} y_d$, $x_z(0) = x_{z0}$. By linearity, we have $x_{z[0,\infty)} = \rho z_{[0,\infty)} + x_{zI[0,\infty)}$, where $x_{zI[0,\infty)}$ satisfies $\dot{x}_{zI} = A_z x_{zI}$, $x_{zI}(0) = x_{z0}$. Hence, we have $\left\| x_{[0,\infty)} \right\|_\infty \ge \left\| x_{z[0,\infty)} \right\|_\infty = +\infty$ for some $\rho \in (0,1)$. This is a contradiction. This completes the proof of the claim. ∎

Back to proof of the lemma. Let $P_z := \begin{bmatrix} A_{z1} & A_z A_{z1} & \cdots & A_z^{n_z-1} A_{z1} \end{bmatrix}$ and $n_{zc} := \operatorname{rank}(P_z)$. We will further distinguish 3 exhaustive and mutually exclusive cases: Case 4.1: $n_{zc} = 0$; Case 4.2: $1 \le n_{zc} < n_z$; Case 4.3: $n_{zc} = n_z$.

Case 4.1: $n_{zc} = 0$. Then, we have $A_{z1} = \mathbf{0}_{n_z\times 1}$. Then, the extended zero dynamics is given by

$$\dot{x}_z = A_z x_z + D_z w; \qquad x_z(0) = x_{z0} \tag{4.5}$$

By the assumption, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_{z0} \in \mathcal{D}_{z0}$ with $|x_{z0}| \le c_w$, let $x_0 = \begin{bmatrix} x'_{z0} & \mathbf{0}_{1\times r} \end{bmatrix}' \in \mathcal{D}_{z0} \times \mathbb{R}^r = \mathcal{D}_0$ and $\check{x}_0 = x_0$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, let $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)} \in \mathcal{C}_r$ with $\left\| Y_{d[0,\infty)} \right\|_\infty = 0$ and $|Y_{d0}| = 0$, then, there exists a unique solution $X_{[0,\infty)}$ to the closed-loop system with $\left\| x_{[0,\infty)} \right\|_\infty \le c_c$. This implies that $\left\| x_{z[0,\infty)} \right\|_\infty \le \left\| x_{[0,\infty)} \right\|_\infty \le c_c$ and $x_{z[0,\infty)}$ is the unique solution to (4.5). Since the extended zero dynamics (4.5) is independent of $x_{1[0,\infty)}$, then the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

Case 4.2: $1 \leq n_{zc} < n_z$. Without loss of generality, assume that (3.2a) is partitioned into controllable and uncontrollable parts (with respect to x_1). We have $x_z = \begin{bmatrix} x'_{zc} & x'_{z\bar{c}} \end{bmatrix}'$

$$\begin{bmatrix} \dot{x}_{zc} \\ \dot{x}_{z\bar{c}} \end{bmatrix} = \begin{bmatrix} A_{zc} & A_{zc\bar{c}} \\ \mathbf{0} & A_{z\bar{c}} \end{bmatrix} \begin{bmatrix} x_{zc} \\ x_{z\bar{c}} \end{bmatrix} + \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} x_1 + \begin{bmatrix} D_{zc} \\ D_{z\bar{c}} \end{bmatrix} w \tag{4.6}$$

where x_{zc} is n_{zc}-dimensional; $x_{z\bar{c}}$ is $n_{z\bar{c}} := n_z - n_{zc}$ dimensional; and the pair (A_{zc}, A_{zc1}) is controllable.

By Claim 4.1.2 and Lemma A.3, $\exists k \in [0, \infty) \subset \mathbb{R}$, $\exists \lambda \in (0, \infty) \subset \mathbb{R}$, such that $\left\| \mathrm{e}^{A_z t} A_{z1} \right\|_{2,2} \leq k \mathrm{e}^{-\lambda t}$, $\forall t \in [0, \infty)$. Note that, $\forall t \in [0, \infty)$,

$$\left\| \mathrm{e}^{A_z t} A_{z1} \right\|_{2,2} = \left\| \begin{bmatrix} \mathrm{e}^{A_{zc} t} & \star \\ \mathbf{0} & \mathrm{e}^{A_{z\bar{c}} t} \end{bmatrix} \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} = \left\| \mathrm{e}^{A_{zc} t} A_{zc1} \right\|_{2,2} \leq k \mathrm{e}^{-\lambda t}$$

Hence, we have that the system $\dot{z}_c = A_{zc} z_c + A_{zc1} v$, $z_c(0) = \mathbf{0}_{n_{zc} \times 1}$ is bounded input and bounded state stable by Lemma A.3. By the controllability of the pair (A_{zc}, A_{zc1}) and Lemma A.7, we have that A_{zc} is Hurwitz.

We will show that the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ by an argument of contradiction. Suppose S_P is not minimum phase. $\exists c_w \in [0, \infty) \subset \mathbb{R}$, $\forall i \in \mathbb{N}$, $\exists x_{z0(i)} \in \mathcal{D}_{z0}$ with $|x_{z0(i)}| \leq c_w$, $\exists w_{(i)[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{(i)[0,\infty)} \right\|_\infty \leq c_w$, $\exists x_{1(i)[0,\infty)} \in \mathcal{C}_r$ with $\left\| x^{[r]}_{1(i)[0,\infty)} \right\|_\infty \leq c_w$, such that $\left\| x_{z(i)[0,\infty)} \right\|_\infty > i$, where $x_{z(i)[0,\infty)}$ is the solution to (4.6) with initial condition specified by $x_{z0(i)}$ and inputs $x_{1(i)[0,\infty)}$ and $w_{(i)[0,\infty)}$. Partition $x_{z(i)}$ as $\begin{bmatrix} x'_{zc(i)} & x'_{z\bar{c}(i)} \end{bmatrix}'$. Then, we have $\limsup_{i\to\infty} \left\| x_{z\bar{c}(i)[0,\infty)} \right\|_\infty = +\infty$ since A_{zc} is Hurwitz ($\left\| x_{z\bar{c}(i)[0,\infty)} \right\|_\infty =: \bar{c}_{wi}$, then, $\left\| x_{zc(i)[0,\infty)} \right\|_\infty \leq k_1 c_w + k_2 \sqrt{\bar{c}^2_{wi} + 2c^2_w}$, for some $k_1, k_2 \in [0, \infty) \subset \mathbb{R}$ that is independent of $i \in \mathbb{N}$). Let $x_0 = x_{0(i)} := \begin{bmatrix} x'_{z0(i)} & \mathbf{0}_{1\times r} \end{bmatrix}' \in \mathcal{D}_0$, $w_{[0,\infty)} = w_{(i)[0,\infty)}$, $\check{x}_0 = x_0$, and $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)}$ with $\left\| Y_{d[0,\infty)} \right\|_\infty = 0$ and $|Y_{d0}| = 0$. By 1., $\exists c_c \in [0, \infty) \subset \mathbb{R}$, where c_c depends only on c_w and S_C, such that the closed-loop system admits unique solution $X_{\{i\}[0,\infty)}$ with continuous signals $u_{\{i\}[0,\infty)}$ and $y_{\{i\}[0,\infty)}$ and $\left\| x_{\{i\}[0,\infty)} \right\|_\infty \leq c_c$. Note that $x_{\{i\}[0,\infty)}$ is the unique solution to (3.2) with initial condition $x_{0(i)}$ and inputs $u_{\{i\}[0,\infty)}$ and $w_{(i)[0,\infty)}$. Then, the component $x_{z\bar{c}\{i\}[0,\infty)}$ of $x_{\{i\}[0,\infty)}$ is the unique solution to $x_{z\bar{c}}$ dynamics in (4.6), which is independent of $x_{1\{i\}[0,\infty)}$. We have $x_{z\bar{c}\{i\}[0,\infty)} = x_{z\bar{c}(i)[0,\infty)}$. Hence, $\limsup_{i\to\infty} \left\| x_{\{i\}[0,\infty)} \right\|_\infty \geq \limsup_{i\to\infty} \left\| x_{z\bar{c}\{i\}[0,\infty)} \right\|_\infty = \limsup_{i\to\infty} \left\| x_{z\bar{c}(i)[0,\infty)} \right\|_\infty = +\infty$. This is a contradiction. Hence, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

Case 4.3: $n_{zc} = n_z$. Then, the pair (A_z, A_{z1}) is controllable. By Claim 4.1.2 and Lemma A.7, we have that the matrix A_z is Hurwitz. Then, by Lemma 3.10, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

This completes the proof of the lemma. □

5. Conclusions. In this paper, we have obtained the extended zero dynamics canonical form for finite-dimensional continuous-time SISO LTI systems with a finite relative degree r. Based on the canonical form, we introduced the definition of extended zero dynamics for the system, which is the zero dynamics defined in [4] together with driving terms involving the noiseless output of the system and the disturbance input. Then, we generalized the concept of minimum phase for such systems that accommodates the disturbance input and subsumes the classical definition of minimum phase. A system is said to be minimum phase with respect to the set of admissible initial conditions and the set of admissible disturbance input waveforms if the extended zero dynamics satisfies that, for any bounded admissible initial condition, any bounded noiseless output, and any bounded admissible disturbance input waveform, the zero dynamics state trajectory is bounded or if the extended zero dynamics is absent. If the extended zero dynamics is asymptotically stable, then the system is clearly minimum phase. The converse holds under the additional assumption that the system is stabilizable from the control input. For a system to be minimum phase, it is necessary that the transfer function from the control input to the output has all zeros with negative real parts. The converse holds when the system is both controllable (from the control input) and observable. An example is presented to illustrate the generalized concept of minimum phase that the extended zero dynamics does not need to be asymptotically stable or bounded input and bounded state stable in the usual sense. It is further shown that the generalized minimum phase property is necessary for the achievement of perfect tracking of any bounded reference trajectories with bounded derivatives up to rth order without any disturbances and the existence of bounded state trajectory for any admissible bounded initial condition, any admissible bounded disturbance waveform, and any bounded reference trajectory with bounded derivatives up to rth order in model reference control of the system.

