Further Properties of the Generalized Minimum Phase Concept for Finite-Dimensional Continuous-Time Square MIMO LTI Systems

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Abstract: In Başar and Pan (2019), we generalized the definitions of extended zero dynamics, minimum phase, and extended zero dynamics canonical form to square MIMO LTI systems with additive disturbances. It was shown that the extended zero dynamics is invariant under the application of dynamic extension to its input, and therefore, a minimum phase system remains minimum phase after a finite number of steps of dynamic extension. In this paper, we show that the generalized minimum phase assumption is necessary in model reference control of square MIMO LTI systems. We prove the (strict) observer canonical form representation for MIMO LTI systems. The strict observer canonical form of an LTI system is guaranteed to exist if the system admits uniform observability indices and the extended zero dynamics canonical form requires the system to admit uniform vector relative degree, both of these canonical forms being needed in robust adaptive control design and analysis for MIMO LTI systems. Toward this end, we need to be able to extend the given system to one with uniform vector relative degree and with uniform observability indices without changing its minimum phase property. We present two lemmas in this respect that fully resolve this issue.

Keywords: Minimum phase, zero dynamics canonical form, extended zero dynamics, extended zero dynamics canonical form, (strict) observer canonical form, dynamic extension, observability indices, vector relative degree.

1. INTRODUCTION

The minimum phase concept is of paramount importance in model reference control for linear and nonlinear systems (Ioannou and Sun, 1996; Isidori, 1995; Liberzon et al., 2002; Sontag and Wang, 1997; Pan and Başar, 2018). In Başar and Pan (2019), we generalized the minimum phase concept (Pan and Başar, 2018) that was defined for SISO LTI systems with well-defined relative degree and additive disturbance inputs to the case of multiple-input and multiple-output (MIMO) LTI systems with additive disturbance inputs. We prove in this paper that the minimum phase concept defined in Başar and Pan (2019) is necessary for model reference control for systems to achieve the following: 1) the system states remain bounded when the disturbance input waveform is admissible and bounded, the initial condition is admissible and bounded, and the reference trajectory is bounded with bounded derivatives up to nth order, where n is the dimension of the system; 2) the perfect tracking of any given bounded reference trajectory with bounded derivatives up to nth order when the disturbance waveform is identically equal to zero, the extended zero dynamics admits any admissible initial condition, and the initial conditions for the rest of the closed-loop system states are appropriately chosen. We proved in Başar and Pan (2019) that the extended zero dynamics of the system is invariant (modulo linear state transformation) if we apply a step of dynamic extension (Isidori, 1995) to the system. Therefore, our concept of minimum phase can be checked for square MIMO LTI systems that are right invertible. We further introduced the extended zero dynamics canonical form for square MIMO LTI systems with uniform vector relative degree. This extended zero dynamics canonical form is essential in the proof of robust adaptive control design (Pan and Başar, 2000), which requires that the system admit uniform vector relative degree. Another essential ingredient in that proof is the strict observer canonical form for the MIMO LTI system, which is defined in this paper to be the observer canonical form that further satisfies the property that the input-free outputs of the system are a subset of the state variables of the representation. This form can be achieved if the system admits uniform observability indices (Chen, 1984). Thus, starting with a square MIMO LTI system that is minimum phase with respect to any admissible initial condition and any admissible disturbance waveforms, we have to obtain a true system representation that admits uniform vector relative degree and uniform observability indices in order to be able to apply the (appropriately vectorized version of) robust adaptive control design of Pan and Başar (2000) to the system. This can be achieved by dynamic extensions that are independent of the unknown parameters in the system to an extended system with vector relative degree. After this, we may alternatively achieve the requirement of uniform vector relative degree by appropriately integrating the output channels of the system, and thus leading to an extended system that admits the extended zero dynamics canonical form. Then, we may be able to investigate admissibility of the strict observer canonical form of the extended system. In the case this is not so, we must further extend the system by adding dummy states (which are always zero under admissible initial conditions and are independent of the control and disturbance inputs) such that the further extended system admits uniform observability indices, and therefore the strict observer canonical form. This process is formally established and proved in this paper, yielding a minimum phase system that is in line with the control design methodology of Pan and Başar (2000).

The balance of the paper is as follows. In the next section, we provide the notations used in the paper. In Section 3, we prove the necessity of the minimum phase assumption as defined in the paper for model reference control design. Then, in Section 4, we define the strict observer canonical form and prove its existence under the assumption that the system admits uniform observability indices. In Section 5, we present two lemmas which establish the possibility of extending a minimum phase square MIMO LTI system with vector relative degree to an extended system which admits both the extended zero dynamics canonical form and the strict observer canonical form representations, and is minimum phase. The paper ends with some concluding remarks and an appendix that contains some technical results needed in the main body of the paper.

2. NOTATIONS

We let $\mathbb{R}$ denote the real line; let $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; and $\mathbb{C}$ be the set of complex numbers. Unless specified otherwise, all signals, constants, and matrices are real. For a function f, we say that it belongs to $\mathcal{C}$ if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously differentiable (Fréchet differentiability), $k \in \mathbb{N} \cup \{\infty\}$. We say that a function is L_∞ if it is bounded. For any matrix A, A' denotes its transpose. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, $|z|$ denotes the Euclidean norm $\sqrt{z'z}$. For $n \in \mathbb{Z}_+$, I_n denotes the $n \times n$-dimensional identity matrix. For $n \in \mathbb{Z}_+$ and $n \times n$-dimensional matrix A, we set $A^0 = I_n$. For any matrix M, $\|M\|_{p,p}$ denotes its p-induced norm, $1 \le p \le \infty$. For any $m, n \in \mathbb{Z}_+$, $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. For any $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, n\}$, $e_{n,k}$ denotes the kth n-dimensional unit vector, i. e., $[\mathbf{0}_{1\times(k-1)}\ 1\ \mathbf{0}_{1\times(n-k)}]'$. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0, t_f), \mathbb{R}^p)$, where $t_f \in (0, \infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$. For a sufficiently smooth signal v, $v^{(i)}$ denotes the ith order derivative of v, $v^{[i]}$ denotes $\left[v'\ (v^{(1)})' \cdots (v^{(i)})'\right]'$, $i \in \mathbb{Z}_+$. We will denote constants or matrices of no specific interest or relevance to the analysis by $\star$. We will denote $m \times n$-dimensional matrices of no specific interest or relevance to the analysis by $\star_{m\times n}$. Let A, B be sets and $A \subseteq B$, then the indicator function of the set A in B is $\chi_{A,B} : B \to \{0, 1\}$ defined by $\chi_{A,B}(x) = \begin{cases} 1 & \text{If } x \in A \\ 0 & \text{If } x \in B \setminus A \end{cases}, \forall x \in B$.

3. NECESSITY OF MINIMUM PHASE PROPERTY IN MODEL REFERENCE CONTROL

We state and prove the following result on the necessity of the generalized minimum phase condition in model reference control of a square MIMO LTI system:

$$\dot{x} = Ax + Bu + Dw; \quad x(0) = x_0 \in \mathcal{D}_0 \tag{1a}$$
$$y = Cx + Fu + Ew \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $u \in \mathbb{R}^m$ is the control input, $m \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the output; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$ (Pan and Başar, 2018), A, B, D, C, F, and E are constant matrices of appropriate dimensions.

Proposition 1. Consider the system (1). Let (6) of Başar and Pan (2019) admit maximal solution $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, and $A_{z1} \in \mathbb{R}^{s\times m}$ that leads to the following extended zero-dynamics for (1), in $x_z = Kx$ coordinates:

$$\dot{x}_z = A_z x_z + A_{z1} y + (KD - A_{z1}E)w; \tag{2}$$
$$x_z(0) = Kx_0 \in K(\mathcal{D}_0) =: \mathcal{D}_{z0}$$

Assume that the identically $\mathbf{0}_{q\times 1}$ function belongs to $\mathcal{W}_d$. Let $y_{d[0,\infty)} \in \mathcal{C}_n([0, \infty), \mathbb{R}^m)$ be the reference trajectory and $Y_d := y_d^{[n]}$, and $Y_{d0} := y_d^{[n-1]}(0) \in \mathbb{R}^{nm}$. Assume there exists a finite-dimensional model reference controller, S_C

$$\dot{\xi} = f(\xi, y, Y_d); \qquad \xi(0) = \xi_0(\check{x}_0, Y_{d0}) \tag{3a}$$
$$u = h(\xi, y, Y_d) \tag{3b}$$

where $\xi \in \mathcal{D}_\xi \subseteq \mathbb{R}^{\bar{n}}$, $\bar{n} \in \mathbb{Z}_+$; $\check{x}_0 \in \mathcal{D}_0$ is an estimate of x_0; $\xi_0 : \mathcal{D}_0 \times \mathbb{R}^{mn} \to \mathcal{D}_\xi$; such that $\forall c_w \ge 0$, $\exists c_c \ge 0$; $\forall Y_{d0} \in \mathbb{R}^{mn}$ with $|Y_{d0}| \le c_w$; $\forall y_{d[0,\infty)} \in \mathcal{C}_n$ with $\left\|Y_{d[0,\infty)}\right\|_\infty \le c_w$, we have

(a) $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$; $\forall \check{x}_0 \in \mathcal{D}_0$ with $|\check{x}_0| \le c_w$; $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\|w_{[0,\infty)}\right\|_\infty \le c_w$, there exists a unique solution $X_{[0,\infty)} := (x_{[0,\infty)}, \xi_{[0,\infty)})$ to the closed-loop system S such that $\left\|x_{[0,\infty)}\right\|_\infty \le c_c$, $\xi(t) \in \mathcal{D}_\xi$, $\forall t \in [0, \infty)$, such that $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$;

(b) $\forall x_{z0} \in \mathcal{D}_{z0}$ with $|x_{z0}| \le c_w$, $\exists x_0 \in \mathcal{D}_0$ with $Kx_0 = x_{z0}$, choosing $\check{x}_0 = x_0 \in \mathcal{D}_0$ and $w_{[0,\infty)}$ equals to the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$, then $y_{[0,\infty)} = y_{d[0,\infty)}$.

