Generalized Minimum Phase Property for Series Interconnected SISO LTI Systems

Zigang Pan* Tamer Başar**

***Abstract*— In [1], we had introduced a generalized concept of minimum phase for finite-dimensional continuous-time single-input and single-output linear time-invariant systems with additive disturbances. In this paper, we study the minimum phase property of series interconnected systems. With two minimum phase systems in series interconnection with additional output feedback, we prove that the composite system is again minimum phase under the generalized concept.**

***Index Terms*— continuous-time systems, extended zero dynamics canonical form, minimum phase, extended zero dynamics.**

I. Introduction

The minimum phase property is of paramount importance in model reference control theory, having attracted sustained research attention [2], [3], [4]. In an earlier paper [1], we introduced a generalization of the minimum phase concept for SISO LTI systems with additive distubance inputs in an attempt to make it necessary for the solvability of the output feedback model reference control problem. When the system has a finite relative degree (RD) r, then it may be transformed into the extended zero dynamics canonical form (EZDCF) representation. Based on this canonical form representation, the extended zero dynamics (EZD) for the system is defined, which is simply the zero dynamics as defined in [3] together with driving terms including the (noiseless) output and the disturbance input of the system. The original system is said to be minimum phase with respect to the given set of admissible initial conditions and the given set of admissible disturbance waveforms if the EZD is absent or satisfies the properties that the zero dynamics state is bounded for any bounded admissible initial condition (for the EZD), any bounded noiseless output waveform, and any bounded admissible disturbance waveform. The relationship of the generalized concept of minimum phase with that introduced in [3] and [2] has been investigated. Furthermore, the generalized minimum phase property is proved to be necessary in model reference control of the system. In an accompanying paper [5], on which this paper is dependent upon, submitted also to this conference, we proved that a composite system comprised of a minimum phase system in feedback connection with another linear system satisfying a certain boundedness condition is itself a minimum phase system, and we further established an inversion result for such systems.

In this paper, we further investigate the properties of minimum phase systems using the definition in [1]. We prove that a composite system consisting of two minimum phase systems in series interconnection with additional output feedback is again minimum phase.

The balance of the paper is as follows. In the next section, we list the notations used in the paper. Then, in Section III, we present the main result of the paper. The paper ends with some concluding remarks in Section IV.

II. Notations

Let $\mathbb{R}$ denote the real line; $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Unless specified, all signals, constants, and matrices are real. For a continuous function f, we say that it belongs to $\mathcal{C}$. We say that a function is L_∞ if it is bounded. For any matrix A, A' denotes its transpose. For any $z \in \mathbb{R}^n$, $|z|$ denotes $\sqrt{z'z}$. I_n denotes the $n \times n$-dimensional identity matrix. For any matrix A, $A^0 = I$. For any matrix M, $\|M\|_{p,p}$ denotes its p-induced norm, $1 \leq p \leq \infty$. $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f), \mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$.

III. Main Result

We first present a technical lemma which establishes the minimum phase property for SISO LTI system in a general canonical form, which arises in interconnected systems.

Lemma 1: Consider a SISO LTI system: [1]

$$\dot{x}_{za} = A_{oa}x_{za} + A_{oa1}x_1 + \cdots + A_{oar}x_r + A_{oa\,r+1}b_0u + D_{oa}w \tag{1a}$$

$$\dot{x}_{zb} = A_{oba}x_{za} + A_{ob}x_{zb} + A_{ob1}x_1 + \cdots + A_{obr}x_r + A_{ob\,r+1}b_0u + D_{ob}w \tag{1b}$$

$$\dot{x}_i = A_{ioa}x_{za} + A_{iob}x_{zb} + a_{i1}x_1 + \cdots + a_{ii}x_i + x_{i+1} + D_iw;\ i = 1, \ldots, r-1 \tag{1c}$$

$$\dot{x}_r = A_{roa}x_{za} + A_{rob}x_{zb} + a_{r1}x_1 + \cdots + a_{rr}x_r + b_0u + D_rw \tag{1d}$$

$$y = x_1 + Ew \tag{1e}$$

where $x_{za} \in \mathbb{R}^{m_a}$, $m_a \in \mathbb{N}$; $x_{zb} \in \mathbb{R}^{m_b}$, $m_b \in \mathbb{N}$; $x_i \in \mathbb{R}$, $i = 1, \ldots, r$, with $m_a + m_b + r = n$ and $r \in \mathbb{N}$; $b_0 \neq 0$; $y \in \mathbb{R}$ is the output; $u \in \mathbb{R}$ is the control input; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; and all matrices are constant. Let $x = [x_{za}'\, x_{zb}'\, x_1 \cdots x_r]'$, $x(0) = x_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ be a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$ (see Definition 2 of [1]).

Assume that $\exists m_{oa}, m_{ob} \in \{1, \ldots, r+1\}$ with $m_{oa} \leq m_{ob}$, such that, $A_{oaj} = \mathbf{0}_{m_a\times 1}$, $j = m_{oa}+1, \ldots, r+1$; $A_{joa} = \mathbf{0}_{1\times m_a}$, $j = 1, \ldots, m_{oa}-1$; $A_{obj} = \mathbf{0}_{m_b\times 1}$, $j = m_{ob}+1, \ldots, r+1$; and $A_{job} = \mathbf{0}_{1\times m_b}$, $j = 1, \ldots, m_{ob}-1$.

Assume that the system (1a) and (1b) satisfy $\forall c_w \geq 0$, $\exists c_c \geq 0$, $\forall x_{z0} := [x_{za0}'\, x_{zb0}']' \in P_z(\mathcal{D}_0)$ with $|x_{z0}| \leq c_w$, where $P_z : \mathbb{R}^n \to \mathbb{R}^{m_a+m_b}$ is the projection of $\mathbb{R}^n$ onto the first $m_a + m_b$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \leq c_w$, $i = 1, \ldots, m_{ob}$, ($x_{r+1} := b_0u$ for notational consistency) we have $\|x_{z[0,\infty)}\|_\infty \leq c_c$, where $x_z := [x_{za}'\, x_{zb}']'$. Then, the system (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits RD r from u to y.

Proof: We will prove the lemma by mathematical induction on $m_{ob} - m_{oa} \in \{0, \ldots, r\}$.

1° $m_{ob} - m_{oa} = 0$. This is proved by Lemma 1 of [5].

2° Assume that the result holds for $m_{ob} - m_{oa} = k \in \{0, \ldots, r-1\}$.

3° Consider $m_{ob} - m_{oa} = k+1 \in \{1, \ldots, r\}$. Then, $m_{oa} \in \{1, \ldots, r\}$ and $m_{ob} \in \{2, \ldots, r+1\}$. Introduce the coordinate transformation

$$\bar{x} = T_1^{-1}x = \left[x_{za}'\ x_{zb}' - A_{obm_{ob}}'x_{m_{ob}-1}\ x_1 \cdots x_r \right]'$$

[1]This result is along the lines of Lemma 1 of [5], but is more general than that one (which corresponds to the special case of $m_b = m_a$), tailored to the class of systems studied in this paper.

* Present address: 4797 Bordeaux Lane, Mason, OH 45040, USA. Tel: 513-398-7825; Fax: 513-398-7825; Email: zigangpan2002@mac.com.

** Coordinated Science Laboratory, University of Illinois, 1308 West Main Street, Urbana, IL 61801, USA. Tel: 217-333-3607; Fax: 217-244-1653; Email: basar1@illinois.edu.

Then, in $\bar{x} := [x'_{za}\, \bar{x}'_{zb}\, x_1 \cdots x_r]'$ coordinates, the system (1) admits the following state space representation

$$\begin{aligned}
\dot{x}_{za} &= A_{oa}x_{za} + A_{oa1}x_1 + \cdots + A_{oam_{oa}}x_{m_{oa}} + D_{oa}w && \text{(2a)}\\
\dot{\bar{x}}_{zb} &= A_{oba}x_{za} + A_{ob}\bar{x}_{zb} + A_{ob}A_{obm_{ob}}x_{m_{ob}-1} + A_{ob1}x_1 + \\
&\quad \cdots + A_{ob\,m_{ob}-1}x_{m_{ob}-1} + D_{ob}w - A_{obm_{ob}}(A_{m_{ob}-1\,oa}\\
&\quad \cdot x_{za} + a_{m_{ob}-1\,1}x_1 + \cdots + a_{m_{ob}-1\,m_{ob}-1}x_{m_{ob}-1}\\
&\quad + D_{m_{ob}-1}w) =: \bar{A}_{oba}x_{za} + A_{ob}\bar{x}_{zb} + \bar{A}_{ob1}x_1 + \cdots\\
&\quad + \bar{A}_{ob\,m_{ob}-1}x_{m_{ob}-1} + \bar{D}_{ob}w && \text{(2b)}\\
\dot{x}_i &= A_{ioa}x_{za} + a_{i1}x_1 + \cdots + a_{ii}x_i + x_{i+1} + D_iw; && \text{(2c)}\\
&\quad i = 1, \ldots, m_{ob} - 1\\
\dot{x}_i &= A_{ioa}x_{za} + A_{iob}\bar{x}_{zb} + A_{iob}A_{obm_{ob}}x_{m_{ob}-1} + a_{i1}x_1\\
&\quad + \cdots + a_{ii}x_i + x_{i+1} + D_iw\\
&=: A_{ioa}x_{za} + A_{iob}\bar{x}_{zb} + \bar{a}_{i1}x_1 + \cdots + \bar{a}_{ii}x_i + x_{i+1}\\
&\quad + D_iw \quad i = m_{ob}, \ldots, r && \text{(2d)}\\
y &= x_1 + Ew && \text{(2e)}
\end{aligned}$$