We envision that this generalized definition of minimum phase will lead to broader results for model reference adaptive control. Future research on this topic is under way to investigate the minimum phase properties for feedback interconnected systems and to investigate the boundedness of the inverse of a minimum phase system [6].

Appendix A. A number of useful results. We first present two lemmas that are standard in linear system theory [2].

LEMMA A.1. *Consider finite-dimensional continuous-time LTI system:*

$$\dot{x} = Ax + Bu; \qquad x(0) = x_0 \tag{A.1a}$$

$$y = Cx + Ku \tag{A.1b}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; $x_0 \in \mathbb{R}^n$; u is the p-dimensional input, $p \in \mathbb{Z}_+$; and y is the m-dimensional output, $m \in \mathbb{Z}_+$; A, B, C, and K are constant matrices of appropriate dimensions. Let $u_{[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}^p)$. Then, $y_{[0,\infty)} \in \mathcal{C}([0,\infty), \mathbb{R}^m)$ and $x_{[0,\infty)} \in \mathcal{C}_1([0,\infty), \mathbb{R}^n)$ are given by, $\forall t \in [0,\infty) \subset \mathbb{R}$,

$$x(t) = \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A\,(t-\tau)} B u(\tau)\, \mathrm{d}\tau \tag{A.2a}$$

$$y(t) = C\mathrm{e}^{At} x_0 + \int_0^t C\mathrm{e}^{A\,(t-\tau)} B u(\tau)\, \mathrm{d}\tau + Ku(t) \tag{A.2b}$$

LEMMA A.2. *Let $n \in \mathbb{Z}_+$ and A be an $n \times n$-dimensional Hurwitz matrix. Then, $\exists k \in [0,\infty) \subset \mathbb{R}$ and $\lambda \in (0,\infty) \subset \mathbb{R}$ such that $\left\| \mathrm{e}^{At} \right\|_{2,2} \leq k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty) \subset \mathbb{R}$.*

We present here a lemma on the bounded input and bounded output stability of LTI systems.

LEMMA A.3. *Consider finite-dimensional continuous-time LTI system:*

$$S: \quad \begin{cases} \dot{x} = Ax + Bu; \qquad x(0) = x_0 = \mathbf{0}_{n\times 1} \\ y = Cx + Du \end{cases} \tag{A.3}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; u is the p-dimensional input, $p \in \mathbb{Z}_+$; and y is the m-dimensional output, $m \in \mathbb{Z}_+$; A, B, C, and D are constant matrices of appropriate dimensions. Then, the following statements are equivalent.

1. *There exist $k \in [0,\infty) \subset \mathbb{R}$ and $\lambda \in (0,\infty) \subset \mathbb{R}$ such that $\|Ce^{At}B\|_{2,2} \leq k e^{-\lambda t}$, $\forall t \in [0,\infty)$.*
2. *Viewed as a linear operator $S : \mathcal{L}_q([0,\infty),\mathbb{R}^p) \rightarrow \mathcal{L}_q([0,\infty),\mathbb{R}^m)$, where $q \in [1,\infty] \subset \mathbb{R}_e$, the system S is bounded.*
3. *$\forall c_u \in [0,\infty) \subset \mathbb{R}$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$ with $\|u_{[0,t_f)}\|_\infty \leq c_u$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, such that $\|y_{[0,t_f)}\|_\infty \leq c_c$.*
4. *$\forall c_u \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$ with $\|u_{[0,t_f)}\|_\infty \leq c_u$, we have $\|y_{[0,t_f)}\|_\infty \leq c_c$.*

Proof. The proof of the lemma is standard and follows the path 1 ⇒ 2 ⇒ 4 ⇒ 3 ⇒ 1. The proof is omitted here. □

DEFINITION A.4. *For the finite-dimensional continuous-time LTI system (A.3) under the assumptions of Lemma A.3, we say that S is* bounded input and bounded output stable *(BIBO) if any of 1 — 4 holds. If, in addition, $C = I_n$ and $D = \mathbf{0}_{m\times p}$, then we say that S is* bounded input and bounded state stable *(BIBS).*

Next, we present a canonical form for a finite-dimensional SISO LTI system that is both controllable and observable.

LEMMA A.5. *Consider a finite-dimensional continuous-time SISO LTI system*

$$\dot{x} = Ax + Bu \tag{A.4a}$$
$$y = Cx \tag{A.4b}$$

where x is the n-dimensional state vector; u is a scalar control input; y is a scalar output; and $n \in \mathbb{N}$. Assume the triple (A, B, C) is controllable and observable. Let the transfer function from u to y be given by, $b_0 \neq 0$,

$$H(s) = C\,(sI_n - A)^{-1}B = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{\sum_{i=0}^m b_i s^{m-i}}{s^n + \sum_{i=1}^n a_i s^{n-i}} \tag{A.5}$$

which admits relative degree $r = n - m \in \{1, \ldots, n\}$. Then, there exists a unique coordinate transformation $\bar{x} = Tx$, within the class of $\mathcal{C}_1$ diffeomorphisms, where T is invertible, such that, in the $\bar{x}$ coordinate, the system (A.4) admits the state space representation

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{A.6a}$$
$$y = \bar{C}\bar{x} \tag{A.6b}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}; \quad \bar{A}_{11} = \left[\begin{array}{c|c} \bar{a}_1 & \\ \vdots & I_{r-1} \\ \bar{a}_{r-1} & \\ \hline \bar{a}_r & \mathbf{0}_{1\times(r-1)} \end{array}\right]$$

$$\bar{A}_{12} = \begin{cases} \begin{bmatrix} & \mathbf{0} & \\ 1 & & \end{bmatrix} & m \geq 1 \\ [\ \ ] & m = 0 \end{cases} \qquad \bar{A}_{21} = \left[\begin{array}{c|c} \bar{a}_{r+1} & \\ \vdots & \mathbf{0}_{(n-r)\times(r-1)} \\ \bar{a}_n & \end{array}\right]$$

$$\bar{A}_{22} = \begin{cases} \left[\begin{array}{c|c} -b_1/b_0 & \\ \vdots & I_{m-1} \\ -b_{m-1}/b_0 & \\ \hline -b_m/b_0 & \mathbf{0}_{1\times(m-1)} \end{array}\right] & m \geq 1 \\ [\ ] & m = 0 \end{cases} \qquad \begin{array}{l} \bar{B} = \left[\begin{array}{ccc} \mathbf{0}_{1\times(r-1)} & b_0 & \mathbf{0}_{1\times m} \end{array}\right]' \\ \bar{C} = \left[\begin{array}{cc} 1 & \mathbf{0}_{1\times(n-1)} \end{array}\right] \end{array}$$

and the constants $\bar{a}_1, \ldots, \bar{a}_n$ *are uniquely determined by*

$$\text{(A.7)} \qquad s^n + \sum_{i=1}^{n} a_i s^{n-i} = \big(s^r - \sum_{i=1}^{r} \bar{a}_i s^{r-i}\big)\big(\sum_{i=0}^{m} b_i s^{m-i}\big)/b_0 - \sum_{i=r+1}^{n} \bar{a}_i s^{n-i}$$

Proof. We will first show the existence of a real transformation $\bar{x} = Tx$ that transforms the system into the canonical form (A.6). Since the pair (A, C) is observable, there exists a real coordinate transformation $z = T_0 x$, such that the system (A.4) admits the observer canonical form representation in the z coordinates,

$$\dot{z} = \left[\begin{array}{c|c} -a_1 & \\ \vdots & I_{n-1} \\ -a_{n-1} & \\ \hline -a_n & \mathbf{0}_{1\times(n-1)} \end{array}\right] z + \left[\begin{array}{c} \mathbf{0}_{(r-1)\times 1} \\ b_0 \\ \vdots \\ b_m \end{array}\right] u =: A_0 z + B_0 u$$
$$y = \bar{C} z$$

Now, we will distinguish 2 exhaustive and mutually exclusive cases: Case 1: $r = n$; Case 2: $r < n$.

Case 1: $r = n$. In this case, the system representation in z coordinate is in the canonical form (A.6). Then, the desired T is T_0 for this case.

Case 2: $r < n$. In this case, we will introduce a sequence of r state transformations that will transform the system into the representation (A.6).

By the first transformation, we will transform B_0 into $\bar{B}$. Let $z_1 = T_1 z$, where T_1 is the same as I_n except the rth column, which equals to

$$\left[\begin{array}{ccccc} \mathbf{0}_{1\times(r-1)} & 1 & -b_1/b_0 & \cdots & -b_m/b_0 \end{array}\right]'$$

Then, its inverse T_1^{-1} is also the same as I_n except the rth column, which equals to $\left[\begin{array}{ccccc} \mathbf{0}_{1\times(r-1)} & 1 & b_1/b_0 & \cdots & b_m/b_0 \end{array}\right]'$. This transforms the system (A.4) into

$$\dot{z}_1 = A_1 z_1 + \bar{B} u$$
$$y = \bar{C} z_1$$

where the matrix $A_1 = T_1 A_0 T_1^{-1}$. Now, we will further distinguish 2 exhaustive and mutually exclusive sub-cases: Case 2a: $r = 1$; Case 2b: $r > 1$.

