GENERALIZED MINIMUM PHASE PROPERTY FOR FINITE-DIMENSIONAL CONTINUOUS-TIME SQUARE MIMO LTI SYSTEMS WITH ADDITIVE DISTURBANCES

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Abstract. In this paper, we further generalize the definition of the extended zero dynamics to finite-dimensional continuous-time square MIMO LTI systems with additive disturbances that are right invertible; and then introduce the concept of minimum phase for this class of systems. It is shown that the extended zero dynamics is invariant under the application of dynamic extension to its input, and therefore, a minimum phase system remains minimum phase after finite number of steps of dynamic extension. We further introduce the extended zero dynamics canonical form for square MIMO LTI systems with uniform vector relative degree. We prove that a system is minimum phase according to our definition if its zero dynamics is asymptotically stable. The converse of the statement holds if the system is stabilizable from the control input. We show that the minimum phase assumption is necessary in the model reference control of the square MMO LTI system that achieves the following two properties 1) the system states remain bounded when the disturbance input waveform is admissible and bounded, the initial condition is admissible and bounded, and the reference trajectory is bounded with bounded derivatives up to nth order, where n is the dimension of the system; 2) the perfect tracking of any given bounded reference trajectory with bounded derivatives up to nth order when the disturbance waveform is identically equal to zero, the extended zero dynamics admits any admissible initial condition, and appropriately choosing the initial condition for the rest of the closed-loop system states.

This extended zero dynamics canonical form and the strict observer canonical form are needed for the true system in the robust adaptive control design for the system and stability analysis of the resulting closed-loop system. The strict observer canonical form of an LTI system is the observer canonical form of the system that further satisfies that the input-free outputs of the system are a subset of the state variables of the representation. The strict observer canonical form of an LTI system is guaranteed to exist if the system admits uniform observability indices. Toward the end of robust adaptive control for the system, we need to be able to extend the given square MIMO LTI system to one with uniform vector relative degree and with uniform observability indices without changing its minimum phase property. We present two lemmas in this respect that fully resolve this issue. Thus, the robust adaptive control design for a square MIMO LTI system can be carried out if the given system is minimum phase and can admit vector relative degree after finite number of steps of dynamic extension that are independent of the unknown parameters in the system.

Key words. Minimum phase, zero dynamics canonical form, extended zero dynamics, extended zero dynamics canonical form, (strict) observer canonical form, dynamic extension, observability indices, vector relative degree.

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1. Introduction.

The minimum phase concept for linear system is of paramount importance in model reference control for the linear system. The classical definition of the minimum phase property for SISO linear system is that its transfer function admits all zeros in $\mathbb{C}_-$ [2]. Then, this concept is generalized to affine nonlinear systems as asymptotic stability property of the zero dynamics of the system. When the system is free of disturbances, the zero dynamics is defined in [3], which is the dynamics of the system when the input is designed to keep the output of the system to be identically zero. When the system is further subject to additive disturbance input, the minimum phase property is an assumption on the extended zero dynamics of the system. The

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extended zero dynamics of the system is simply the zero dynamics of the system as defined in [3] together with driving terms involving the output of the system and the additive disturbance inputs (see [7] for single-input and single-output (SISO) case). Interest in the generalization of the minimum phase concept to nonlinear system had been studied earlier in [4], which is in line with the development of [11]. Continued interest in the minimum phase concept illustrates the importance of this study.

In this paper, we investigate the generalization of the minimum phase concept [7] that is defined for SISO LTI systems with well-defined relative degree and additive disturbance inputs to the case of multiple-input and multiple-output (MIMO) LTI systems with additive disturbance inputs. The objective of this research is to define the minimum phase concept such that it is necessary and sufficient for model reference control for the system to achieve 1) the system states remain bounded when the disturbance input waveform is admissible and bounded, the initial condition is admissible and bounded, and the reference trajectory is bounded with bounded derivatives up to nth order, where n is the dimension of the system; 2) the perfect tracking of any given bounded reference trajectory with bounded derivatives up to nth order when the disturbance waveform is identically equal to zero, the extended zero dynamics admits any admissible initial condition, and appropriately choosing the initial condition for the rest of the closed-loop system states. Motivated by the result of [7] and [3], we define the extended zero dynamics of any square MIMO LTI system as the maximal dimension linear subdynamics that is driven by the output of the system and the disturbance inputs but independent of the rest of the system states or the control input. It is keen to observe that the definition may not apply to every square MIMO LTI system. For square MIMO LTI systems with vector relative degree, then the zero dynamics canonical form of the system as defined in [3] reveals the extended zero dynamics of the system. We prove that the extended zero dynamics of the system is invariant (modulo linear state transformation) if we apply a step of dynamic extension ([3]) to the system. Therefore, for a square MIMO LTI system that, after finite number of dynamic extensions, admits vector relative degree, the extended zero dynamics of the system exists and is amenable for our definition of minimum phase concept. This means that our concept of minimum phase applies to square MIMO LTI systems that are right invertible. We further introduce the extended zero dynamics canonical form for square MIMO LTI systems with uniform vector relative degree. This extended zero dynamics canonical form is essential in the proof of robust adaptive control design [6]. Thus, starting with a square MIMO LTI system that is minimum phase with respect to the admissible initial condition and admissible disturbance waveform, we must obtain a true system representation that admits both the extended zero dynamics canonical form representation and the strict observer canonical form representation in order to apply the (appropriately vectorized version of) robust adaptive control design of [6] to the system. The strict observer canonical form of the system is defined to be the observer canonical form that further satisfies that the input-free outputs of the system are a subset of the state variables of the representation. For the system to admit the extended zero dynamics canonical form, it must admit uniform vector relative degree. This can be achieved by dynamic extensions that are independent of the unknown parameters in the system. This assumption on the system may be too restrictive. Actually, we just need the system to admit vector relative degree (not necessarily uniform) after finite number of dynamic extensions that are independent of the unknown parameters in the system. After that, we can further achieve the requirement of uniform vector relative degree by appropriately integrating the output channels of the system, and thus leading to an extended system that admits the extended zero dynamics canonical form. Then, we may investigate the admissibility of the strict observer canonical form of the extended system. In case that the extended system does not admit the strict observer canonical form, we must further extend the system by adding dummy states (which are always zero under admissible initial condition and are independent of the control and disturbance inputs) such that the further extended system admits uniform observability indices [1]. Such a procedure is presented and will keep the uniform vector relative property and the minimum phase property of the extended system intact. Thus, we arrive at the further extended system that admits both the extended zero dynamics canonical form representation and the strict observer canonical form representation. The observable part of this further extended system may serve as the foundation for the design model as depicted in [6]. Therefore, the appropriately vectorized SISO robust adaptive control design can be carried out for the further extended system to yield the desired tracking and disturbance attenuation objectives.

The balance of the paper is as follows. In the next section, we give the notations used in the paper. In Section 3, we first present the zero dynamics canonical form representation of a square MIMO LTI system with vector relative degree. Then, we introduce the definition of the extended zero dynamics of a square MIMO LTI system and the concept of minimum phase for the system based on the property of its extended zero dynamics. Then, we present the extended zero dyanmics canonical form representation for a square MIMO LTI system with uniform vector relative degree in four exhaustive cases based on the relationship of the relative degree r, the number of output channels m, and the dimension of the system n. We show here that the system is minimum phase according to our definition if its zero dynamics (as defined in [3]) is asymptotically stable; the converse of this statement holds if the system is further stabilizable from the control inputs. In Section 4, we prove the necessity of the minimum phase assumption as defined in this paper to the model reference control design of the system. Then, in Section 5, we present two lemmas that establishes the possibility of extending a minimum phase square MIMO LTI system with vector relative degree to an extended system which admits both the extended zero dynamics canonical form and the strict observer canonical form representations and is minimum phase. The paper ends with some concluding remarks and two appendices that contain the background material on the observer canonical form representation of a MIMO LTI system and some technical results that are needed in the main results of the paper.

2. Notations. We let $\mathbb{R}$ denote the real line; let $\mathbb{R}_e$ denote the extended real line, $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ to be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; and $\mathbb{C}$ to be the set of complex numbers. Unless specified, all signals, constants, and matrices are real. For a function f, we say that it belongs to $\mathcal{C}$ (or $\mathcal{C}_0$) if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously differentiable (Fréchet differentiability), which is equivalent to that all partial derivatives up to kth order are continuous when $\mathrm{dom}\,(f)$ is open, $k \in \mathbb{N} \cup \{\infty\}$. We say that a function is L_2 if it is square integrable; and that it is L_∞ if it is bounded. We will write $\mathcal{C}_k(A, B)$ and $\mathrm{L}_p(A, B)$ to denote set of functions of A to B which are k-times continuously differentiable and set of functions of A to B which have a finite L_p norm, respectively, $p \in [1, \infty] \subset \mathbb{R}_e$. For any matrix A, A' denotes its transpose. For any $m, n \in \mathbb{Z}_+$ and any $m \times n$-dimensional matrix M, $\mathcal{R}\,(M)$ denotes the range space of M and $\mathcal{N}\,(M)$ denotes the null space of M. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, $|z|$

denotes $\sqrt{z'z}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, and any $n \times n$-dimensional symmetric matrix M, $|z|_M^2 := z'Mz$. For $n \times n$-dimensional symmetric matrices M_1 and M_2, where $n \in \mathbb{Z}_+$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{Z}_+$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. For $n \in \mathbb{Z}_+$, I_n denotes the $n \times n$-dimensional identity matrix. For $n \in \mathbb{Z}_+$ and $n \times n$-dimensional matrix A, we set $A^0 = I_n$. For any matrix M, $\|M\|_{p,p}$ denotes its p-induced norm, $1 \leq p \leq \infty$. For any $m, n \in \mathbb{Z}_+$, $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are zeros. For any $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$, $e_{n,k}$ denotes the kth n-dimensional unit vector, i.e., $\begin{bmatrix} \mathbf{0}_{1\times(k-1)} & 1 & \mathbf{0}_{1\times(n-k)} \end{bmatrix}'$. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f), \mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$ and $\|u_{[0,t_f)}\|_q = \left(\int_0^{t_f} |u(t)|^q \,\mathrm{d}t\right)^{1/q}$, $q \in [1,\infty)$. For a sufficiently smooth signal v, $v^{(i)}$ denotes the ith order derivative of v, $v^{[i]}$ denotes $\begin{bmatrix} v' & (v^{(1)})' & \cdots & (v^{(i)})' \end{bmatrix}'$, $i \in \mathbb{Z}_+$. For a $\mathcal{C}_\infty$ vector field f and a $\mathcal{C}_\infty$ function h, $L_f h$ denotes the derivative of h along f, which equals to $\frac{\partial h}{\partial x}(x)f(x)$ in local coordinates. $L_f^{k+1}h = L_f(L_f^k h)$, $k \in \mathbb{N}$; $L_f^0 h = h$. $\forall \lambda \in \mathbb{C}$, $\mathrm{Re}\,(\lambda)$ and $\mathrm{Im}\,(\lambda)$ denote the real part and the imaginary part of λ, respectively. We will denote constants or matrices of no specific interest or relevance to the analysis by $\star$. We will denote $m \times n$-dimensional matrices of no specific interest or relevance to the analysis by $\star_{m\times n}$. Let A, B be sets and $A \subseteq B$, the indicator function of set A in B is $\chi_{A,B} : B \to \{0,1\}$ defined by $\chi_{A,B}(x) = \begin{cases} 1 & \text{If } x \in A \\ 0 & \text{If } x \in B \setminus A \end{cases}$, $\forall x \in B$.

3. Definition of minimum phase property. In this section, we will introduce a generalized definition of the minimum phase property for finite-dimensional continuous time square MIMO LTI systems that can be dynamically extended ([3, Chapter 5]) to admit vector relative degree. We first present a canonical form that reveals the extended zero dynamics for square MIMO LTI systems with vector relative degree.

LEMMA 3.1. *Consider a finite-dimensional continuous-time MIMO square LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{3.1a}$$

$$y = Cx + Fu + Ew \tag{3.1b}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; u is the m-dimensional control input, $m \in \mathbb{Z}_+$; y is the m-dimensional output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; A, B, D, C, F, and E are constant matrices of appropriate dimensions. Let the system have vector relative degree $r_1, \ldots, r_m \in \{0, \ldots, n\} := \bar{J}$ from u to y, that is, $F_{i,} = C_{i,}B = \cdots = C_{i,}A^{r_i-2}B = \mathbf{0}_{1\times m}$, $i = 1, \ldots, m$, where $F_{i,}$ and $C_{i,}$ is the ith row vector of the matrices F and C, respectively, and $\begin{bmatrix} C_{1,}A^{r_1-1}B \\ \vdots \\ C_{m,}A^{r_m-1}B \end{bmatrix} =: B_0$ *is an invertible matrix (for those $i = 1, \ldots, m$ with $r_i = 0$, the corresponding row in B_0 is replaced by $F_{i,}$). The matrix B_0 is said to be the* high frequency gain matrix. *Then, there exists an invertible matrix T_o such that, in $\bar{x} := T_o^{-1}x = \begin{bmatrix} x_z' & x_{1,1} & \ldots & x_{1,r_1} & \ldots & x_{m,1} & \ldots & x_{m,r_m} \end{bmatrix}'$ coordinates, the system (3.1) admits the state space representation*

$$\dot{x}_z = A_z x_z + \sum_{i=1}^{m} A_{z1,i} y_i + D_z w \tag{3.2a}$$

(3.2b) $\dot{x}_{i,j} = x_{i,j+1} + D_{i,j}w; \qquad i = 1, \ldots, m$ with $r_i > 0, j = 1, \ldots, r_i - 1$

(3.2c) $\dot{x}_{i,r_i} = A_i\bar{x} + C_{i,}A^{r_i-1}Bu + D_{i,r_i}w \qquad i = 1, \ldots, m$ with $r_i > 0$

(3.2d) $y_i = x_{i,1} + E_{i,}w \qquad i = 1, \ldots, m$ with $r_i > 0$

(3.2e) $y_i = \bar{C}_{i,}\bar{x} + F_{i,}u + E_{i,}w \qquad i = 1, \ldots, m$ with $r_i = 0$

where x_z is $(n - \sum_{i=1}^{m} r_i)$-dimensional; $x_{i,j}$, $i = 1, \ldots, m$ with $r_i > 0$, $j = 1, \ldots, r_i$, are scalars. The representation (3.2) is called the zero dynamics canonical form *of system (3.1). (Note here that (3.2) is not the extended zero dynamics canonical form.) The dynamics (3.2a) is said to be the* extended zero dynamics *of system (3.1).*

Proof. It is proved in [3, Chapter 5], that the following row vectors are linearly independent:

$$Q_{i,j} := C_{i,}A^{j-1}, i = 1, \ldots, m \text{ with } r_i > 0, j = 1, \ldots, r_i$$

Let K_1 be any $(n - \sum_{i=1}^{m} r_i) \times n$-dimensional real matrix such that $T_1 =: \begin{bmatrix} K_1' & Q_{1,1}' & \cdots & Q_{1,r_1}' & \cdots & Q_{m,1}' & \cdots Q_{m,r_m}' \end{bmatrix}'$ is an invertible matrix. Let $\hat{x} := T_1 x =: (\hat{x}_z, x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{m,1}, \ldots, x_{m,r_m})$, where $\hat{x}_z$ is $(n - \sum_{i=1}^{m} r_i)$-dimensional, and the rest are scalars. In $\hat{x}$ coordinates, the system (3.1) admits the state space representation