with $\bar{x}(0) = \bar{x}_0 = [x'_{za0}\, \bar{x}'_{zb0}\, x_{1,0} \cdots x_{r,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_0) \neq \emptyset$. Clearly, (2) is in the form of (1) with $\bar{m}_{oa} = m_{oa}$ and $\bar{m}_{ob} = m_{ob} - 1$ such that $\bar{m}_{ob} - \bar{m}_{oa} = k$. We will apply the inductive assumption to show that (2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD r. Toward this end, $\forall c_w \geq 0$, $\forall \bar{x}_{z0} := [x'_{za0}\, \bar{x}'_{zb0}]' \in P_z(\bar{\mathcal{D}}_0)$ with $|\bar{x}_{z0}| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \leq c_w$, $i = 1, \ldots, \bar{m}_{ob}$. By Lemma 5 of [5], $\exists c_1 \in [0, \infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_0$, and $\exists \acute{x}_{1,0}, \ldots, \acute{x}_{r,0} \in \mathbb{R}$ such that $\acute{x}_0 := [\bar{x}'_{z0}\, \acute{x}_{1,0} \cdots \acute{x}_{r,0}]' \in \bar{\mathcal{D}}_0$ and $|\acute{x}_0| \leq c_1 c_w$. Then, $[\acute{x}'_{z0}\, \acute{x}_{1,0} \cdots \acute{x}_{r,0}]' = T_1\acute{x}_0 \in \mathcal{D}_0$ where $\acute{x}_{z0} := \left[x'_{za0}\, \bar{x}'_{zb0} + A'_{ob\,m_{ob}}\acute{x}_{m_{ob}-1,0}\right]' \in P_z(\mathcal{D}_0)$ with $|\acute{x}_{z0}| \leq |T_1\acute{x}_0| \leq \|T_1\|_{2,2}|\acute{x}_0| \leq \|T_1\|_{2,2}c_1c_w =: c_1c_2c_w$. By the structure of (2a) and (2b), which exhibits that the dynamics of x_{za} is identical to (1a) and is not affected by x_{zb} or $x_{m_{ob}}$, we may apply the assumption of the lemma to conclude that $\exists c_{c1} \geq 0$, which depends only on c_w, such that $\|x_{za[0,\infty)}\|_\infty \leq c_{c1}$.

Let $\bar{x}_{m_{ob}} = -A_{m_{ob}-1\,oa}x_{za} - a_{m_{ob}-1\,1}x_1 - \cdots - a_{m_{ob}-1\,m_{ob}-1}x_{m_{ob}-1} - D_{m_{ob}-1}w$. Let $\bar{x}_z = [x'_{za}\, \bar{x}'_{zb}]'$. Then, $\bar{x}_{z[0,\infty)}$ may be generated by

$$\begin{aligned}
\dot{\eta}_1 &= \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix}\eta_1 + \begin{bmatrix} A_{oa1} \\ A_{ob1} \end{bmatrix}x_1 + \cdots + \begin{bmatrix} A_{oa\bar{m}_{ob}} \\ A_{ob\bar{m}_{ob}} \end{bmatrix}x_{\bar{m}_{ob}}\\
&\quad + \begin{bmatrix} \mathbf{0}_{m_a\times 1} \\ A_{obm_{ob}} \end{bmatrix}\bar{x}_{m_{ob}} + \begin{bmatrix} D_{oa} \\ D_{ob} \end{bmatrix}w;\\
&\quad \eta_1(0) = \acute{x}_{z0} = \begin{bmatrix} x_{za0} \\ \bar{x}_{zb0} + A_{obm_{ob}}\acute{x}_{\bar{m}_{ob},0} \end{bmatrix}\\
\dot{\eta}_2 &= \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix}\eta_2;\ \eta_2(0) = \begin{bmatrix} \mathbf{0}_{m_a\times 1} \\ -A_{obm_{ob}} \end{bmatrix}\acute{x}_{\bar{m}_{ob},0}\\
\dot{\eta}_3 &= \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix}\eta_3 + \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix}\begin{bmatrix} \mathbf{0}_{m_a\times 1} \\ A_{obm_{ob}} \end{bmatrix}x_{m_{ob}-1};\\
&\quad \eta_3(0) = \mathbf{0}_{(m_a+m_b)\times 1}\\
\bar{x}_{z[0,\infty)} &= \eta_{1[0,\infty)} + \eta_{2[0,\infty)} + \eta_{3[0,\infty)}
\end{aligned}$$

Note that $\bar{x}_{m_{ob}[0,\infty)} \in \mathcal{C}$ and $\|\bar{x}_{m_{ob}[0,\infty)}\|_\infty \leq \|A_{\bar{m}_{ob}\,oa}\|_{2,2}c_{c1} + |a_{\bar{m}_{ob}\,1}|\,c_w + \cdots + |a_{\bar{m}_{ob}\,\bar{m}_{ob}}|\,c_w + \|D_{\bar{m}_{ob}}\|_{2,2}c_w =: c_{w1}$. By the assumption of the lemma, $\exists c_{c2} \geq 0$, which depends only c_{w1}, $c_1c_2c_w$, and c_w, such that $\|\eta_{1[0,\infty)}\|_\infty \leq c_{c2}$.

By the assumption of the lemma and Lemma 9 of [1], we have the following system

$$\dot{\xi} = \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix}\xi + \begin{bmatrix} \mathbf{0}_{m_a\times 1} \\ A_{obm_{ob}} \end{bmatrix}v;\ \xi(0) = \mathbf{0}_{(m_a+m_b)\times 1}$$

is bounded input bounded state (BIBS) stable. For the dynamics of η_2, by Lemma 3 of [5], $\exists c_{c3} \geq 0$, which does not depend on any other constants, such that $\|\eta_{2[0,\infty)}\|_\infty \leq c_{c3}c_1c_w$.

Again by Lemma 3 of [5], the dynamics for η_3 is BIBS stable. Then, by Lemma 6 of [1], $\exists c_{c4} \geq 0$, which does not depend on any other constants, such that $\|\eta_{3[0,\infty)}\|_\infty \leq c_{c4}c_w$. Hence, we have $\|\bar{x}_{z[0,\infty)}\|_\infty \leq c_{c2} + c_{c3}c_1c_w + c_{c4}c_w$. This proves that (2) satisfies the inductive assumption. Hence, (2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD r from u to y. This completes the induction process and the proof of the lemma. ■

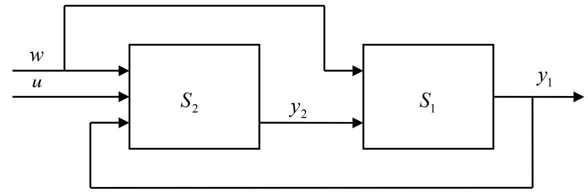

Fig. 1. Block diagram of two series interconnected systems.

We next present a result which shows that the composite system consisting of two series interconnected minimum phase finite-dimensional continuous-time SISO LTI systems with additional output feedback is again a minimum phase system. The block diagram of the system is shown in Figure 1.

Theorem 1: Consider two SISO LTI systems

$$S_1 : \begin{cases} \dot{x} = A_1x + B_1y_2 + D_1w; \quad x(0) = x_0 \\ y_1 = C_1x + K_1y_2 + E_1w \end{cases} \tag{3}$$

$$S_2 : \begin{cases} \dot{\eta} = A_2\eta + B_2u + D_2w + A_{21}y_1;\ \eta(0) = \eta_0 \\ y_2 = C_2\eta + K_2u + E_2w + K_{21}y_1 \end{cases} \tag{4}$$

where $x \in \mathbb{R}^{n_1}$ is the state for S_1, $n_1 \in \mathbb{Z}_+$; $y_2 \in \mathbb{R}$ is the control input of S_1, and is also the output of S_2; $y_1 \in \mathbb{R}$ is the output of S_1, and is also an input to S_2; $u \in \mathbb{R}$ is the control input for S_2; $\eta \in \mathbb{R}^{n_2}$ is the state for S_2, $n_2 \in \mathbb{Z}_+$; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; all matrices are constant; $x_0 \in \mathcal{D}_{x0}$, $\mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$ is a subspace; $\eta_0 \in \mathcal{D}_{\eta 0}$, $\mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$ is a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$.

Assume that S_1 admits RD $r_1 \in \{0, 1, \ldots, n_1\}$ from y_2 to y_1 and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$. Assume that S_2 admits RD $r_2 \in \{0, 1, \ldots, n_2\}$ from u to y_2 and is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty), \mathbb{R})$, where w and y_1 are viewed as disturbance inputs. Let S be the composite system with output y_1, control input u, disturbance input w, and state $[x'\,\eta']'$. Assume that the composite system is well posed, that is $\hat{K} := 1 - K_1K_{21} \neq 0$.

Then, S admits RD $r_1 + r_2$ from u to y_1; and is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

Proof: We will distinguish 18 exhaustive and mutually exclusive cases, which are listed in Table I.