Case 2a: $r = 1$. In this case, A_1 is in the same form as $\bar{A}$. Hence, the desired $T = T_1 T_0$ for this sub-case.

Case 2b: $r > 1$. In this case, A_1 is the same as the matrix $\bar{A}$ except the 1st and rth columns, which are $\left[\begin{array}{cccc} a_{11} & a_{21} & \cdots & a_{n1} \end{array}\right]'$ and $\left[\begin{array}{ccccc} \mathbf{0}_{1\times(r-2)} & 1 & d_{11} & \cdots & d_{m+1\,1} \end{array}\right]'$, respectively, and a_{j1}, $j = 1, \ldots, n$ and d_{j1}, $j = 1, \ldots, m+1$, are some constants in $\mathbb{R}$. We will continue in this case to introduce additional $r - 1$ transformations.

By the ith $(2 \leq i \leq r-1)$ transformation, we will transform the $(r-i+2)$nd column of the matrix A_i into the $(r-i+1)$st unit vector. Let $z_i = T_i z_{i-1}$, where T_i is the same as I_n except the $(r-i+1)$st column, which equals to

$$\left[\begin{array}{ccccc} \mathbf{0}_{1\times(r-i)} & 1 & -d_{1\,i-1} & \cdots & -d_{m+i-1\,i-1} \end{array}\right]'$$

Then, its inverse T_i^{-1} is also the same as I_n except the $(r-i+1)$st column, which equals to $\left[\begin{array}{cccc} \mathbf{0}_{1\times(r-i)} & 1 & d_{1\,i-1} & \cdots & d_{m+i-1\,i-1} \end{array}\right]'$. This transforms the system (A.4) into

$$\begin{aligned} \dot{z}_i &= A_i z_i + \bar{B} u \\ y &= \bar{C} z_i \end{aligned}$$

where the matrix $A_i = T_i A_{i-1} T_i^{-1}$ is the same as the matrix $\bar{A}$ except the 1st and $(r-i+1)$st columns, which are

$$\left[\begin{array}{cccc} a_{1i} & a_{2i} & \cdots & a_{ni} \end{array}\right]' \text{ and } \left[\begin{array}{ccccc} \mathbf{0}_{1\times(r-i-1)} & 1 & d_{1i} & \cdots & d_{m+i\,i} \end{array}\right]'$$

respectively, and a_{ji}, $j = 1, \ldots, n$ and d_{ji}, $j = 1, \ldots, m+i$, are some constants in $\mathbb{R}$.

By the rth transformation, which is the last one, we will transform the 2nd column of the matrix A_{r-1} into the 1st unit vector. Let $z_r = T_r z_{r-1}$, where T_r is the same as I_n except the 1st column, which equals to $\left[\begin{array}{cccc} 1 & -d_{1\,r-1} & \cdots & -d_{n-1\,r-1} \end{array}\right]'$. Then, its inverse T_r^{-1} is also the same as I_n except the 1st column, which equals to $\left[\begin{array}{cccc} 1 & d_{1\,r-1} & \cdots & d_{n-1\,r-1} \end{array}\right]'$. This transforms the system (A.4) into

$$\begin{aligned} \dot{z}_r &= A_r z_r + \bar{B} u \\ y &= \bar{C} z_r \end{aligned}$$

where the matrix $A_r = T_r A_{r-1} T_r^{-1}$ is the same as the matrix $\bar{A}$. Hence, the desired state transformation is given by $\bar{x} = z_r = T_r \cdots T_1 T_0 x =: Tx$ for this sub-case.

Thus, in all cases, we have shown the existence of a real transformation $\bar{x} = Tx$ that transforms the system into the canonical form (A.6).

Next, we show that the constants $\bar{a}_1, \ldots, \bar{a}_n$ are uniquely determined by (A.7). Note that

$$H(s) = \bar{C}\,(sI_n - \bar{A})^{-1}\bar{B} = b_0 M_{1r}(s)$$

where M_{1r} is the $1r$-element of $(sI_n - \bar{A})^{-1}$. By, Cramer's rule, the following holds,

$$H(s) = \frac{b_0\,(-1)^{r+1}\det(\bar{\Delta}_{r1})}{\det(sI_n - \bar{A})}$$

where $\bar{\Delta}_{r1}$ is the matrix obtained from $sI_n - \bar{A}$ by removing the rth row and the 1st column. It is clear that

$$\det(\bar{\Delta}_{r1}) = \left\{ \begin{array}{ll} (-1)^{r-1}\det(\bar{\Delta}) & m \geq 1 \\ (-1)^{r-1} & m = 0 \end{array} \right. = (-1)^{r-1}/b_0 \sum_{i=0}^{m} b_i s^{m-i}$$

where $\bar{\Delta} = sI_m - \bar{A}_{22}$.

Let $\Delta_n = sI_n - \bar{A}$. We will distinguish 3 exhaustive and mutually exclusive cases: Case A: $m = 0$; Case B: $m = 1$; Case C: $m \geq 2$.

Case A: $m = 0$. In this case, we have

$$\det(\Delta_n) = s^n - \sum_{i=1}^{n} \bar{a}_i s^{n-i} = \big(s^r - \sum_{i=1}^{r} \bar{a}_i s^{r-i}\big)\big(\sum_{i=0}^{m} b_i s^{m-i}\big)/b_0 - \sum_{i=r+1}^{n} \bar{a}_i s^{n-i}$$

Case B: $m = 1$. In this case, by expanding the last row of Δ_n using the Laplace Theorem, we have

$$\begin{aligned} \det(\Delta_n) &= (-\bar{a}_n)\,(-1)^{n+1}\det(\bar{\Delta}_{n,n1}) + (s + b_1/b_0)\det(sI_r - \bar{A}_{11}) \\ &= \big(s^r - \sum_{i=1}^{r} \bar{a}_i s^{r-i}\big)\big(\sum_{i=0}^{m} b_i s^{m-i}\big)/b_0 - \sum_{i=r+1}^{n} \bar{a}_i s^{n-i} \end{aligned}$$

where $\bar{\Delta}_{n,n1}$ is a matrix obtained from Δ_n by removing the nth row and the 1st column.

Case C: $m \geq 2$. By expanding the last row of Δ_n using the Laplace Theorem, we have

$$\begin{aligned}\det(\Delta_n) = (-\bar{a}_n)\,(-1)^{n+1}\det(\bar{\Delta}_{n,n1}) + (b_m/b_0)\,(-1)^{n+r+1}\det(\bar{\Delta}_{n,n\,r+1})\\ +s\det(\bar{\Delta}_{n,nn})\end{aligned}$$

where $\bar{\Delta}_{n,ij}$ denotes the matrix obtained from Δ_n by removing the ith row and the jth column, $i, j = 1, \ldots, n$. It is clear that

$$\det(\bar{\Delta}_{n,n1}) = (-1)^{n-1}$$

$$\det(\bar{\Delta}_{n,n\,r+1}) = (-1)^{m-1}\det(sI_r - \bar{A}_{11}) = (-1)^{m-1}\left(s^r - \sum_{i=1}^{r}\bar{a}_i s^{r-i}\right)$$

Let $\Delta_{n-1} = \bar{\Delta}_{n,nn}$, which is a sub-matrix of Δ_n consisting of the first $n-1$ rows and first $n-1$ columns. Then, we have

$$\det(\Delta_n) = -\bar{a}_n + (b_m/b_0)\left(s^r - \sum_{i=1}^{r}\bar{a}_i s^{r-i}\right) + s\det(\Delta_{n-1})$$

Repeated application of the Laplace Theorem by expanding the last row of Δ_i, $i = n-1, \ldots, r+2$, we have

$$\det(\Delta_i) = -\bar{a}_i + (b_{i-r}/b_0)\,(s^r - \bar{a}_1 s^{r-1} - \cdots - \bar{a}_r) + s\det(\Delta_{i-1})$$

where Δ_{i-1} is the matrix obtained from Δ_i by removing the last row and the last column. The matrix Δ_i is a sub-matrix of Δ_n consisting of the first i rows and first i columns.

Apply the Laplace Theorem to $\det(\Delta_{r+1})$ by expanding its last row, we have

$$\det(\Delta_{r+1}) = (-\bar{a}_{r+1})\,(-1)^{r+2}\det(\bar{\Delta}_{r+1,r+1\,1}) + (s + b_1/b_0)\det(sI_r - \bar{A}_{11})$$

where $\bar{\Delta}_{r+1,r+1\,1}$ is a matrix obtained from Δ_{r+1} by removing the $(r+1)$st row and the 1st column. It is clear $\det(\bar{\Delta}_{r+1,r+1\,1}) = (-1)^r$. Then,

$$\det(\Delta_{r+1}) = -\bar{a}_{r+1} + (s + b_1/b_0)\left(s^r - \sum_{i=1}^{r}\bar{a}_i s^{r-i}\right)$$

Based on the recursive formula for $\det(\Delta_i)$, we conclude

$$\det(\Delta_n) = \left(s^r - \sum_{i=1}^{r}\bar{a}_i s^{r-i}\right)\left(\sum_{i=0}^{m} b_i s^{m-i}\right)/b_0 - \sum_{i=r+1}^{n}\bar{a}_i s^{n-i}$$

In all three cases, we obtain $\det(\Delta_n) = \left(s^r - \sum_{i=1}^{r}\bar{a}_i s^{r-i}\right)\left(\sum_{i=0}^{m} b_i s^{m-i}\right)/b_0 - \sum_{i=r+1}^{n}\bar{a}_i s^{n-i}$. Then,

$$H(s) = \frac{\sum_{i=0}^{m} b_i s^{m-i}}{\det(\Delta_n)}$$

Compare the above formula with (A.5), we obtain (A.7). Clearly, the constants $\bar{a}_1$, $\ldots$, $\bar{a}_n$ are uniquely determined by (A.7) since $s^r - \sum_{i=1}^{r}\bar{a}_i s^{r-i}$ and $-\sum_{i=r+1}^{n}\bar{a}_i s^{n-i}$

are the quotient and remainder of $s^n + \sum_{i=1}^n a_i s^{n-i}$ divided by $\left(\sum_{i=0}^m b_i s^{m-i}\right)/b_0$, respectively.