(3.3a) $$\dot{\hat{x}}_z = \hat{A}_z\hat{x}_z + \sum_{i=1}^{m}\sum_{j=1}^{r_i}\hat{A}_{zj,i}x_{i,j} + \hat{B}_z u + \hat{D}_z w$$

(3.3b) $\dot{x}_{i,j} = x_{i,j+1} + D_{i,j}w; \qquad i = 1, \ldots, m$ with $r_i > 0, j = 1, \ldots, r_i - 1$

(3.3c) $\dot{x}_{i,r_i} = \hat{A}_i\hat{x} + C_{i,}A^{r_i-1}Bu + D_{i,r_i}w \qquad i = 1, \ldots, m$ with $r_i > 0$

(3.3d) $y_i = x_{i,1} + E_{i,}w \qquad i = 1, \ldots, m$ with $r_i > 0$

(3.3e) $y_i = \hat{C}_{i,}\hat{x} + F_{i,}u + E_{i,}w \qquad i = 1, \ldots, m$ with $r_i = 0$

By (3.3c) and (3.3e), we may solve for $u = \hat{K}_1 y + \tilde{K}_1\hat{x} + \hat{H}w + \sum_{i=1,r_i>0}^{m}\tilde{H}_i\dot{x}_{i,r_i}$, where $\hat{K}_1$ has nonzero column i only for y_i with $r_i = 0$. Substitute this into (3.3a), we have

$$\dot{\hat{x}}_z = \hat{A}_z\hat{x}_z + \sum_{i=1}^{m}\sum_{j=1}^{r_i}\hat{A}_{zj,i}x_{i,j} + \hat{B}_z(\hat{K}_1 y + \tilde{K}_1\hat{x} + \hat{H}w + \sum_{i=1,r_i>0}^{m}\tilde{H}_i\dot{x}_{i,r_i}) + \hat{D}_z w$$

Introduce the coordinate transformation $\tilde{x}_z = \hat{x}_z - \sum_{i=1,r_i>0}^{m}\hat{B}_z\tilde{H}_i x_{i,r_i}$. Then, in $\tilde{x} := (\tilde{x}_z, x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{m,1}, \ldots x_{m,r_m})$ coordinates, we have

(3.4a) $$\dot{\tilde{x}}_z = \tilde{A}_z\tilde{x}_z + \sum_{i=1}^{m}\sum_{j=1}^{r_i}\tilde{A}_{zj,i}x_{i,j} + \sum_{i=1,r_i=0}^{m}\tilde{A}_{zi}y_i + \tilde{D}_z w$$

(3.4b) $\dot{x}_{i,j} = x_{i,j+1} + D_{i,j}w; \qquad i = 1, \ldots, m$ with $r_i > 0, j = 1, \ldots, r_i - 1$

(3.4c) $\dot{x}_{i,r_i} = \tilde{A}_i\tilde{x} + C_{i,}A^{r_i-1}Bu + D_{i,r_i}w \qquad i = 1, \ldots, m$ with $r_i > 0$

(3.4d) $y_i = x_{i,1} + E_{i,}w \qquad i = 1, \ldots, m$ with $r_i > 0$

(3.4e) $y_i = \tilde{C}_{i,}\tilde{x} + F_{i,}u + E_{i,}w \qquad i = 1, \ldots, m$ with $r_i = 0$

Note that $x_{i,r_i} = \dot{x}_{i,r_i-1} - D_{i,r_i-1}w$, $i = 1, \ldots, m$ with $r_i \geq 2$. Then, we can substitute these into (3.4a) and introduce a state transformation: $\tilde{x}_z - \sum_{i=1,r_i\geq 2}^{m}\tilde{A}_{zr_i,i}x_{i,r_i-1} =:$

$\breve{x}_z$ to arrive at the following representation in $(\breve{x}_z, x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{m,1}, \ldots x_{m,r_m})$ $=: \breve{x}$ coordinates for (3.1)

$$\dot{\breve{x}}_z = \breve{A}_z \breve{x}_z + \sum_{i=1}^{m} \sum_{j=1}^{0\vee(r_i-1)} \breve{A}_{zj,i} x_{i,j} + \sum_{i=1, r_i=0}^{m} \breve{A}_{zi} y_i + \breve{D}_z w \tag{3.5a}$$

$$\dot{x}_{i,j} = x_{i,j+1} + D_{i,j} w; \qquad i = 1, \ldots, m \text{ with } r_i > 0, j = 1, \ldots, r_i - 1 \tag{3.5b}$$

$$\dot{x}_{i,r_i} = \breve{A}_i \breve{x} + C_{i,} A^{r_i - 1} B u + D_{i,r_i} w \qquad i = 1, \ldots, m \text{ with } r_i > 0 \tag{3.5c}$$

$$y_i = x_{i,1} + E_{i,} w \qquad i = 1, \ldots, m \text{ with } r_i > 0 \tag{3.5d}$$

$$y_i = \breve{C}_{i,} \breve{x} + F_{i,} u + E_{i,} w \qquad i = 1, \ldots, m \text{ with } r_i = 0 \tag{3.5e}$$

Recursively, we can eliminate all $x_{i,j}$'s in $\breve{x}_z$ dynamics for all $j > 1$. Then, note that $x_{i,1} = y_i - E_{i,} w$, $i = 1, \ldots, m$ with $r_i > 0$. Then, we arrive the canonical form (3.2).

This completes the proof of the lemma. □

For square MIMO LTI systems without vector relative degree, but can be dynamically extended to one with vector relative degree, we will show (which will be proved later in the paper) that the extended zero dynamics of such system is identical to the extended zero dynamics for the dynamically extended system with vector relative degree. Instead of obtaining the canonical forms for this more general class of systems, we seek to directly obtain the extended zero dynamics of a given system by noting the structure of the (3.2a).

Consider a finite-dimensional continuous-time square MIMO LTI system

$$\dot{x} = Ax + Bu + Dw; \quad x(0) = x_0 \tag{3.6a}$$

$$y = Cx + Fu + Ew \tag{3.6b}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; u is the m-dimensional control input, $m \in \mathbb{Z}_+$; y is the m-dimensional output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; A, B, D, C, F, and E are constant matrices of appropriate dimensions, $x_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$ (see [7]). For MIMO systems, we tend to require $\mathcal{D}_0$ be a subspace of $\mathbb{R}^n$ rather than just a nonempty subset of $\mathbb{R}^n$. We will make our assumption explicit in the various results if we do require $\mathcal{D}_0$ to be a subspace.

If we can find a full row rank real matrix K, which is $s \times n$-dimensional, and real matrices $A_z \in \mathbb{R}^{s\times s}$ and $A_{z1} \in \mathbb{R}^{s\times m}$ such that

$$KA = A_z K + A_{z1} C \tag{3.7a}$$

$$A_{z1} F = KB \tag{3.7b}$$

Then, define $x_z := Kx$, and it satisfies the dynamic equation

$$\begin{aligned}\dot{x}_z &= K(Ax + Bu + Dw) = A_z K x + A_{z1} C x + KBu + KDw \\ &= A_z x_z + A_{z1} y + (KD - A_{z1} E) w\end{aligned}$$

This looks like the extended zero dynamics we seek. Let K, A_z, A_{z1} be a solution to (3.7) and $\bar{K}$, $\bar{A}_z$, $\bar{A}_{z1}$ be another solution to (3.7), then it is easy to check that $\begin{bmatrix} \bar{K} \\ K \end{bmatrix}$, $\begin{bmatrix} \bar{A}_z & \mathbf{0} \\ \mathbf{0} & A_z \end{bmatrix}$, $\begin{bmatrix} \bar{A}_{z1} \\ A_{z1} \end{bmatrix}$ is also a solution to (3.7), except that this solution might not be full row rank for $\begin{bmatrix} \bar{K} \\ K \end{bmatrix}$, even though $\bar{K}$ and K are. Now, let $\tilde{K}$ be a matrix that consists of a set of maximal linearly independent row vectors in $\begin{bmatrix} \bar{K} \\ K \end{bmatrix}$. Then, there exists real matrix T such that $\begin{bmatrix} \bar{K} \\ K \end{bmatrix} = T\tilde{K}$. Clearly, T is of full column rank. Now, it is easy to check that $\tilde{K}$, $\tilde{A}_z := (T'T)^{-1}T' \begin{bmatrix} \bar{A}_z & \mathbf{0} \\ \mathbf{0} & A_z \end{bmatrix} T$, $\tilde{A}_{z1} := (T'T)^{-1}T' \begin{bmatrix} \bar{A}_{z1} \\ A_{z1} \end{bmatrix}$ is a full row rank solution to (3.7). Thus, (3.7) admits a maximal solution K, A_z, A_{z1}, in the sense that K is of full row rank and any other solution to (3.7), $\tilde{K}$, $\tilde{A}_z$ and $\tilde{A}_{z1}$, we have all row vectors of $\tilde{K}$ are linearly dependent on row vectors of K. Thus, $\tilde{K} = T_z K$. Thus, if $\tilde{K}$ is also a maximal solution, then T_z is an invertible real matrix. Thus, these two solutions will yield extended zero dynamics that is similar to each other.

We introduce the following definition of minimum phase for continuous-time square MIMO LTI systems.

DEFINITION 3.2. *Consider a finite-dimensional continuous-time square MIMO LTI system (3.6). Let $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, $A_{z1} \in \mathbb{R}^{s\times m}$ be a maximal solution to (3.7). Then, $x_z := Kx$ satisfies the dynamics*

$$\dot{x}_z = A_z x_z + A_{z1} y + (KD - A_{z1}E)w; \quad x_z(0) = Kx_0 \in K(\mathcal{D}_0) \tag{3.8}$$

This is said to be the extended zero dynamics *of (3.6). (Note that $s = 0$ is also a possible solution, which corresponding to the case when the extended zero dynamics is absent) We will say that (3.6) is* minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ *if the extended zero dynamics is absent; or if (3.8) satisfies $\forall c_w \in \overline{\mathbb{R}_+}$, $\exists c_c \in \overline{\mathbb{R}_+}$, $\forall x_z(0) = Kx_0 \in K(\mathcal{D}_0)$ with $|x_z(0)| \le c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\left\| y_{[0,\infty)} \right\|_\infty \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, we have $\left\| x_{z[0,\infty)} \right\|_\infty \le c_c$.*

For square MIMO LTI systems with vector relative degree, then Lemma 3.1 guarantees that it admits the zero dynamics canonical form (3.2). Therefore, the maximal solution to (3.7) for the system is K, which is equal to the matrix consisting of the first $n - \sum_{i=1}^m r_i$ rows of the matrix T_o^{-1}, A_z, and $A_{z1} := \begin{bmatrix} A_{z1,1} & \cdots A_{z1,m} \end{bmatrix}$ as in Lemma 3.1.

It is straightforward to see that Definition 3.2 subsumes Definition 3.7 of [7] for SISO systems, since SISO systems considered in that paper is assumed to admit (vector) relative degree.

In the following, we will show that for a square MIMO LTI systems, a step of dynamic extension [3, Chapter 5] doesn't alter the extended zero dynamics of the system. Therefore, the original system (before dynamic extension) is minimum phase if, and only if, the dynamically extended system is minimum phase.

Consider a finite-dimensional continuous-time square MIMO LTI system (3.6). Let $r_i \in \mathbb{Z}_+$, $i = 1, \dots, m$, be such that $F_{i,} = C_{i,}B = \cdots = C_{i,}A^{r_i-2}B = \mathbf{0}_{1\times m}$, $i = 1, \dots, m$, and $C_{i,}A^{r_i-1}B \neq \mathbf{0}_{1\times m}$, (or $F_{i,} \neq \mathbf{0}_{1\times m}$ if $r_i = 0$), where $F_{i,}$ and $C_{i,}$ is the ith row vector of the matrices F and C, respectively. Let $H := \begin{bmatrix} C_{1,}A^{r_1-1}B \\ \vdots \\ C_{m,}A^{r_m-1}B \end{bmatrix}$ (if $r_i = 0$, then the ith row of H is $F_{i,}$). H may or may not be invertible. Assume that (3.6) admits a maximal solution $K \in \mathbb{R}^{s\times n}$, A_z, A_{z1} to (3.7).

Let the elements of H be $(H_{i,j})_{m\times m}$ and $H_{i_0,j_0} \neq 0$. Let us do a step of dynamic extension for u_{j_0} with pivot H_{i_0,j_0} as

$$u_j = v_j; \quad \forall j = 1, \dots, m \text{ with } j \neq j_0; \tag{3.9a}$$

$$u_{j_0} = H_{i_0,j_0}^{-1}(\xi - \sum_{\substack{j=1\\ j\neq j_0}}^{m} H_{i_0,j} v_j) \tag{3.9b}$$

$$\dot{\xi} = v_{j_0}; \quad \xi(0) \in \mathbb{R} \tag{3.9c}$$

The composition of (3.6) and (3.9) defines the extended system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = A_e \begin{bmatrix} x \\ \xi \end{bmatrix} + B_e v + D_e w \quad \begin{bmatrix} x(0) \\ \xi(0) \end{bmatrix} \in \mathcal{D}_0 \times \mathbb{R} =: \bar{\mathcal{D}}_0 \tag{3.10a}$$

$$y = C_e \begin{bmatrix} x \\ \xi \end{bmatrix} + F_e v + E w \tag{3.10b}$$

where $v = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}$, $B_{,j}$ is the jth column of B, $F_{,j}$ is the jth column of F, $C_{i,}$ is the ith row of C,

$$A_e = \begin{bmatrix} A & H_{i_0,j_0}^{-1} B_{,j_0} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}; \quad D_e = \begin{bmatrix} D \\ \mathbf{0}_{1\times q} \end{bmatrix};$$

$$B_e = \begin{bmatrix} B_{e,1} & \cdots B_{e,m} \end{bmatrix}; \quad B_{e,j} = \begin{bmatrix} B_{,j} - \frac{H_{i_0,j}}{H_{i_0,j_0}} B_{,j_0} \\ 0 \end{bmatrix}, \ \forall j = 1, \ldots, m \text{ with } j \neq j_0;$$

$$B_{e,j_0} = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ 1 \end{bmatrix}; \quad C_e = \begin{bmatrix} C & \frac{1}{H_{i_0,j_0}} F_{,j_0} \end{bmatrix}$$

$$F_e = \begin{bmatrix} F_{e,1} & \cdots & F_{e,m} \end{bmatrix}; \quad F_{e,j} = F_{,j} - \frac{H_{i_0,j}}{H_{i_0,j_0}} F_{,j_0}, \ \forall j = 1, \ldots, m$$

It is simple algebra to verify that $\bar{K} := \begin{bmatrix} K & \mathbf{0}_{s\times 1} \end{bmatrix}$, A_z, A_{z1} is a full row rank solution to (3.7) for the extended system (3.10). Let $\hat{K} \in \mathbb{R}^{\hat{s}\times(n+1)}$, $\hat{A}_z$, and $\hat{A}_{z1}$ be any maximal solution to (3.7) for the extended system (3.10). Then, $\hat{s} \geq s$. Partition $\hat{K} = \begin{bmatrix} \hat{K}_1 & \hat{K}_2 \end{bmatrix}$, where $\hat{K}_2$ is a column vector. Substitute this structure into (3.7) for the extended system (3.10), we obtain

$$\hat{K} A_e = \hat{A}_z \hat{K} + \hat{A}_{z1} C_e; \quad \hat{A}_{z1} F_e = \hat{K} B_e$$

which yields

$$\begin{bmatrix} \hat{K}_1 A & \hat{K}_1 B_{,j_0}/H_{i_0,j_0} \end{bmatrix} = \begin{bmatrix} \hat{A}_z \hat{K}_1 + \hat{A}_{z1} C & \hat{A}_z \hat{K}_2 + \hat{A}_{z1} F_{,j_0}/H_{i_0,j_0} \end{bmatrix};$$

$$\hat{A}_{z1}(F_{,j} - F_{,j_0}\frac{H_{i_0,j}}{H_{i_0,j_0}}) = \hat{K}_1(B_{,j} - B_{,j_0}\frac{H_{i_0,j}}{H_{i_0,j_0}}); \ j = 1, \ldots, m \text{ with } j \neq j_0; \ \mathbf{0}_{\hat{s}\times 1} = \hat{K}_2$$

By straightforward algebra, we have $\hat{K}_1$, $\hat{A}_z$, and $\hat{A}_{z1}$ satisfies (3.7) for the original system (3.6) and $\hat{K} = \begin{bmatrix} \hat{K}_1 & \mathbf{0}_{\hat{s}\times 1} \end{bmatrix}$, which implies that $\hat{K}_1$ has full row rank. This shows that $\hat{s} \leq s$ by the maximality of K, A_z, and A_{z1} solution to (3.7) for the original system. Hence, we have $s = \hat{s}$ and $\hat{K} = \begin{bmatrix} T_z K & \mathbf{0}_{s\times 1} \end{bmatrix}$, $\hat{A}_z = T_z A_z T_z^{-1}$, $\hat{A}_{z1} = T_z A_{z1}$, where T_z is a real $s \times s$-dimensional invertible matrix. The extended zero dynamics for the extended system (3.10) is

$$\dot{x}_z = T_z A_z T_z^{-1} x_z + T_z A_{z1} y + T_z(KD - A_{z1}E)w; \quad x_z(0) = T_z K x_0 \in T_z K(\mathcal{D}_0)$$

which is identical (modulo linear transformations) to the extended zero dynamics (3.8) for the original system (3.6). Hence, the process of dynamic extension doesn't alter the extended zero dynamics or the minimum phase property of the system.