Case 1: $0 = r_1 < n_1$ and $0 = r_2 < n_2$. By Lemma 3 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} = \bar{A}_1x + \bar{B}_1(y_1 - E_1w) + D_1w \\ \quad =: \bar{A}_1x + \bar{B}_1y_1 + \bar{D}_1w;\ x(0) = x_0 \\ y_1 = C_1x + K_1y_2 + E_1w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta} = \bar{A}_2\eta + \bar{B}_2(y_2 - E_2w - K_{21}y_1) + D_2w + A_{21}y_1 \\ \quad =: \bar{A}_2\eta + \bar{B}_2y_2 + \bar{D}_2w + \bar{A}_{21}y_1;\ \eta(0) = \eta_0 \\ y_2 = C_2\eta + K_2u + E_2w + K_{21}y_1 \end{cases}$$

Note that

$$y_1 = C_1x + K_1\left(C_2\eta + K_2u + E_2w + K_{21}y_1\right) + E_1w \ \Rightarrow$$
$$y_1 = \hat{K}^{-1}\left(C_1x + K_1C_2\eta + K_1K_2u + (K_1E_2 + E_1)w\right)$$
$$y_2 = K_1^{-1}\left(y_1 - C_1x - E_1w\right)$$

Then, S admits the following representation:

$$\begin{aligned}
\dot{x} &= \bar{A}_1x + \bar{B}_1y_1 + \bar{D}_1w; \qquad x(0) = x_0 && \text{(5a)}\\
\dot{\eta} &= \bar{A}_2\eta + \bar{B}_2K_1^{-1}\left(y_1 - C_1x - E_1w\right) + \bar{D}_2w + \bar{A}_{21}y_1 && \text{(5b)}\\
&= -\bar{B}_2K_1^{-1}C_1x + \bar{A}_2\eta + (\bar{A}_{21} + \bar{B}_2K_1^{-1})y_1\\
&\quad + (\bar{D}_2 - \bar{B}_2K_1^{-1}E_1)w;\ \eta(0) = \eta_0\\
y_1 &= \hat{K}^{-1}\left(C_1x + K_1C_2\eta + K_1K_2u + (K_1E_2 + E_1)w\right) && \text{(5c)}
\end{aligned}$$

Clearly, (5) admits RD $0 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1K_2 \neq 0$ and is in the EZDCF (6) of [1] with (5a)

TABLE I

18 EXHAUSTIVE AND MUTUALLY EXCLUSIVE CASES FOR THEOREM 1.

Case 1:	$0=r_1<n_1,\ 0=r_2<n_2$
Case 2:	$0=r_1<n_1,\ 1=r_2<n_2$
Case 3:	$0=r_1<n_1,\ 2\le r_2<n_2$
Case 4:	$0=r_1<n_1,\ 1=r_2=n_2$
Case 5:	$0=r_1<n_1,\ 2\le r_2=n_2$
Case 6:	$1\le r_1<n_1,\ 0=r_2<n_2$
Case 7:	$1\le r_1=n_1,\ 0=r_2<n_2$
Case 8:	$1\le r_1=n_1,\ 1\le r_2=n_2$
Case 9:	$1\le r_1<n_1,\ 1\le r_2=n_2$
Case 10:	$1\le r_1=n_1,\ 1\le r_2<n_2$
Case 11:	$1\le r_1<n_1,\ 1\le r_2<n_2$
Case 12:	$0=r_1<n_1,\ 0=r_2=n_2$
Case 13:	$1\le r_1=n_1,\ 0=r_2=n_2$
Case 14:	$1\le r_1<n_1,\ 0=r_2=n_2$
Case 15:	$0=r_1=n_1,\ 0=r_2<n_2$
Case 16:	$0=r_1=n_1,\ 1\le r_2=n_2$
Case 17:	$0=r_1=n_1,\ 1\le r_2<n_2$
Case 18:	$0=r_1=n_1=r_2=n_2$

and (5b) defining the EZD of S.

Next, we show that S is minimum phase with respect to $\mathcal{D}_{x0}\times\mathcal{D}_{\eta0}$ and $\mathcal{W}_d$. $\forall c_w\ge 0$, $\forall\,[x_0'\,\eta_0']'\in\mathcal{D}_{x0}\times\mathcal{D}_{\eta0}$ with $\left|[x_0'\,\eta_0']'\right|\le c_w$, $\forall w_{[0,\infty)}\in\mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty\le c_w$, $\forall y_{1[0,\infty)}\in\mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty\le c_w$, let $x_{[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (5a) and (5b). Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then $\exists c_{c1}\ge 0$, which depends only on c_w, such that $\|x_{[0,\infty)}\|_\infty\le c_{c1}$. Note that $y_{2[0,\infty)}\in\mathcal{C}$ with $\|y_{2[0,\infty)}\|_\infty\le|K_1^{-1}|c_w+\|K_1^{-1}C_1\|_{2,2}c_{c1}+\|K_1^{-1}E_1\|_{2,2}c_w=:\bar{c}_w\ge 0$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta0}$ and $\mathcal{W}_d\times\mathcal{C}([0,\infty),\mathbb{R})$, then $\exists c_{c2}\ge 0$, which depends only on $\bar{c}_w$ and c_w, such that $\|\eta_{[0,\infty)}\|_\infty\le c_{c2}$. Therefore, $\|\xi_{[0,\infty)}\|_\infty\le\sqrt{c_{c1}^2+c_{c2}^2}$, where $\xi=[x'\,\eta']'$. This proves that (5) is minimum phase with respect to $\mathcal{D}_{x0}\times\mathcal{D}_{\eta0}$ and $\mathcal{W}_d$. This proves Case 1.

Case 2: $0=r_1<n_1$ and $1=r_2<n_2$. By Lemmas 3 and 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1\ne 0$, $b_2\ne 0$,

$$S_1:\begin{cases}\dot{x}=\bar{A}_1x+\bar{B}_1(y_1-E_1w)+D_1w\\ \quad=:\bar{A}_1x+\bar{B}_1y_1+\bar{D}_1w;\qquad x(0)=x_0\\ y_1=C_1x+K_1y_2+E_1w\end{cases}$$

$$S_2:\begin{cases}\dot{\eta}_z=A_{2,z}\eta_z+A_{2,z1}\eta_1+D_{2,z}w+A_{21,z}y_1\\ \dot{\eta}_1=A_{2,1z}\eta_z+a_{2,11}\eta_1+b_2u+D_{2,1}w+A_{21,1}y_1\\ y_2=\eta_1+E_2w+K_{21}y_1\end{cases}$$

where $\eta=[\eta_z'\,\eta_1]'$; $\eta_z\in\mathbb{R}^{n_2-r_2}$; $\eta_1\in\mathbb{R}$, $\eta(0)=\eta_0:=[\eta_{x0}'\,\eta_{1,0}]'\in\mathcal{D}_{\eta0}$. Note that

$$y_1=C_1x+K_1\left(\eta_1+E_2w+K_{21}y_1\right)+E_1w\qquad\Rightarrow$$
$$y_1=\hat{K}^{-1}\left(C_1x+K_1\eta_1+(E_1+K_1E_2)w\right)=:\bar{\eta}_1+\bar{E}_1w$$

Then, $\eta_1=K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)$, and S admits representation, in $\xi=[x'\,\eta_z'\,\bar{\eta}_1]'=T_1^{-1}[x'\,\eta_z'\,\eta_1]'$ coordinates with $\xi(0)=\xi_0:=[x_0'\,\eta_{z0}'\,\bar{\eta}_{1,0}]'\in\bar{\mathcal{D}}_0:=T_1^{-1}(\mathcal{D}_{x0}\times\mathcal{D}_{\eta0})$,

$$\begin{aligned}\dot{x}&=\bar{A}_1x+\bar{B}_1y_1+\bar{D}_1w\\ &=\bar{A}_1x+\bar{B}_1\bar{\eta}_1+(\bar{D}_1+\bar{B}_1\bar{E}_1)w &\text{(6a)}\\ \dot{\eta}_z&=A_{2,z}\eta_z+A_{2,z1}K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)+D_{2,z}w\\ &\quad+A_{21,z}(\bar{\eta}_1+\bar{E}_1w) &\text{(6b)}\\ &=-A_{2,z1}K_1^{-1}C_1x+A_{2,z}\eta_z+(A_{2,z1}K_1^{-1}\hat{K}\\ &\quad+A_{21,z})\bar{\eta}_1+(D_{2,z}+A_{21,z}\bar{E}_1)w\\ \dot{\bar{\eta}}_1&=\hat{K}^{-1}\big(C_1(\bar{A}_1x+\bar{B}_1\bar{\eta}_1+(\bar{D}_1+\bar{B}_1\bar{E}_1)w)\\ &\quad+K_1\big(A_{2,1z}\eta_z+a_{2,11}K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)+b_2u\\ &\quad+D_{2,1}w+A_{21,1}(\bar{\eta}_1+\bar{E}_1w)\big)\big) &\text{(6c)}\\ &=:\bar{A}_{2,1x}x+\bar{A}_{2,1z}\eta_z+\bar{a}_{2,11}\bar{\eta}_1+\hat{K}^{-1}K_1b_2u+\bar{D}_{2,1}w\\ y_1&=\bar{\eta}_1+\bar{E}_1w &\text{(6d)}\end{aligned}$$

Clearly, (6) admits RD $1=r_1+r_2$ from u to y_1 since $\hat{K}^{-1}K_1b_2\ne 0$; and is in EZDCF (2) of [1] with (6a) and (6b) defining the EZD for S.