Then, (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

Proof We will consider separately two exhaustive and mutually exclusive cases: Case 1: $0 = s$; Case 2: $s > 0$.

Case 1: $0 = s$. Clearly, the system S_P is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof for this case.

Case 2: $s > 0$. We need the following result.

Claim 1. The system $\dot{z} = A_z z + A_{z1} v$; $z(0) = \mathbf{0}_{s\times 1}$ is bounded input and bounded state (BIBS) stable.

Proof of claim: Suppose that the claim is false. By Corollary 2 of Pan and Başar (2018), $\exists v_{[0,\infty)} \in \mathcal{C}_n$ such that $\left\|v^{[n]}_{[0,\infty)}\right\|_\infty =: c_v > 0$ and $\left\|z_{[0,\infty)}\right\|_\infty = +\infty$. Choose $w_{[0,\infty)}$ to be the identically $\mathbf{0}_{q\times 1}$ function in $\mathcal{W}_d$. Let $y_{d[0,\infty)} = v_{[0,\infty)}$. Fix $x_{z0} = \mathbf{0}_{s\times 1} \in \mathcal{D}_{z0}$ and choose $x_0 \in \mathcal{D}_0$ with $Kx_0 = x_{z0}$ according (b). Set $\check{x}_0 = x_0$ and $c_w = \max\{c_v, |x_0|\} \in \mathbb{R}$. By (a), $\exists c_c \ge 0$, where c_c depends only on c_w and S_C, such that the closed-loop system S admits a unique solution $X_{[0,\infty)}$ with $\left\|x_{[0,\infty)}\right\|_\infty \le c_c$, and $u_{[0,\infty)}, y_{[0,\infty)} \in \mathcal{C}$. By (b), we have $y_{[0,\infty)} = y_{d[0,\infty)} = v_{[0,\infty)}$. Note that $x_{z[0,\infty)}$ is the unique solution to $\dot{x}_z = A_z x_z + A_{z1} y_d$, $x_z(0) = Kx_0 = x_{z0} = \mathbf{0}_{s\times 1}$. Then, we have $x_{z[0,\infty)} = z_{[0,\infty)}$. Hence, we have $\left\|x_{z[0,\infty)}\right\|_\infty = +\infty$. This is a contradiction, thus proving the claim. □

Let $\left[A_{z1}\ A_z A_{z1} \cdots A_z^{s-1} A_{z1}\right] =: P_z$ and $n_{zc} := \mathrm{rank}(P_z)$. We will further distinguish 3 exhaustive and mutually exclusive cases: Case 2.1: $n_{zc} = 0$; Case 2.2: $1 \leq n_{zc} < s$; Case 2.3: $n_{zc} = s$.

Case 2.1: $n_{zc} = 0$. Then, we have $A_{z1} = \mathbf{0}_{s\times m}$, and the extended zero dynamics is given by

$$\dot{x}_z = A_z x_z + KDw; \qquad x_z(0) = x_{z0} \tag{4}$$

$\forall c_w \geq 0$, by Lemma 4, $\exists c_1 \geq 0$, $\forall x_{z0} \in \mathcal{D}_{z0}$ with $|x_{z0}| \leq c_w$, $\exists x_0 \in \mathcal{D}_0$ with $Kx_0 = x_{z0}$ and $|x_0| \leq c_1 c_w$, by (a), $\exists c_c \geq 0$ that depends only on c_1 and c_w, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\| w_{[0,\infty)} \|_\infty \leq c_w$, let $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)} \in \mathcal{C}_n$ with $\| Y_{d[0,\infty)} \|_\infty = 0$ and $|Y_{d0}| = 0$ and $\check{x}_0 = x_0$, then, there exists a unique solution $X_{[0,\infty)}$ to the closed-loop system with $\| x_{[0,\infty)} \|_\infty \leq c_c$. This implies that $\| x_{z[0,\infty)} \|_\infty \leq \| K \|_{2,2} \| x_{[0,\infty)} \|_\infty \leq \| K \|_{2,2}\, c_c$ and $x_{z[0,\infty)}$ is the unique solution to (4). Since the extended zero dynamics (4) is independent of $y_{[0,\infty)}$, then (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This proves this sub-case.

Case 2.2: $1 \leq n_{zc} < s$. Without loss of generality, assume that (2) is partitioned into controllable and uncontrollable parts (with respect to y). We have $x_z = [x'_{zc}\ x'_{z\bar{c}}]'$

$$\begin{bmatrix} \dot{x}_{zc} \\ \dot{x}_{z\bar{c}} \end{bmatrix} = \begin{bmatrix} A_{zc} & A_{zc\bar{c}} \\ \mathbf{0} & A_{z\bar{c}} \end{bmatrix} \begin{bmatrix} x_{zc} \\ x_{z\bar{c}} \end{bmatrix} + \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} y + \begin{bmatrix} D_{zc} \\ D_{z\bar{c}} \end{bmatrix} w \tag{5}$$

where $x_{zc} \in \mathbb{R}^{n_{zc}}$; $x_{z\bar{c}} \in \mathbb{R}^{n_{z\bar{c}}}$, $n_{z\bar{c}} := s - n_{zc}$; and the pair (A_{zc}, A_{zc1}) is controllable. By Lemma 6 of Pan and Başar (2018) and Claim 1, $\exists k \geq 0$, $\exists \lambda > 0$, such that $\| \mathrm{e}^{A_z t} A_{z1} \|_{2,2} \leq k\mathrm{e}^{-\lambda t}$, $\forall t \geq 0$. Note that, $\forall t \geq 0$,

$$\begin{aligned} \| \mathrm{e}^{A_z t} A_{z1} \|_{2,2} &= \left\| \begin{bmatrix} \mathrm{e}^{A_{zc} t} & \star \\ \mathbf{0} & \mathrm{e}^{A_{z\bar{c}} t} \end{bmatrix} \begin{bmatrix} A_{zc1} \\ \mathbf{0} \end{bmatrix} \right\|_{2,2} \\ &= \| \mathrm{e}^{A_{zc} t} A_{zc1} \|_{2,2} \leq k\mathrm{e}^{-\lambda t} \end{aligned}$$

Hence, we have that the system $\dot{z}_c = A_{zc} z_c + A_{zc1} v$, $z_c(0) = \mathbf{0}_{n_{zc}\times 1}$ is BIBS stable by Lemma 6 of Pan and Başar (2018). By the controllability of the pair (A_{zc}, A_{zc1}) and Lemma 7 of Pan and Başar (2018), we have that A_{zc} is Hurwitz.