REMARK 3.1. *Based on Theorem 2.1 of [10] and Proposition 5.4.1 of [3], we can prove that a square MIMO LTI system is right invertible if, and only if, it can be dynamically extended to admit vector relative degree. For such a system, we may determine the extended zero dynamics by Lemma 3.1 for the dynamically extended system. This extended zero dynamics is itself the one for the original system. The preceding discussion also gives a way to compute the solution K, A_z and A_{z1} to (3.7) for the original system given the solution to (3.7) for the dynamically extended system. For square MIMO LTI systems that are not right invertible, the Definition 3.2 applies to these systems, but we haven't been able to offer a way to calculate its extended zero dynamics, which may not be useful in the first place.*

Next, we present the extended zero dynamics canonical form representation for a finite-dimensional continuous-time square MIMO LTI system with uniform vector relative degree in four different cases.

LEMMA 3.3. *Consider a finite-dimensional continuous-time square MIMO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{3.11a}$$

$$y = Cx + Ew \tag{3.11b}$$

where x is the n-dimensional state, $n \in \mathbb{N}$; u is the m-dimensional control input, $m \in \mathbb{Z}_+$; y is the m-dimensional output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; A, B, D, C, and E are constant matrices of appropriate dimensions. Let the system have uniform vector relative degree $r \in \mathbb{N}$, $r < n/m$, from u to y, that is, $CB = \cdots = CA^{r-2}B = \mathbf{0}_{m\times m}$ and $CA^{r-1}B =: B_0$ is an invertible matrix. Then, there exists an real invertible matrix T_o such that, in $\begin{bmatrix} x_z' & x_1 & \cdots & x_r \end{bmatrix}' = T_o^{-1}x$ coordinates, the system (3.11) admits the state space representation

$$\dot{x}_z = A_z x_z + A_{z1}x_1 + D_z w \tag{3.12a}$$

$$\dot{x}_i = A_{i1}x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \tag{3.12b}$$

$$\dot{x}_r = A_{rz}x_z + A_{r1}x_1 + B_0 u + D_r w \tag{3.12c}$$

$$y = x_1 + Ew \tag{3.12d}$$

where x_z is $(n-rm)$-dimensional; x_i, $i = 1, \ldots, r$, are m-dimensional; B_0 is the high-frequency gain matrix of the system. The representation (3.12) is called the extended zero dynamics canonical form *of system (3.11). Note that $x_1 = y - Ew$ from (3.12d) and therefore (3.12a) can be written as*

$$\dot{x}_z = A_z x_z + A_{z1}y + (D_z - A_{z1}E)w$$

which is the extended zero dynamics of (3.11) as defined in Definition 3.2.

Proof. We will follow the argument for SISO system [7], and adapt it to our present problem. Let $V = \begin{bmatrix} B & \cdots & A^{r-1}B \end{bmatrix}_{n\times(rm)}$ and $U = \begin{bmatrix} C \\ \vdots \\ CA^{r-1} \end{bmatrix}_{(rm)\times n}$.

Then, we have

$$UV = \begin{bmatrix} \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & CA^{r-1}B \\ \vdots & \iddots & \iddots & \star_{m\times m} \\ \mathbf{0}_{m\times m} & \iddots & \iddots & \vdots \\ CA^{r-1}B & \star_{m\times m} & \cdots & \star_{m\times m} \end{bmatrix}_{(rm)\times(rm)}$$

which is clearly invertible. Hence, U and V are of rank rm. Note that $\mathrm{rank}(V) = rm$ implies that $\dim(\mathcal{N}(V')) = n - rm$. Hence, there exists a real nonsingular $(n-rm) \times n$-dimensional matrix K such that $V'K' = \mathbf{0}_{(rm)\times(n-rm)}$ and $\mathrm{rank}(K) = n - rm$. Define $\bar{U} = \begin{bmatrix} K \\ U \end{bmatrix}_{n\times n}$ and $\bar{V} = \begin{bmatrix} K' & V \end{bmatrix}_{n\times n}$. Then, we have $\bar{U}\bar{V} = \begin{bmatrix} KK' & \mathbf{0}_{(n-rm)\times(rm)} \\ UK' & UV \end{bmatrix}$. Since K is nonsingular, then KK' is invertible. Hence, $\bar{U}\bar{V}$ is block lower triangular and is invertible. Then, we have $\bar{U}$ and $\bar{V}$ as invertible matrices.

Let $x_z = Kx$ and $z_i = CA^{i-1}x$, $i = 1, \ldots, r$. Consider the coordinate transformation $z := \begin{bmatrix} x_z' & z_1' & \cdots & z_r' \end{bmatrix}' = \bar{U}x$. In z coordinates, system (3.11) admits the state space representation

$$\begin{aligned} \dot{z} &= \bar{U}A\bar{U}^{-1}z + \bar{U}Bu + \bar{U}Dw =: \tilde{A}z + \tilde{B}u + \tilde{D}w \\ y &= C\bar{U}^{-1}z + Ew =: \tilde{C}z + Ew \end{aligned}$$

Note that

$$\begin{aligned} C &= \begin{bmatrix} \mathbf{0}_{m\times(n-rm)} & I_m & \mathbf{0}_{m\times(rm-m)} \end{bmatrix}\bar{U} \\ &\Rightarrow \quad \tilde{C} = \begin{bmatrix} \mathbf{0}_{m\times(n-rm)} & I_m & \mathbf{0}_{m\times(rm-m)} \end{bmatrix} \\ \tilde{A} &= \bar{U}A\bar{U}^{-1} =: \begin{bmatrix} \tilde{A}_z & \tilde{A}_{z1} & \cdots & \tilde{A}_{zr} \\ \tilde{A}_{1z} & \tilde{A}_{11} & \cdots & \tilde{A}_{1r} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{rz} & \tilde{A}_{r1} & \cdots & \tilde{A}_{rr} \end{bmatrix} \\ \tilde{B} &= \bar{U}B = \begin{bmatrix} K \\ U \end{bmatrix} B = \begin{bmatrix} \mathbf{0}_{(n-rm)\times m} \\ \mathbf{0}_{m\times m} \\ \vdots \\ \mathbf{0}_{m\times m} \\ B_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-m)\times m} \\ B_0 \end{bmatrix} \end{aligned}$$

where $\tilde{A}_z$ is $(n-rm)\times(n-rm)$-dimensional; $\tilde{A}_{ij}$, $i,j = 1,\ldots,r$, are $m\times m$-dimensional. Next, we calculate the matrix $\tilde{A}$, by the equality $\bar{U}A = \tilde{A}\bar{U}$. Thus, we have

$$\bar{U}A = \begin{bmatrix} KA \\ CA \\ \vdots \\ CA^r \end{bmatrix} = \begin{bmatrix} \tilde{A}_zK + \sum_{i=1}^r \tilde{A}_{zi}CA^{i-1} \\ \tilde{A}_{1z}K + \sum_{i=1}^r \tilde{A}_{1i}CA^{i-1} \\ \vdots \\ \tilde{A}_{rz}K + \sum_{i=1}^r \tilde{A}_{ri}CA^{i-1} \end{bmatrix} = \tilde{A}\bar{U}$$

Equating CA^j and $\tilde{A}_{jz}K + \sum_{i=1}^r \tilde{A}_{ji}CA^{i-1}$, $j = 1,\ldots,r-1$, we have $\tilde{A}_{jz} = \mathbf{0}_{m\times(n-rm)}$, $\tilde{A}_{j\,j+1} = I_m$, and $\tilde{A}_{ji} = \mathbf{0}_{m\times m}$, when $j = 1,\ldots,r-1$ and $i = 1,\ldots,r$, and $i \neq j+1$.

Equating KA and $\tilde{A}_zK + \sum_{i=1}^r \tilde{A}_{zi}CA^{i-1}$, we have the following line of argument. Note that

$$\begin{aligned} KAV &= K\begin{bmatrix} AB & \cdots & A^rB \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-rm)\times m} & \cdots & \mathbf{0}_{(n-rm)\times m} & KA^rB \end{bmatrix} \\ &= KA^rBB_0^{-1}CV \end{aligned}$$

Therefore, we have $(KA - KA^rBB_0^{-1}C)V = \mathbf{0}_{(n-rm)\times(rm)}$. Then, $V'(A'K' - C'(B_0')^{-1}B'A'^rK') = \mathbf{0}_{(rm)\times(n-rm)}$. Denote the row vectors of K by K_i, $i = 1,\ldots,n-rm$. Then, the column vectors of $A'K' - C'(B_0')^{-1}B'A'^rK'$, that is $A'K_i' - C'(B_0')^{-1}B'A'^rK_i'$, $i = 1,\ldots,n-rm$, are in the null space of V', and therefore in the span of K'. Hence, there exists $(n-rm)\times(n-rm)$-dimensional real matrix $\tilde{\tilde{A}}_z$ such that $A'K' - C'(B_0')^{-1}B'A'^rK' = K'\tilde{\tilde{A}}_z$, which implies $KA = \tilde{\tilde{A}}_z'K + KA^rBB_0^{-1}C$. Then, we have $\tilde{A}_z = \tilde{\tilde{A}}_z'$, $\tilde{A}_{z1} = KA^rBB_0^{-1}$, and $\tilde{A}_{zj} = \mathbf{0}_{(n-rm)\times m}$, $j = 2,\ldots,r$.

Hence, in z coordinates, system (3.11) may be represented by

$$\begin{aligned}
\dot{x}_z &= \tilde{A}_z x_z + \tilde{A}_{z1} z_1 + \tilde{D}_z w \\
\dot{z}_i &= z_{i+1} + \tilde{D}_i w; \qquad i = 1,\ldots,r-1 \\
\dot{z}_r &= \tilde{A}_{rz} x_z + \sum_{i=1}^{r} \tilde{A}_{ri} z_i + B_0 u + \tilde{D}_r w \\
y &= z_1 + Ew
\end{aligned}$$

Let $z_f = \begin{bmatrix} z_1' & \cdots & z_r' \end{bmatrix}'$. Then, the dynamics for z_f is

$$(3.13a)\quad \dot{z}_f = \left[\begin{array}{c|c} \mathbf{0}_{(rm-m)\times m} & I_{rm-m} \\ \hline \tilde{A}_{r1} & \tilde{A}_{r2} \ \cdots \ \tilde{A}_{rr} \end{array}\right] z_f + \begin{bmatrix} \mathbf{0}_{(rm-m)\times m} \\ B_0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix} x_z + \begin{bmatrix} \tilde{D}_1 \\ \vdots \\ \tilde{D}_r \end{bmatrix} w =: A_f z_f + B_f u + A_{fz} x_z + D_f w$$

$$(3.13b)\quad y = \begin{bmatrix} I_m & \mathbf{0}_{m\times(rm-m)} \end{bmatrix} z_f + Ew =: C_f z_f + Ew$$

It is clear that the dynamics (3.13) with inputs u, x_z, w, and output y is observable with uniform observability indices r and admits uniform vector relative degree r with respect to the input u. By Corollary A.4, there exists an real invertible coordinate transformation $x_f = T_f^{-1} z_f$ that transforms (3.13) into the observer canonical form.

$$\begin{aligned}
\dot{x}_f &= \left[\begin{array}{c|c} \hat{A}_{11} & \\ \vdots & I_{rm-m} \\ \hat{A}_{r-1,1} & \\ \hline \hat{A}_{r1} & \mathbf{0}_{m\times(rm-m)} \end{array}\right] x_f + \begin{bmatrix} \mathbf{0}_{(rm-m)\times m} \\ B_0 \end{bmatrix} u \\
&\quad + T_f^{-1} \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix} x_z + T_f^{-1} D_f w \\
y &= \begin{bmatrix} I_m & \mathbf{0}_{m\times(rm-m)} \end{bmatrix} x_f + Ew
\end{aligned}$$

Note that $T_f^{-1} \begin{bmatrix} \mathbf{0}_{m\times(rm-m)} & B_0' \end{bmatrix}' = \begin{bmatrix} \mathbf{0}_{m\times(rm-m)} & B_0' \end{bmatrix}'$ implies that

$$T_f^{-1} \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix}$$

Partition x_f as $\begin{bmatrix} x_1 & \cdots & x_r \end{bmatrix}'$, with x_i being m-dimensional, $i = 1,\ldots,r$. Clearly, $y = z_1 + Ew = x_1 + Ew$, then, we have $z_1 = x_1$. Then, the system (3.11) admits the following state space representation, in $x := \begin{bmatrix} x_z' & x_1 & \cdots & x_r \end{bmatrix}'$ coordinates,

$$\dot{x}_z = \tilde{A}_z x_z + \tilde{A}_{z1} x_1 + \tilde{D}_z w$$

$$\begin{aligned}\dot{x}_i &= \hat{A}_{i1}x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1\\ \dot{x}_r &= \tilde{A}_{rz}x_z + \hat{A}_{r1}x_1 + B_0 u + D_r w\\ y &= x_1 + Ew\end{aligned}$$

where $\begin{bmatrix} D_1' & \cdots & D_r' \end{bmatrix}' = T_f^{-1} D_f$. Clearly, the above is in the form of (3.12). Hence, the desired matrix $T_o = \bar{U}^{-1} \begin{bmatrix} I_{n-rm} & \mathbf{0} \\ \mathbf{0} & T_f \end{bmatrix}$.