Finally, we prove the uniqueness of the transformation $\bar{x} = Tx$ within the class of all $\mathcal{C}_1$ diffeomorphisms. Let $\bar{x} = T(x)$ be any $\mathcal{C}_1$ diffeomorphism that leads to the state space representation (A.6). Denote the elements of $\bar{x}$ by $\bar{x} = \left[\begin{array}{ccc} \bar{x}_1 & \cdots & \bar{x}_n \end{array}\right]'$. Then, we have $\bar{x}_i = T_i(x)$, $i = 1, 2, \ldots, n$. Note that $y = \bar{C}\bar{x} = \bar{x}_1 = Cx$, then $T_1(x) = Cx =: M_1 x$. We will obtain a recursive formula for $T_i(x)$, $i = 1, \ldots, r$. When $i = 1$, we have $\bar{x}_1 = T_1(x) = M_1 x$. Assume that we have shown that $\bar{x}_i = T_i(x) = M_i x$, $i = 1, \ldots, k$, where $1 \le k < r$. Then, take derivative of $\bar{x}_k$, we have

$$\dot{\bar{x}}_k = \bar{a}_k \bar{x}_1 + \bar{x}_{k+1} = M_k \left(Ax + Bu\right)$$

This implies that

$$\bar{x}_{k+1} = T_{k+1}(x) = M_k A x - \bar{a}_k M_1 x =: M_{k+1} x$$

and $M_k B = 0$. The above induction argument shows that $\bar{x}_k = T_k(x) = M_k x$, $k = 1, \ldots, r$, and M_k satisfies the recursive formula

$$M_1 = C; \qquad M_{k+1} = M_k A - \bar{a}_k M_1, \quad k = 1, \ldots, r-1$$

Now, we will distinguish 4 exhaustive and mutually exclusive cases: Case I: $n = 1$; Case II: $r = n \ge 2$; Case III: $r = n - 1 \ge 1$; Case IV: $1 \le r \le n - 2$.

Case I: $n = 1$. Clearly, $T(x) = T_1(x) = M_1 x$ is unique. Case II: $r = n \ge 2$. We have $T_i(x) = M_i x$, $i = 1, \ldots, n$, such that $M_1 = C$; $M_{k+1} = M_k A - \bar{a}_k M_1$, $k = 1, \ldots, n-1$. This shows that $T(x)$ is uniquely determined in this case.

Case III: $1 \le r = n - 1$. Take the derivative of $\bar{x}_r$ to yield,

$$\dot{\bar{x}}_{n-1} = \bar{a}_{n-1} \bar{x}_1 + \bar{x}_n + b_0 u = M_{n-1} \left(Ax + Bu\right)$$

This implies that

$$\bar{x}_n = T_n(x) = M_{n-1} A x - \bar{a}_{n-1} M_1 x =: M_n x$$

and $M_n B = b_0$. Hence, $\bar{x}_k = T_k(x) = M_k x$, $k = 1, \ldots, n$, and M_k satisfies

$$M_1 = C; \qquad M_{k+1} = M_k A - \bar{a}_k M_1, \quad k = 1, \ldots, n-1$$

This shows that $T(x)$ is unique in this case.

Case IV: $1 \le r \le n - 2$. Take the derivative of $\bar{x}_r$ to yield

$$\dot{\bar{x}}_r = \bar{a}_r \bar{x}_1 + \bar{x}_{r+1} + b_0 u = M_r \left(Ax + Bu\right)$$

Then, we have $\bar{x}_{r+1} = T_{r+1}(x) = M_r A x - \bar{a}_r M_1 x =: M_{r+1} x$ and $M_r B = b_0$.

Assume that we have shown that $\bar{x}_i = T_i(x) = M_i x$, $i = 1, \ldots, k$, where $r + 1 \le k < n$. Take the derivative of $\bar{x}_k$ to yield

$$\dot{\bar{x}}_k = \bar{a}_k \bar{x}_1 - b_{k-r}/b_0 \bar{x}_{r+1} + \bar{x}_{k+1} = M_k \left(Ax + Bu\right)$$

This implies that

$$\bar{x}_{k+1} = T_{k+1}(x) = M_k A x - \bar{a}_k M_1 x + b_{k-r}/b_0 M_{r+1} x =: M_{k+1} x,$$

and $M_k B = 0$. The above induction argument shows that $\bar{x}_k = T_k(x) = M_k x$, $k = 1, \dots, n$, and M_k satisfies

$$M_1 = C; \qquad M_{k+1} = M_k A - \bar{a}_k M_1,\ k = 1, \dots, r$$
$$M_{k+1} = M_k A - \bar{a}_k M_1 + b_{k-r}/b_0 M_{r+1}, \quad k = r+1, \dots, n-1$$

Hence, $T(x)$ is uniquely determined in this case.

In all cases, we have shown the uniqueness of $T(x)$.

This completes the proof of the lemma. ∎

Here, we present the real Jordan canonical form for a square matrix.

LEMMA A.6. *Consider an $n \times n$-dimensional real matrix A, $n \in \mathbb{Z}_+$, with l distinct real eigenvalues λ_i, $i = 1, \dots, l$, $l \in \mathbb{Z}_+$, and k pairs of distinct complex conjugate eigenvalues μ_j, $\bar{\mu}_j$, $j = 1, \dots, k$, $k \in \mathbb{Z}_+$. Then, there exist an $n \times n$ real invertible matrix $\hat{T}$ such that A can be transformed into the real Jordan canonical form representation, $J_R = \hat{T}^{-1} A \hat{T}$,*

$$J_R = \begin{bmatrix} A_1 &&&&&&& \\ & \ddots &&&&&& \\ && A_l &&&&& \\ &&& \mathrm{Re}\,(B_1) & \mathrm{Im}\,(B_1) &&& \\ &&& -\mathrm{Im}\,(B_1) & \mathrm{Re}\,(B_1) &&& \\ &&&&& \ddots && \\ &&&&&& \mathrm{Re}\,(B_k) & \mathrm{Im}\,(B_k) \\ &&&&&& -\mathrm{Im}\,(B_k) & \mathrm{Re}\,(B_k) \end{bmatrix} \tag{A.8}$$

where the matrices A_i, $i = 1, \dots, l$, and B_j, $j = 1, \dots, k$, admit the following structures

$$A_i = \begin{bmatrix} A_{i1} &&& \\ & \ddots && \\ && \ddots & \\ &&& A_{il_i} \end{bmatrix}; \quad B_j = \begin{bmatrix} B_{j1} &&& \\ & \ddots && \\ && \ddots & \\ &&& B_{jk_j} \end{bmatrix}; \ l_i \in \mathbb{N}; \ k_j \in \mathbb{N}$$

$$A_{ih} = \begin{bmatrix} \lambda_i & 1 && \\ & \ddots & \ddots & \\ && \ddots & 1 \\ &&& \lambda_i \end{bmatrix}_{n_{ih} \times n_{ih}}; \qquad h = 1, \dots, l_i; \quad n_{ih} \in \mathbb{N}$$

$$B_{jq} = \begin{bmatrix} \mu_j & 1 && \\ & \ddots & \ddots & \\ && \ddots & 1 \\ &&& \mu_j \end{bmatrix}_{m_{jq} \times m_{jq}}; \qquad q = 1, \dots, k_j; \quad m_{jq} \in \mathbb{N}$$

$$\sum_{i=1}^{l} \sum_{h=1}^{l_i} n_{ih} + 2 \sum_{j=1}^{k} \sum_{q=1}^{k_j} m_{jq} = n$$

Proof. Define

$$n_i = \sum_{h=1}^{l_i} n_{ih}, \quad m_j = \sum_{q=1}^{k_j} m_{jq}; \quad n_i \in \mathbb{N}; \quad m_j \in \mathbb{N}; \quad i = 1, \ldots, l; \quad j = 1, \ldots, k$$

Let $v_1^{\langle i \rangle}, \ldots, v_{n_i}^{\langle i \rangle}$ be a set of generalized eigenvectors associated with λ_i, $i = 1, \ldots, l$, as determined by procedures described in [2]; Let $u_1^{\langle j \rangle}, \ldots, u_{m_j}^{\langle j \rangle} \in \mathbb{C}^n$ be a set of generalized eigenvectors associated with μ_j, $j = 1, \ldots, k$, as determined by procedures described in [2]; Then, a set of generalized eigenvectors associated with $\bar{\mu}_j$ are $\bar{u}_1^{\langle j \rangle}, \ldots, \bar{u}_{m_j}^{\langle j \rangle} \in \mathbb{C}^n$, which are the complex conjugate of $u_l^{\langle j \rangle}$, $l = 1, \ldots, m_j$, $j = 1, \ldots, k$.