We will show that (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ by an argument of contradiction. Suppose (1) is not minimum phase. Then, $\exists c_w \geq 0$, $\forall i \in \mathbb{N}$, $\exists x_{z0(i)} \in \mathcal{D}_{z0}$ with $|x_{z0(i)}| \leq c_w$, $\exists w_{(i)[0,\infty)} \in \mathcal{W}_d$ with $\| w_{(i)[0,\infty)} \|_\infty \leq c_w$, $\exists y_{(i)[0,\infty)} \in \mathcal{C}$ with $\| y_{(i)[0,\infty)} \|_\infty \leq c_w$, such that $\| x_{z(i)[0,\infty)} \|_\infty > i$, where $x_{z(i)[0,\infty)}$ is the solution to (5) with initial condition specified by $x_{z0(i)}$ and inputs $y_{(i)[0,\infty)}$ and $w_{(i)[0,\infty)}$. Partition $x_{z(i)}$ as $[x'_{zc(i)}\ x'_{z\bar{c}(i)}]'$. Then, we have $\limsup_{i\to\infty} \| x_{z\bar{c}(i)[0,\infty)} \|_\infty = \infty$ since A_{zc} is Hurwitz ($\| x_{z\bar{c}(i)[0,\infty)} \|_\infty =: \bar{c}_{wi}$, then, $\| x_{zc(i)[0,\infty)} \|_\infty \leq k_1 c_w + k_2 \sqrt{\bar{c}_{wi}^2 + 2c_w^2}$, for some $k_1, k_2 \geq 0$ that is independent of $i \in \mathbb{N}$). By Lemma 4, there exists $c_1 \geq 0$ such that we may set $x_0 = x_{0(i)} \in \mathcal{D}_0$ with $Kx_{0(i)} = x_{z0(i)}$ and $|x_0| \leq c_1 |x_{z0(i)}| \leq c_1 c_w$. Let $w_{[0,\infty)} = w_{(i)[0,\infty)}$, $\check{x}_0 = x_0$, and $y_{d[0,\infty)} = \mathbf{0}_{[0,\infty)}$ with $\| Y_{d[0,\infty)} \|_\infty = 0$ and $|Y_{d0}| = 0$. By (a), $\exists c_c \geq 0$, where c_c depends only on c_w and c_1, such that the closed-loop system admits unique solution $X_{\{i\}[0,\infty)}$ with continuous signals $u_{\{i\}[0,\infty)}$ and $y_{\{i\}[0,\infty)}$ and $\| x_{\{i\}[0,\infty)} \|_\infty \leq c_c$, $\forall i \in \mathbb{N}$. Note that $x_{\{i\}[0,\infty)}$ is the unique solution to (1) with initial condition $x_{0(i)}$ and inputs $u_{\{i\}[0,\infty)}$ and $w_{(i)[0,\infty)}$. Then, the component $x_{z\bar{c}\{i\}[0,\infty)}$ of $x_{\{i\}[0,\infty)}$ is the unique solution to $x_{z\bar{c}}$ dynamics in (5), which is independent of $y_{\{i\}[0,\infty)}$. We have $x_{z\bar{c}\{i\}[0,\infty)} = x_{z\bar{c}(i)[0,\infty)}$. Hence, $\limsup_{i\to\infty} \| x_{\{i\}[0,\infty)} \|_\infty = \infty$ since $\limsup_{i\to\infty} \| x_{z\bar{c}\{i\}[0,\infty)} \|_\infty = \limsup_{i\to\infty} \| x_{z\bar{c}(i)[0,\infty)} \|_\infty = \infty$. This is a contradiction. Thus, (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This proves this sub-case.

Case 2.3: $n_{zc} = s$. Then, the pair (A_z, A_{z1}) is controllable. By Lemma 7 of Pan and Başar (2018) and Claim 1, we have that the matrix A_z is Hurwitz. Then, by Lemma 6 of Başar and Pan (2019), (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This proves this sub-case.

This completes the proof of the lemma. □

4. OBSERVER CANONICAL FORM REPRESENTATION

First, we present a lemma and a corollary that defines the observer canonical form representation of an LTI system.

Lemma 1. Consider an LTI system

$$\dot{x} = Ax \tag{6a}$$

$$y = Cx \tag{6b}$$

where $x \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the output, $m \in \mathbb{Z}_+$; and A and C are constant matrices of appropriate dimensions. Let the row vectors of C be $c_1, \ldots, c_m$. Seek linearly independent row vectors from the list P that includes row vectors $c_1, \ldots, c_m, c_1 A, \ldots, c_m A, \ldots, c_1 A^{n-1}, \ldots, c_m A^{n-1}$ in the listed order, to result in row vectors $q_1, \ldots, q_{n_O}$, where $n_O \in \mathbb{Z}_+$ and $n_O \leq n$. Then,

(i) there exist observability indices $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ such that the list $\bar{P}$ of row vectors $c_1, \ldots, c_1 A^{\nu_1 - 1}, c_2, \ldots, c_2 A^{\nu_2 - 1}, \ldots, c_m, \ldots, c_m A^{\nu_m - 1}$ is a permutation of $q_1, \ldots, q_{n_O}$; $n_O = \sum_{i=1}^m \nu_i$; and $\nu := \max_{1\leq i\leq m} \nu_i$ is said to be the *observability index* of (6).

Let $Q_i := [c'_i \cdots (A')^{\nu_i - 1} c'_i]'$, $1 \leq i \leq m$ (note that Q_i is an empty matrix if $\nu_i = 0$). Clearly, $Q_i \in \mathbb{R}^{\nu_i \times n}$. Let $Q := [Q'_1 \cdots Q'_m\ q'_{n_O+1} \cdots q'_n]'$, where $q_{n_O+1}, \ldots, q_n$ are n-dimensional row vectors selected arbitrarily so that the matrix Q is invertible. Let $T := Q^{-1} =: [T_1 \cdots T_m\ t_{n_O+1} \cdots t_n]$, where $T_i \in \mathbb{R}^{n\times\nu_i}$, $1 \leq i \leq m$, and $t_i \in \mathbb{R}^n$, $n_O + 1 \leq i \leq n$; and $e_i := t_{i,\nu_i}$, $1 \leq i \leq m$, be the last column vector of T_i whenever $\nu_i > 0$. Let $S_i = [A^{\nu_i - 1} e_i \cdots e_i] \in \mathbb{R}^{n\times\nu_i}$, $1 \leq i \leq m$; and $S := [S_1 \cdots S_m\ t_{n_O+1} \cdots t_n] \in \mathbb{R}^{n\times n}$. Then, the following statements hold.

(ii) $$QS = \begin{bmatrix} W_{1,1} & \cdots & W_{1,m} & \mathbf{0}_{\nu_1\times(n-n_O)} \\ \vdots & & \vdots & \vdots \\ W_{m,1} & \cdots & W_{m,m} & \mathbf{0}_{\nu_m\times(n-n_O)} \\ \star_{(n-n_O)\times\nu_1} & \cdots & \star_{(n-n_O)\times\nu_m} & I_{n-n_O} \end{bmatrix},$$

where $W_{i,j} \in \mathbb{R}^{\nu_i\times\nu_j}$, $1 \leq i \leq m$, $1 \leq j \leq m$, with elements $w_{i,j,k,l}$, $1 \leq k \leq \nu_i$, $1 \leq l \leq \nu_j$, such that $w_{i,i,k,k} = 1$, $1 \leq i \leq m$, $1 \leq k \leq \nu_i$; $w_{i,j,k,l} = 0$, $1 \leq i \leq m$, $1 \leq j \leq m$, $1 \leq k \leq \nu_i$, $k+1 \leq l \leq \nu_j$; and $w_{i,j,k,k} = 0$, $1 \leq i \leq m$, $i+1 \leq j \leq m$, $1 \leq k \leq \min\{\nu_i, \nu_j\}$.

(iii) S is an invertible matrix.

(iv) In $\bar{x} := S^{-1}x$ coordinates, the system (6) admits the *interweaved observer canonical form* representation

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_{1,1} & \cdots & \bar{A}_{1,m} & \mathbf{0}_{\nu_1\times(n-n_O)} \\ \vdots & & \vdots & \vdots \\ \bar{A}_{m,1} & \cdots & \bar{A}_{m,m} & \mathbf{0}_{\nu_m\times(n-n_O)} \\ \bar{A}_{\bar{o},1} & \cdots & \bar{A}_{\bar{o},m} & \bar{A}_{\bar{o}} \end{bmatrix} \bar{x} =: \bar{A}\bar{x} \tag{7a}$$

$$y = \left[\bar{C}_1 \cdots \bar{C}_m \; \mathbf{0}_{m\times(n-n_O)}\right] \bar{x} =: \bar{C}\bar{x} \tag{7b}$$

where $\bar{A}_{i,j} \in \mathbb{R}^{\nu_i\times\nu_j}$, $1 \le i \le m$, $1 \le j \le m$, with elements $a_{i,j,k,l}$, $1 \le k \le \nu_i$, $1 \le l \le \nu_j$, that satisfy $a_{i,j,k,l} = 0$, if $i \neq j$ and $l \ge 2$, $a_{i,i,k,k+1} = 1$, $\forall 1 \le k \le \nu_i - 1$, and $a_{i,i,k,l} = 0$, if $l \ge 2$ and $k+1 \neq l$; and $\bar{A}_{\bar{o},j} \in \mathbb{R}^{(n-n_O)\times\nu_j}$, $1 \le j \le m$, with elements $a_{\bar{o},j,k,l}$, $1 \le k \le n - n_0$, $1 \le l \le \nu_j$, that satisfy $a_{\bar{o},j,k,l} = 0$ if $l \ge 2$; $\bar{A}_{\bar{o}} \in \mathbb{R}^{(n-n_O)\times(n-n_O)}$; $\bar{C}_j \in \mathbb{R}^{m\times\nu_j}$, $1 \le j \le m$, with elements $c_{j,k,l}$, $1 \le k \le m$, $1 \le l \le \nu_j$, that satisfy $c_{j,k,l} = 0$, if $l \ge 2$, $c_{j,j,1} = 1$, and $c_{j,k,1} = 0$ if $k < j$. We will further say that the interweaved observer canonical form (7) is *strict* if the matrix $\bar{C}$ further satisfies that $c_{j,k,1} = 0$ if $j \neq k$, $1 \le j \le m$, $1 \le k \le m$.

Proof (i) Let the characteristic polynomial of A be $p(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$, where $\alpha_i \in \mathbb{R}$, $1 \le i \le n$. By the Cayley-Hamilton Theorem, we have $A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I_n = \mathbf{0}_{n\times n}$. Then, $A^n = -\alpha_1 A^{n-1} - \cdots - \alpha_n I_n$. Fix any row vector $c_i A^j$, with $i \in \{1, \ldots, m\}$ and $j \in \{0, \ldots, n-1\}$. If $j < n-1$ and $c_i A^j$ is linearly dependent on preceding row vectors in the list P, then $c_i A^{j+1} = (c_i A^j)\cdot A$ is linearly dependent on preceding row vectors in the list P. If $j = n-1$, then $c_i A^n = -\alpha_1 c_i A^{n-1} - \cdots - \alpha_n c_i$ is again linearly dependent on row vectors in the list P. Thus, (i) holds. Furthermore, we may prove the following claim.