This completes the proof of the lemma. □

REMARK 3.2. *We observe in the previous lemma that the zero dynamics of the system (3.11) according to [3] is exactly $\dot{x}_z = A_z x_z$. The extended zero dynamics is simply the zero dynamics together with driving terms which include the output of the system and the disturbance input.*

LEMMA 3.4. *Consider a finite-dimensional continuous-time square MIMO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{3.14a}$$

$$y = Cx + Ew \tag{3.14b}$$

where x is the n-dimensional state, $n \in \mathbb{N}$; u is the m-dimensional control input, $m \in \mathbb{N}$; y is the m-dimensional output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; A, B, D, C, and E are constant matrices of appropriate dimensions. Let the system have relative degree $r = n/m \in \mathbb{N}$, from u to y, that is, $CB = \cdots = CA^{r-2}B = \mathbf{0}_{m\times m}$ and $CA^{r-1}B =: B_0$ is invertible. Then, there exists an invertible matrix T_o such that, in $\begin{bmatrix} x_1 & \cdots & x_r \end{bmatrix}' = T_o^{-1}x$ coordinates, the system (3.14) admits the state space representation

$$\dot{x}_i = A_{i1}x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \tag{3.15a}$$

$$\dot{x}_r = A_{r1}x_1 + B_0 u + D_r w \tag{3.15b}$$

$$y = x_1 + Ew \tag{3.15c}$$

where x_i, $i = 1, \ldots, r$, are m-dimensional; B_0 is the high-frequency gain matrix of the system. The representation (3.15) is called the extended zero dynamics canonical form *of system (3.14) (which is also the observer canonical form). The* extended zero dynamics *for the system is clearly absent.*

Proof. Define $V = \begin{bmatrix} B & \cdots & A^{r-1}B \end{bmatrix}_{n\times n}$ and $U = \begin{bmatrix} C \\ \vdots \\ CA^{r-1} \end{bmatrix}_{n\times n}$. Then, we have

$$UV = \begin{bmatrix} \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & CA^{r-1}B \\ \vdots & \iddots & \iddots & \star_{m\times m} \\ \mathbf{0}_{m\times m} & \iddots & \iddots & \vdots \\ CA^{r-1}B & \star_{m\times m} & \cdots & \star_{m\times m} \end{bmatrix}_{n\times n}$$

which is clearly invertible. Hence, U and V are invertible. Then, the system (3.14) is observable, and is controllable from u. Clearly, the observability indices of (3.14) all equal to r. By Corollary A.4, there exists a real invertible transformation $T_o^{-1}x =:$

$\begin{bmatrix} x_1' & \cdots & x_r' \end{bmatrix}'$, where x_i is m-dimensional, $i = 1, \ldots, r$, that transforms the system into observer canonical form.

$$\begin{aligned} \dot{x}_i &= A_{i1}x_1 + x_{i+1} + B_i u + D_i w; \qquad i = 1, \ldots, r-1 \\ \dot{x}_r &= A_{r1}x_1 + B_r u + D_r w \\ y &= x_1 + Ew \end{aligned}$$

Because the system (3.14) admits uniform vector relative degree r from u to y, then, we have $B_1 = \cdots = B_{r-1} = \mathbf{0}_{m\times m}$. It is straightforward to obtain that $B_r = CA^{r-1}B = B_0$. This completes the proof of the lemma. □

LEMMA 3.5. *Consider a finite-dimensional continuous-time square MIMO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{3.16a}$$

$$y = Cx + B_0 u + Ew \tag{3.16b}$$

where x is the n-dimensional state, $n \in \mathbb{N}$; u is the m-dimensional control input, $m \in \mathbb{Z}_+$; y is the m-dimensional output; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; and A, B, D, C, B_0, and E are constant matrices of appropriate dimensions. Let B_0 be invertible, i. e., that the system admits uniform vector relative degree 0 *from u to y, and B_0 is the high-frequency gain matrix of the system. Then, the system (3.16) admits the following representation:*

$$\dot{x} = (A - BB_0^{-1}C)x + BB_0^{-1}(y - Ew) + Dw =: \hat{A}x + \hat{B}(y - Ew) + Dw \tag{3.17a}$$

$$y = Cx + B_0 u + Ew \tag{3.17b}$$

The representation (3.17) is called the extended zero dynamics canonical form *of (3.16). The dynamics (3.17a) is called the* extended zero dynamics *of (3.16).*

Proof. Note that $u = B_0^{-1}(y - Cx - Ew)$. Substitution of this equality into (3.16a) immediately leads to (3.17a). Hence, (3.17) is a representation of (3.16).

By Lemma 3.1, we identify that (3.17a) is the extended zero dynamics for (3.16) according to Definition 3.2.

This completes the proof of the lemma. □

DEFINITION 3.6. *Consider a finite-dimensional continuous-time MIMO LTI system*

$$y = B_0 u + Ew \tag{3.18}$$

where u is the p-dimensional control input, $p \in \mathbb{Z}_+$; y is the m-dimensional output, $m \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; and B_0 and E are constant matrices of appropriate dimensions. Let B_0 be of full row rank, which implies that the system admits uniform vector relative degree 0 *from u to y, and B_0 is the high-frequency gain matrix of the system. Then, (3.18) is called the* extended zero dynamics canonical form. *Clearly, the* extended zero dynamics *is absent in this case.*

LEMMA 3.7. *Consider the system (3.6) of Definition 3.2. Assume that it admits the extended zero dynamics (3.8). Then, (3.6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ implies that the system*

$$\dot{z} = A_z z + A_{z1} v; \qquad z(0) = \mathbf{0}_{s\times 1} \tag{3.19}$$

is bounded input and bounded state stable.

Proof. This is a direct consequence of Lemma 9 of [7]. This completes the proof of the lemma. ∎

Next, we present a result that links the generalized minimum phase property to the asymptotic stability property of the extended zero dynamics.

LEMMA 3.8. *Consider the system (3.6) of Definition 3.2. Let the system admit the extended zero dynamics (3.8). Then, the system is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ if the matrix A_z is Hurwitz. On the other hand, if the system is stabilizable from u and is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, then the matrix A_z is Hurwitz.*

Proof. If the matrix A_z is Hurwitz, then obviously the system (3.6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ according to Definition 3.2.

On the other hand, if the system (3.6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and is stabilizable from u. By the stabilizability of the pair (A, B), we have the pair (A_z, A_{z1}) is stabilizable. This is because the uncontrollable part of x_z dynamics from y must be part of the uncontrollable part of (3.6) from u, and therefore Hurwitz. By Lemma 3.7, the following system

$$\dot{x}_u = A_z x_u + A_{z1} v; \qquad x_u(0) = \mathbf{0}_{s\times 1}$$

is bounded input and bounded state stable. Since the triple (A_z, A_{z1}, I_s) is stabilizable and detectable, then, by Corollary 1 of [7], A_z is Hurwitz.

This completes the proof of the lemma. ∎

REMARK 3.3. *We conclude, based on the previous lemma, that if a finite-dimensional continuous-time square MIMO LTI system is minimum phase according to [3], then it is also minimum phase according to the generalized definition; on the other hand, if it is minimum phase according to the generalized definition, and it is stabilizable from u, then it is minimum phase according to [3].*

4. Necessity of minimum phase property in model reference control. We state and prove the following result on the necessity of the generalized minimum phase condition in model reference control of finite-dimensional continuous-time square MIMO LTI systems.

PROPOSITION 4.1. *Consider the system (3.6) of Definition 3.2. Let (3.7) admit maximal solution $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, and $A_{z1} \in \mathbb{R}^{s\times m}$ for system (3.6). Assume that*

(i) the identically $\mathbf{0}_{q\times 1}$ function belongs to $\mathcal{W}_d$;

(ii) the set $\mathcal{D}_0$ satisfies $\mathcal{D}_{z0} := K(\mathcal{D}_0)$, where $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace.

Let $y_{d[0,\infty)} \in \mathcal{C}_n([0,\infty), \mathbb{R}^m)$ be the reference trajectory and $Y_d := \begin{bmatrix} y_d' & (y_d^{(1)})' & \cdots & (y_d^{(n)})' \end{bmatrix}'$, and $Y_{d0} := \begin{bmatrix} (y_d(0))' & (y_d^{(1)}(0))' & \cdots & (y_d^{(n-1)}(0))' \end{bmatrix}' \in \mathbb{R}^{nm}$. Assume that there exists a finite-dimensional model reference controller, S_C

$$\dot{\xi} = f(\xi, y, Y_d); \qquad \xi(0) = \xi_0(\check{x}_0, Y_{d0}) \tag{4.1a}$$

$$u = h(\xi, y, Y_d) \tag{4.1b}$$

where ξ is $\bar{n}$-dimensional, $\bar{n} \in \mathbb{Z}_+$; $\mathcal{D}_\xi \subseteq \mathbb{R}^{\bar{n}}$ is nonempty; $\check{x}_0 \in \mathcal{D}_0$ is an estimate of x_0; $\xi_0 : \mathcal{D}_0 \times \mathbb{R}^{mn} \to \mathcal{D}_\xi$; such that $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$; $\forall Y_{d0} \in \mathbb{R}^{mn}$ with $|Y_{d0}| \le c_w$; $\forall y_{d[0,\infty)} \in \mathcal{C}_n$ with $\left\| Y_{d[0,\infty)} \right\|_\infty \le c_w$, we have

1. *$\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$; $\forall \check{x}_0 \in \mathcal{D}_0$ with $|\check{x}_0| \le c_w$; $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$; there exists a unique solution $X_{[0,\infty)} := (x_{[0,\infty)}, \xi_{[0,\infty)})$ to the*

closed-loop system S such that $\left\| x_{[0,\infty)} \right\|_\infty \leq c_c$, $\xi(t) \in \mathcal{D}_\xi$, $\forall t \in [0,\infty)$, such that $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$;

2. *$\forall x_{z0} \in \mathcal{D}_{z0}$, $\exists x_0 \in \mathcal{D}_0$ with $Kx_0 = x_{z0}$, set $\check{x}_0 = x_0 \in \mathcal{D}_0$ and $w_{[0,\infty)}$ equals to the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$, then $y(t) = y_d(t)$, $\forall t \in [0,\infty) \subset \mathbb{R}$.*

Then, the system (3.6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

Proof. We will consider separately two exhaustive and mutually exclusive cases: Case 1: $0 = s$; Case 2: $s > 0$.

Case 1: $0 = s$. Clearly, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof for this case.

Case 2: $s > 0$. We need the following result.

Claim 4.1.1. *The system $\dot{z} = A_z z + A_{z1} v$; $z(0) = \mathbf{0}_{s\times 1}$ is bounded input and bounded state stable.*

Proof. Suppose that the claim is false. By Corollary 2 of [7], $\exists v_{[0,\infty)} \in \mathcal{C}_n$ such that $\left\| v^{[n]}_{[0,\infty)} \right\|_\infty =: c_v \in (0,\infty) \subset \mathbb{R}$ and $\left\| z_{[0,\infty)} \right\|_\infty = +\infty$. Choose $w_{[0,\infty)}$ to be the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$. Let $y_{d[0,\infty)} = v_{[0,\infty)}$. Fix $x_{z0} = \mathbf{0}_{s\times 1} \in \mathcal{D}_{z0}$ and choose $x_0 \in \mathcal{D}_0$ with $Kx_0 = x_{z0}$ according 2.. Set $\check{x}_0 = x_0$ and $c_w = \max\{c_v, |x_0|\} \in \mathbb{R}$. By 1., $\exists c_c \in [0,\infty) \subset \mathbb{R}$, where c_c depends only on c_w and S_C, such that the closed-loop system S admits a unique solution $X_{[0,\infty)}$ with $\left\| x_{[0,\infty)} \right\|_\infty \leq c_c$, and $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$. By 2., we have $y_{[0,\infty)} = y_{d[0,\infty)} = v_{[0,\infty)}$. Note that $x_{z[0,\infty)}$ is the unique solution to $\dot{x}_z = A_z x_z + A_{z1} y_d$, $x_z(0) = Kx_0 = x_{z0} = \mathbf{0}_{s\times 1}$. Then, we have $x_{z[0,\infty)} = z_{[0,\infty)}$. Hence, we have $\left\| x_{z[0,\infty)} \right\|_\infty = +\infty$. This is a contradiction. Hence, the claim is proved. □

Back to proof of the lemma. Let $P_z := \begin{bmatrix} A_{z1} & A_z A_{z1} & \cdots & A_z^{s-1} A_{z1} \end{bmatrix}$ and $n_{zc} := \mathrm{rank}(P_z)$. We will further distinguish 3 exhaustive and mutually exclusive cases: Case 2.1: $n_{zc} = 0$; Case 2.2: $1 \leq n_{zc} < s$; Case 2.3: $n_{zc} = s$.

Case 2.1: $n_{zc} = 0$. Then, we have $A_{z1} = \mathbf{0}_{s\times m}$. Then, the extended zero dynamics is given by

$$\dot{x}_z = A_z x_z + KDw; \qquad x_z(0) = x_{z0} \tag{4.2}$$

By the assumption, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_{z0} \in \mathcal{D}_{z0}$ with $|x_{z0}| \leq c_w$, let $x_0 \in \mathcal{D}_0$ with $Kx_0 = x_{z0}$, and $\check{x}_0 = x_0$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, let $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)} \in \mathcal{C}_n$ with $\left\| Y_{d[0,\infty)} \right\|_\infty = 0$ and $|Y_{d0}| = 0$, then, there exists a unique solution $X_{[0,\infty)}$ to the closed-loop system with $\left\| x_{[0,\infty)} \right\|_\infty \leq c_c$. This implies that $\left\| x_{z[0,\infty)} \right\|_\infty \leq \left\| x_{[0,\infty)} \right\|_\infty \leq c_c$ and $x_{z[0,\infty)}$ is the unique solution to (4.2). Since the extended zero dynamics (4.2) is independent of $y_{[0,\infty)}$, then the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

Case 2.2: $1 \leq n_{zc} < s$. Without loss of generality, assume that (3.12a) is partitioned into controllable and uncontrollable parts (with respect to x_1). We have $x_z = \begin{bmatrix} x'_{zc} & x'_{z\bar{c}} \end{bmatrix}'$

$$\begin{bmatrix} \dot{x}_{zc} \\ \dot{x}_{z\bar{c}} \end{bmatrix} = \begin{bmatrix} A_{zc} & A_{zc\bar{c}} \\ \mathbf{0} & A_{z\bar{c}} \end{bmatrix} \begin{bmatrix} x_{zc} \\ x_{z\bar{c}} \end{bmatrix} + \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} x_1 + \begin{bmatrix} D_{zc} \\ D_{z\bar{c}} \end{bmatrix} w \tag{4.3}$$

where x_{zc} is n_{zc}-dimensional; $x_{z\bar{c}}$ is $n_{z\bar{c}} := s - n_{zc}$ dimensional; and the pair (A_{zc}, A_{zc1}) is controllable.