Now, we will show that (6) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w\ge 0$, $\forall\xi_{z0}:=[x_0'\,\eta_{z0}']'\in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}|\le c_w$, where $P_z:\mathbb{R}^{n_1+n_2}\to\mathbb{R}^{n_1+n_2-1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_1+n_2-1 coordinates, $\forall w_{[0,\infty)}\in\mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty\le c_w$, $\forall\bar{\eta}_{1[0,\infty)}\in\mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty\le c_w$, let $x_{[0,\infty)}$ and $\eta_{z[0,\infty)}$ be the solution to (6a) and (6b). By Lemma 5 of [5], $\exists c_1\in[0,\infty)\subset\mathbb{R}$, which depends only on $\mathcal{D}_{x0}\times\mathcal{D}_{\eta0}$, such that $\exists\acute{\xi}_0:=\left[\xi_{z0}'\,\acute{\bar{\eta}}_{1,0}\right]'\in\bar{\mathcal{D}}_0$ with $|\acute{\xi}_0|\le c_1c_w$. Then, $\xi_{z0}=P_zT_1\acute{\xi}_0\in P_z(\mathcal{D}_{x0}\times\mathcal{D}_{\eta0})=\mathcal{D}_{x0}\times P_{\eta z}(\mathcal{D}_{\eta0})$ and $|\xi_{z0}|\le c_w$, where $P_{\eta z}:\mathbb{R}^{n_2}\to\mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first n_2-1 coordinates. Hence, $x_0\in\mathcal{D}_{x0}$ and $\eta_{z0}\in P_{\eta z}(\mathcal{D}_{\eta0})$. Note that $y_{1[0,\infty)}\in\mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty\le c_w+\|\bar{E}_1\|_{2,2}c_w=:c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then $\exists c_{c1}\ge 0$, which depends only on c_w and c_{w1}, such that $\|x_{[0,\infty)}\|_\infty\le c_{c1}$. Note that $\eta_{1[0,\infty)}\in\mathcal{C}$ with $\|\eta_{1[0,\infty)}\|_\infty\le|K_1^{-1}\hat{K}|c_w+\|K_1^{-1}C_1\|_{2,2}c_{c1}=:c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta0}$ and $\mathcal{W}_d\times\mathcal{C}([0,\infty),\mathbb{R})$, then $\exists c_{c2}\ge 0$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{z[0,\infty)}\|_\infty\le c_{c2}$. Therefore, $\|\xi_{z[0,\infty)}\|_\infty\le\sqrt{c_{c1}^2+c_{c2}^2}$, where $\xi_z=[x'\,\eta_z']'$. This proves that (6) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Proposition 1 of [1]. This proves Case 2.

Case 3: $0=r_1<n_1$ and $2\le r_2<n_2$. By Lemmas 3 and 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1\ne 0$, $b_2\ne 0$,

$$S_1:\begin{cases}\dot{x}=\bar{A}_1x+\bar{B}_1(y_1-E_1w)+D_1w\\ \quad=:\bar{A}_1x+\bar{B}_1y_1+\bar{D}_1w;\ x(0)=x_0\\ y_1=C_1x+K_1y_2+E_1w\end{cases}$$

$$S_2:\begin{cases}\dot{\eta}_z=A_{2,z}\eta_z+A_{2,z1}\eta_1+D_{2,z}w+A_{21,z}y_1\\ \dot{\eta}_i=a_{2,i1}\eta_1+\eta_{i+1}+D_{2,i}w+A_{21,i}y_1;\ 1\le i<r_2\\ \dot{\eta}_{r_2}=A_{2,r_2z}\eta_z+a_{2,r_21}\eta_1+b_2u+D_{2,r_2}w+A_{21,r_2}y_1\\ y_2=\eta_1+E_2w+K_{21}y_1\end{cases}$$

where $\eta=[\eta_z'\,\eta_1\cdots\eta_{r_2}]'$; $\eta_z\in\mathbb{R}^{n_2-r_2}$; $\eta_i\in\mathbb{R}$, $1\le i\le r_2$; $\eta(0)=\eta_0:=[\eta_{z0}'\,\eta_{1,0}\cdots\eta_{r_2,0}]'\in\mathcal{D}_{\eta0}$. Note that

$$y_1=C_1x+K_1\left(\eta_1+E_2w+K_{21}y_1\right)+E_1w\qquad\Rightarrow$$
$$y_1=\hat{K}^{-1}\left(C_1x+K_1\eta_1+(E_1+K_1E_2)w\right)=:\bar{\eta}_1+\bar{E}_1w$$

Then, $\eta_1=K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)$. Define $\bar{\eta}_i=\hat{K}^{-1}K_1\eta_i$, $i=2,\ldots,r_2$. Then, S admits the following representation, in $\xi=T_1^{-1}[x'\,\eta']'=[x'\,\eta_z'\,\bar{\eta}_1\cdots\bar{\eta}_{r_2}]'$ coordinates with $\xi(0)=\xi_0:=[x_0'\,\eta_{z0}'\,\bar{\eta}_{1,0}\cdots\bar{\eta}_{r_2,0}]'\in\bar{\mathcal{D}}_0:=T_1^{-1}(\mathcal{D}_{x0}\times\mathcal{D}_{\eta0})\ne\emptyset$,

$$\begin{aligned}\dot{x}&=\bar{A}_1x+\bar{B}_1y_1+\bar{D}_1w\\ &=\bar{A}_1x+\bar{B}_1\bar{\eta}_1+(\bar{D}_1+\bar{B}_1\bar{E}_1)w &\text{(7a)}\\ \dot{\eta}_z&=A_{2,z}\eta_z+A_{2,z1}K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)+D_{2,z}w\\ &\quad+A_{21,z}(\bar{\eta}_1+\bar{E}_1w)\\ &=-A_{2,z1}K_1^{-1}C_1x+A_{2,z}\eta_z+(A_{2,z1}K_1^{-1}\hat{K}\\ &\quad+A_{21,z})\bar{\eta}_1+(D_{2,z}+A_{21,z}\bar{E}_1)w &\text{(7b)}\\ \dot{\bar{\eta}}_1&=\hat{K}^{-1}\left(C_1(\bar{A}_1x+\bar{B}_1\bar{\eta}_1+(\bar{D}_1+\bar{B}_1\bar{E}_1)w)+K_1\right.\\ &\quad\cdot\left(a_{2,11}K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)+\eta_2+D_{2,1}w+A_{21,1}(\bar{\eta}_1\right.\\ &\quad\left.\left.+\bar{E}_1w)\right)\right)=:\bar{A}_{2,1x}x+\bar{a}_{2,11}\bar{\eta}_1+\bar{\eta}_2+\bar{D}_{2,1}w &\text{(7c)}\\ \dot{\bar{\eta}}_i&=\hat{K}^{-1}K_1\left(a_{2,i1}K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)+\eta_{i+1}+D_{2,i}w\right.\\ &\quad\left.+A_{21,i}(\bar{\eta}_1+\bar{E}_1w)\right) &\text{(7d)}\\ &=:\bar{A}_{2,ix}x+\bar{a}_{2,i1}\bar{\eta}_1+\bar{\eta}_{i+1}+\bar{D}_{2,i}w;\ 2\le i<r_2\\ \dot{\bar{\eta}}_{r_2}&=\hat{K}^{-1}K_1\left(A_{2,r_2z}\eta_z+a_{2,r_21}K_1^{-1}(\hat{K}\bar{\eta}_1-C_1x)+b_2u\right.\\ &\quad\left.+D_{2,r_2}w+A_{21,r_2}(\bar{\eta}_1+\bar{E}_1w)\right) &\text{(7e)}\\ &=:\bar{A}_{2,r_2x}x+\bar{A}_{2,r_2z}\eta_z+\bar{a}_{2,r_21}\bar{\eta}_1+\hat{K}^{-1}K_1b_2u+\bar{D}_{2,r_2}w\end{aligned}$$

$$y_1 = \bar{\eta}_1 + \bar{E}_1 w \tag{7f}$$

Clearly, (7) admits RD $r_2 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1 b_2 \neq 0$; and is in the form (1) of [5] with $m_o = 1$.

Now, we will show that (7) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by applying Lemma 1 of [5]. $\forall c_w \geq 0$, $\forall \xi_{z0} := [x_0' \; \eta_{z0}']' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1+n_2-r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{[0,\infty)}$ and $\eta_{z[0,\infty)}$ be the solution to (7a) and (7b). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \left[\xi_{z0}' \; \acute{\bar{\eta}}_{1,0} \cdots \acute{\bar{\eta}}_{r_2,0}\right]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \leq c_1 c_w$. Then, $\xi_{z0} = P_z T_1 \acute{\xi}_0 \in P_z(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) = \mathcal{D}_{x0} \times P_{\eta z}(\mathcal{D}_{\eta 0})$ and $|\xi_{z0}| \leq c_w$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 - r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - r_2$ coordinates. Hence, $x_0 \in \mathcal{D}_{x0}$ and $\eta_{z0} \in P_{\eta z}(\mathcal{D}_{\eta 0})$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty \leq c_w + \|\bar{E}_1\|_{2,2} c_w =: c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \geq 0$, which depends only on c_w and c_{w1}, such that $\|x_{[0,\infty)}\|_\infty \leq c_{c1}$. Note that $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\|\eta_{1[0,\infty)}\|_\infty \leq |K_1^{-1}\hat{K}| c_w + \|K_1^{-1} C_1\|_{2,2} c_{c1} =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty), \mathbb{R})$, then, $\exists c_{c2} \geq 0$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{z[0,\infty)}\|_\infty \leq c_{c2}$. Therefore, $\|\xi_{z[0,\infty)}\|_\infty \leq \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = [x' \; \eta_z']'$. Then, (7) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This proves Case 3.

Case 4: $0 = r_1 < n_1$ and $1 = r_2 = n_2$. By Lemmas 3 and 2 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} = \bar{A}_1 x + \bar{B}_1(y_1 - E_1 w) + D_1 w \\ \quad =: \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0 \\ y_1 = C_1 x + K_1 y_2 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_1 = a_{2,11}\eta_1 + b_2 u + D_{2,1} w + A_{21,1} y_1 \\ y_2 = \eta_1 + E_2 w + K_{21} y_1 \end{cases}$$

where $\eta = \eta_1$ is a scalar. Note that

$$y_1 = C_1 x + K_1(\eta_1 + E_2 w + K_{21} y_1) + E_1 w \qquad \Rightarrow$$
$$y_1 = \hat{K}^{-1}(C_1 x + K_1 \eta_1 + (E_1 + K_1 E_2) w) =: \bar{\eta}_1 + \bar{E}_1 w$$

Then, $\eta_1 = K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x)$, and S admits the following representation, in $\xi = [x' \; \bar{\eta}_1]' = T_1^{-1}[x' \; \eta_1]'$ coordinates with $\xi(0) = \xi_0 := [x_0' \; \bar{\eta}_{1,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta_0})$,

$$\dot{x} = \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w = \bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w \tag{8a}$$
$$\begin{aligned} \dot{\bar{\eta}}_1 = {} & \hat{K}^{-1}\big(C_1(\bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w) + K_1\big(a_{2,11} \cdot \\ & K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x) + b_2 u + D_{2,1} w + A_{21,1}(\bar{\eta}_1 + \bar{E}_1 w)\big)\big) \\ =: {} & \bar{A}_{2,1x} x + \bar{a}_{2,11}\bar{\eta}_1 + \hat{K}^{-1} K_1 b_2 u + \bar{D}_{2,1} w \end{aligned} \tag{8b}$$
$$y_1 = \bar{\eta}_1 + \bar{E}_1 w \tag{8c}$$

Clearly, (8) admits RD $1 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1 b_2 \neq 0$; and is in EZDCF (2) of [1]. Hence, (8a) defines the EZD for S.