Claim 2. $\forall 1 \le i \le m$, $\forall \tau \in \mathbb{Z}_+$, we have

$$c_i A^{\nu_i+\tau} = \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ ((k<\nu_i+\tau) \text{ or } (k=\nu_i+\tau) \text{ and } (j<i))}} \beta_{j,k} c_j A^k$$

where $\beta_{j,k}$'s are constants.

Proof of claim: We will prove the above claim by mathematical induction on τ.

1° Let $\tau = 0$. $\forall 1 \le i \le m$, by the preceding argument, $c_i A^{\nu_i}$ can be expressed as a linear combination of row vectors $c_j A^k \in \bar{P}$ with $k < \nu_i$, or $k = \nu_i$ and $j < i$. Hence, the result holds for $\tau = 0$.

2° Assume the result holds $\forall 0 \le \tau \le \sigma$ with $\sigma \in \mathbb{Z}_+$.

3° Consider the case when $\tau = \sigma + 1$.

$$\begin{aligned} c_i A^{\nu_i+\tau} &= (c_i A^{\nu_i+\sigma}) \cdot A \\ &= \Big(\sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ (k<\nu_i+\sigma \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} c_j A^k \Big) \cdot A \\ &= \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ (k<\nu_i+\sigma \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} c_j A^{k+1} \end{aligned}$$

where the second equality follows from the inductive assumption. Since $c_j A^k \in \bar{P}$, we have $k < \nu_j$ and $k+1 \le \nu_j$. When $k+1 = \nu_j$, by 1°, $c_j A^{k+1} = c_j A^{\nu_j} = \sum_{\substack{(c_{\bar{j}} A^{\bar{k}} \in \bar{P}) \text{ and} \\ ((\bar{k}<\nu_j) \text{ or } (\bar{k}=\nu_j) \text{ and } (\bar{j}<j))}} \bar{\beta}_{\bar{j},\bar{k}} c_{\bar{j}} A^{\bar{k}}$. Then,

$$\begin{aligned} c_i A^{\nu_i+\tau} = & \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and } (k+1<\nu_j) \text{ and} \\ ((k<\nu_i+\sigma) \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} c_j A^{k+1} \\ & + \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and } (k+1=\nu_j) \text{ and} \\ ((k<\nu_i+\sigma) \text{ or } (k=\nu_i+\sigma) \text{ and } (j<i))}} \beta_{j,k} \cdot \\ & \sum_{\substack{(c_{\bar{j}} A^{\bar{k}} \in \bar{P}) \text{ and} \\ ((\bar{k}<\nu_j) \text{ or } (\bar{k}=\nu_j) \text{ and } (\bar{j}<j))}} \bar{\beta}_{\bar{j},\bar{k}} c_{\bar{j}} A^{\bar{k}} \\ = & \sum_{\substack{(c_j A^k \in \bar{P}) \text{ and} \\ ((k<\nu_i+\tau) \text{ or } (k=\nu_i+\tau) \text{ and } (j<i))}} \tilde{\beta}_{j,k} c_j A^k \end{aligned}$$

Hence, the result holds for $\tau = \sigma + 1$. This completes the proof of the claim. □

(ii) Note that $QT = I_n$. Then, $Qt_i = e_{n,i}$, $n_O + 1 \le i \le n$. Clearly, $W_{i,j} = Q_i S_j$, $1 \le i \le m$, $1 \le j \le m$. $\forall 1 \le i \le m$, $\forall 1 \le k \le \nu_i$, $w_{i,i,k,k} = (c_i A^{k-1})\cdot(A^{\nu_i-k} e_i) = (c_i A^{\nu_i-1})\cdot e_i$, which equals to the product of the $(\sum_{l=1}^{i} \nu_l)$'s row of Q and $(\sum_{l=1}^{i} \nu_l)$'s column of T. Hence, $w_{i,i,k,k} = 1$.

$\forall 1 \le i \le m$, $\forall 1 \le j \le m$, $\forall 1 \le k \le \nu_i$, $\forall k+1 \le l \le \nu_j$, $w_{i,j,k,l} = (c_i A^{k-1}) \cdot (A^{\nu_j - l} e_j) = (c_i A^{\nu_j - l + k - 1}) \cdot e_j$. If $\nu_j - l + k < \nu_i$, then $w_{i,j,k,l}$ equals to the product of the $(\sum_{\tau=1}^{i-1} \nu_\tau + \nu_j - l + k)$'s row of Q and the $(\sum_{\tau=1}^{j} \nu_\tau)$'s column of T. Hence, $w_{i,j,k,l} = 0$ since $QT = I_n$. On the other hand, if $\nu_j - l + k \ge \nu_i$, then $\nu_j > \nu_i$ and $j \neq i$; by Claim 2, $w_{i,j,k,l} = c_i A^{\nu_j - l + k - 1} e_j = \sum_{\substack{(c_\tau A^\sigma \in \bar{P}) \text{ and} \\ ((\sigma<\nu_j-l+k-1) \text{ or } (\sigma=\nu_j-l+k-1) \text{ and } (\tau<i))}} \beta_{\tau,\sigma} c_\tau A^\sigma e_j$. Since $QT = I_n$, then $w_{i,j,k,l} = 0$. Hence, $w_{i,j,k,l} = 0$, $\forall 1 \le i \le m$, $\forall 1 \le j \le m$, $\forall 1 \le k \le \nu_i$, $\forall k+1 \le l \le \nu_j$.

$\forall 1 \le i \le m$, $\forall i+1 \le j \le m$, $1 \le k \le \min\{\nu_i, \nu_j\}$, $w_{i,j,k,k} = (c_i A^{k-1}) \cdot (A^{\nu_j - k} e_j) = c_i A^{\nu_j - 1} e_j$. If $\nu_j \le \nu_i$, then $w_{i,j,k,k}$ equals to the product of the $(\sum_{\tau=1}^{i-1} \nu_\tau + \nu_j)$'s row of Q and the $(\sum_{\tau=1}^{j} \nu_\tau)$'s column of T, which equals to 0 since $i \neq j$ and $QT = I_n$. On the other hand, if $\nu_j > \nu_i$, by Claim 2, $w_{i,j,k,k} = c_i A^{\nu_j - 1} e_j = \sum_{\substack{(c_\tau A^\sigma \in \bar{P}) \text{ and} \\ ((\sigma<\nu_j-1) \text{ or } (\sigma=\nu_j-1) \text{ and } (\tau<i))}} \beta_{\tau,\sigma} c_\tau A^\sigma e_j$. Since $QT = I_n$, then $w_{i,j,k,k} = 0$. Hence, $w_{i,j,k,k} = 0$, $\forall 1 \le i \le m$, $\forall i+1 \le j \le m$, $1 \le k \le \min\{\nu_i, \nu_j\}$.

This proves (ii).

(iii) By (ii), the matrix QS is invertible, which implies that S is invertible.

(iv) In $\bar{x}$ coordinates, $\bar{A} = S^{-1} A S$ and $\bar{C} = CS$. Then, $S\bar{A} = AS$. $\forall 1 \le i \le m$, $\forall 2 \le j \le \nu_i$, the $(\sum_{l=1}^{i-1} \nu_l + j)$th column of S is $A^{\nu_i - j} e_i$. Then, the $(\sum_{l=1}^{i-1} \nu_l + j)$th column of $AS = S\bar{A}$ is $A^{\nu_i - j + 1} e_i = S e_{n, \sum_{l=1}^{i-1} \nu_l + j - 1}$. Hence, the $(\sum_{l=1}^{i-1} \nu_l + j)$th column of $\bar{A}$ is $e_{n, \sum_{l=1}^{i-1} \nu_l + j - 1}$. This proves that the first n_O columns of $\bar{A}$ are as desired.

Since $QT = I_n$, then $c_i A^j t_l = 0$, $\forall 1 \le i \le m$, $\forall 0 \le j \le \nu_i - 1$, $\forall n_O + 1 \le l \le n$. Fix any $n_O + 1 \le l \le n$. By Claim 2, we have $c_i A^{\nu_i} t_l = 0$, $\forall 1 \le i \le m$. Then, $c_i A^j t_l = 0$, $\forall 1 \le i \le m$, $\forall 0 \le j \le \nu_i$. Therefore, $(c_i A^j) \cdot (A t_l) = 0$, $\forall 1 \le i \le m$, $\forall 0 \le j \le \nu_i - 1$. This leads to $QAt_l = \left[\mathbf{0}_{1\times n_O} \; \star_{1\times(n-n_O)}\right]'$ and $At_l = T\left[\mathbf{0}_{1\times n_O} \; \star_{1\times(n-n_O)}\right]' = S\left[\mathbf{0}_{1\times n_O} \; \star_{1\times(n-n_O)}\right]'$. Hence, the last $n - n_O$ columns of $\bar{A}$ are as desired.