By Claim 4.1.1 and Lemma 6 of [7], $\exists k \in [0,\infty) \subset \mathbb{R}$, $\exists \lambda \in (0,\infty) \subset \mathbb{R}$, such that $\left\| \mathrm{e}^{A_z t} A_{z1} \right\|_{2,2} \leq k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty)$. Note that, $\forall t \in [0,\infty)$,

$$\left\| \mathrm{e}^{A_z t} A_{z1} \right\|_{2,2} = \left\| \begin{bmatrix} \mathrm{e}^{A_{zc} t} & \star \\ \mathbf{0} & \mathrm{e}^{A_{z\bar{c}} t} \end{bmatrix} \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} = \left\| \mathrm{e}^{A_{zc} t} A_{zc1} \right\|_{2,2} \leq k\mathrm{e}^{-\lambda t}$$

Hence, we have that the system $\dot{z}_c = A_{zc}z_c + A_{zc1}v$, $z_c(0) = \mathbf{0}_{n_{zc}\times 1}$ is bounded input and bounded state stable by Lemma 6 of [7]. By the controllability of the pair (A_{zc}, A_{zc1}) and Lemma A.7 of [7], we have that A_{zc} is Hurwitz.

We will show that the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ by an argument of contradiction. Suppose S_P is not minimum phase. $\exists c_w \in [0,\infty) \subset \mathbb{R}$, $\forall i \in \mathbb{N}$, $\exists x_{z0(i)} \in \mathcal{D}_{z0}$ with $|x_{z0(i)}| \leq c_w$, $\exists w_{(i)[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{(i)[0,\infty)} \right\|_\infty \leq c_w$, $\exists y_{(i)[0,\infty)} \in \mathcal{C}$ with $\left\| y_{(i)[0,\infty)} \right\|_\infty \leq c_w$, such that $\left\| x_{z(i)[0,\infty)} \right\|_\infty > i$, where $x_{z(i)[0,\infty)}$ is the solution to (4.3) with initial condition specified by $x_{z0(i)}$ and inputs $y_{(i)[0,\infty)}$ and $w_{(i)[0,\infty)}$. Partition $x_{z(i)}$ as $\begin{bmatrix} x'_{zc(i)} & x'_{z\bar{c}(i)} \end{bmatrix}'$. Then, we have $\limsup_{i\to\infty} \left\| x_{z\bar{c}(i)[0,\infty)} \right\|_\infty = +\infty$ since A_{zc} is Hurwitz ($\left\| x_{z\bar{c}(i)[0,\infty)} \right\|_\infty =: \bar{c}_{wi}$, then, $\left\| x_{zc(i)[0,\infty)} \right\|_\infty \leq k_1 c_w + k_2\sqrt{\bar{c}_{wi}^2 + 2c_w^2}$, for some $k_1, k_2 \in [0,\infty) \subset \mathbb{R}$ that is independent of $i \in \mathbb{N}$). By Lemma B.1, there exists $c_1 \in \overline{\mathbb{R}_+}$ such that we may set $x_0 = x_{0(i)} \in \mathcal{D}_0$ with $Kx_{0(i)} = x_{z0(i)}$ and $|x_0| \leq c_1 \left| x_{z0(i)} \right| \leq c_1 c_w$. Let $w_{[0,\infty)} = w_{(i)[0,\infty)}$, $\check{x}_0 = x_0$, and $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)}$ with $\left\| Y_{d[0,\infty)} \right\|_\infty = 0$ and $|Y_{d0}| = 0$. By 1., $\exists c_c \in [0,\infty) \subset \mathbb{R}$, where c_c depends only on c_w, c_1, and S_C, such that the closed-loop system admits unique solution $X_{\{i\}[0,\infty)}$ with continuous signals $u_{\{i\}[0,\infty)}$ and $y_{\{i\}[0,\infty)}$ and $\left\| x_{\{i\}[0,\infty)} \right\|_\infty \leq c_c$. Note that $x_{\{i\}[0,\infty)}$ is the unique solution to (3.6) with initial condition $x_{0(i)}$ and inputs $u_{\{i\}[0,\infty)}$ and $w_{(i)[0,\infty)}$. Then, the component $x_{z\bar{c}\{i\}[0,\infty)}$ of $x_{\{i\}[0,\infty)}$ is the unique solution to $x_{z\bar{c}}$ dynamics in (4.3), which is independent of $y_{\{i\}[0,\infty)}$. We have $x_{z\bar{c}\{i\}[0,\infty)} = x_{z\bar{c}(i)[0,\infty)}$. Hence, $\limsup_{i\to\infty} \left\| x_{\{i\}[0,\infty)} \right\|_\infty \geq \limsup_{i\to\infty} \left\| x_{z\bar{c}\{i\}[0,\infty)} \right\|_\infty = \limsup_{i\to\infty} \left\| x_{z\bar{c}(i)[0,\infty)} \right\|_\infty = +\infty$. This is a contradiction. Hence, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

Case 2.3: $n_{zc} = s$. Then, the pair (A_z, A_{z1}) is controllable. By Claim 4.1.1 and Lemma 7 of [7], we have that the matrix A_z is Hurwitz. Then, by Lemma 3.8, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for this sub-case.

This completes the proof of the lemma. □

5. Extension of square MIMO LTI system into one with uniform vector relative degree and uniform observability indices. We present here a lemma that establishes the invariance of the minimum phase property under the application of an output integration operation for a MIMO square LTI system with vector relative degree. Thus, a minimum phase finite-dimensional continuous time MIMO linear time-invariant system will remain so after finite number of steps of dynamic extension or output integration to a system with uniform vector relative degree.

Lemma 5.1. *Consider the finite-dimensional continuous time square MIMO LTI system (3.6) under the specification of Definition 3.2. Let the system (3.6) admits vector relative degree $r_1, \ldots, r_m \in \{0, \ldots, n\}$ with respect to the control inputs; $y := (y_1, \ldots, y_m)$; and $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace. Assume that (3.6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. $\forall i_0 \in \{1, \ldots, m\}$, we introduce an integrator at y_{i_0}*

$$\dot{\xi} = y_{i_0} = C_{i_0,}x + K_{i_0,}u + E_{i_0,}w; \quad \xi(0) = \alpha \in \mathbb{R} \tag{5.1a}$$

$$\bar{y}_{i_0} = \xi \tag{5.1b}$$

to result in the composite system with state vector $\bar{x} := (x, \xi) \in \mathbb{R}^{n+1}$ and output $\bar{y} = (y_1, \ldots, y_{i_0-1}, \bar{y}_{i_0}, y_{i_0+1}, \ldots, y_m)$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + \bar{D}w; \qquad \bar{x}(0) = (x_0, \alpha) \tag{5.2a}$$

$$\bar{y} = \bar{C}\bar{x} + \bar{F}u + \bar{E}w \tag{5.2b}$$

where u and w are the same as in (3.6); $\bar{A}$, $\bar{B}$, $\bar{D}$, $\bar{C}$, $\bar{K}$, and $\bar{E}$ are constant matrices of appropriate dimensions; and $\bar{x}_0 \in \bar{\mathcal{D}}_0 := \mathcal{D}_0 \times \mathbb{R}$, $\bar{\mathcal{D}}_0 \subseteq \mathbb{R}^{n+1}$ is a subspace. Then, the extended system admits vector relative degree $r_1, \ldots, r_{i_0-1}, r_{i_0}+1, r_{i_0+1}, \ldots, r_m$ with respect to u, and is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

Proof. It is easy to see, with the introduction of the integration on y_{i_0} as in (5.1), the extended system (5.2) admits vector relative degree $r_1, \ldots, r_{i_0-1}, r_{i_0} + 1, r_{i_0+1}, \ldots, r_m$ with respect to u. We need to show that (5.2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

Let $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, $A_{z1} \in \mathbb{R}^{s\times m}$ be the maximal solution to (3.7) for system (3.6); and thus (3.6) admits extended zero dynamics (3.8). By Lemma 3.1 and Definition 3.2, we have $s = n - \sum_{i=1}^{m} r_i$. Let $C_{i,}$ be the ith row vector of the matrix C; $F_{i,}$ be the ith row vector of the matrix F; $E_{i,}$ be the ith row vector of the matrix E. Then, we have

$$\bar{A} := \begin{bmatrix} A & \mathbf{0}_{n\times 1} \\ C_{i_0,} & 0 \end{bmatrix}; \quad \bar{B} := \begin{bmatrix} B \\ F_{i_0,} \end{bmatrix}; \quad \bar{D} := \begin{bmatrix} D \\ E_{i_0,} \end{bmatrix}$$

$$\bar{C} := \begin{bmatrix} C_{1,} & 0 \\ \vdots & \vdots \\ C_{i_0-1,} & 0 \\ \mathbf{0}_{1\times n} & 1 \\ C_{i_0+1,} & 0 \\ \vdots & \vdots \\ C_{m,} & 0 \end{bmatrix}; \quad \bar{F} := \begin{bmatrix} F_{1,} \\ \vdots \\ F_{i_0-1,} \\ \mathbf{0}_{1\times m} \\ F_{i_0+1,} \\ \vdots \\ F_{m,} \end{bmatrix}; \quad \bar{E} := \begin{bmatrix} E_{1,} \\ \vdots \\ E_{i_0-1,} \\ \mathbf{0}_{1\times q} \\ E_{i_0+1,} \\ \vdots \\ E_{m,} \end{bmatrix}$$

It is easy to check that (3.7) admits solution $\bar{K} := \begin{bmatrix} K & -A_{z1,i_0} \end{bmatrix}$, A_z, $\bar{A}_{z1} := \begin{bmatrix} A_{z1,1} & \cdots & A_{z1,i_0-1} & A_z A_{z1,i_0} & A_{z1,i_0+1} & \cdots A_{z1,m} \end{bmatrix}$ for the extended system (5.2), where $A_{z1,i}$ is the ith column vector of the matrix A_{z1}. This is then the maximal solution by Lemma 3.1 and Definition 3.2, since $\bar{K}$ is $s \times (n+1)$ dimensional and $s = n - \sum_{i=1}^{m} r_i$. This implies that the extended system (5.2) admits the extended zero dynamics

$$\begin{aligned} \dot{\bar{x}}_z &= A_z \bar{x}_z + \bar{A}_{z1}\bar{y} + (KD - A_{z1}E)w; \\ \bar{x}_{z0} &= \bar{K}\bar{x}_0 = Kx_0 - \alpha A_{z1,i_0} \in \bar{K}(\bar{\mathcal{D}}_0) =: \bar{\mathcal{D}}_{z0} \end{aligned} \tag{5.3}$$

By the assumption, (3.6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, and therefore, its extended zero dynamics (3.8) satisfies the requirements of Definition 3.2. By Lemma 3.7, the following system is bounded input bounded state stable:

$$\dot{z} = A_z z + A_{z1} v; \quad z(0) = \mathbf{0}_{s\times 1}$$

By Lemma 6 of [7], there exists $k \in \overline{\mathbb{R}_+}$ and $\lambda \in \mathbb{R}_+$ such that $\left\| \mathrm{e}^{A_z t} A_{z1} \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \in \overline{\mathbb{R}_+}$. Then, we have $\left\| \mathrm{e}^{A_z t} A_{z1,i} \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \in \overline{\mathbb{R}_+}$, $\forall i = 1, \ldots, m$. By Lemma 6 of [7], the following system

$$\dot{z}_i = A_z z_i + A_{z1,i} v_i; \quad z_i(0) = \mathbf{0}_{s\times 1}$$

is bounded input bounded state stable. By Lemma B.2, we have

$$\dot{\bar{z}}_{i_0} = A_z z_{i_0} + A_z A_{z1,i_0} v_{i_0}; \quad \bar{z}_{i_0}(0) = \mathbf{0}_{s\times 1}$$

is bounded input bounded state stable. Thus, we have that the following system

$$\dot{\eta} = A_z \eta + \bar{A}_{z1} v; \quad \eta(0) = \mathbf{0}_{s\times 1}$$

is bounded input bounded state stable. By the assumption, we have $\forall c_w \in \overline{\mathbb{R}_+}$, $\exists c_{c1}(c_w) \in \overline{\mathbb{R}_+}$, $\forall x_{z0} \in \mathcal{D}_{z0}$ with $|x_{z0}| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\| w_{[0,\infty)} \|_\infty \leq c_w$, we have $\| \lambda_{[0,\infty)} \|_\infty \leq c_{c1}(c_w)$, where λ is the unique solution to the following system

$$\dot{\lambda} = A_z \lambda + (KD - A_{z1}E)w; \quad \lambda(0) = x_{z0}$$

By the bounded input bounded state stability of the η system, we have that $\exists c_{c2}(c_w) \in \overline{\mathbb{R}_+}$ such that $\forall v_{[0,\infty)} \in \mathcal{C}$ with $\| v_{[0,\infty)} \|_\infty \leq c_w$, we have $\| \eta_{[0,\infty)} \|_\infty \leq c_{c2}(c_w)$. Consider the extended zero dynamics (5.3) of the extended system (5.2). Fix any $\bar{x}_{z0} = \bar{K}\bar{x}_0 = Kx_0 - \alpha A_{z1,i_0} \in \bar{\mathcal{D}}_{z0}$ with $|\bar{x}_{z0}| \leq c_w$. By Lemma B.1, $\exists c_1 \in \overline{\mathbb{R}_+}$ such that $\exists \bar{x}_0 = (x_0, \alpha) \in \bar{\mathcal{D}}_0$ with $|\bar{x}_0| \leq c_1 c_w$ such that $\bar{K}\bar{x}_0 = \bar{x}_{z0}$. The solution to (5.3) is $\bar{x}_{z[0,\infty)}$ with initial condition $\bar{x}_{z0}$ and input waveforms $\bar{y}_{[0,\infty)} := v_{[0,\infty)}$ and $w_{[0,\infty)}$ can be expressed as $\bar{x}_{z[0,\infty)} = \eta_{[0,\infty)} + \lambda_{[0,\infty)} + \delta_{[0,\infty)}$, where $\lambda(0) = Kx_0 = x_{z0}$ and $\delta_{[0,\infty)}$ is the solution to the system $\dot{\delta} = A_z\delta$, $\delta(0) = -\alpha A_{z1,i_0}$. Thus, we have $\| \bar{x}_{z[0,\infty)} \|_\infty \leq \| \eta_{[0,\infty)} \|_\infty + \| \lambda_{[0,\infty)} \|_\infty + \| \delta_{[0,\infty)} \|_\infty = c_{c1}(\| K \|_{2,2}\, c_1 c_w) + c_{c2}(c_w) + kc_1 c_w\, |A_{z1,i_0}| =: c_c(c_w) \in \overline{\mathbb{R}_+}$. Hence, the extended system (5.2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

This completes the proof of the lemma. □

Toward the end of applying the vectorized version of SISO robust adaptive control design ([6]), we need the following lemma that allows us to further extend the square MIMO LTI system with uniform vector relative degree to one with uniform observability indices, without changing the fact that the system admits uniform vector relative degree and the fact that the system is minimum phase with respect to its admissible initial conditions and admissible disturbance waveforms.