Now, we will show that (8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \geq 0$, $\forall x_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|x_0| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_1 coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{[0,\infty)}$ be the solution to (8a). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \left[x_0' \; \acute{\bar{\eta}}_{1,0}\right]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \leq c_1 c_w$. Then, $x_0 = P_z T_1 \acute{\xi}_0 \in P_z(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) = \mathcal{D}_{x0}$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty \leq c_w + \|\bar{E}_1\|_{2,2} c_w =: c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then $\exists c_{c1} \geq 0$, which depends only on c_w and c_{w1}, such that $\|x_{[0,\infty)}\|_\infty \leq c_{c1}$. By Proposition 1 of [1], (8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This proves Case 4.

Case 5: $0 = r_1 < n_1$ and $2 \leq r_2 = n_2$. By Lemmas 3 and 2 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} = \bar{A}_1 x + \bar{B}_1(y_1 - E_1 w) + D_1 w \\ \quad =: \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0 \\ y_1 = C_1 x + K_1 y_2 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i} w + A_{21,i} y_1; \; 1 \leq i < r_2 \\ \dot{\eta}_{r_2} = a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2} w + A_{21,r_2} y_1 \\ y_2 = \eta_1 + E_2 w + K_{21} y_1 \end{cases}$$

where $\eta = [\eta_1 \cdots \eta_{r_2}]'$; $\eta_i \in \mathbb{R}$, $1 \leq i \leq r_2$; and $\eta(0) = \eta_0 := [\eta_{1,0} \cdots \eta_{r_2,0}]' \in \mathcal{D}_{\eta 0}$. Note that

$$y_1 = C_1 x + K_1(\eta_1 + E_2 w + K_{21} y_1) + E_1 w \qquad \Rightarrow$$
$$y_1 = \hat{K}^{-1}(C_1 x + K_1 \eta_1 + (E_1 + K_1 E_2) w) =: \bar{\eta}_1 + \bar{E}_1 w$$

Then, $\eta_1 = K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x)$. Define $\bar{\eta}_i = \hat{K}^{-1}K_1 \eta_i$, $2 \leq i \leq r_2$. Then, S admits the following representation, in $\xi = T_1^{-1}[x' \; \eta']' = [x' \; \bar{\eta}_1 \cdots \bar{\eta}_{r_2}]'$ coordinates with $\xi(0) = \xi_0 := [x_0' \; \bar{\eta}_{1,0} \cdots \bar{\eta}_{r_2,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\dot{x} = \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w = \bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w \tag{9a}$$
$$\begin{aligned} \dot{\bar{\eta}}_1 = {} & \hat{K}^{-1}\big(C_1(\bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w) + K_1\big(a_{2,11} \\ & K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x) + \eta_2 + D_{2,1} w + A_{21,1}(\bar{\eta}_1 + \bar{E}_1 w)\big)\big) \\ =: {} & \bar{A}_{2,1x} x + \bar{a}_{2,11}\bar{\eta}_1 + \bar{\eta}_2 + \bar{D}_{2,1} w \end{aligned} \tag{9b}$$
$$\begin{aligned} \dot{\bar{\eta}}_i = {} & \hat{K}^{-1}K_1\big(a_{2,i1}K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x) + \eta_{i+1} + D_{2,i} w \\ & + A_{21,i}(\bar{\eta}_1 + \bar{E}_1 w)\big) \\ =: {} & \bar{A}_{2,ix} x + \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i} w; \; 2 \leq i < r_2 \end{aligned} \tag{9c}$$
$$\begin{aligned} \dot{\bar{\eta}}_{r_2} = {} & \hat{K}^{-1}K_1\,\big(a_{2,r_2 1}K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x) + b_2 u + D_{2,r_2} w \\ & + A_{21,r_2}(\bar{\eta}_1 + \bar{E}_1 w)\big) \\ =: {} & \bar{A}_{2,r_2 x} x + \bar{a}_{2,r_2 1}\bar{\eta}_1 + \hat{K}^{-1}K_1 b_2 u + \bar{D}_{2,r_2} w \end{aligned} \tag{9d}$$
$$y_1 = \bar{\eta}_1 + \bar{E}_1 w \tag{9e}$$

Clearly, (9) admits RD $r_2 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1 b_2 \neq 0$; and is in the form (1) of [5] with $m_o = 1$.

Now, we will show that (9) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by applying Lemma 1 of [5]. $\forall c_w \geq 0$, $\forall x_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|x_0| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_1 coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{[0,\infty)}$ be the solution to (9a). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \left[x_0' \; \acute{\bar{\eta}}_{1,0} \cdots \acute{\bar{\eta}}_{r_2,0}\right]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \leq c_1 c_w$. Then, $x_0 = P_z T_1 \acute{\xi}_0 \in P_z(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) = \mathcal{D}_{x0}$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty \leq c_w + \|\bar{E}_1\|_{2,2} c_w =: c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then $\exists c_{c1} \geq 0$, which depends only on c_w and c_{w1}, such that $\|x_{[0,\infty)}\|_\infty \leq c_{c1}$. Then, (9) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This proves Case 5.

Case 6: $1 \leq r_1 < n_1$ and $0 = r_2 < n_2$. By Lemmas 1 and 3 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x}_z = A_{1,z} x_z + A_{1,z1} x_1 + D_{1,z} w \\ \dot{x}_i = a_{1,i1} x_1 + x_{i+1} + D_{1,i} w; \; 1 \leq i < r_1 \\ \dot{x}_{r_1} = A_{1,r_1 z} x_z + a_{1,r_1 1} x_1 + b_1 y_2 + D_{1,r_1} w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta} = \bar{A}_2 \eta + \bar{B}_2(y_2 - E_2 w - K_{21} y_1) + D_2 w + A_{21} y_1 \\ \quad =: \bar{A}_2 \eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21} y_1; \; \eta(0) = \eta_0 \\ y_2 = C_2 \eta + K_2 u + E_2 w + K_{21} y_1 \end{cases}$$

where $x = [x_z' \; x_1 \cdots x_{r_1}]'$; $x_z \in \mathbb{R}^{n_1 - r_1}$; $x_i \in \mathbb{R}$, $1 \leq i \leq r_1$; $x(0) = x_0 := [x_{z0}' \; x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$. The composite system $\bar{S}$ with state vector $[x' \; \eta']'$, y_1 as output, and y_2 as input admits the following representation, in $\xi :=$

$[x_z'\,\eta'\,x_1 \cdots x_{r_1}]' = T_1^{-1}[x'\,\eta']'$ coordinates with $\xi(0) = \xi_0 := [x_{z0}'\,\eta_0'\,x_{1,0} \cdots x_{r_1,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\dot{x}_z = \bar{A}_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \tag{10a}$$

$$\begin{aligned}\dot{\eta} &= \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1 \\ &= \bar{A}_2\eta + \bar{A}_{21}x_1 + \bar{B}_2 b_1^{-1}(b_1 y_2) + (\bar{D}_2 + \bar{A}_{21}E_1)w\end{aligned} \tag{10b}$$

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \le i < r_1 \tag{10c}$$

$$\dot{x}_{r_1} = A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \tag{10d}$$

$$y_1 = x_1 + E_1 w \tag{10e}$$

(10) is in the form (1) with $m_{oa} = 1$ and $m_{ob} = r_1 + 1$.

We will apply Lemma 1 to prove that (10) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \ge 0$, $\forall \xi_{z0} := [x_{z0}'\,\eta_0']' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1+n_2-r_1$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \le c_w$, $1 \le i \le r_1 + 1$ ($x_{r_1+1} := b_1 y_2$ for notational consistency), let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (10a) and (10b). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := [\xi_{z0}'\,\acute{x}_{1,0} \cdots \acute{x}_{r_1,0}]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \le c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $[x_{z0}'\,\acute{x}_{1,0} \cdots \acute{x}_{r_1,0}]' \in \mathcal{D}_{x0}$, and $\eta_0 \in \mathcal{D}_{\eta 0}$. Hence, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then $\exists c_{c1} \ge 0$, which depends only on c_w, such that $\|x_{z[0,\infty)}\|_\infty \le c_{c1}$. Note that $y_{2[0,\infty)} \in \mathcal{C}$ with $\|y_{2[0,\infty)}\|_\infty \le |b_1^{-1}|c_w =: c_{w1}$; $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty \le c_w + \|E_1\|_{2,2}c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then $\exists c_{c2} \ge 0$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{[0,\infty)}\|_\infty \le c_{c2}$. Then, $\|\xi_{z[0,\infty)}\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = [x_z'\,\eta']'$. By Lemma 1, (10) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r_1 = r_1 + r_2$ from y_2 to y_1.