Since $QT = I_n$ and by Claim 2, we have $c_i t_l = 0$, $\forall 1 \le i \le m$, $\forall n_O + 1 \le l \le n$. Hence, the last $n - n_O$ columns of $\bar{C}$ are as desired. Clearly, $\bar{C}_j = CS_j$, $\forall 1 \le j \le m$. $\forall 1 \le k \le m$, $\forall 1 \le j \le m$, $\forall 2 \le l \le \nu_j$, we have $c_{j,k,l} = c_k A^{\nu_j - l} e_j = 0$, by the arguments shown in the 6th from the last paragraph. $\forall 1 \le j \le m$ with $\nu_j \ge 1$, we have $c_{j,j,1} = c_j A^{\nu_j - 1} e_j = 1$, since $QT = I_n$. $\forall 1 \le j \le m$, $\forall 1 \le k \le j-1$, $c_{j,k,1} = c_k A^{\nu_j - 1} e_j = 0$, by the arguments shown in the 5th from the last paragraph. Hence, $\bar{C}$ admits the structure as desired.

This completes the proof of the lemma. □

Corollary 1. Consider a MIMO LTI system

$$\dot{x} = Ax + Bu + Dw \tag{8a}$$
$$y = Cx + Ku + Ew \tag{8b}$$

where $x \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $u \in \mathbb{R}^p$ is the control input, $p \in \mathbb{Z}_+$; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the output, $m \in \mathbb{Z}_+$; and all matrices are constant. Let $\nu \in \{0, \ldots, n\}$ be the observability index of the system and $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ be the observability indices for the output channels. Then, there exists an invertible matrix T such that in $\bar{x} = [x'_{\bar{o}}\, x'_1 \cdots x'_\nu]' = T^{-1}x$ coordinates, we have $\nu = \max_{1 \le i \le m} \nu_i$, $n_O := \sum_{i=1}^m \nu_i \le n$, $x_{\bar{o}} \in \mathbb{R}^{n-n_O}$, $x_i \in \mathbb{R}^{n_i}$ with $n_i := \sum_{l=1}^m \chi_{\{t \ge i\}, \mathbb{Z}}(\nu_l)$, $1 \le i \le \nu$, $m \ge n_1 \ge n_2 \ge \cdots \ge n_\nu \ge 0$, $n_\nu > 0$ if $\nu > 0$, $n_O = \sum_{i=1}^\nu n_i$, and the system (8) admits the *observer canonical form* representation

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + \hat{A}_{\bar{o},1} x_1 + \hat{B}_{\bar{o}} u + \hat{D}_{\bar{o}} w \tag{9a}$$
$$\dot{x}_i = \hat{A}_{i,1} x_1 + \hat{A}_{i,i+1} x_{i+1} + \hat{B}_i u + \hat{D}_i w;\ 1 \le i < \nu \tag{9b}$$
$$\dot{x}_\nu = \hat{A}_{\nu,1} x_1 + \hat{B}_\nu u + \hat{D}_\nu w \tag{9c}$$
$$y = \hat{C}_1 x_1 + Ku + Ew \tag{9d}$$

where $\hat{C}_1$ is composed of the first column vectors of $\bar{C}_j$, $1 \le j \le m$ with $\nu_j > 0$, which are defined in (7), and is therefore of full column rank; $\hat{A}_{\bar{o}} = \bar{A}_{\bar{o}}$, which is defined in (7); and $\hat{A}_{i,i+1} \in \mathbb{R}^{n_i \times n_{i+1}}$, $1 \le i < \nu$, whose column vectors are a subset of the column vectors of I_{n_i} and is of full column rank. We will further say that the observer canonical form (9) is *strict* if $\hat{C}_1$ consists of a subset of column vectors of I_m.

Proof This follows directly from Lemma 1, and is therefore omitted. □

Next, we specialize the above canonical form to the case where the observability indices are uniform, and the case where the observability indices are uniform and the system admits uniform vector relative degree from the control input to the output.

Corollary 2. Consider a MIMO LTI system (8). Let $\nu \in \{0, \ldots, n\}$ be the observability index of the system and $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ be the observability indices for the output channels. Assume that $\nu = \nu_1 = \cdots = \nu_m$. Then, there exists an invertible matrix T such that in $\tilde{x} = [x'_{\bar{o}}\, \tilde{x}'_1 \cdots \tilde{x}'_\nu]' = T^{-1}x$ coordinates, we have $n_O := m\nu$, $x_{\bar{o}} \in \mathbb{R}^{n-n_O}$, $\tilde{x}_i \in \mathbb{R}^m$, $1 \le i \le \nu$, and (8) admits the *strict observer canonical form* representation, if $\nu > 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + \tilde{A}_{\bar{o},1} \tilde{x}_1 + \tilde{B}_{\bar{o}} u + \tilde{D}_{\bar{o}} w \tag{10a}$$
$$\dot{\tilde{x}}_i = \tilde{A}_{i,1} \tilde{x}_1 + \tilde{x}_{i+1} + \tilde{B}_i u + \tilde{D}_i w;\ 1 \le i < \nu \tag{10b}$$
$$\dot{\tilde{x}}_\nu = \tilde{A}_{\nu,1} \tilde{x}_1 + \tilde{B}_\nu u + \tilde{D}_\nu w \tag{10c}$$
$$y = \tilde{x}_1 + Ku + Ew \tag{10d}$$

or if $\nu = 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + \hat{B}_{\bar{o}} u + \hat{D}_{\bar{o}} w \tag{11a}$$
$$y = Ku + Ew \tag{11b}$$

Proof The result is trivial if $\nu = 0$. When $\nu > 0$, by Corollary 1, there exists an invertible matrix T_1 such that in $\bar{x} = [x'_{\bar{o}}\, x'_1 \cdots x'_\nu]' = T_1^{-1}x$ coordinates, we have $x_i \in \mathbb{R}^m$, $\forall 1 \le i \le \nu$, (8) admits representation (9), $\hat{A}_{i,i+1} = I_m$ by possibly reordering state variables within x_{i+1}, $\forall 1 \le i < \nu$, and $\hat{C}_1$ is a lower triangular matrix with 1's on the diagonal. Then, $\hat{C}_1$ is invertible. Let $\tilde{x} := [x'_{\bar{o}}\, \tilde{x}'_1 \cdots \tilde{x}'_\nu]' = \left[x'_{\bar{o}}\, (\hat{C}_1 x_1)' \cdots (\hat{C}_1 x_\nu)'\right]' =: T^{-1}x$. Then, (8) admits representation (10). This proves the corollary. □

Corollary 3. Consider a MIMO LTI system (8). Let $\nu \in \{0, \ldots, n\}$ be the observability index of the system and $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ be the observability indices for the output channels. Assume that $\nu = \nu_1 = \cdots = \nu_m$ and the system admits vector relative degree $r_1, \ldots, r_m \in \{0, \ldots, n\}$ with respect to the control input u such that $r = r_1 = \cdots = r_m$. Then, $r \le \nu$, there exists an invertible matrix T such that in $\bar{x} = [x'_{\bar{o}}\, x'_1 \cdots x'_\nu]' = T^{-1}x$ coordinates, we have $n_O := m\nu$, $x_{\bar{o}} \in \mathbb{R}^{n-n_O}$, $x_i \in \mathbb{R}^m$, $1 \le i \le \nu$, and (8) admits the *strict observer canonical form* representation with $B_0 := K$, if $\nu > 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + A_{\bar{o},1} x_1 + B_{\bar{o}} u + D_{\bar{o}} w \tag{12a}$$
$$\dot{x}_i = A_{i,1} x_1 + x_{i+1} + B_i u + D_i w;\ 1 \le i < \nu \tag{12b}$$
$$\dot{x}_\nu = A_{\nu,1} x_1 + B_\nu u + D_\nu w \tag{12c}$$
$$y = x_1 + B_0 u + Ew \tag{12d}$$

or if $\nu = 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}} x_{\bar{o}} + B_{\bar{o}} u + D_{\bar{o}} w \tag{13a}$$
$$y = B_0 u + Ew \tag{13b}$$

where $B_i = \mathbf{0}_{m \times p}$, $\forall 0 \le i < r$, and B_r is of rank m.

Proof This follows immediately from Corollary 2 and the definition of vector relative degree by Isidori (1995). □

5. EXTENSION OF A SQUARE MIMO LTI SYSTEM INTO ONE WITH UNIFORM VECTOR RELATIVE DEGREE AND UNIFORM OBSERVABILITY INDICES

We present here a lemma that establishes the invariance of the minimum phase property under the application of an output integration operation for a MIMO square LTI system with vector relative degree. Thus, a minimum phase MIMO LTI system will remain so after a finite number of steps of dynamic extension or output integration to a system with uniform vector relative degree.