Define

$$T = [v_1^{\langle 1 \rangle} \ldots v_{n_1}^{\langle 1 \rangle} \ldots \ldots v_1^{\langle l \rangle} \ldots v_{n_l}^{\langle l \rangle}\; u_1^{\langle 1 \rangle} \ldots u_{m_1}^{\langle 1 \rangle}\; \bar{u}_1^{\langle 1 \rangle} \ldots \bar{u}_{m_1}^{\langle 1 \rangle} \ldots \ldots u_1^{\langle k \rangle} \ldots u_{m_k}^{\langle k \rangle} \\ \bar{u}_1^{\langle k \rangle} \ldots \bar{u}_{m_k}^{\langle k \rangle}]$$

This set of eigenvectors is linearly independent and can be used as a basis. Then, the representation of A with respect to T is the Jordan canonical form representation

$$J = T^{-1}AT = \begin{bmatrix} A_1 & & & & & & & \\ & \ddots & & & & & & \\ & & A_l & & & & & \\ & & & B_1 & & & & \\ & & & & \bar{B}_1 & & & \\ & & & & & \ddots & & \\ & & & & & & B_k & \\ & & & & & & & \bar{B}_k \end{bmatrix} \tag{A.9}$$

where $\bar{B}_j$ is the complex conjugate of B_j, $j = 1, \ldots, k$. Therefore, J is also a complex matrix, it can be transformed into a real matrix by the following similarity transformation

$$\begin{aligned} J_R &= Q^{-1}JQ \\ &= \begin{bmatrix} A_1 & & & & & & & \\ & \ddots & & & & & & \\ & & A_l & & & & & \\ & & & \mathrm{Re}\,(B_1) & \mathrm{Im}\,(B_1) & & & \\ & & & -\mathrm{Im}\,(B_1) & \mathrm{Re}\,(B_1) & & & \\ & & & & & \ddots & & \\ & & & & & & \mathrm{Re}\,(B_k) & \mathrm{Im}\,(B_k) \\ & & & & & & -\mathrm{Im}\,(B_k) & \mathrm{Re}\,(B_k) \end{bmatrix} \end{aligned}$$

where

$$Q = \begin{bmatrix} I_{n_1} & & & & & & & \\ & \ddots & & & & & & \\ & & I_{n_l} & & & & & \\ & & & 0.5I_{m_1} & -0.5\mathrm{i}I_{m_1} & & & \\ & & & 0.5I_{m_1} & 0.5\mathrm{i}I_{m_1} & & & \\ & & & & & \ddots & & \\ & & & & & & 0.5I_{m_k} & -0.5\mathrm{i}I_{m_k} \\ & & & & & & 0.5I_{m_k} & 0.5\mathrm{i}I_{m_k} \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} I_{n_1} & & & & & & & \\ & \ddots & & & & & & \\ & & I_{n_l} & & & & & \\ & & & I_{m_1} & I_{m_1} & & & \\ & & & \mathrm{i}I_{m_1} & -\mathrm{i}I_{m_1} & & & \\ & & & & & \ddots & & \\ & & & & & & I_{m_k} & I_{m_k} \\ & & & & & & \mathrm{i}I_{m_k} & -\mathrm{i}I_{m_k} \end{bmatrix}$$

where i is the complex unit.

Taking $\hat{T} = TQ$, then we have

$$\begin{aligned} \hat{T} = [v_1^{\langle 1\rangle} \dots v_{n_1}^{\langle 1\rangle} \dots \dots v_1^{\langle l\rangle} \dots v_{n_l}^{\langle l\rangle} \ \mathrm{Re}\left(u_1^{\langle 1\rangle}\right) \dots \mathrm{Re}\left(u_{m_1}^{\langle 1\rangle}\right) \ \mathrm{Im}\left(u_1^{\langle 1\rangle}\right) \dots \\ \mathrm{Im}\left(u_{m_1}^{\langle 1\rangle}\right) \ \dots \dots \mathrm{Re}\left(u_1^{\langle k\rangle}\right) \dots \ \mathrm{Re}\left(u_{m_k}^{\langle k\rangle}\right) \mathrm{Im}\left(u_1^{\langle k\rangle}\right) \dots \mathrm{Im}\left(u_{m_k}^{\langle k\rangle}\right)] \end{aligned}$$

Clearly, $\hat{T}$ is a real matrix.

This completes the proof of the lemma. □

Next, we present a lemma and a corollary that deduces the asymptotic stability of a LTI system based on its bounded input and bounded output stability.

Lemma A.7. *Consider a finite-dimensional continuous-time LTI system:*

$$S: \quad \begin{cases} \dot{x} = Ax + Dw; & x(0) = x_0 = \mathbf{0}_{n\times 1} \\ z = Cx \end{cases} \tag{A.10}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; z is the m-dimensional output, $m \in \mathbb{Z}_+$; and the matrices A, D, and C are constant matrices of appropriate dimensions. Assume that the system is bounded input and bounded output stable from w to z; and it is controllable from w and observable from z. Then, the matrix A is Hurwitz.

Proof. We will show this by an argument of contradiction. Suppose A is not Hurwitz. Then, A admits an eigenvalue $\lambda \in \mathbb{C}$ with nonnegative real part. We will distinguish two exhaustive and mutually exclusive cases: Case 1: $\lambda \in \mathbb{R}$; Case 2: $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Case 1: $\lambda \in \mathbb{R}$. Without loss of generality, assume that the matrix A is given in the real Jordan canonical form (A.8) with λ being the eigenvalue of the first Jordan block; that is $\lambda = \lambda_1$. By the controllability of the system, there exists $w_{[0,1]} \in \mathcal{C}$ such that $w(0) = \mathbf{0}_{q\times 1}$, $w(1) = \mathbf{0}_{q\times 1}$, and transfer the state from $x(0) = \mathbf{0}_{n\times 1}$ to $x(1) = \begin{bmatrix} 1 & \mathbf{0}_{1\times(n-1)} \end{bmatrix}'$. Since the pair (A, C) is observable, then $C = \begin{bmatrix} c_1 & \star_{m\times(n-1)} \end{bmatrix}$ with $c_1 \neq \mathbf{0}_{m\times 1}$. Set $w_{[1,\infty)}$ to be the identically $\mathbf{0}_{q\times 1}$ function. Then, on the interval $[1, \infty)$, $z_{[1,\infty)}$ is generated by

$$\begin{aligned} \dot{x}_1 &= \lambda x_1; \qquad x_1(1) = 1 \\ z &= c_1 x_1 \end{aligned}$$

Since $\lambda \in [0, \infty) \subset \mathbb{R}$, then $\left\| z_{[1,\infty)} \right\|_2 = \infty$. Hence, we have $w_{[0,\infty)} \in \mathcal{C}$, $\left\| w_{[0,\infty)} \right\|_2 = \left\| w_{[0,1]} \right\|_2 < +\infty$, and $\left\| z_{[0,\infty)} \right\|_2 = +\infty$. Thus, the system (A.10) viewed as a linear operator $S : \mathcal{L}_2([0, \infty), \mathbb{R}^q) \to \mathcal{L}_2([0, \infty), \mathbb{R}^m)$ is not bounded. This contradicts the assumption that (A.10) is bounded input and bounded output stable. Hence, we have arrived at a contradiction in this case.

Case 2: $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Without loss of generality, assume that A is given in the real Jordan canonical form (A.8) with λ being the eigenvalue of the B_1, that is $\lambda = \mu_1$. By the controllability of the system, there exists $w_{[0,1]} \in \mathcal{C}$ such that $w(0) = \mathbf{0}_{q\times 1}$, $w(1) = \mathbf{0}_{q\times 1}$, and transfer the state from $x(0) = \mathbf{0}_{n\times 1}$ to $x(1) = \begin{bmatrix} \mathbf{0}_{1\times(\sum_{i=1}^{l} n_i)} & 1 & \mathbf{0}_{1\times(n-\sum_{i=1}^{l} n_i - 1)} \end{bmatrix}'$. Since the pair (A, C) is observable, then $C = \begin{bmatrix} \star_{m\times(\sum_{i=1}^{l} n_i)} & c_1 & \star_{m\times(m_1-1)} & c_2 & \star_{m\times(n-\sum_{i=1}^{l} n_i - m_1 - 1)} \end{bmatrix}$ with c_1 and c_2 not both being zero vectors. Set $w_{[1,\infty)}$ to be the identically $\mathbf{0}_{q\times 1}$ function. Then, on the interval $[1, \infty)$, $z_{[1,\infty)}$ is generated by

$$\begin{aligned} \dot{x}_1 &= \operatorname{Re}(\lambda)\, x_1 + \operatorname{Im}(\lambda)\, x_2; \qquad x_1(1) = 1 \\ \dot{x}_2 &= -\operatorname{Im}(\lambda)\, x_1 + \operatorname{Re}(\lambda)\, x_2; \qquad x_2(1) = 0 \\ z &= c_1 x_1 + c_2 x_2 \end{aligned}$$

Let $a = \operatorname{Re}(\lambda) \geq 0$ and $b = \operatorname{Im}(\lambda) \neq 0$. Then $x_1(t) = \mathrm{e}^{a\,(t-1)} \cos(b\,(t-1))$ and $x_2(t) = -\mathrm{e}^{a\,(t-1)} \sin(b\,(t-1))$, $\forall t \in [1, \infty)$. Thus, $\left\| z_{[1,\infty)} \right\|_2 = +\infty$. Hence, we have $w_{[0,\infty)} \in \mathcal{C}$, $\left\| w_{[0,\infty)} \right\|_2 = \left\| w_{[0,1]} \right\|_2 < \infty$, and $\left\| z_{[0,\infty)} \right\|_2 = +\infty$. Thus, the system (A.10) viewed as a linear operator $S : \mathcal{L}_2([0, \infty), \mathbb{R}^q) \to \mathcal{L}_2([0, \infty), \mathbb{R}^m)$ is not bounded. This contradicts the assumption that (A.10) is bounded input and bounded output stable. Hence, we have arrived at a contradiction in this case.