LEMMA 5.2. *Consider the finite-dimensional continuous time MIMO square LTI system*

$$\dot{x} = Ax + Bu + Dw; \qquad x(0) = x_0 \tag{5.4a}$$

$$y = Cx + Fu + Ew \tag{5.4b}$$

where x is the n-dimensional state vector, $n \in \mathbb{Z}_+$; u is the m-dimensional control input, $m \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; y is the m-dimensional output; A, B, D, C, F, and E are constant matrices of appropriate dimensions, $x_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $w_{[0,\infty)} \in \mathcal{W}_d$, and $\mathcal{W}_d$ is of class $\mathcal{B}_q$. Let system (5.4) be minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. Let $\nu_1, \ldots, \nu_m \in \{0, \ldots, n\}$ be the observability indices for the output channels, $\nu := \max_i \nu_i$ be the observability index, and the system admits uniform vector relative degree $r = r_1 = \cdots = r_m \in \{0, \ldots, n\}$ with respect to the control inputs. Without loss of generality, assume that matrices A and C are given in the interweaved observer canonical form as (A.2). We may extend the system (5.4) to an $(n + m\nu - \sum_{l=1}^{m} \nu_l)$-dimensional system

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + \bar{D}w; \qquad \bar{x}(0) = \bar{x}_0 \tag{5.5a}$$

$$y = \bar{C}\bar{x} + Fu + Ew \tag{5.5b}$$

where $\bar{x}$ is the $(n + m\nu - \sum_{l=1}^{m} \nu_l)$-dimensional state vector; u and w are the same as in (5.4); $\bar{A}$, $\bar{B}$, $\bar{D}$, and $\bar{C}$ are constant matrices of appropriate dimensions; and

$\bar{x}_0 \in \bar{\mathcal{D}}_0 := \mathcal{D}_0 \times \left\{ \mathbf{0}_{m\nu - \sum_{l=1}^m \nu_l} \right\}$, $\bar{\mathcal{D}}_0 \subseteq \mathbb{R}^{n+m\nu-\sum_{l=1}^m \nu_l}$ is a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$; such that $\forall x_0 \in \mathcal{D}_0$, let $\bar{x}_0 = (x_0, \mathbf{0}_{m\nu-\sum_{l=1}^m \nu_l}) \in \bar{\mathcal{D}}_0$, we have $\bar{y}_{[0,t_f)} = y_{[0,t_f)}$ for both of the systems with the same input waveforms $u_{[0,t_f)}$ and $w_{[0,t_f)}$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, and (5.5) admits uniform observability indices $\nu = \nu_1 = \cdots = \nu_m$ and uniform vector relative degree r with respect to the control input, and (5.5) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

Proof. It is proved in [3, Chapter 5], that the following row vectors are linearly independent:

$$Q_{i,j} := C_{i,}A^{j-1}, i = 1,\ldots,m \text{ with } j = 1,\ldots,r$$

Thus, by Lemma A.1, the first mr vectors in the list P are linearly independent, and therefore are the first mr row vectors of $q_1,\ldots,q_{n_O}$. This implies that the observability indices $\nu_i \geq r$, $i = 1,\ldots,m$ and $\nu \geq r$.

Let $i_l := \sum_{k=1}^l \nu_k$, $l = 1,\ldots,m$; the ith element of x be x_i, $i = 1,\ldots,n$; and introduce the $(\nu-\nu_l)$-dimensional dynamics with state $\xi_l := (\xi_{l,1},\ldots,\xi_{l,\nu-\nu_l})$: $\dot{\xi}_{l,k} = \xi_{l,k+1}$, $k = 1,\ldots,\nu-\nu_l-1$, $\xi_l(0) = \mathbf{0}_{\nu-\nu_l}$, $l = 1,\ldots,m$. Let $\bar{x} := (x,\xi_1,\ldots,\xi_m)$, which is $(n+m\nu-\sum_{l=1}^m \nu_l)$-dimensional, and modify x_{i_l} dynamics to $\dot{x}_{i_l} = A_{i_l,}x + B_{i_l,}u + D_{i_l,}w + \xi_{l,1}$, $l = 1,\ldots,m$ with $\nu_l > 0$, where $A_{i,}$ is the ith row vector of the matrix A; $B_{i,}$ is the ith row vector of the matrix B; $D_{i,}$ is the ith row vector of the matrix D, or modify the measurement channel $y_l = C_{l,}x + F_{l,}u + E_{l,}w + \xi_{l,1}$, $l = 1,\ldots,m$ with $\nu_l = 0$, where $C_{l,}$ is the lth row vector of the matrix C; $F_{l,}$ is the lth row vector of the matrix F; $E_{l,}$ is the lth row vector of the matrix E. Then, the extended system (5.5) admits the following system matrices

$$\bar{A} := \begin{bmatrix} A & \tilde{G} \\ \mathbf{0}_{(m\nu-\sum_{l=1}^m \nu_l)\times n} & \hat{G} \end{bmatrix}; \quad \bar{B} := \begin{bmatrix} B \\ \mathbf{0}_{(m\nu-\sum_{l=1}^m \nu_l)\times m} \end{bmatrix}$$

$$\bar{D} := \begin{bmatrix} D \\ \mathbf{0}_{(m\nu-\sum_{l=1}^m \nu_l)\times q} \end{bmatrix}; \bar{C} := \begin{bmatrix} C & \tilde{H} \end{bmatrix}$$

$$\tilde{G} := \begin{bmatrix} \tilde{G}_1 & \cdots & \tilde{G}_m \end{bmatrix} \text{ with } \tilde{G}_l \in \mathbb{R}^{n\times(\nu-\nu_l)},$$

$$\tilde{G}_l = \begin{cases} \begin{bmatrix} e_{n,i_l} & \mathbf{0}_{n\times(\nu-\nu_l-1)} \end{bmatrix} & \text{if } \nu > \nu_l > 0 \\ \mathbf{0}_{n\times(\nu-\nu_l)} & \text{if } \nu > \nu_l = 0 \end{cases}, l = 1,\ldots,m;$$

$$\hat{G} := \text{block diagonal}\left(\hat{G}_1,\ldots,\hat{G}_m\right) \text{ with } \hat{G}_l \in \mathbb{R}^{(\nu-\nu_l)\times(\nu-\nu_l)}, l = 1,\ldots,m;$$

$$\tilde{H} := \begin{bmatrix} \tilde{H}_1 & \cdots \tilde{H}_m \end{bmatrix} \text{ with } \tilde{H}_l \in \mathbb{R}^{m\times(\nu-\nu_l)}$$

$$\tilde{H}_l = \begin{cases} \mathbf{0}_{m\times(\nu-\nu_l)} & \text{if } \nu > \nu_l > 0 \\ \begin{bmatrix} e_{m,l} & \mathbf{0}_{m\times(\nu-\nu_l-1)} \end{bmatrix} & \text{if } \nu > \nu_l = 0 \end{cases}, l = 1,\ldots,m$$

where $\dot{\xi}_l =: \hat{G}_l\xi_l$, $l = 1,\ldots,m$.

Define $\hat{x}_l := (x_{i_{l-1}+1},\ldots,x_{i_l},\xi_l) \in \mathbb{R}^\nu$, $l = 1,\ldots,m$; $\hat{x}_{\bar{o}} := (x_{i_m+1},\ldots,x_n)$; and $\hat{x} := (\hat{x}_1,\ldots,\hat{x}_m,\hat{x}_{\bar{o}})$. Then, it is easy to see that, in $\hat{x}$ coordinates, the extended system (5.5) admits the state space representation

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u + \hat{D}w; \quad \hat{x}(0) =: \hat{x}_0 \\ y &= \hat{C}\hat{x} + Fu + Ew \end{aligned}$$

where $i_m = n_O = \sum_{l=1}^{m} \nu_l$,

$$\hat{A} = \begin{bmatrix} \hat{A}_{1,1} & \cdots & \hat{A}_{1,m} & \mathbf{0}_{\nu\times(n-n_O)} \\ \vdots & & \vdots & \vdots \\ \hat{A}_{m,1} & \cdots & \hat{A}_{m,m} & \mathbf{0}_{\nu\times(n-n_O)} \\ \hat{A}_{\bar{o},1} & \cdots & \hat{A}_{\bar{o},m} & \hat{A}_{\bar{o}} \end{bmatrix}; \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \vdots \\ \hat{B}_m \\ \hat{B}_{\bar{o}} \end{bmatrix};$$

$$\hat{C} = \begin{bmatrix} \hat{C}_1 & \cdots & \hat{C}_m & \mathbf{0}_{m\times(n-n_O)} \end{bmatrix}$$

where $\hat{A}_{i,j}$, $i = 1,\ldots,m$, $j = 1,\ldots,m$, is a $\nu\times\nu$-dimensional matrix with elements $\hat{a}_{i,j,k,l}$, $k = 1,\ldots,\nu$, $l = 1,\ldots,\nu$, that satisfies $\hat{a}_{i,j,k,l} = 0$, if $i \neq j$ and $l \geq 2$, $\hat{a}_{i,i,k,k+1} = 1$, $\forall k = 1,\ldots,\nu_i - 1$, and $\hat{a}_{i,i,k,l} = 0$, if $l \geq 2$ and $k+1 \neq l$; and $\hat{A}_{\bar{o},j}$, $j = 1,\ldots,m$, is an $(n-n_O)\times\nu$-dimensional matrix with elements $\hat{a}_{\bar{o},j,k,l}$, $k = 1,\ldots,n-n_O$, $l = 1,\ldots,\nu$, that satisfies $\hat{a}_{\bar{o},j,k,l} = 0$ if $l \geq 2$; $\hat{A}_{\bar{o}}$ is an $(n-n_O)\times(n-n_O)$-dimensional matrix; $\hat{C}_j$, $j = 1,\ldots,m$, is an $m\times\nu$-dimensional matrix with elements $\hat{c}_{j,k,l}$, $k = 1,\ldots,m$, $l = 1,\ldots,\nu$, that satisfies $\hat{c}_{j,k,l} = 0$, if $l \geq 2$, $\hat{c}_{j,j,1} = 1$, and $\hat{c}_{j,k,1} = 0$ if $k < j$; partition $B = \begin{bmatrix} B_1' & \cdots & B_m' & B_{\bar{o}}' \end{bmatrix}'$ and the partitioning is compatible with that of A and C, and B_i, $i = 1,\ldots,m$, is $\nu_i\times m$-dimensional matrix with elements $b_{i,k,l}$, $k = 1,\ldots,\nu_i$, $l = 1,\ldots,m$; $\hat{B}_i$, $i = 1,\ldots,m$, is a $\nu\times m$-dimensional matrix with elements $\hat{b}_{i,k,l}$, $k = 1,\ldots,\nu$, $l = 1,\ldots,m$, that satisfies $\hat{b}_{i,k,l} = 0$, if $k > \nu_i$, $l = 1,\ldots,m$, and $\hat{b}_{i,k,l} = b_{i,k,l}$, if $1 \leq k \leq \nu_i$, $l = 1,\ldots,m$. Thus, based on the structure of $\hat{A}$ and $\hat{C}$, we conclude that (5.5) admits uniform observability indices ν as desired. Since (5.4) admits uniform vector relative degree r, then the matrix B must satisfy $b_{i,k,l} = 0$, if $k < r$, $i = 1,\ldots,m$, $l = 1,\ldots,m$. Thus, the extended system (5.5) in $\hat{x}$ coordinates satisfies that $\hat{b}_{i,k,l} = 0$ if $k < r$, $i = 1,\ldots,m$, $l = 1,\ldots,m$. Thus, the extended system (5.5) admits uniform vector relative degree r.

Let the system (5.4) admit maximal solution $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, and $A_{z1} \in \mathbb{R}^{s\times m}$ to (3.7). By Lemma 3.1 and Definition 3.2, we have $s = n - mr$. It is easy to show that $\bar{K} :=$ block diagonal $\left(K, I_{m\nu-\sum_{i=1}^m \nu_i}\right) \in \mathbb{R}^{(s+m\nu-\sum_{i=1}^m \nu_i)\times(n+m\nu-\sum_{i=1}^m \nu_i)}$, $\bar{A}_z := \begin{bmatrix} A_z & K\tilde{G} - A_{z1}\tilde{H} \\ \mathbf{0}_{(m\nu-\sum_{i=1}^m \nu_i)\times s} & \hat{G} \end{bmatrix} \in \mathbb{R}^{(s+m\nu-\sum_{i=1}^m \nu_i)\times(s+m\nu-\sum_{i=1}^m \nu_i)}$, and $\bar{A}_{z1} := \begin{bmatrix} A_{z1} \\ \mathbf{0}_{(m\nu-\sum_{i=1}^m \nu_i)\times m} \end{bmatrix} \in \mathbb{R}^{(s+m\nu-\sum_{i=1}^m \nu_i)\times m}$ satisfies the equation (3.7) for the extended system (5.5). The matrix $\bar{K}$ is of full row rank. By Lemma 3.1and Definition 3.2, set of solution $\bar{K}$, $\bar{A}_z$, and $\bar{A}_{z1}$ is maximal. Therefore, the extended zero dynamics of (5.5) is

$$\begin{aligned} \dot{x}_z &= A_z x_z + (K\tilde{G} - A_{z1}\tilde{H})\xi + A_{z1}y + (KD - A_{z1}E)w; \quad x_z(0) = Kx_0 \in K(\mathcal{D}_0) \\ \dot{\xi} &= \hat{G}\xi; \quad \xi(0) = \mathbf{0}_{m\nu-\sum_{i=1}^m \nu_i} \end{aligned}$$

with state vector $\bar{x}_z := (x_z, \xi)$, and $\bar{\mathcal{D}}_0 := \mathcal{D}_0 \times \left\{\mathbf{0}_{m\nu-\sum_{i=1}^m \nu_i}\right\} \subseteq \mathbb{R}^{s+m\nu-\sum_{i=1}^m \nu_i}$ is a subspace. By the assumption that (5.4) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, we can readily conclude that the extended system (5.5) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

This completes the proof of the lemma. □

6. Conclusions. In this paper, we generalize the definition of extended zero dynamics [7] to square MIMO LTI systems that are right invertible. Such a system is said to be minimum phase with respect to admissible initial conditions and admissible disturbance waveforms if the extended zero dynamics is absent or admits bounded state trajectory under admissible initial state, arbitrary bounded continuous output waveform, and admissible bounded disturbance waveform. We prove that the extended zero dynamics of a square MIMO LTI system is invariant (modulo linear state transformation) under a step of dynamic extension [3] for one of its inputs. We also prove that a square MIMO LTI system is minimum phase if its zero dynamics is asymptotically stable; the converse holds if the system is further stabilizable from the control input. We present the extended zero dynamics canonical form for a square MIMO LTI system with uniform vector relative degree, which show difference in four exhaustive cases that mimic the SISO case [7]. We prove that the minimum phase assumption is necessary in the model reference control of the square MIMO LTI system that achieves the following two properties: 1) the system states remain bounded when the disturbance input waveform is admissible and bounded, the initial condition is admissible and bounded, and the reference trajectory is bounded with bounded derivatives up to nth order, where n is the dimension of the system; 2) the perfect tracking of any given bounded reference trajectory with bounded derivatives up to nth order when the disturbance waveform is identically equal to zero, the extended zero dynamics admits any admissible initial condition, and appropriately choosing the initial condition for the rest of the closed-loop system states.