Lemma 2. Consider the square MIMO LTI system (1), which admits vector relative degree $r_1, \ldots, r_m \in \{0, \ldots, n\}$ with respect to the control inputs. Let $y := (y_1, \ldots, y_m)$, and $\mathcal{D}_0 \subseteq \mathbb{R}^n$ be a subspace. Assuming that (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. $\forall i_0 \in \{1, \ldots, m\}$, we introduce an integrator at y_{i_0}

$$\dot{\xi} = y_{i_0} = C_{i_0,} x + F_{i_0,} u + E_{i_0,} w;\ \xi(0) = \alpha \in \mathbb{R} \tag{14a}$$

$$\bar{y}_{i_0} = \xi \tag{14b}$$

to result in the extended system with state $\bar{x} := (x, \xi) \in \mathbb{R}^{n+1}$ and output $\bar{y} = (y_1, \ldots, y_{i_0-1}, \bar{y}_{i_0}, y_{i_0+1}, \ldots, y_m)$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + \bar{D}w; \qquad \bar{x}(0) = (x_0, \alpha) \tag{15a}$$

$$\bar{y} = \bar{C}\bar{x} + \bar{F}u + \bar{E}w \tag{15b}$$

where u and w are the same as in (1); $C_{i,}$, $F_{i,}$ and $E_{i,}$ are the ith row vectors of the matrices C, F, and E, respectively, $1 \le i \le m$; and $\bar{x}_0 \in \bar{\mathcal{D}}_0 := \mathcal{D}_0 \times \mathbb{R}$, where $\bar{\mathcal{D}}_0 \subseteq \mathbb{R}^{n+1}$ is a subspace. Then, the extended system admits vector relative degree $r_1, \ldots, r_{i_0-1}, r_{i_0} + 1, r_{i_0+1}, \ldots, r_m$ with respect to u, and is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

Proof It is easy to see, with the introduction of the integration on y_{i_0} as in (14), that the extended system (15) admits vector relative degree $r_1, \ldots, r_{i_0-1}, r_{i_0} + 1, r_{i_0+1}, \ldots, r_m$ with respect to u. We need to show that (15) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

Let $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, $A_{z1} \in \mathbb{R}^{s\times m}$ be the maximal solution to (6) of Başar and Pan (2019) for (1); and thus (1) admits extended zero dynamics (2). By Lemma 1 and Definition 1 of Başar and Pan (2019), we have $s = n - \sum_{i=1}^m r_i$. Then,

$$\bar{A} := \begin{bmatrix} A & \mathbf{0}_{n\times 1} \\ C_{i_0,} & 0 \end{bmatrix}; \quad \bar{B} := \begin{bmatrix} B \\ F_{i_0,} \end{bmatrix}; \quad \bar{D} := \begin{bmatrix} D \\ E_{i_0,} \end{bmatrix}$$

$$\bar{C} := \begin{bmatrix} C_{1,} & 0 \\ \vdots & \vdots \\ C_{i_0-1,} & 0 \\ \mathbf{0}_{1\times n} & 1 \\ C_{i_0+1,} & 0 \\ \vdots & \vdots \\ C_{m,} & 0 \end{bmatrix}; \quad \bar{F} := \begin{bmatrix} F_{1,} \\ \vdots \\ F_{i_0-1,} \\ \mathbf{0}_{1\times m} \\ F_{i_0+1,} \\ \vdots \\ F_{m,} \end{bmatrix}; \quad \bar{E} := \begin{bmatrix} E_{1,} \\ \vdots \\ E_{i_0-1,} \\ \mathbf{0}_{1\times q} \\ E_{i_0+1,} \\ \vdots \\ E_{m,} \end{bmatrix}$$

It is easy to check that (6) of Başar and Pan (2019) admits the solution $[\, K \; -A_{z1,i_0} \,] =: \bar{K}$, A_z, $\bar{A}_{z1} =: [\, A_{z1,1} \cdots A_{z1,i_0-1} \; A_z A_{z1,i_0} \; A_{z1,i_0+1} \cdots A_{z1,m} \,]$ for the extended system (15), where $A_{z1,i}$ is the ith column vector of the matrix A_{z1}. This is then the maximal solution by Lemma 1 and Definition 1 of Başar and Pan (2019), since $\bar{K} \in \mathbb{R}^{s\times(n+1)}$ and $s = n - \sum_{i=1}^m r_i$. This implies that the extended system (15) admits the extended zero dynamics

$$\dot{\bar{x}}_z = A_z \bar{x}_z + \bar{A}_{z1}\bar{y} + (KD - A_{z1}E)w; \tag{16}$$
$$\bar{x}_{z0} = \bar{K}\bar{x}_0 = Kx_0 - \alpha A_{z1,i_0} \in \bar{K}(\bar{\mathcal{D}}_0) =: \bar{\mathcal{D}}_{z0}$$

By (1) be minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, its extended zero dynamics (2) satisfies the requirements of Definition 1 of Başar and Pan (2019). By Lemma 5 of Başar and Pan (2019), the following system is BIBS stable:

$$\dot{z} = A_z z + A_{z1} v; \quad z(0) = \mathbf{0}_{s\times 1}$$

By Lemma 6 of Pan and Başar (2018), there exists $k \ge 0$ and $\lambda > 0$ such that $\left\| \mathrm{e}^{A_z t} A_{z1} \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \ge 0$. Then, we have $\left\| \mathrm{e}^{A_z t} A_{z1,i} \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \ge 0$, $\forall 1 \le i \le m$. By Lemma 6 of Pan and Başar (2018), the following system

$$\dot{z}_i = A_z z_i + A_{z1,i} v_i; \quad z_i(0) = \mathbf{0}_{s\times 1}$$

is BIBS stable. By Lemma 5, we have

$$\dot{\bar{z}}_{i_0} = A_z z_{i_0} + A_z A_{z1,i_0} v_{i_0}; \quad \bar{z}_{i_0}(0) = \mathbf{0}_{s\times 1}$$

is BIBS stable. Thus, we have that the following system

$$\dot{\eta} = A_z \eta + \bar{A}_{z1} v; \quad \eta(0) = \mathbf{0}_{s\times 1}$$

is BIBS stable. Then, $\exists c_{c2}(c_w) \ge 0$ such that $\forall v_{[0,\infty)} \in \mathcal{C}$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$, we have $\left\| \eta_{[0,\infty)} \right\|_\infty \le c_{c2}(c_w)$. By the assumption, we have $\forall c_w \ge 0$, $\exists c_{c1}(c_w) \ge 0$, $\forall x_{z0} \in \mathcal{D}_{z0}$ with $|\, x_{z0} \,| \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, we have $\left\| \lambda_{[0,\infty)} \right\|_\infty \le c_{c1}(c_w)$, where λ is the unique solution to the following system

$$\dot{\lambda} = A_z \lambda + (KD - A_{z1}E)w; \quad \lambda(0) = x_{z0}$$

Consider the extended zero dynamics (16) of the extended system (15). Fix any $\bar{x}_{z0} = \bar{K}\bar{x}_0 = Kx_0 - \alpha A_{z1,i_0} \in \bar{\mathcal{D}}_{z0}$ with $|\, \bar{x}_{z0} \,| \le c_w$. By Lemma 4, $\exists c_1 \ge 0$ such that $\exists \bar{x}_0 = (x_0, \alpha) \in \bar{\mathcal{D}}_0$ with $|\, \bar{x}_0 \,| \le c_1 c_w$ such that $\bar{K}\bar{x}_0 = \bar{x}_{z0}$. The solution to (16) is $\bar{x}_{z[0,\infty)}$ with initial condition $\bar{x}_{z0}$ and input waveforms $\bar{y}_{[0,\infty)} := v_{[0,\infty)}$ and $w_{[0,\infty)}$ can be expressed as $\bar{x}_{z[0,\infty)} = \eta_{[0,\infty)} + \lambda_{[0,\infty)} + \delta_{[0,\infty)}$, where $\delta_{[0,\infty)}$ is the solution to the system $\dot{\delta} = A_z \delta$, $\delta(0) = -\alpha A_{z1,i_0}$. Thus, we have $\left\| \bar{x}_{z[0,\infty)} \right\|_\infty \le \left\| \eta_{[0,\infty)} \right\|_\infty + \left\| \lambda_{[0,\infty)} \right\|_\infty + \left\| \delta_{[0,\infty)} \right\|_\infty = c_{c1}(\| K \|_{2,2}\, c_1 c_w) + c_{c2}(c_w) + k c_1 c_w \,|\, A_{z1,i_0} \,| =: c_c(c_w) \ge 0$. Hence, the extended system (15) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This completes the proof of the lemma. □

Toward the end of applying the vectorized version of SISO robust adaptive control design (Pan and Başar, 2000), we need the following lemma that allows us to further extend the square MIMO LTI system with uniform vector relative degree to one with uniform observability indices, without changing the fact that the system admits uniform vector relative degree and the fact that the system is minimum phase with respect to its admissible initial conditions and admissible disturbance waveforms.