In both cases, we have arrived at a contradiction. Hence, the hypothesis does not hold. Then, A is Hurwitz.

This completes the proof of the lemma. $\square$

Corollary A.8. *Consider the system (A.10). Assume that the system is bounded input and bounded output stable from w to z; and it is stabilizable from w and detectable from z. Then, the matrix A is Hurwitz.*

Proof. Let $Q = \begin{bmatrix} C' & A'C' & \cdots & A'^{n-1}C' \end{bmatrix}'$ and $\operatorname{rank}(Q) = n_o \in \{0, 1, \ldots, n\}$. We will distinguish 3 exhaustive and mutually exclusive cases: Case 1: $n_o = 0$; Case 2: $1 \leq n_o < n$; Case 3: $n_o = n$.

Case 1: $n_o = 0$. Then, $C = \mathbf{0}_{m\times n}$. Since the pair (A, C) is detectable, then, the matrix

$$\begin{bmatrix} C \\ \lambda I_n - A \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times n} \\ \lambda I_n - A \end{bmatrix}$$

has rank n, $\forall \lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) \geq 0$. Thus, all eigenvalues of A has negative real parts. Hence, A is Hurwitz. This case is proved.

Case 2: $1 \leq n_o < n$. We will partition the system (A.10) into observable and unobservable parts. There exists a real invertible matrix T such that, in $\left[\begin{array}{cc} x_o' & x_{\bar{o}}' \end{array}\right]' = T^{-1}x$ coordinate, the system (A.10) admits the state space representation:

$$\dot{x}_o = A_o x_o + D_o w \tag{A.11a}$$

$$\dot{x}_{\bar{o}} = A_{\bar{o}o} x_o + A_{\bar{o}} x_{\bar{o}} + D_{\bar{o}} w \tag{A.11b}$$

$$z = C_o x_o \tag{A.11c}$$

where x_o is n_o-dimensional; $x_{\bar{o}}$ is $n_{\bar{o}}$-dimensional, $n_{\bar{o}} = n - n_o \in \mathbb{N}$; and the pair (A_o, C_o) is observable. By the detectability of the pair (A, C), the matrix $A_{\bar{o}}$ is Hurwitz. Since the pair (A, D) is stabilizable, then, the matrix

$$\left[\begin{array}{cc|c} \lambda I_{n_o} - A_o & \mathbf{0} & D_o \\ -A_{\bar{o}o} & \lambda I_{n_{\bar{o}}} - A_{\bar{o}} & D_{\bar{o}} \end{array}\right]$$

has rank n, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) \geq 0$. Therefore, the matrix $\left[\begin{array}{cc} \lambda I_{n_o} - A_o & D_o \end{array}\right]$ has rank n_o, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) \geq 0$. Hence, the pair (A_o, D_o) is stabilizable. Let $P_o = \left[\begin{array}{cccc} D_o & A_o D_o & \cdots & A_o^{n_o-1} D_o \end{array}\right]$ and $\mathrm{rank}(P_o) = n_{co} \in \{0, \ldots, n_o\}$. We will further distinguish three exhaustive and mutually exclusive cases: Case 2a: $n_{co} = 0$; Case 2b: $1 \leq n_{co} < n_o$; Case 2c: $n_{co} = n_o$.

Case 2a: $n_{co} = 0$. Then, $D_o = \mathbf{0}_{n_o \times q}$. Since the pair (A_o, D_o) is stabilizable, then, the matrix $\left[\begin{array}{cc} \lambda I_{n_o} - A_o & \mathbf{0}_{n_o \times q} \end{array}\right]$ has rank n_o, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) \geq 0$. Hence, the matrix A_o is Hurwitz. Then, we have $A = T \left[\begin{array}{cc} A_o & \mathbf{0} \\ A_{\bar{o}o} & A_{\bar{o}} \end{array}\right] T^{-1}$ is Hurwitz. This sub-case is proved.

Case 2b: $1 \leq n_{co} < n_o$. We will further partition the observable part (A.11a) into controllable and uncontrollable parts. There exists a real invertible matrix T_1 such that, in $\left[\begin{array}{cc} x_{co}' & x_{\bar{c}o}' \end{array}\right]' = T_1^{-1} x_o$ coordinates, the system (A.11a) and (A.11c) admits the state space representation

$$\begin{aligned} \dot{x}_{co} &= A_{co} x_{co} + A_{c\bar{c}o} x_{\bar{c}o} + D_{co} w \\ \dot{x}_{\bar{c}o} &= A_{\bar{c}o} x_{\bar{c}o} \\ z &= C_{co} x_{co} + C_{\bar{c}o} x_{\bar{c}o} \end{aligned}$$

where x_{co} is n_{co}-dimensional; $x_{\bar{c}o}$ is $n_{\bar{c}o}$-dimensional, $n_{\bar{c}o} = n_o - n_{co} \in \mathbb{N}$; and the pair (A_{co}, D_{co}) is controllable. By the stabilizability of the pair (A_o, D_o), the matrix $A_{\bar{c}o}$ is Hurwitz. By the observability of the pair (A_o, C_o), the matrix

$$\left[\begin{array}{c} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n_o-1} \end{array}\right] T_1 = \left[\begin{array}{cc} C_{co} & C_{\bar{c}o} \\ C_{co} A_{co} & \star_{m \times n_{\bar{c}o}} \\ \vdots & \vdots \\ C_{co} A_{co}^{n_o-1} & \star_{m \times n_{\bar{c}o}} \end{array}\right]$$

has rank n_o. Therefore, the pair (A_{co}, C_{co}) is observable. In $\left[\begin{array}{ccc} x_{co}' & x_{\bar{c}o}' & x_{\bar{o}}' \end{array}\right]' = \bar{T}^{-1}x$ coordinates, the system (A.10) admits the state space representation

$$\left[\begin{array}{c} \dot{x}_{co} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{o}} \end{array}\right] = \left[\begin{array}{ccc} A_{co} & A_{c\bar{c}o} & \mathbf{0} \\ \mathbf{0} & A_{\bar{c}o} & \mathbf{0} \\ A_{\bar{o}co} & A_{\bar{o}\bar{c}o} & A_{\bar{o}} \end{array}\right] \left[\begin{array}{c} x_{co} \\ x_{\bar{c}o} \\ x_{\bar{o}} \end{array}\right] + \left[\begin{array}{c} D_{co} \\ \mathbf{0} \\ D_{\bar{o}} \end{array}\right] w$$

$$z = \left[\begin{array}{ccc} C_{co} & C_{\bar{c}o} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} x_{co} \\ x_{\bar{c}o} \\ x_{\bar{o}} \end{array}\right]$$

Since (A.10) is bounded input and bounded output stable, then, by Lemma A.3, $\exists k \in [0,\infty) \subset \mathbb{R}$ and $\exists \lambda \in (0,\infty) \subset \mathbb{R}$ such that $\left\| C\mathrm{e}^{At} D \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty)$. Note that, $\forall t \in [0,\infty)$,

$$\begin{aligned}\left\| C\mathrm{e}^{At} D \right\|_{2,2} &= \left\| \begin{bmatrix} C_{co} & C_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_{co}t} & \star & \mathbf{0} \\ \mathbf{0} & \mathrm{e}^{A_{\bar{c}o}t} & \mathbf{0} \\ \star & \star & \mathrm{e}^{A_{\bar{o}}t} \end{bmatrix} \begin{bmatrix} D_{co} \\ \mathbf{0} \\ D_{\bar{o}} \end{bmatrix} \right\|_{2,2} \\ &= \left\| C_{co}\mathrm{e}^{A_{co}t} D_{co} \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}\end{aligned}$$

Hence, by Lemma A.3, the system

$$\begin{aligned}\dot{\bar{x}} &= A_{co}\bar{x} + D_{co}w; \qquad \bar{x}(0) = \mathbf{0}_{n_{co}\times 1} \\ y &= C_{co}\bar{x}\end{aligned}$$

is bounded input and bounded output stable. This coupled with the controllability and observability of the triple (A_{co}, D_{co}, C_{co}), by Lemma A.7, then the matrix A_{co} is Hurwitz. Hence, the matrix $A = \bar{T} \begin{bmatrix} A_{co} & A_{c\bar{o}} & \mathbf{0} \\ \mathbf{0} & A_{\bar{c}o} & \mathbf{0} \\ A_{\bar{o}co} & A_{\bar{o}\bar{c}o} & A_{\bar{o}} \end{bmatrix} \bar{T}^{-1}$ is Hurwitz. This sub-case is proved.