This extended zero dynamics canonical form and the strict observer canonical form are needed for the true system in the robust adaptive control design for the system and stability analysis of the resulting closed-loop system [6]. The strict observer canonical form of an LTI system is the observer canonical form of the system that further satisfies that the input-free outputs of the system are a subset of the state variables of the representation. The strict observer canonical form of an LTI system is guaranteed to exist if the system admits uniform observability indices. Toward the end of robust adaptive control for the system, we need to be able to extend the given square MIMO LTI system to one with uniform vector relative degree and with uniform observability indices without changing its minimum phase property. We present two lemmas in Section 5 that fully resolve this issue. Thus, the robust adaptive control design for a square MIMO LTI system can be carried out if the given system is minimum phase and can admit vector relative degree after finite number of steps of dynamic extension that are independent of the unknown parameters in the system. Once, we arrive at a continuous-time square MIMO LTI system with vector relative degree, which is further minimum phase, we can further apply parameter independent dynamic extensions or to add integrators on its outputs to arrive at a square MIMO LTI system with uniform vector relative degree, without changing its minimum phase property. Then, this extended system admits the extended zero dynamics canonical form. To apply robust adaptive control design (according to [6]) on this system, the system must also admit strict observer canonical form for the estimation part of the design to be carried out. Whenever this condition doesn't hold, we can always further extend the extended system by adding dummy states (which are always zero under admissible initial condition and are independent of the inputs) such that the further extended system admits uniform observability indices, without changing the fact that the system admits uniform vector relative degree and the fact that the system is minimum phase with respect to its admissible initial conditions and admissible disturbance waveforms. The observable part of this further extended MIMO LTI system then may serve as the true system in the robust adaptive control design using the vectorized version of [6]. Thus, the robust adaptive control design of a minimum phase square MIMO LTI system is solved.

Further study of the minimum phase property of composite systems that consists of interconnected LTI systems under suitable assumptions on the component system is fruitful. For SISO composite systems, comprehensive results have already been obtained in [9, 8]. The MIMO version of these results will be studied in the future.

Appendix A. Observer canonical form representation. First, we present a lemma and a corollary that defines the observer canonical form representation of an LTI system.

LEMMA A.1. *Consider a finite-dimensional continuous-time LTI system*

$$\dot{x} = Ax \tag{A.1a}$$

$$y = Cx \tag{A.1b}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; y is the m-dimensional output, $m \in \mathbb{Z}_+$; A, B, and C are constant matrices of appropriate dimensions. Let the row vectors of C be $c_1, \ldots, c_m$. Seek linearly independent rows vectors from the list P that includes row vectors $c_1, \ldots, c_m, c_1A, \ldots, c_mA, \ldots, c_1A^{n-1}, \ldots, c_mA^{n-1}$ in the listed order, to result in row vectors $q_1, \ldots, q_{n_O}$, where $n_O \in \mathbb{Z}_+$ and $n_O \le n$. Then,

(i) there exists observability indices $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ such that the list $\bar{P}$ of row vectors $c_1, \ldots, c_1A^{\nu_1-1}, c_2, \ldots, c_2A^{\nu_2-1}, \ldots, c_m, \ldots, c_mA^{\nu_m-1}$ is a permutation of $q_1, \ldots, q_{n_O}$; $n_O = \sum_{i=1}^m \nu_i$; and $\nu := \max_{1\le i\le m} \nu_i$ is said to be the observability index *of (A.1).*

Let $Q_i := \begin{cases} \begin{bmatrix} c_i' & \cdots & (A')^{\nu_i-1}c_i' \end{bmatrix}' & \text{if } \nu_i > 0 \\ [] & \text{if } \nu_i = 0 \end{cases}$, $i = 1, \ldots, m$. Clearly, Q_i is a $\nu_i \times n$-dimensional matrix. Let $Q := \begin{bmatrix} Q_1' & \cdots & Q_m' & q_{n_O+1}' & \cdots & q_n' \end{bmatrix}'$, where $q_{n_O+1}, \ldots, q_n$ are n-dimensional row vectors selected arbitrarily so that the matrix Q is invertible. Let $T := Q^{-1} =: \begin{bmatrix} T_1 & \cdots & T_m & t_{n_O+1} & \cdots & t_n \end{bmatrix}$, where T_i, $i = 1, \ldots, m$, are $n \times \nu_i$-dimensional matrices, and t_i, $i = n_O + 1, \ldots, n$, are n-dimensional column vectors; and $e_i := t_{i,\nu_i}$, $i = 1, \ldots, m$, be the last column vector of T_i whenever $\nu_i > 0$. Let $S_i = \begin{bmatrix} A^{\nu_i-1}e_i & \cdots & e_i \end{bmatrix}$, $i = 1, \ldots, m$, be an $n \times \nu_i$-dimensional matrix; and $S := \begin{bmatrix} S_1 & \cdots & S_m & t_{n_O+1} & \cdots t_n \end{bmatrix}$ be an $n \times n$-dimensional matrix. Then, the following statements holds.

(ii) $QS = \begin{bmatrix} W_{1,1} & \cdots & W_{1,m} & \mathbf{0}_{\nu_1\times(n-n_O)} \\ \vdots & & \vdots & \vdots \\ W_{m,1} & \cdots & W_{m,m} & \mathbf{0}_{\nu_m\times(n-n_O)} \\ \star_{(n-n_O)\times\nu_1} & \cdots & \star_{(n-n_O)\times\nu_m} & I_{n-n_O} \end{bmatrix}$*, where $W_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, m$, is a $\nu_i \times \nu_j$-dimensional matrix with elements $w_{i,j,k,l}$, $k = 1, \ldots, \nu_i$, $l = 1, \ldots, \nu_j$, such that $w_{i,i,k,k} = 1$, $i = 1, \ldots, m$, $k = 1, \ldots, \nu_i$; $w_{i,j,k,l} = 0$, $i = 1, \ldots, m$, $j = 1, \ldots, m$, $k = 1, \ldots, \nu_i$, $l = k + 1, \ldots, \nu_j$; and $w_{i,j,k,k} = 0$, $i = 1, \ldots, m$, $j = i + 1, \ldots, m$, $k = 1, \ldots, \min\{\nu_i, \nu_j\}$.*

(iii) S is an invertible matrix.

(iv) In $\bar{x} := S^{-1}x$ coordinates, the system (A.1) admits the interweaved observer canonical form *representation*

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_{1,1} & \cdots & \bar{A}_{1,m} & \mathbf{0}_{\nu_1\times(n-n_O)} \\ \vdots & & \vdots & \vdots \\ \bar{A}_{m,1} & \cdots & \bar{A}_{m,m} & \mathbf{0}_{\nu_m\times(n-n_O)} \\ \bar{A}_{\bar{o},1} & \cdots & \bar{A}_{\bar{o},m} & \bar{A}_{\bar{o}} \end{bmatrix} \bar{x} =: \bar{A}\bar{x} \tag{A.2a}$$

(A.2b) $$y = \begin{bmatrix} \bar{C}_1 & \cdots & \bar{C}_m & \mathbf{0}_{m\times(n-n_O)} \end{bmatrix} \bar{x} =: \bar{C}\bar{x}$$

where $\bar{A}_{i,j}$, $i = 1,\ldots,m$, $j = 1,\ldots,m$, is a $\nu_i \times \nu_j$-dimensional matrix with elements $a_{i,j,k,l}$, $k = 1,\ldots,\nu_i$, $l = 1,\ldots,\nu_j$, that satisfies $a_{i,j,k,l} = 0$, if $i \neq j$ and $l \geq 2$, $a_{i,i,k,k+1} = 1$, $\forall k = 1,\ldots,\nu_i - 1$, and $a_{i,i,k,l} = 0$, if $l \geq 2$ and $k+1 \neq l$; and $\bar{A}_{\bar{o},j}$, $j = 1,\ldots,m$, is an $(n-n_O)\times\nu_j$-dimensional matrix with elements $a_{\bar{o},j,k,l}$, $k = 1,\ldots,n-n_O$, $l = 1,\ldots,\nu_j$, that satisfies $a_{\bar{o},j,k,l} = 0$ if $l \geq 2$; $\bar{A}_{\bar{o}}$ is an $(n-n_O)\times(n-n_O)$-dimensional matrix; $\bar{C}_j$, $j = 1,\ldots,m$, is an $m \times \nu_j$-dimensional matrix with elements $c_{j,k,l}$, $k = 1,\ldots,m$, $l = 1,\ldots,\nu_j$, that satisfies $c_{j,k,l} = 0$, if $l \geq 2$, $c_{j,j,1} = 1$, and $c_{j,k,1} = 0$ if $k < j$. We will further say that the interweaved observer canonical form (A.2) is a strict interweaved observer canonical form *if the matrix $\bar{C}$ further satisfies that $c_{j,k,1} = 0$ if $j \neq k$, $j = 1,\ldots,m$, $k = 1,\ldots,m$.*

Proof. (i) Let the characteristic polynomial of A be $p(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$, where $\alpha_i \in \mathbb{R}$, $i = 1,\ldots,n$. By Cayley-Hamilton Theorem, we have $A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I_n = \mathbf{0}_{n\times n}$. Then, $A^n = -\alpha_1 A^{n-1} - \cdots - \alpha_n I_n$. Fix any row vector $c_i A^j$, with $i \in \{1,\ldots,m\}$ and $j \in \{0,\ldots,n-1\}$. If $j < n-1$ and $c_i A^j$ is linearly dependent on preceding row vectors in the list P, then $c_i A^{j+1} = (c_i A^j)\cdot A$ is linearly dependent on preceding row vectors in the list P. If $j = n-1$, then $c_i A^n = -\alpha_1 c_i A^{n-1} - \cdots - \alpha_n c_i$ is again linearly dependent on row vectors in the list P. Thus, (i) holds. Furthermore, we may prove the following claim.

Claim A.1.1. *$\forall i = 1,\ldots,m$, $\forall \tau \in \mathbb{Z}_+$, we have*

$$c_i A^{\nu_i+\tau} = \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ ((k<\nu_i+\tau) \text{ or } (k=\nu_i+\tau) \text{ and } (j<i))}} \beta_{j,k} c_j A^k$$

where $\beta_{j,k}$'s are constants.

Proof. We will prove the above claim by mathematical induction on τ.

1° Let $\tau = 0$. $\forall i = 1,\ldots,m$, by the preceding argument, $c_i A^{\nu_i}$ can be expressed as a linear combination of row vectors $c_j A^k \in \bar{P}$ with $k < \nu_i$, or $k = \nu_i$ and $j < i$. Hence, the result holds for $\tau = 0$.

2° Assume the result holds $\forall \tau = 0,\ldots,\sigma$ with $\sigma \in \mathbb{Z}_+$.

3° Consider the case when $\tau = \sigma + 1$.

$$\begin{aligned} c_i A^{\nu_i+\tau} &= (c_i A^{\nu_i+\sigma})\cdot A = \Big(\sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ (k<\nu_i+\sigma \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} c_j A^k\Big)\cdot A \\ &= \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ (k<\nu_i+\sigma \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} c_j A^{k+1} \end{aligned}$$

where the second equality follows from the inductive assumption. Since $c_j A^k \in \bar{P}$, we have $k < \nu_j$ and $k+1 \leq \nu_j$. When $k+1 = \nu_j$, by 1°, $c_j A^{k+1} = c_j A^{\nu_j} = \sum_{\substack{(c_{\bar{j}} A^{\bar{k}} \in \bar{P}) \text{ and} \\ ((\bar{k}<\nu_j) \text{ or } (\bar{k}=\nu_j) \text{ and } (\bar{j}<j))}} \bar{\beta}_{\bar{j},\bar{k}} c_{\bar{j}} A^{\bar{k}}$. Then,

$$\begin{aligned} c_i A^{\nu_i+\tau} = & \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and } (k+1<\nu_j) \text{ and} \\ ((k<\nu_i+\sigma) \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} c_j A^{k+1} \\ & + \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and } (k+1=\nu_j) \text{ and} \\ ((k<\nu_i+\sigma) \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} \sum_{\substack{(c_{\bar{j}} A^{\bar{k}} \in \bar{P}) \text{ and} \\ ((\bar{k}<\nu_j) \text{ or } (\bar{k}=\nu_j) \text{ and } (\bar{j}<j))}} \bar{\beta}_{\bar{j},\bar{k}} c_{\bar{j}} A^{\bar{k}} \end{aligned}$$

$$= \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ ((k<\nu_i+\tau) \text{ or } (k=\nu_i+\tau) \text{ and } (j<i))}} \tilde{\beta}_{j,k} c_j A^k$$

Hence, the result holds for $\tau = \sigma + 1$.

This completes the induction process. This completes the proof of the claim. □

(ii) Note that $QT = I_n$. Then, $Qt_i = e_{n,i}$, $i = n_O + 1, \ldots, n$. Clearly, $W_{i,j} = Q_i S_j$, $i = 1, \ldots, m$, $j = 1, \ldots, m$. $\forall i = 1, \ldots, m$, $\forall k = 1, \ldots, \nu_i$, $w_{i,i,k,k} = (c_i A^{k-1}) \cdot (A^{\nu_i - k} e_i) = (c_i A^{\nu_i - 1}) \cdot e_i$, which equals to the product of the $(\sum_{l=1}^{i} \nu_l)$'s row of Q and $(\sum_{l=1}^{i} \nu_l)$'s column of T. Hence, $w_{i,i,k,k} = 1$ since $QT = I_n$.

$\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, m$, $\forall k = 1, \ldots, \nu_i$, $\forall l = k+1, \ldots, \nu_j$, $w_{i,j,k,l} = (c_i A^{k-1}) \cdot (A^{\nu_j - l} e_j) = (c_i A^{\nu_j - l + k - 1}) \cdot e_j$. If $\nu_j - l + k < \nu_i$, then $w_{i,j,k,l}$ equals to the product of the $(\sum_{\tau=1}^{i-1} \nu_\tau + \nu_j - l + k)$'s row of Q and the $(\sum_{\tau=1}^{j} \nu_\tau)$'s column of T. Hence, $w_{i,j,k,l} = 0$ since $QT = I_n$. On the other hand, if $\nu_j - l + k \geq \nu_i$, then $\nu_j > \nu_i$ and $j \neq i$; by Claim A.1.1, $w_{i,j,k,l} = c_i A^{\nu_j - l + k - 1} e_j = \sum_{\substack{(c_\tau A^\sigma \in \bar{P}) \text{ and} \\ ((\sigma<\nu_j-l+k-1) \text{ or } (\sigma=\nu_j-l+k-1) \text{ and } (\tau<i))}} \beta_{\tau,\sigma} c_\tau A^\sigma e_j$. Since $QT = I_n$, then $w_{i,j,k,l} = 0$. Hence, $w_{i,j,k,l} = 0$, $\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, m$, $\forall k = 1, \ldots, \nu_i$, $\forall l = k+1, \ldots, \nu_j$.

$\forall i = 1, \ldots, m$, $\forall j = i+1, \ldots, m$, $k = 1, \ldots, \min\{\nu_i, \nu_j\}$, $w_{i,j,k,k} = (c_i A^{k-1}) \cdot (A^{\nu_j - k} e_j) = c_i A^{\nu_j - 1} e_j$. If $\nu_j \leq \nu_i$, then $w_{i,j,k,k}$ equals to the product of the $(\sum_{\tau=1}^{i-1} \nu_\tau + \nu_j)$'s row of Q and the $(\sum_{\tau=1}^{j} \nu_\tau)$'s column of T, which equals to 0 since $i \neq j$ and $QT = I_n$. On the other hand, if $\nu_j > \nu_i$, by Claim A.1.1, $w_{i,j,k,k} = c_i A^{\nu_j - 1} e_j = \sum_{\substack{(c_\tau A^\sigma \in \bar{P}) \text{ and} \\ ((\sigma<\nu_j-1) \text{ or } (\sigma=\nu_j-1) \text{ and } (\tau<i))}} \beta_{\tau,\sigma} c_\tau A^\sigma e_j$. Since $QT = I_n$, then $w_{i,j,k,k} = 0$. Hence, $w_{i,j,k,k} = 0$, $\forall i = 1, \ldots, m$, $\forall j = i+1, \ldots, m$, $k = 1, \ldots, \min\{\nu_i, \nu_j\}$.