Lemma 3. Consider the square MIMO LTI system (1). Let system (1) be minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. Let $\nu_1, \ldots, \nu_m \in \{0, \ldots, n\}$ be the observability indices for the output channels, $\nu := \max_i \nu_i$ be the observability index, and the system admit uniform vector relative degree $r = r_1 = \cdots = r_m \in \{0, \ldots, n\}$ with respect to the control inputs. Without loss of generality, assume that matrices A and C are given in the interweaved observer canonical form as (7). We may extend the system (1) to an $(n + m\nu - \sum_{l=1}^m \nu_l)$-dimensional LTI system

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + \bar{D}w; \qquad \bar{x}(0) = \bar{x}_0 \tag{17a}$$

$$\bar{y} = \bar{C}\bar{x} + Fu + Ew \tag{17b}$$

where $\bar{x} \in \mathbb{R}^{n+m\nu-\sum_{l=1}^m \nu_l}$; u and w are the same as in (1); and $\bar{x}_0 \in \bar{\mathcal{D}}_0 := \mathcal{D}_0 \times \left\{ \mathbf{0}_{m\nu - \sum_{l=1}^m \nu_l} \right\}$, where $\bar{\mathcal{D}}_0 \subseteq \mathbb{R}^{n+m\nu-\sum_{l=1}^m \nu_l}$ is a subspace; such that $\forall x_0 \in \mathcal{D}_0$, letting $\bar{x}_0 = (x_0, \mathbf{0}_{m\nu-\sum_{l=1}^m \nu_l}) \in \bar{\mathcal{D}}_0$, we have $\bar{y}_{[0,t_f)} = y_{[0,t_f)}$ for both of the systems with the same input waveforms $u_{[0,t_f)}$ and $w_{[0,t_f)}$, $\forall t_f \in (0, \infty] \subset \mathbb{R}_e$, and (17) admits uniform observability indices $\nu = \nu_1 = \cdots = \nu_m$ and uniform vector relative degree r with respect to the control input, and (17) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

Proof It is proved in Isidori (1995, Chapter 5), that the following row vectors are linearly independent:

$$Q_{i,j} := C_{i,} A^{j-1}, 1 \le i \le m \text{ with } j = 1, \ldots, r$$

Thus, by Lemma 1, the first mr vectors in the list P are linearly independent, and therefore are the first mr row vectors of $q_1, \ldots, q_{n_O}$. This implies that the observability indices $\nu_i \geq r$, $1 \leq i \leq m$ and $\nu \geq r$.

Let $i_l := \sum_{k=1}^{l} \nu_k$, $1 \leq l \leq m$; the ith element of x be x_i, $1 \leq i \leq n$; and introduce the $(\nu - \nu_l)$-dimensional dynamics with state $\xi_l := (\xi_{l,1}, \ldots, \xi_{l,\nu-\nu_l})$: $\dot{\xi}_{l,k} = \xi_{l,k+1}$, $k = 1, \ldots, \nu - \nu_l - 1$, $\xi_l(0) = \mathbf{0}_{\nu-\nu_l}$, $1 \leq l \leq m$. Let $\bar{x} := (x, \xi_1, \ldots, \xi_m)$, which is $(n + m\nu - \sum_{l=1}^{m} \nu_l)$-dimensional, and modify x_{i_l} dynamics to $\dot{x}_{i_l} = A_{i_l,}x + B_{i_l,}u + D_{i_l,}w + \xi_{l,1}$, $1 \leq l \leq m$ with $\nu_l > 0$, where $A_{i,}$, $B_{i,}$, and $D_{i,}$ are the ith row vector of the matrix A, B, and D, respectively; or modify the measurement channel $y_l = C_{l,}x + F_{l,}u + E_{l,}w + \xi_{l,1}$, $1 \leq l \leq m$ with $\nu_l = 0$, where $C_{l,}$, $F_{l,}$, and $E_{l,}$ are the lth row vector of the matrix C, F, and E, respectively. Then, the extended system (17) admits the following system matrices

$$\bar{A} := \begin{bmatrix} A & \tilde{G} \\ \mathbf{0}_{(m\nu - \sum_{l=1}^{m} \nu_l) \times n} & \hat{G} \end{bmatrix}; \quad \bar{B} := \begin{bmatrix} B \\ \mathbf{0}_{(m\nu - \sum_{l=1}^{m} \nu_l) \times m} \end{bmatrix}$$

$$\bar{D} := \begin{bmatrix} D \\ \mathbf{0}_{(m\nu - \sum_{l=1}^{m} \nu_l) \times q} \end{bmatrix}; \bar{C} := \begin{bmatrix} C \, \tilde{H} \end{bmatrix}$$

$$\tilde{G} := \begin{bmatrix} \tilde{G}_1 \cdots \tilde{G}_m \end{bmatrix} \text{ with } \tilde{G}_l \in \mathbb{R}^{n \times (\nu - \nu_l)},$$

$$\tilde{G}_l = \begin{cases} \begin{bmatrix} e_{n,i_l} \, \mathbf{0}_{n \times (\nu - \nu_l - 1)} \end{bmatrix} & \text{if } \nu > \nu_l > 0 \\ \mathbf{0}_{n \times (\nu - \nu_l)} & \text{if } \nu > \nu_l = 0 \end{cases}, 1 \leq l \leq m;$$

$$\hat{G} := \text{block diagonal} \left(\hat{G}_1, \ldots, \hat{G}_m \right) \text{ with}$$

$$\hat{G}_l \in \mathbb{R}^{(\nu - \nu_l) \times (\nu - \nu_l)}, 1 \leq l \leq m;$$

$$\tilde{H} := \begin{bmatrix} \tilde{H}_1 \cdots \tilde{H}_m \end{bmatrix} \text{ with } \tilde{H}_l \in \mathbb{R}^{m \times (\nu - \nu_l)}$$

$$\tilde{H}_l = \begin{cases} \mathbf{0}_{m \times (\nu - \nu_l)} & \text{if } \nu > \nu_l > 0 \\ \begin{bmatrix} e_{m,l} \, \mathbf{0}_{m \times (\nu - \nu_l - 1)} \end{bmatrix} & \text{if } \nu > \nu_l = 0 \end{cases}, 1 \leq l \leq m$$

where $\dot{\xi}_l =: \hat{G}_l \xi_l$, $l = 1, \ldots, m$.

Define $\hat{x}_l := (x_{i_{l-1}+1}, \ldots, x_{i_l}, \xi_l) \in \mathbb{R}^{\nu}$, $1 \leq l \leq m$; $\hat{x}_{\bar{o}} := (x_{i_m+1}, \ldots, x_n)$; and $\hat{x} := (\hat{x}_1, \ldots, \hat{x}_m, \hat{x}_{\bar{o}})$. Then, it is easy to see that, in $\hat{x}$ coordinates, the extended system (17) admits the state space representation

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u + \hat{D}w; \quad \hat{x}(0) =: \hat{x}_0$$
$$\bar{y} = \hat{C}\hat{x} + Fu + Ew$$

where $i_m = n_O = \sum_{l=1}^{m} \nu_l$,

$$\hat{A} = \begin{bmatrix} \hat{A}_{1,1} & \cdots & \hat{A}_{1,m} & \mathbf{0}_{\nu \times (n - n_O)} \\ \vdots & & \vdots & \vdots \\ \hat{A}_{m,1} & \cdots & \hat{A}_{m,m} & \mathbf{0}_{\nu \times (n - n_O)} \\ \hat{A}_{\bar{o},1} & \cdots & \hat{A}_{\bar{o},m} & \hat{A}_{\bar{o}} \end{bmatrix}; \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \vdots \\ \hat{B}_m \\ \hat{B}_{\bar{o}} \end{bmatrix};$$

$$\hat{C} = \begin{bmatrix} \hat{C}_1 \cdots \hat{C}_m \, \mathbf{0}_{m \times (n - n_O)} \end{bmatrix}$$

where $\hat{A}_{i,j} \in \mathbb{R}^{\nu \times \nu}$, $1 \leq i \leq m$, $1 \leq j \leq m$, with elements $\hat{a}_{i,j,k,l}$, $1 \leq k \leq \nu$, $1 \leq l \leq \nu$, that satisfies $\hat{a}_{i,j,k,l} = 0$, if $i \neq j$ and $l \geq 2$, $\hat{a}_{i,i,k,k+1} = 1$, $\forall 1 \leq k \leq \nu_i - 1$, and $\hat{a}_{i,i,k,l} = 0$, if $l \geq 2$ and $k + 1 \neq l$; and $\hat{A}_{\bar{o},j} \in \mathbb{R}^{(n-n_O) \times \nu}$, $1 \leq j \leq m$, with elements $\hat{a}_{\bar{o},j,k,l}$, $1 \leq k \leq n - n_0$, $1 \leq l \leq \nu$, that satisfies $\hat{a}_{\bar{o},j,k,l} = 0$ if $l \geq 2$; $\hat{A}_{\bar{o}} \in \mathbb{R}^{(n-n_O) \times (n-n_O)}$; $\hat{C}_j \in \mathbb{R}^{m \times \nu}$, $1 \leq j \leq m$, with elements $\hat{c}_{j,k,l}$, $1 \leq k \leq m$, $1 \leq l \leq \nu$, that satisfies $\hat{c}_{j,k,l} = 0$, if $l \geq 2$, $\hat{c}_{j,j,1} = 1$, and $\hat{c}_{j,k,1} = 0$ if $k < j$; partition $B = [B_1' \cdots B_m' \, B_{\bar{o}}']'$ compatibly with that of A and C, and $B_i \in \mathbb{R}^{\nu_i \times m}$, $1 \leq i \leq m$, with elements $b_{i,k,l}$, $1 \leq k \leq \nu_i$, $1 \leq l \leq m$; $\hat{B}_i \in \mathbb{R}^{\nu \times m}$, $1 \leq i \leq m$, with elements $\hat{b}_{i,k,l}$, $k = 1, \ldots, \nu$, $1 \leq l \leq m$, that satisfies $\hat{b}_{i,k,l} = 0$, if $k > \nu_i$, $l = 1, \ldots, m$, and $\hat{b}_{i,k,l} = b_{i,k,l}$, if $1 \leq k \leq \nu_i$, $1 \leq l \leq m$. Thus, based on the structure of $\hat{A}$ and $\hat{C}$, we conclude that (17) admits uniform observability indices ν as desired. Since (1) admits uniform vector relative degree r, then the matrix B must satisfy $b_{i,k,l} = 0$, if $k < r$, $1 \leq i \leq m$, $1 \leq l \leq m$. Thus, the extended system (17) in $\hat{x}$ coordinates satisfies $\hat{b}_{i,k,l} = 0$ if $k < r$, $1 \leq i \leq m$, $1 \leq l \leq m$. Thus, the extended system (17) admits uniform vector relative degree r.