By Lemma 1 of [1], $\bar{S}$ admits EZDCF, in $\hat{\xi} = [\hat{\xi}_z'\,\hat{\xi}_1 \cdots \hat{\xi}_{r_1}]' = T_2^{-1}[x'\,\eta']'$ coordinates with $\hat{\xi}_z \in \mathbb{R}^{n_2+n_1-r_1}$; $\hat{\xi}_i \in \mathbb{R}$, $1 \le i \le r_1$; $b \ne 0$; $\hat{\xi}(0) = \hat{\xi}_0 := [\hat{\xi}_{z0}'\,\hat{\xi}_{1,0} \cdots \hat{\xi}_{r_1,0}]' \in \hat{\mathcal{D}}_0 := T_2^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\dot{\hat{\xi}}_z = A_z\hat{\xi}_z + A_{z1}\hat{\xi}_1 + D_z w \tag{11a}$$

$$\dot{\hat{\xi}}_i = a_{i1}\hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w;\ 1 \le i < r_1 \tag{11b}$$

$$\dot{\hat{\xi}}_{r_1} = A_{r_1 z}\hat{\xi}_z + a_{r_1 1}\hat{\xi}_1 + b y_2 + D_{r_1}w \tag{11c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{11d}$$

In $\hat{\xi}$ coordinates, we have $y_2 = \hat{C}_z\hat{\xi}_z + \hat{C}_1\hat{\xi}_1 + \cdots + \hat{C}_{r_1}\hat{\xi}_{r_1} + K_2 u + (E_2 + K_{21}E_1)w$. Then, S with state vector $[x'\,\eta']'$, y_1 as output, and u as input admits the following representation, in $\hat{\xi}$ coordinates, with $bK_2 \ne 0$,

$$\dot{\hat{\xi}}_z = A_z\hat{\xi}_z + A_{z1}\hat{\xi}_1 + D_z w \tag{12a}$$

$$\dot{\hat{\xi}}_i = a_{i1}\hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w; \qquad 1 \le i < r_1 \tag{12b}$$

$$\dot{\hat{\xi}}_{r_1} = \hat{A}_{r_1 z}\hat{\xi}_z + \hat{a}_{r_1 1}\hat{\xi}_1 + \cdots + \hat{a}_{r_1 r_1}\hat{\xi}_{r_1} + bK_2 u + \hat{D}_{r_1}w \tag{12c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{12d}$$

Clearly, (12) is in the form of (1) of [5] with $m_o = 1$. By the fact that (10) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$, then (11) is minimum phase with respect to $\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$. By Proposition 1 of [1], $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall \hat{\xi}_{z0} \in P_z(\hat{\mathcal{D}}_0)$ with $|\hat{\xi}_{z0}| \le c_w$, $\forall \hat{\xi}_{1[0,\infty)} \in \mathcal{C}$ with $\|\hat{\xi}_{1[0,\infty)}\|_\infty \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, we have $\|\hat{\xi}_{z[0,\infty)}\|_\infty \le c_c$, where $\hat{\xi}_{z[0,\infty)}$ is the solution to (11a) with initial condition $\hat{\xi}_{z0}$ and inputs $\hat{\xi}_{1[0,\infty)}$ and $w_{[0,\infty)}$. Then, the assumption for Lemma 1 of [5] is satisfied for (12). Therefore, (12) is minimum phase with respect to $\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r_1 = r_1 + r_2$ from u to y_1. This proves Case 6.

Case 7: $1 \le r_1 = n_1$ and $0 = r_2 < n_2$. By Lemmas 2 and 3 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \ne 0$, $K_2 \ne 0$,

$$S_1 : \begin{cases} \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \le i < r_1 \\ \dot{x}_{r_1} = a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta} = \bar{A}_2\eta + \bar{B}_2(y_2 - E_2 w - K_{21}y_1) + D_2 w + A_{21}y_1 \\ \quad =: \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1;\ \eta(0) = \eta_0 \\ y_2 = C_2\eta + K_2 u + E_2 w + K_{21}y_1 \end{cases}$$

where $x = [x_1 \cdots x_{r_1}]'$; $x_i \in \mathbb{R}$, $1 \le i \le r_1$; $x(0) = x_0 := [x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$. The composite system $\bar{S}$ with state vector $[x'\,\eta']'$, y_1 as output, and y_2 as input admits the following representation, in $\xi := [\eta'\,x_1 \cdots x_{r_1}]' = T_1^{-1}[x'\,\eta']'$ coordinates with $\xi(0) = \xi_0 := [\eta_0'\,x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{\eta 0} \times \mathcal{D}_{x0} =: \bar{\mathcal{D}}_0$,

$$\begin{aligned}\dot{\eta} &= \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1 \\ &= \bar{A}_2\eta + \bar{A}_{21}x_1 + \bar{B}_2 b_1^{-1}(b_1 y_2) + (\bar{D}_2 + \bar{A}_{21}E_1)w\end{aligned} \tag{13a}$$

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \le i < r_1 \tag{13b}$$

$$\dot{x}_{r_1} = a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \tag{13c}$$

$$y_1 = x_1 + E_1 w \tag{13d}$$

Clearly, (13) is in the form (1) of [5] with $m_o = r_1 + 1$.

We will apply Lemma 1 of [5] to prove that (13) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD r_1. $\forall c_w \ge 0$, $\forall \eta_0 \in P_z(\bar{\mathcal{D}}_0) = \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_2 coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \le c_w$, $1 \le i \le r_1 + 1$ ($x_{r_1+1} := b_1 y_2$ for notational consistency), let $\eta_{[0,\infty)}$ be the solution to (13a). Note that $y_{2[0,\infty)} \in \mathcal{C}$ with $\|y_{2[0,\infty)}\|_\infty \le |b_1^{-1}|c_w =: c_{w1}$; $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty \le c_w + \|E_1\|_{2,2}c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then $\exists c_{c2} \ge 0$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{[0,\infty)}\|_\infty \le c_{c2}$. Then, (13) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r_1 = r_1 + r_2$ from y_2 to y_1.

By Lemma 1 of [1], $\bar{S}$ admits the following EZDCF, in $\hat{\xi} = [\hat{\xi}_z'\,\hat{\xi}_1 \cdots \hat{\xi}_{r_1}]' = T_2^{-1}[x'\,\eta']'$ coordinates, where $\hat{\xi}_z \in \mathbb{R}^{n_2}$; $\hat{\xi}_i \in \mathbb{R}$, $1 \le i \le r_1$; $b \ne 0$; $\hat{\xi}(0) = \hat{\xi}_0 := [\hat{\xi}_{z0}'\,\hat{\xi}_{1,0} \cdots \hat{\xi}_{r_1,0}]' \in \hat{\mathcal{D}}_0 := T_2^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\dot{\hat{\xi}}_z = A_z\hat{\xi}_z + A_{z1}\hat{\xi}_1 + D_z w \tag{14a}$$

$$\dot{\hat{\xi}}_i = a_{i1}\hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w;\ 1 \le i < r_1 \tag{14b}$$

$$\dot{\hat{\xi}}_{r_1} = A_{r_1 z}\hat{\xi}_z + a_{r_1 1}\hat{\xi}_1 + b y_2 + D_{r_1}w \tag{14c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{14d}$$

In $\hat{\xi}$ coordinates, we have $y_2 = \hat{C}_z\hat{\xi}_z + \hat{C}_1\hat{\xi}_1 + \cdots + \hat{C}_{r_1}\hat{\xi}_{r_1} + K_2 u + (E_2 + K_{21}E_1)w$. Then, S with state vector $[x'\,\eta']'$, y_1 as output, and u as input admits the following representation, in $\hat{\xi}$ coordinates, with $bK_2 \ne 0$,

$$\dot{\hat{\xi}}_z = A_z\hat{\xi}_z + A_{z1}\hat{\xi}_1 + D_z w \tag{15a}$$

$$\dot{\hat{\xi}}_i = a_{i1}\hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w; \qquad 1 \le i < r_1 \tag{15b}$$

$$\dot{\hat{\xi}}_{r_1} = \hat{A}_{r_1 z}\hat{\xi}_z + \hat{a}_{r_1 1}\hat{\xi}_1 + \cdots + \hat{a}_{r_1 r_1}\hat{\xi}_{r_1} + bK_2 u + \hat{D}_{r_1}w \tag{15c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{15d}$$

Clearly, (15) is in the form of (1) of [5] with $m_o = 1$. By the fact that (13) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$, then (14) is minimum phase with respect to

$\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$. By Proposition 1 of [1], $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall \hat{\xi}_{z0} \in P_z(\hat{\mathcal{D}}_0)$ with $|\hat{\xi}_{z0}| \leq c_w$, $\forall \hat{\xi}_{1[0,\infty)} \in \mathcal{C}$ with $\|\hat{\xi}_{1[0,\infty)}\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, we have $\|\hat{\xi}_{z[0,\infty)}\|_\infty \leq c_c$, where $\hat{\xi}_{z[0,\infty)}$ is the solution to (14a) with initial condition $\hat{\xi}_{z0}$ and inputs $\hat{\xi}_{1[0,\infty)}$ and $w_{[0,\infty)}$. Then, the assumption for Lemma 1 of [5] is satisfied for (15). Hence, (15) is minimum phase with respect to $\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r_1 = r_1 + r_2$ from u to y_1. This proves Case 7.

Case 8: $1 \leq r_1 = n_1$ and $1 \leq r_2 = n_2$. By Lemma 2 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \neq 0$, $b_2 \neq 0$, with $x = [x_1 \cdots x_{r_1}]'$; $x_i \in \mathbb{R}$, $1 \leq i \leq r_1$; $\eta = [\eta_1 \cdots \eta_{r_2}]'$; $\eta_i \in \mathbb{R}$, $1 \leq i \leq r_2$.

$$S_1: \begin{cases} \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \leq i < r_1 \\ \dot{x}_{r_1} = a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2: \begin{cases} \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1;\ 1 \leq i < r_2 \\ \dot{\eta}_{r_2} = a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2 w + K_{21}y_1 \end{cases}$$

It is straightforward to check that S admits RD $r_1 + r_2$ from u to y_1 since $b_1 b_2 \neq 0$. Since the RD equals to the dimension of system, which implies that the EZD is absent by Lemma 2 of [1], then S is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This proves Case 8.