Case 2c: $n_{co} = n_o$. Then, it means the pair (A_o, D_o) is controllable. Hence, the triple (A_o, D_o, C_o) is controllable and observable. Since the system (A.10) is bounded input and bounded output stable, then, by Lemma A.3, $\exists k \in [0,\infty) \subset \mathbb{R}$ and $\exists \lambda \in (0,\infty) \subset \mathbb{R}$ such that $\left\| C\mathrm{e}^{At} D \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty)$. Note that, $\forall t \in [0,\infty)$,

$$\left\| C\mathrm{e}^{At} D \right\|_{2,2} = \left\| \begin{bmatrix} C_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_o t} & \mathbf{0} \\ \star & \mathrm{e}^{A_{\bar{o}}t} \end{bmatrix} \begin{bmatrix} D_o \\ D_{\bar{o}} \end{bmatrix} \right\|_{2,2} = \left\| C_o \mathrm{e}^{A_o t} D_o \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$$

Hence, by Lemma A.3, the system

$$\begin{aligned}\dot{\bar{x}} &= A_o\bar{x} + D_o w; \qquad \bar{x}(0) = \mathbf{0}_{n_o\times 1} \\ y &= C_o\bar{x}\end{aligned}$$

is bounded input and bounded output stable. This coupled with the controllability and observability of the triple (A_o, D_o, C_o), by Lemma A.7, then the matrix A_o is Hurwitz. Hence, the matrix $A = T \begin{bmatrix} A_o & \mathbf{0} \\ A_{\bar{o}o} & A_{\bar{o}} \end{bmatrix} T^{-1}$ is Hurwitz. This sub-case is proved.

This completes the proof for Case 2.

Case 3: $n_o = n$. Then, the pair (A, C) is observable. Let $\mathrm{rank}(P) = n_c$, where $P = \begin{bmatrix} D & AD & \cdots & A^{n-1}D \end{bmatrix}$. We will further distinguish 3 exhaustive and mutually exclusive cases: Case 3a: $n_c = 0$; Case 3b: $1 \le n_c < n$; Case 3c: $n_c = n$.

Case 3a: $n_c = 0$. Then, $D = \mathbf{0}_{n\times q}$. Since the pair (A, D) is stabilizable, then the matrix $\begin{bmatrix} \lambda I_n - A & D \end{bmatrix}$ has rank n, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) \ge 0$. Hence the matrix A has all eigenvalues with negative real parts. Hence, A is Hurwitz. This sub-case is proved.

Case 3b: $1 \le n_c < n$. We will partition (A.10) into controllable and uncontrollable parts. There exists a real invertible matrix T such that, in $\begin{bmatrix} x_c' & x_{\bar{c}}' \end{bmatrix}' = T^{-1}x$

coordinates, the system (A.10) admits the state space representation:

$$\begin{aligned}\dot{x}_c &= A_c x_c + A_{c\bar{c}} x_{\bar{c}} + D_c w\\ \dot{x}_{\bar{c}} &= A_{\bar{c}} x_{\bar{c}}\\ z &= C_c x_c + C_{\bar{c}} x_{\bar{c}}\end{aligned}$$

where x_c is n_c-dimensional and $x_{\bar{c}}$ is $n_{\bar{c}}$-dimensional, $n_{\bar{c}} = n - n_c \in \mathbb{N}$; the pair (A_c, D_c) is controllable. By the stabilizability of the pair (A, D), the matrix $A_{\bar{c}}$ is Hurwitz. By the observability of the pair (A, C), we have

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T = \begin{bmatrix} C_c & C_{\bar{c}} \\ C_c A_c & \star_{m\times n_{\bar{c}}} \\ \vdots & \vdots \\ C_c A_c^{n-1} & \star_{m\times n_{\bar{c}}} \end{bmatrix}$$

has rank n. Hence, the pair (A_c, C_c) is observable. Hence, the triple (A_c, D_c, C_c) is controllable and observable.

Since (A.10) is bounded input and bounded output stable, then, by Lemma A.3, $\exists k \in [0, \infty) \subset \mathbb{R}$ and $\exists \lambda \in (0, \infty) \subset \mathbb{R}$ such that $\left\| C\mathrm{e}^{At} D \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \in [0, \infty)$. Note that, $\forall t \in [0, \infty)$,

$$\left\| C\mathrm{e}^{At} D \right\|_{2,2} = \left\| \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_c t} & \star \\ \mathbf{0} & \mathrm{e}^{A_{\bar{c}} t} \end{bmatrix} \begin{bmatrix} D_c \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} = \left\| C_c \mathrm{e}^{A_c t} D_c \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$$

Hence, by Lemma A.3, the system

$$\begin{aligned}\dot{\bar{x}} &= A_c \bar{x} + D_c w; \qquad \bar{x}(0) = \mathbf{0}_{n_c\times 1}\\ y &= C_c \bar{x}\end{aligned}$$

is bounded input and bounded output stable. This coupled with the controllability and observability of the triple (A_c, D_c, C_c), by Lemma A.7, then the matrix A_c is Hurwitz. Hence, the matrix $A = T \begin{bmatrix} A_c & A_{c\bar{c}} \\ \mathbf{0} & A_{\bar{c}} \end{bmatrix} T^{-1}$ is Hurwitz. This sub-case is proved.

Case 3c: $n_c = n$. Hence, the system (A.10) is controllable and observable. By Lemma A.7, the matrix A is Hurwitz. This sub-case is proved.

This completes the proof of the corollary. □

Next, we present a technical lemma and a technical corollary that are essential for the proof of Lemma 3.9.

LEMMA A.9. *Consider a finite-dimensional single-input LTI system:*

$$\dot{z} = Az + Bv; \qquad z(0) = \mathbf{0}_{n\times 1} \tag{A.12}$$

where z is the n-dimensional state, $n \in \mathbb{Z}_+$; v is the scalar control input; and A and B are constant matrices of appropriate dimensions. Assume that (A.12) is controllable but is not bounded input and bounded state stable. Let $r \in \mathbb{Z}_+$. Then, $\exists v_{[0,\infty)} \in \mathcal{C}_r$, such that $\left\| v_{[0,\infty)}^{[r]} \right\|_\infty \in (0, \infty) \subset \mathbb{R}$ and $\left\| z_{[0,\infty)} \right\|_\infty = +\infty$.

Proof. Without loss of generality, assume that A is given in the real Jordan canonical form (A.8). Partition B accordingly. Clearly, the matrix A is not Hurwitz.

Then, A admits an eigenvalue $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) \geq 0$. We will distinguish 2 exhaustive and mutually exclusive cases: Case 1: $\lambda \in \mathbb{R}$; Case 2: $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Case 1: $\lambda \in \mathbb{R}$. Without loss of generality, assume λ is the eigenvalue of the first Jordan block; that is $\lambda = \lambda_1$. Then, the element of z, z_a, corresponding to the last row of the first Jordan block, satisfies the dynamics: $\dot{z}_a = \lambda z_a + b_a v$, $z_a(0) = 0$, where $b_a \neq 0$ since the pair (A, B) is controllable. Let $v_{[0,\infty)}$ be given by a constant function with value 1. Then, $\left\| v^{[r]}_{[0,\infty)} \right\|_\infty = 1$ and $z_a(t) = \int_0^t \mathrm{e}^{\lambda\,(t-\tau)} b_a \,\mathrm{d}\tau = \begin{cases} \frac{b_a}{\lambda}\mathrm{e}^{\lambda\tau}|_0^t = \frac{b_a}{\lambda}(\mathrm{e}^{\lambda t} - 1) & \lambda > 0 \\ b_a t & \lambda = 0 \end{cases}$, $\forall t \in [0, \infty)$. Hence, $\left\| z_{[0,\infty)} \right\|_\infty \geq \left\| z_{a[0,\infty)} \right\|_\infty = \infty$. This completes the proof for this case.

Case 2: $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Without loss of generality, assume $\lambda = a + b\mathrm{i}$ is the eigenvalue associated with the first complex Jordan block; that is $\lambda = \mu_1$. Note that $a \geq 0$ and $b \neq 0$. Then, the element of z, z_b, corresponding to the last row of the first complex Jordan block, satisfies the dynamics

$$\begin{aligned} \dot{z}_b &= a z_b + b z_c + b_b v; \qquad z_b(0) = 0 \\ \dot{z}_c &= -b z_b + a z_c + b_c v; \qquad z_c(0) = 0 \end{aligned}$$

where b_b and b_c are not both 0s, since the pair (A, B) is controllable. The state transition matrix for the above 2nd order system is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{at}\cos(bt) & \mathrm{e}^{at}\sin(bt) \\ -\mathrm{e}^{at}\sin(bt) & \mathrm{e}^{at}\cos(bt) \end{bmatrix}$$