Let the system (1) admit maximal solution $K \in \mathbb{R}^{s \times n}$, $A_z \in \mathbb{R}^{s \times s}$, and $A_{z1} \in \mathbb{R}^{s \times m}$ to (6) of Başar and Pan (2019). By Lemma 1 and Definition 1 of Başar and Pan (2019), we have $s = n - mr$. It is easy to show that

$$\bar{K} := \text{block diagonal} \left(K, I_{m\nu - \sum_{i=1}^{m} \nu_i} \right)$$
$$\in \mathbb{R}^{(s + m\nu - \sum_{i=1}^{m} \nu_i) \times (n + m\nu - \sum_{i=1}^{m} \nu_i)}$$
$$\bar{A}_z := \begin{bmatrix} A_z & K\tilde{G} - A_{z1}\tilde{H} \\ \mathbf{0}_{(m\nu - \sum_{i=1}^{m} \nu_i) \times s} & \hat{G} \end{bmatrix}$$
$$\in \mathbb{R}^{(s + m\nu - \sum_{i=1}^{m} \nu_i) \times (s + m\nu - \sum_{i=1}^{m} \nu_i)}$$
$$\bar{A}_{z1} := \begin{bmatrix} A_{z1} \\ \mathbf{0}_{(m\nu - \sum_{i=1}^{m} \nu_i) \times m} \end{bmatrix} \in \mathbb{R}^{(s + m\nu - \sum_{i=1}^{m} \nu_i) \times m}$$

satisfies the equation (6) of Başar and Pan (2019) for the extended system (17). The matrix $\bar{K}$ is of full row rank. By Lemma 1 and Definition 1 of Başar and Pan (2019), the set of solutions $\bar{K}$, $\bar{A}_z$, and $\bar{A}_{z1}$ is maximal. Therefore, the extended zero dynamics of (17) is

$$\dot{x}_z = A_z x_z + (K\tilde{G} - A_{z1}\tilde{H})\xi + A_{z1}y + (KD - A_{z1}E)w;$$
$$x_z(0) = Kx_0 \in K(\mathcal{D}_0)$$
$$\dot{\xi} = \hat{G}\xi; \quad \xi(0) = \mathbf{0}_{m\nu - \sum_{i=1}^{m} \nu_i}$$

with state vector $\bar{x}_z := (x_z, \xi)$, and $\bar{\mathcal{D}}_0 := \mathcal{D}_0 \times \left\{ \mathbf{0}_{m\nu - \sum_{i=1}^{m} \nu_i} \right\} \subseteq \mathbb{R}^{s + m\nu - \sum_{i=1}^{m} \nu_i}$ is a subspace. By the assumption that (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, we can readily conclude that the extended system (17) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$.

This completes the proof of the lemma. □

6. CONCLUSIONS

In this paper, we proved that the minimum phase condition is necessary for model reference control of square MIMO LTI systems to achieve the following two properties: 1) the system states remain bounded when the disturbance input waveform is admissible and bounded, the initial condition is admissible and bounded, and the reference trajectory is bounded with bounded derivatives up to nth order, where n is the dimension of the system; 2) the perfect tracking of any given bounded reference trajectory with bounded derivatives up to nth order when the disturbance waveform is identically equal to zero, the extended zero dynamics admits any admissible initial condition, and by appropriately choosing the initial condition for the rest of the closed-loop system states. We further proved the existence of observer canonical form, and defined and proved the existence of the strict observer canonical form. The strict observer canonical form of an

LTI system is guaranteed to exist if the system admits uniform observability indices.

The extended zero dynamics canonical form and the strict observer canonical form are needed for the true system in robust adaptive control design and in stability analysis of the resulting closed-loop system (Pan and Başar, 2000). In order to apply an appropriately vectorized verion of robust adaptive control design for the system (as delineated in Pan and Başar (2000)), we need to be able to extend the given square MIMO LTI system to one with uniform vector relative degree and with uniform observability indices without changing its minimum phase property. We presented two lemmas in Section 5 that fully resolve this issue. Thus, robust adaptive control design for a square MIMO LTI system can be carried out if the given system is minimum phase and can admit vector relative degree after a finite number of steps of dynamic extension that are independent of the unknown parameters in the system. Once, we arrive at a square MIMO LTI system with vector relative degree, which is further minimum phase, we can further apply parameter independent dynamic extensions or add integrators on its outputs to arrive at a square MIMO LTI system with uniform vector relative degree, without changing its minimum phase property. Then, this extended system admits the extended zero dynamics canonical form. We can always further extend the extended system by adding dummy states (which are always zero under admissible initial conditions and are independent of the inputs) such that the further extended system admits uniform observability indices, without changing the fact that the system admits uniform vector relative degree and the fact that the system is minimum phase with respect to its admissible initial conditions and admissible disturbance waveforms. The observable part of this further extended MIMO LTI system may then serve as the true system in the robust adaptive control design using the vectorized version of Pan and Başar (2000). This thus completely resolves robust adaptive control design of a minimum phase square MIMO LTI system.

Further study of the minimum phase property of composite systems that consist of interconnected LTI systems under suitable assumptions on the component system is a fruitful research direction. For SISO composite systems, comprehensive results have already been obtained in Pan and Başar (2019b,a). The MIMO version of these results are currently under study.

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Appendix A. USEFUL RESULTS

Lemma 4. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{K}$ and $\mathcal{D}_0 \subseteq \mathcal{X}$ be a closed subspace, P be a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$, and $P(\mathcal{D}_0) \subseteq \mathcal{Y}$ be closed. Then, $\exists c \geq 0$, $\forall c_w \geq 0$, $\forall b \in P(\mathcal{D}_0) \subseteq \mathcal{Y}$ with $\| b \|_{\mathcal{Y}} \leq c_w$, $\exists x \in \mathcal{D}_0 \subseteq \mathcal{X}$ such that $b = Px$ and $\| x \|_{\mathcal{X}} \leq c c_w$.

Proof This follows from the Generalized Inverse Function Theorem (Luenberger, 1969, pp. 240). □

Lemma 5. Consider a LTI system:

$$\dot{z} = Az + Bv; \qquad z(0) = \mathbf{0}_{n\times 1} \tag{A.1}$$

where z is the n-dimensional state, $n \in \mathbb{Z}_+$; and v is the p-dimensional input, $p \in \mathbb{Z}_+$. Assume that (A.1) is BIBS stable. Then, the following statements hold.

(i) For system $\dot{\eta} = A\eta$, $\eta(0) = B\xi$, where $\xi \in \mathbb{R}^p$, there exist $k \geq 0$ and $\lambda > 0$ such that $\forall \xi \in \mathbb{R}^p$ with $|\xi| \leq c_w \geq 0$, we have $|\eta(t)| \leq c_w k e^{-\lambda t}$, $\forall t \in [0,\infty)$.

(ii) The system $\dot{x} = Ax + ABu$, $x(0) = \mathbf{0}_{n\times 1}$, is BIBS stable.

Proof By Lemma 6 of Pan and Başar (2018), $\exists k \geq 0$ and $\exists \lambda \in (0,\infty) \subset \mathbb{R}$ such that $\left\| e^{At} B \right\|_{2,2} \leq k e^{-\lambda t}$, $\forall t \geq 0$. For (i), by Chen (1984), $|\eta(t)| = \left| e^{At} B \xi \right| \leq k e^{-\lambda t} c_w$, $\forall t \geq 0$. For (ii), we note that $e^{At} AB = \left(\sum_{i=0}^{\infty} \frac{1}{i!}(At)^i\right) AB = A \cdot \left(\sum_{i=0}^{\infty} \frac{1}{i!}(At)^i\right) B = A e^{At} B$, $\forall t \in \mathbb{R}$. Then, $\left\| e^{At} AB \right\|_{2,2} = \left\| A e^{At} B \right\|_{2,2} \leq \| A \|_{2,2} \cdot \left\| e^{At} B \right\|_{2,2} \leq \| A \|_{2,2} k e^{-\lambda t}$, $\forall t \geq 0$. By Lemma 6 of Pan and Başar (2018), the result holds.

This completes the proof of the lemma. □