This proves (ii).

(iii) By (ii), the matrix QS is invertible, which implies that S is invertible.

(iv) In $\bar{x}$ coordinates, $\bar{A} = S^{-1} A S$ and $\bar{C} = CS$. Then, $S\bar{A} = AS$. $\forall i = 1, \ldots, m$, $\forall j = 2, \ldots, \nu_i$, the $(\sum_{l=1}^{i-1} \nu_l + j)$th column of S is $A^{\nu_i - j} e_i$. Then, the $(\sum_{l=1}^{i-1} \nu_l + j)$th column of $AS = S\bar{A}$ is $A^{\nu_i - j + 1} e_i = S e_{n, \sum_{l=1}^{i-1} \nu_l + j - 1}$. Hence, the $(\sum_{l=1}^{i-1} \nu_l + j)$th column of $\bar{A}$ is $e_{n, \sum_{l=1}^{i-1} \nu_l + j - 1}$. This proves that the first n_O columns of $\bar{A}$ are as desired.

Since $QT = I_n$, then $c_i A^j t_l = 0$, $\forall i = 1, \ldots, m$, $\forall j = 0, \ldots, \nu_i - 1$, $\forall l = n_O + 1, \ldots, n$. Fix any $l = n_O + 1, \ldots, n$. By Claim A.1.1, we have $c_i A^{\nu_i} t_l = 0$, $\forall i = 1, \ldots, m$. Then, $c_i A^j t_l = 0$, $\forall i = 1, \ldots, m$, $\forall j = 0, \ldots, \nu_i$. Therefore, $(c_i A^j) \cdot (A t_l) = 0$, $\forall i = 1, \ldots, m$, $\forall j = 0, \ldots, \nu_i - 1$. This leads to $QAt_l = \begin{bmatrix} \mathbf{0}_{1 \times n_O} & \star_{1 \times (n - n_O)} \end{bmatrix}'$ and $At_l = T \begin{bmatrix} \mathbf{0}_{1 \times n_O} & \star_{1 \times (n - n_O)} \end{bmatrix}' = S \begin{bmatrix} \mathbf{0}_{1 \times n_O} & \star_{1 \times (n - n_O)} \end{bmatrix}'$. Hence, the last $n - n_O$ columns of $\bar{A}$ are as desired.

Since $QT = I_n$ and by Claim A.1.1, we have $c_i t_l = 0$, $\forall i = 1, \ldots, m$, $\forall l = n_O + 1, \ldots, n$. Hence, the last $n - n_O$ columns of $\bar{C}$ are as desired. Clearly, $\bar{C}_j = CS_j$, $\forall j = 1, \ldots, m$. $\forall k = 1, \ldots, m$, $\forall j = 1, \ldots, m$, $\forall l = 2, \ldots, \nu_j$, we have $c_{j,k,l} = c_k A^{\nu_j - l} e_j = 0$, by the arguments shown in the 6th paragraph preceding this paragraph. $\forall j = 1, \ldots, m$ with $\nu_j \geq 1$, we have $c_{j,j,1} = c_j A^{\nu_j - 1} e_j = 1$, since $QT = I_n$. $\forall j = 1, \ldots, m$, $\forall k = 1, \ldots, j-1$, $c_{j,k,1} = c_k A^{\nu_j - 1} e_j = 0$, by the arguments shown in the 5th paragraph preceding this paragraph. Hence, $\bar{C}$ admits the structure as desired.

This completes the proof of the lemma. □

COROLLARY A.2. *Consider a finite-dimensional continuous time MIMO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{A.3a}$$

(A.3b) $$y = Cx + Ku + Ew$$

where x is the n-dimensional state vector, $n \in \mathbb{Z}_+$; u is the p-dimensional control input, $p \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; y is the m-dimensional output, $m \in \mathbb{Z}_+$; A, B, D, C, K, and E are constant matrices of appropriate dimensions. Let $\nu \in \{0, \ldots, n\}$ be the observability index of the system and $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ be the observability indices for the output channels. Then, there exists an invertible matrix T such that in $\bar{x} = \begin{bmatrix} x'_{\bar{o}} & x'_1 & \cdots & x'_\nu \end{bmatrix}' = T^{-1}x$ coordinates, we have $\nu = \max_{1\le i\le m} \nu_i$, $n_O := \sum_{i=1}^m \nu_i \le n$, $x_{\bar{o}}$ is $(n - n_O)$-dimensional, x_i is $(n_i := \sum_{l=1}^m \chi_{\{t\ge i\},\mathbb{Z}}(\nu_l))$-dimensional, $i = 1, \ldots, \nu$, $m \ge n_1 \ge n_2 \ge \cdots \ge n_\nu \ge 0$, $n_\nu > 0$ if $\nu > 0$, $n_O = \sum_{i=1}^\nu n_i$, and the system (A.3) admits the observer canonical form *representation*

(A.4a) $$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + \hat{A}_{\bar{o},1} x_1 + \hat{B}_{\bar{o}} u + \hat{D}_{\bar{o}} w$$

(A.4b) $$\dot{x}_i = \hat{A}_{i,1} x_1 + \hat{A}_{i,i+1} x_{i+1} + \hat{B}_i u + \hat{D}_i w; \quad i = 1, \ldots, \nu - 1$$

(A.4c) $$\dot{x}_\nu = \hat{A}_{\nu,1} x_1 + \hat{B}_\nu u + \hat{D}_\nu w$$

(A.4d) $$y = \hat{C}_1 x_1 + Ku + Ew$$

where all matrices are constant and of appropriate dimensions; $\hat{C}_1$ is composed of the first column vectors of $\bar{C}_j$, $j = 1, \ldots, m$ with $\nu_j > 0$, which are defined in (A.2), and is therefore of full column rank; $\hat{A}_{\bar{o}} = \bar{A}_{\bar{o}}$, which is defined in (A.2); and $\hat{A}_{i,i+1}$, $i = 1, \ldots, \nu - 1$, is an $n_i \times n_{i+1}$-dimensional matrix whose column vectors are a subset of the column vectors of I_{n_i} and is of full column rank. We will further say that the observer canonical form (A.4) is a strict observer canonical form *if $\hat{C}_1$ consists of a subset of column vectors of I_m.*

Proof. This follows directly from Lemma A.1, and is therefore omitted. ∎

Next, we specialize the above canonical form to the case where the observability indices are uniform, and the case where the observability indices are uniform and when the system admits uniform vector relative degree from the control input to the output.

Corollary A.3. *Consider a finite-dimensional continuous time MIMO LTI system*

(A.5a) $$\dot{x} = Ax + Bu + Dw$$

(A.5b) $$y = Cx + Ku + Ew$$

where x is the n-dimensional state vector, $n \in \mathbb{Z}_+$; u is the p-dimensional control input, $p \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; y is the m-dimensional output, $m \in \mathbb{Z}_+$; A, B, D, C, K, and E are constant matrices of appropriate dimensions. Let $\nu \in \{0, \ldots, n\}$ be the observability index of the system and $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ be the observability indices for the output channels. Assume that $\nu = \nu_1 = \cdots = \nu_m$. Then, there exists an invertible matrix T such that in $\tilde{x} = \begin{bmatrix} x'_{\bar{o}} & \tilde{x}'_1 & \cdots & \tilde{x}'_\nu \end{bmatrix}' = T^{-1}x$ coordinates, we have $n_O := m\nu$, $x_{\bar{o}}$ is $(n - n_O)$-dimensional, $\tilde{x}_i$ is m-dimensional, $i = 1, \ldots, \nu$, and the system (A.5) admits the strict observer canonical form *representation, if $\nu > 0$,*

(A.6a) $$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + \tilde{A}_{\bar{o},1} \tilde{x}_1 + \tilde{B}_{\bar{o}} u + \tilde{D}_{\bar{o}} w$$

(A.6b) $$\dot{\tilde{x}}_i = \tilde{A}_{i,1} \tilde{x}_1 + \tilde{x}_{i+1} + \tilde{B}_i u + \tilde{D}_i w; \quad i = 1, \ldots, \nu - 1$$

(A.6c) $$\dot{\tilde{x}}_\nu = \tilde{A}_{\nu,1} \tilde{x}_1 + \tilde{B}_\nu u + \tilde{D}_\nu w$$

(A.6d) $$y = \tilde{x}_1 + Ku + Ew$$

or if $\nu = 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + \hat{B}_{\bar{o}} u + \hat{D}_{\bar{o}} w \tag{A.7a}$$

$$y = Ku + Ew \tag{A.7b}$$

where all matrices are constant and of appropriate dimensions.

Proof. The result is trivial if $\nu = 0$. When $\nu > 0$, by Corollary A.2, there exists an invertible matrix T_1 such that in $\bar{x} = \begin{bmatrix} x_{\bar{o}}' & x_1' & \cdots & x_\nu' \end{bmatrix}' = T_1^{-1} x$ coordinates, we have x_i are m-dimensional, $\forall i = 1, \ldots, \nu$, the system (A.5) admits state space representation (A.4), $\hat{A}_{i,i+1} = I_m$ with possibly reordering state variables within x_{i+1}, $\forall i = 1, \ldots, \nu - 1$, and $\hat{C}_1$ is a lower triangular matrix with 1's on the diagonal. Then, $\hat{C}_1$ is invertible. Let $\tilde{x} := \begin{bmatrix} x_{\bar{o}}' & \tilde{x}_1' & \cdots & \tilde{x}_\nu' \end{bmatrix}' = \begin{bmatrix} x_{\bar{o}}' & (\hat{C}_1 x_1)' & \cdots & (\hat{C}_1 x_\nu)' \end{bmatrix}' =: T^{-1} x$. Then, the system (A.5) admits state space representation (A.6). This completes the proof of the corollary. □

Corollary A.4. *Consider a finite-dimensional continuous time MIMO LTI system*

$$\dot{x} = Ax + Bu + Dw \tag{A.8a}$$

$$y = Cx + Ku + Ew \tag{A.8b}$$

where x is the n-dimensional state vector, $n \in \mathbb{Z}_+$; u is the p-dimensional control input, $p \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; y is the m-dimensional output, $m \in \mathbb{Z}_+$; A, B, D, C, K, and E are constant matrices of appropriate dimensions. Let $\nu \in \{0, \ldots, n\}$ be the observability index of the system and $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ be the observability indices for the output channels. Assume that $\nu = \nu_1 = \cdots = \nu_m$ and the system admits vector relative degree $r_1, \ldots, r_m \in \{0, \ldots, n\}$ with respect to the control input u such that $r = r_1 = \cdots = r_m$. Then, $r \le \nu$, there exists an invertible matrix T such that in $\bar{x} = \begin{bmatrix} x_{\bar{o}}' & x_1' & \cdots & x_\nu' \end{bmatrix}' = T^{-1} x$ coordinates, we have $n_O := m\nu$, $x_{\bar{o}}$ is $(n - n_O)$-dimensional, x_i is m-dimensional, $i = 1, \ldots, \nu$, and the system (A.8) admits the strict observer canonical form *representation, if $\nu > 0$,*

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + A_{\bar{o},1} x_1 + B_{\bar{o}} u + D_{\bar{o}} w \tag{A.9a}$$

$$\dot{x}_i = A_{i,1} x_1 + x_{i+1} + B_i u + D_i w; \quad i = 1, \ldots, \nu - 1 \tag{A.9b}$$

$$\dot{x}_\nu = A_{\nu,1} x_1 + B_\nu u + D_\nu w \tag{A.9c}$$

$$y = x_1 + Ku + Ew \tag{A.9d}$$

or if $\nu = 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + B_{\bar{o}} u + D_{\bar{o}} w \tag{A.10a}$$

$$y = Ku + Ew \tag{A.10b}$$

where all matrices are constant and of appropriate dimensions, $B_0 := K$, and $B_i = \mathbf{0}_{m \times p}$, $\forall i = 0, \ldots, r - 1$, and B_r is of rank m.

Proof. This follows immediately from Corollary A.3 and the definition of vector relative degree [3]. □

Appendix B. Useful results.

Lemma B.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{K}$ and $\mathcal{D}_0 \subseteq \mathcal{X}$ be a closed subspace, P be a bounded linear operator of $\mathcal{X}$ to $\mathcal{Y}$, and $P(\mathcal{D}_0) \subseteq \mathcal{Y}$ is closed. Then, $\exists c \ge 0$, $\forall c_w \ge 0$, $\forall b \in P(\mathcal{D}_0) \subseteq \mathcal{Y}$ with $\| b \|_{\mathcal{Y}} \le c_w$, $\exists x \in \mathcal{D}_0 \subseteq \mathcal{X}$ such that $b = Px$ and $\| x \|_{\mathcal{X}} \le c c_w$.*

Proof. This is immediate from the fact that the pseudoinverse operator of $P|_{\mathcal{D}_0} : \mathcal{D}_0 \to \mathcal{Y}$ is a bounded linear operator [5]. ∎

LEMMA B.2. *Consider a finite-dimensional continuous-time LTI system:*

$$\dot{z} = Az + Bv; \qquad z(0) = \mathbf{0}_{n\times 1} \tag{B.1}$$

where z is the n-dimensional state, $n \in \mathbb{Z}_+$; and v is the p-dimensional input, $p \in \mathbb{Z}_+$. Assume that the system (B.1) is bounded input and bounded state stable. Then, the following statements hold.

1. *For system $\dot{\eta} = A\eta$, $\eta(0) = B\xi$, where $\xi \in \mathbb{R}^p$, there exist $k \in [0, \infty) \subset \mathbb{R}$ and $\lambda \in (0, \infty) \subset \mathbb{R}$ such that $\forall \xi \in \mathbb{R}^p$ with $|\xi| \le c_w \in [0, \infty) \subset \mathbb{R}$, we have $|\eta(t)| \le c_w k \mathrm{e}^{-\lambda t}$, $\forall t \in [0, \infty)$.*
2. *The system $\dot{x} = Ax + ABu$, $x(0) = \mathbf{0}_{n\times 1}$, is bounded input and bounded state stable.*

Proof. By Lemma 6 of [7], $\exists k \in [0, \infty) \subset \mathbb{R}$ and $\exists \lambda \in (0, \infty) \subset \mathbb{R}$ such that $\left\| \mathrm{e}^{At} B \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \in [0, \infty) \subset \mathbb{R}$.

For the first statement, $|\, \eta(t)\, | = \left| \mathrm{e}^{At} B\xi \right| \le k\mathrm{e}^{-\lambda t} c_w$, $\forall t \in [0, \infty) \subset \mathbb{R}$.

For the second statement, we note that $\mathrm{e}^{At}AB = \left(\sum_{i=0}^{\infty} \frac{1}{i!}(At)^i\right) AB = A \cdot \left(\sum_{i=0}^{\infty} \frac{1}{i!}(At)^i\right) B = A\mathrm{e}^{At}B$, $\forall t \in \mathbb{R}$. Then, $\left\| \mathrm{e}^{At}AB \right\|_{2,2} = \left\| A\mathrm{e}^{At}B \right\|_{2,2} \le \|\, A\,\|_{2,2} \cdot \left\| \mathrm{e}^{At}B \right\|_{2,2} \le \|\, A\,\|_{2,2}\, k\mathrm{e}^{-\lambda t}$, $\forall t \in [0, \infty) \subset \mathbb{R}$. By Lemma 6 of [7], the result holds.

This completes the proof of the lemma. ∎

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