Case 9: $1 \leq r_1 < n_1$ and $1 \leq r_2 = n_2$. By Lemmas 1 and 2 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \neq 0$, $b_2 \neq 0$, with $x = [x_z' \, x_1 \cdots x_{r_1}]'$; $x_z \in \mathbb{R}^{n_1 - r_1}$; $x_i \in \mathbb{R}$, $1 \leq i \leq r_1$; $x(0) = x_0 := [x_{z0}' \, x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$; $\eta = [\eta_1 \cdots \eta_{r_2}]'$; $\eta_i \in \mathbb{R}$, $1 \leq i \leq r_2$; $\eta(0) = \eta_0 := [\eta_{1,0} \cdots \eta_{r_2,0}]' \in \mathcal{D}_{\eta 0}$,

$$S_1: \begin{cases} \dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \\ \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \leq i < r_1 \\ \dot{x}_{r_1} = A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2: \begin{cases} \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1;\ 1 \leq i < r_2 \\ \dot{\eta}_{r_2} = a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2 w + K_{21}y_1 \end{cases}$$

Define $\bar{\eta}_i = b_1 \eta_i$, $1 \leq i \leq r_2$. Then, S admits the following representation, in $\xi = [x_z' \, x_1 \cdots x_{r_1} \, \bar{\eta}_1 \cdots \bar{\eta}_{r_2}]' = T_1^{-1}[x' \, \eta']'$ coordinates, with $\xi(0) = \xi_0 := [x_{z0}' \, x_{1,0} \cdots x_{r_1,0} \, \bar{\eta}_{1,0} \cdots \bar{\eta}_{r_2,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$,

$$\dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \tag{16a}$$
$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \leq i < r_1 \tag{16b}$$
$$\begin{aligned}\dot{x}_{r_1} &= A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1 (b_1^{-1}\bar{\eta}_1 + E_2 w + K_{21}(x_1 \\ &\quad + E_1 w)) + D_{1,r_1}w \\ &=: A_{1,r_1 z}x_z + \bar{a}_{1,r_1 1}x_1 + \bar{\eta}_1 + \bar{D}_{1,r_1}w \end{aligned} \tag{16c}$$
$$\begin{aligned}\dot{\bar{\eta}}_i &= b_1(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}(x_1 + E_1 w)) \\ &=: \bar{A}_{21,i}x_1 + \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w;\ 1 \leq i < r_2 \end{aligned} \tag{16d}$$
$$\begin{aligned}\dot{\bar{\eta}}_{r_2} &= b_1 (a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}(x_1 + E_1 w)) \\ &=: \bar{A}_{21,r_2}x_1 + \bar{a}_{2,r_2 1}\bar{\eta}_1 + b_1 b_2 u + \bar{D}_{2,r_2}w \end{aligned} \tag{16e}$$
$$y_1 = x_1 + E_1 w \tag{16f}$$

Clearly, (16) is in the form (1) of [5] with $m_o = 1$.

We will apply Lemma 1 of [5] to prove this case. $\forall c_w \geq 0$, $\forall x_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|x_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1 - r_1$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall x_{1[0,\infty)} \in \mathcal{C}$ with $\|x_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{z[0,\infty)}$ be the solution to (16a). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := [x_{z0}' \, \acute{\xi}_{1,0} \cdots \acute{\xi}_{n_2+r_1,0}]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \leq c_1 c_w$. Then, $T_1 \acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $[x_{z0}' \, \acute{\xi}_{1,0} \cdots \acute{\xi}_{r_1,0}]' \in \mathcal{D}_{x0}$. Hence, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1 - r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then $\exists c_{c1} \geq 0$, which depends only on c_w, such that $\|x_{z[0,\infty)}\|_\infty \leq c_{c1}$. Then, (16) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r_1 + r_2$ from u to y_1. This proves Case 9.

Case 10: $1 \leq r_1 = n_1$ and $1 \leq r_2 < n_2$. By Lemmas 2 and 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \neq 0$, $b_2 \neq 0$, with $x = [x_1 \cdots x_{r_1}]'$; $x_i \in \mathbb{R}$, $1 \leq i \leq r_1$; $x(0) = x_0 := [x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$; $\eta = [\eta_z' \, \eta_1 \cdots \eta_{r_2}]'$; $\eta_z \in \mathbb{R}^{n_2 - r_2}$; $\eta_i \in \mathbb{R}$, $1 \leq i \leq r_2$; $\eta(0) = \eta_0 := [\eta_{z0}' \, \eta_{1,0} \cdots \eta_{r_2,0}]' \in \mathcal{D}_{\eta 0}$,

$$S_1: \begin{cases} \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \leq i < r_1 \\ \dot{x}_{r_1} = a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2: \begin{cases} \dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\ \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1;\ 1 \leq i < r_2 \\ \dot{\eta}_{r_2} = A_{2,r_2 z}\eta_z + a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2 w + K_{21}y_1 \end{cases}$$

Define $\bar{\eta}_i = b_1 \eta_i$, $1 \leq i \leq r_2$. Then, S admits the following representation, in $\xi = [\eta_z' \, x_1 \cdots x_{r_1} \, \bar{\eta}_1 \cdots \bar{\eta}_{r_2}]' = T_1^{-1}[x' \, \eta']'$ coordinates, with $\xi(0) = \xi_0 = [\eta_{z0}' \, x_{1,0} \cdots x_{r_1,0} \, \bar{\eta}_{1,0} \cdots \bar{\eta}_{r_2,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\begin{aligned}\dot{\eta}_z &= A_{2,z}\eta_z + A_{21,z}x_1 + A_{2,z1}b_1^{-1}\bar{\eta}_1 \\ &\quad + (D_{2,z} + A_{21,z}E_1)w \end{aligned} \tag{17a}$$
$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \leq i < r_1 \tag{17b}$$
$$\begin{aligned}\dot{x}_{r_1} &= (a_{1,r_1 1} + b_1 K_{21})x_1 + \bar{\eta}_1 + (D_{1,r_1} + b_1 E_2 + b_1 K_{21} \\ &\quad \cdot E_1)w =: \bar{a}_{1,r_1 1}x_1 + \bar{\eta}_1 + \bar{D}_{1,r_1}w \end{aligned} \tag{17c}$$
$$\begin{aligned}\dot{\bar{\eta}}_i &= b_1 (a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}(x_1 + E_1 w)) \\ &=: b_1 A_{21,i}x_1 + \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w;\ 1 \leq i < r_2 \end{aligned} \tag{17d}$$
$$\begin{aligned}\dot{\bar{\eta}}_{r_2} &= b_1 (A_{2,r_2 z}\eta_z + a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2} \\ &\quad (x_1 + E_1 w)) =: b_1 A_{2,r_2 z}\eta_z + b_1 A_{21,r_2}x_1 \\ &\quad + \bar{a}_{2,r_2 1}\bar{\eta}_1 + b_1 b_2 u + \bar{D}_{2,r_2}w \end{aligned} \tag{17e}$$
$$y_1 = x_1 + E_1 w \tag{17f}$$

Clearly, (17) is in the form of (1) of [5] with $m_o = r_1 + 1$.

We will apply Lemma 1 of [5] to prove this case. $\forall c_w \geq 0$, $\forall \eta_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2 - r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_2 - r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \leq c_w$, $1 \leq i \leq r_1$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty \leq c_w$, let $\eta_{z[0,\infty)}$ be the solution to (17a). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := [\eta_{z0}' \, \acute{\xi}_{1,0} \cdots \acute{\xi}_{n_1+r_2,0}]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \leq c_1 c_w$. Then, $T_1 \acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $[\eta_{z0}' \, b_1^{-1}\acute{\xi}_{n_1+1,0} \cdots b_1^{-1}\acute{\xi}_{n_1+r_2,0}]' \in \mathcal{D}_{\eta 0}$. Hence, $\eta_{z0} \in P_{\eta z}(\mathcal{D}_{\eta 0})$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 - r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - r_2$ coordinates. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty = \|x_{1[0,\infty)} + E_1 w_{[0,\infty)}\|_\infty \leq c_w + \|E_1\|_{2,2}c_w =: c_{w1}$ and $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\|\eta_{1[0,\infty)}\|_\infty = \|b_1^{-1}\bar{\eta}_{1[0,\infty)}\|_\infty \leq |b_1^{-1}|c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then $\exists c_{c1} \geq 0$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{z[0,\infty)}\|_\infty \leq c_{c1}$. By Lemma 1 of [5], (17) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r_1 + r_2$ from u to y_1. This proves Case 10.