Then, we have

$$\begin{aligned} z_b(t) &= \int_0^t \mathrm{e}^{a\,(t-\tau)}\,(\cos(b\,(t-\tau))b_b + \sin(b\,(t-\tau))b_c) v(\tau)\,\mathrm{d}\tau \\ &= \sqrt{b_b^2 + b_c^2} \int_0^t \mathrm{e}^{a\,(t-\tau)} \sin(b\,(t-\tau) + \phi) v(\tau)\,\mathrm{d}\tau; \qquad \forall t \in [0, \infty) \end{aligned}$$

where $\sin(\phi) = \frac{b_b}{\sqrt{b_b^2 + b_c^2}}$ and $\cos(\phi) = \frac{b_c}{\sqrt{b_b^2 + b_c^2}}$. Choose $v_{[0,\infty)}$ to be $v(t) = \sin(-bt + \phi)$, $\forall t \in [0, \infty)$. Then, $\left\| v^{[r]}_{[0,\infty)} \right\|_\infty \leq 1 + |b| + \cdots + |b|^r$. Furthermore, we have, when $a = 0$,

$$z_b(2k\pi/|b|) = \sqrt{b_b^2 + b_c^2} \int_0^{2k\pi/|b|} (\sin(-b\tau + \phi))^2\,\mathrm{d}\tau = \sqrt{b_b^2 + b_c^2} k\pi/|b|; \quad k = 1, 2, \ldots$$

when $a > 0$,

$$\begin{aligned} z_b(2k\pi/|b|) &= \sqrt{b_b^2 + b_c^2}\mathrm{e}^{2k\pi a/|b|} \int_0^{2k\pi/|b|} \mathrm{e}^{-a\tau}\,(\sin(-b\tau + \phi))^2\,\mathrm{d}\tau \\ &= \sqrt{b_b^2 + b_c^2}\mathrm{e}^{2k\pi a/|b|} \Big(-\frac{1}{2a}\mathrm{e}^{-a\tau}|_0^{2k\pi/|b|} + \frac{\mathrm{e}^{-a\tau}}{2\,(a^2 + 4b^2)}\,(a\cos(-2b\tau + 2\phi) \\ &\quad + 2b\sin(-2b\tau + 2\phi))|_0^{2k\pi/|b|} \Big) \\ &= \sqrt{b_b^2 + b_c^2}\,(\mathrm{e}^{2k\pi a/|b|} - 1) \left(\frac{1}{2a} - \frac{1}{2\,(a^2 + 4b^2)}\,(a\cos(2\phi) + 2b\sin(2\phi)) \right) \\ &= \sqrt{b_b^2 + b_c^2}\,(\mathrm{e}^{2k\pi a/|b|} - 1) \left(\frac{1}{2a} - \frac{1}{2\sqrt{a^2 + 4b^2}} \sin(2\phi + \theta) \right); \qquad k = 1, 2, \ldots \end{aligned}$$

where $\sin(\theta) = \frac{a}{\sqrt{a^2+4b^2}}$ and $\cos(\theta) = \frac{2b}{\sqrt{a^2+4b^2}}$. Hence, we have $\| z_{[0,\infty)} \|_\infty \geq \| z_{b[0,\infty)} \|_\infty = +\infty$. This completes the proof for this case.

This completes the proof for the lemma. $\square$

Corollary A.10. *Consider a finite-dimensional continuous-time LTI system:*

$$\dot{z} = Az + Bv; \qquad z(0) = \mathbf{0}_{n\times 1} \tag{A.13}$$

where z is the n-dimensional state, $n \in \mathbb{Z}_+$; v is the p-dimensional control input, $p \in \mathbb{Z}_+$; and A and B are constant matrices of appropriate dimensions. Assume that (A.13) is not bounded input and bounded state stable. Let $r \in \mathbb{Z}_+$. Then, $\exists v_{[0,\infty)} \in \mathcal{C}_r$, such that $\| v^{[r]}_{[0,\infty)} \|_\infty \in (0,\infty) \subset \mathbb{R}$ and $\| z_{[0,\infty)} \|_\infty = +\infty$.

Proof. Consider the special case $p = 1$. Let $P = \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}$ and $\mathrm{rank}(P) = n_c$. Clearly, $0 < n_c \leq n$ since the system (A.13) is not bounded input and bounded state stable. When $n_c = n$, then, the system (A.13) is controllable. The result holds by Lemma A.9.

When $0 < n_c < n$. Without loss of generality, assume that the system (A.13) is partitioned into controllable and uncontrollable parts:

$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} A_c & A_{c\bar{c}} \\ \mathbf{0} & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_c \\ z_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ \mathbf{0} \end{bmatrix} v; \qquad \begin{bmatrix} z_c(0) \\ z_{\bar{c}}(0) \end{bmatrix} = \mathbf{0}_{n\times 1}$$

where z_c is n_c-dimensional and the pair (A_c, B_c) is controllable. Clearly, $z_{\bar{c}}(t) = \mathbf{0}_{(n-n_c)\times 1}$, $\forall t \in [0,\infty)$. Then, we have

$$\dot{z}_c = A_c z_c + B_c v; \qquad z_c(0) = \mathbf{0}_{n_c\times 1} \tag{A.14}$$

By the fact that (A.13) is not bounded input and bounded state stable, then (A.14) is not bounded input and bounded state stable. Then, by Lemma A.9, $\exists v_{[0,\infty)} \in \mathcal{C}_r$, such that $\| v^{[r]}_{[0,\infty)} \|_\infty \in (0,\infty) \subset \mathbb{R}$, and $\| z_{c[0,\infty)} \|_\infty = +\infty$. Hence, $\| z_{[0,\infty)} \|_\infty = +\infty$. This completes the proof for the special case.

Consider the general case $p \in \mathbb{Z}_+$. Let column vectors of B be $\begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix}$ and the elements of v be $\begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix}$. Since (A.13) is not bounded input and bounded state stable, by Lemma A.3, then, $\exists i_0 \in \{1,\ldots,p\}$, such that the system $\dot{\xi} = A\xi + B_{i_0} v_{i_0}$, $\xi(0) = \mathbf{0}_{n\times 1}$, is not bounded input and bounded state stable. Without loss of generality, assume $i_0 = 1$. By Corollary A.10, $\exists v_{1[0,\infty)} \in \mathcal{C}_r$, such that $\| v^{[r]}_{1[0,\infty)} \|_\infty \in (0,\infty) \subset \mathbb{R}$ and $\| \xi_{[0,\infty)} \|_\infty = +\infty$. Set v_i, $i = 2,\ldots,p$, to constantly zero functions. Then, we have $v_{[0,\infty)} \in \mathcal{C}_r$, such that $\| v^{[r]}_{[0,\infty)} \|_\infty = \| v^{[r]}_{1[0,\infty)} \|_\infty \in (0,\infty) \subset \mathbb{R}$, and $\| z_{[0,\infty)} \|_\infty = \| \xi_{[0,\infty)} \|_\infty = +\infty$.

This completes the proof of this lemma. $\square$

Finally, we present a technical lemma that is needed for the proof of Lemma 3.10.

Lemma A.11. *Consider a finite-dimensional LTI system*

$$\dot{x} = Ax + Bu + Dw; \qquad x(0) = x_0 \tag{A.15}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; u is the p-dimensional control input, $p \in \mathbb{Z}_+$; and w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; and A, B, and D are constant matrices of appropriate dimensions; $x_0 \in \mathbb{R}^n$ is fixed; and $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$. Assume that $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with

$\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ *with* $\left\| u_{[0,\infty)} \right\|_\infty \le c_w$, *then* $\left\| x_{[0,\infty)} \right\|_\infty \le c_c$. *Then, we conclude that the following system*

$$\dot{z} = Az + Bv; \qquad z(0) = \mathbf{0}_{n\times 1} \tag{A.16}$$

is bounded input and bounded state stable.

Proof. We will first show that the system (A.16) satisfies $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall v_{[0,\infty)} \in \mathcal{C}$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$ such that $\left\| z_{[0,\infty)} \right\|_\infty \le c_c$.

We will prove this by using an argument of contradiction. Suppose the result is not true. Then, $\exists c_y \in (0,\infty) \subset \mathbb{R}$, $\exists \bar{v}_{[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{v}_{[0,\infty)} \right\|_\infty \le c_y$ such that $\left\| \bar{z}_{[0,\infty)} \right\|_\infty = \infty$, where $\bar{z}_{[0,\infty)}$ is the solution to (A.16) with $\bar{v}_{[0,\infty)}$ as the input.

Then, $\exists c_w \in [c_y,\infty) \subset \mathbb{R}$, $\exists w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$. Let $\xi_{[0,\infty)}$ be the solution to the following system

$$\dot{\xi} = A\xi + Dw; \qquad \xi(0) = x_0$$

Choose $u_{[0,\infty)} = \rho \bar{v}_{[0,\infty)}$ with $\rho \in (0,1)$. Then, the solution to (A.15) is given by $x_{[0,\infty)} = \xi_{[0,\infty)} + \rho \bar{z}_{[0,\infty)}$. Clearly, $\left\| x_{[0,\infty)} \right\|_\infty = \infty$ for some $\rho \in (0,1)$. This contradicts with the assumption of the lemma. Hence, the result holds.

Then, the system (A.16) is bounded input and bounded state stable, by the causality of the system and Lemma A.3.

This completes the proof of the lemma. $\square$

REFERENCES

[1] R. G. Bartle, *The Elements Of Real Analysis*, John Wiley & Sons, New York, NY, 2nd ed., 1976.

[2] C. T. Chen, *Linear System Theory and Design*, Oxford University Press, New York, NY, 1984.

[3] P. A. Ioannou and J. Sun, *Robust Adaptive Control*, Prentice Hall, Upper Saddle River, NJ, 1996.

[4] Alberto Isidori, *Nonlinear Control Systems*, Springer-Verlag, London, 3rd ed., 1995.

[5] D. Liberzon, A. S. Morse, and E. D. Sontag, *Output-input stability and minimum-phase nonlinear systems*, IEEE Transactions on Automatic Control, 47 (2002), pp. 422–436.

[6] Zigang Pan and Tamer Başar, *Generalized minimum phase property for finite-dimensional continuous-time SISO LTI systems with additive disturbances. Part II: Further properties.* To be submitted to *SIAM J. Control and Optimization*, May 2007.

[7] E. D. Sontag and Y. Wang, *Output-to-state stability and detectability of nonlinear systems*, Systems and Control Letters, 29 (1997), pp. 279–290.