Case 11: $1 \leq r_1 < n_1$ and $1 \leq r_2 < n_2$. By Lemma 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \neq 0$, $b_2 \neq 0$, with $x = [x_z' \, x_1 \cdots x_{r_1}]'$; $x_z \in \mathbb{R}^{n_1 - r_1}$; $x_i \in \mathbb{R}$, $1 \leq i \leq r_1$; $x(0) = x_0 := [x_{z0}' \, x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$;

$\eta = [\eta_z' \, \eta_1 \cdots \eta_{r_2}]'$; $\eta_z \in \mathbb{R}^{n_2 - r_2}$; $\eta_i \in \mathbb{R}$, $1 \le i \le r_2$; $\eta(0) = \eta_0 := [\eta_{z0}' \, \eta_{1,0} \cdots \eta_{r_2,0}]' \in \mathcal{D}_{\eta 0}$,

$$S_1: \begin{cases} \dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \\ \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w;\ 1 \le i < r_1 \\ \dot{x}_{r_1} = A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2: \begin{cases} \dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\ \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1;\ 1 \le i < r_2 \\ \dot{\eta}_{r_2} = A_{2,r_2z}\eta_z + a_{2,r_21}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2 w + K_{21}y_1 \end{cases}$$

Define $\bar{\eta}_i = b_1\eta_i$, $1 \le i \le r_2$. Then, S admits the following representation, in $\xi = [x_z' \, \eta_z' \, x_1 \cdots x_{r_1} \, \bar{\eta}_1 \cdots \bar{\eta}_{r_2}]' = T_1^{-1}[x' \, \eta']'$ coordinates,

$$\dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \tag{18a}$$

$$\dot{\eta}_z = A_{2,z}\eta_z + A_{21,z}x_1 + A_{2,z1}b_1^{-1}\bar{\eta}_1 + (D_{2,z} + A_{21,z}E_1)w \tag{18b}$$

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad 1 \le i < r_1 \tag{18c}$$

$$\begin{aligned} \dot{x}_{r_1} &= A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1(\eta_1 + K_{21}x_1 + (E_2 + K_{21} \cdot E_1)w) + D_{1,r_1}w \\ &=: A_{1,r_1z}x_z + \bar{a}_{1,r_11}x_1 + \bar{\eta}_1 + \bar{D}_{1,r_1}w \end{aligned} \tag{18d}$$

$$\begin{aligned} \dot{\bar{\eta}}_i &= b_1(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}(x_1 + E_1 w)) \\ &=: b_1 A_{21,i}x_1 + \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w;\ 1 \le i < r_2 \end{aligned} \tag{18e}$$

$$\begin{aligned} \dot{\bar{\eta}}_{r_2} &= b_1(A_{2,r_2z}\eta_z + a_{2,r_21}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2} \cdot (x_1 + E_1 w)) =: b_1 A_{2,r_2z}\eta_z + b_1 A_{21,r_2}x_1 \\ &\quad + \bar{a}_{2,r_21}\bar{\eta}_1 + b_1 b_2 u + \bar{D}_{2,r_2}w \end{aligned} \tag{18f}$$

$$y_1 = x_1 + E_1 w \tag{18g}$$

where $\xi(0) = \xi_0 := [x_{z0}' \, \eta_{z0}' \, x_{1,0} \cdots x_{r_1,0} \, \bar{\eta}_{1,0} \cdots \bar{\eta}_{r_2,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$. Clearly, (18) is in the form of (1) with $m_{oa} = 1$ and $m_{ob} = r_1 + 1$.

We will apply Lemma 1 to prove this case. $\forall c_w \ge 0$, $\forall \xi_{z0} := [x_{z0}' \, \eta_{z0}']' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_1-r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1 - r_1 + n_2 - r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \le c_w$, $1 \le i \le r_1$, and $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty \le c_w$, let $x_{z[0,\infty)}$ and $\eta_{z[0,\infty)}$ be the solution to (18a) and (18b). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := [\xi_{z0}' \, \acute{\xi}_{1,0} \cdots \acute{\xi}_{r_1+r_2,0}]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \le c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $[x_{z0}' \, \acute{\xi}_{1,0} \cdots \acute{\xi}_{r_1,0}]' \in \mathcal{D}_{x0}$, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates, $[\eta_{z0}' \, b_1^{-1}\acute{\xi}_{r_1+1,0} \cdots b_1^{-1}\acute{\xi}_{r_1+r_2,0}]' \in \mathcal{D}_{\eta 0}$, and $\eta_{z0} \in P_{\eta z}(\mathcal{D}_{\eta 0})$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - r_2$ coordinates. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, such that $\|x_{z[0,\infty)}\|_\infty \le c_{c1}$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty = \|x_{1[0,\infty)} + E_1 w_{[0,\infty)}\|_\infty \le c_w + \|E_1\|_{2,2}c_w =: c_{w1}$. Note also that $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\|\eta_{1[0,\infty)}\|_\infty = \|b_1^{-1}\bar{\eta}_{1[0,\infty)}\|_\infty \le |b_1^{-1}|c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty), \mathbb{R})$, then, $\exists c_{c2} \ge 0$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{z[0,\infty)}\|_\infty \le c_{c2}$. Therefore, $\|\xi_{z[0,\infty)}\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = [x_z' \, \eta_z']'$. By Lemma 1, (18) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r_1 + r_2$ from u to y_1. This proves Case 11.

Case 12: $0 = r_1 < n_1$ and $0 = r_2 = n_2$. By Lemma 3 and Definition 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \ne 0$, $K_2 \ne 0$,

$$S_1: \begin{cases} \dot{x} = \bar{A}_1 x + \bar{B}_1(y_1 - E_1 w) + D_1 w \\ \quad =: \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w;\ x(0) = x_0 \\ y_1 = C_1 x + K_1 y_2 + E_1 w \end{cases}$$

$$S_2:\ y_2 = K_2 u + E_2 w + K_{21}y_1$$

Then, $\mathcal{D}_{\eta 0} = \mathbb{R}^0$. Note that

$$y_1 = C_1 x + K_1(K_2 u + E_2 w + K_{21}y_1) + E_1 w \quad \Rightarrow$$

$$y_1 = \hat{K}^{-1}(C_1 x + K_1 K_2 u + (E_1 + K_1 E_2)w)$$

Then, S admits the following representation

$$\dot{x} = \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0$$

$$y_1 = \hat{K}^{-1}C_1 x + \hat{K}^{-1}K_1 K_2 u + \hat{K}^{-1}(E_1 + K_1 E_2)w$$

Clearly, the above is in EZDCF (6) of [1] since $\hat{K}^{-1}K_1 K_2 \ne 0$. Hence, S admits RD $0 = r_1 + r_2$ from u to y_1 and is minimum phase with respect to $\mathcal{D}_{x0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$. This proves Case 12.

Case 13: $1 \le r_1 = n_1$ and $0 = r_2 = n_2$. By Lemma 2 and Definition 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \ne 0$, $K_2 \ne 0$, with $x = [x_1 \cdots x_{r_1}]'$; $x_i \in \mathbb{R}$, $1 \le i \le r_1$; $x(0) = x_0 := [x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$; and $\mathcal{D}_{\eta 0} = \mathbb{R}^0$,

$$S_1: \begin{cases} \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad 1 \le i < r_1 \\ \dot{x}_{r_1} = a_{1,r_11}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2:\ y_2 = K_2 u + E_2 w + K_{21}y_1$$

Then, S admits the following representation,

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad 1 \le i < r_1$$

$$\begin{aligned} \dot{x}_{r_1} &= a_{1,r_11}x_1 + b_1(K_2 u + E_2 w + K_{21}(x_1 + E_1 w)) \\ &\quad + D_{1,r_1}w =: \bar{a}_{1,r_11}x_1 + b_1 K_2 u + \bar{D}_{1,r_1}w \end{aligned}$$

$$y_1 = x_1 + E_1 w$$

Clearly, S admits RD $r_1 = r_1 + r_2 = n_1 + n_2$ from u to y_1 since $b_1 K_2 \ne 0$. Then, S is minimum phase with respect to $\mathcal{D}_{x0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since its EZD is absent. This proves Case 13.

Case 14: $1 \le r_1 < n_1$ and $0 = r_2 = n_2$. By Lemma 1 and Definition 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $b_1 \ne 0$, $K_2 \ne 0$, with $x = [x_z' \, x_1 \cdots x_{r_1}]'$; $x_z \in \mathbb{R}^{n_1 - r_1}$; $x_i \in \mathbb{R}$, $1 \le i \le r_1$; $x(0) = x_0 := [x_{z0}' \, x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$; and $\mathcal{D}_{\eta 0} = \mathbb{R}^0$,

$$S_1: \begin{cases} \dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \\ \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad 1 \le i < r_1 \\ \dot{x}_{r_1} = A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 = x_1 + E_1 w \end{cases}$$

$$S_2:\ y_2 = K_2 u + E_2 w + K_{21}y_1$$

Then, S admits the following representation,

$$\dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w$$

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad 1 \le i < r_1$$

$$\begin{aligned} \dot{x}_{r_1} &= A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1(K_2 u + E_2 w + K_{21}(x_1 + E_1 w)) \\ &\quad + D_{1,r_1}w =: A_{1,r_1z}x_z + \bar{a}_{1,r_11}x_1 + b_1 K_2 u + \bar{D}_{1,r_1}w \end{aligned}$$

$$y_1 = x_1 + E_1 w$$

Clearly, S admits RD $r_1 = r_1 + r_2$ from u to y_1 since $b_1 K_2 \ne 0$. Then, S is minimum phase with respect to $\mathcal{D}_{x0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$. This proves Case 14.

Case 15: $0 = r_1 = n_1$ and $0 = r_2 < n_2$. By Definition 1 and Lemma 3 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \ne 0$, $K_2 \ne 0$, with $\mathcal{D}_{x0} = \mathbb{R}^0$; and $\eta_0 \in \mathcal{D}_{\eta 0}$,

$$S_1:\ y_1 = K_1 y_2 + E_1 w$$

$$S_2: \begin{cases} \dot{\eta} = \bar{A}_2\eta + \bar{B}_2(y_2 - E_2 w - K_{21}y_1) + D_2 w + A_{21}y_1 \\ \quad =: \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1;\ \eta(0) = \eta_0 \\ y_2 = C_2\eta + K_2 u + E_2 w + K_{21}y_1 \end{cases}$$

Note that

$$y_1 = K_1(C_2\eta + K_2 u + E_2 w + K_{21}y_1) + E_1 w \quad \Rightarrow$$

$$y_1 = \hat{K}^{-1}\left(K_1C_2\eta + K_1K_2u + (K_1E_2+E_1)w\right)$$
$$y_2 = K_1^{-1}(y_1 - E_1w)$$

Hence, S admits the following representation,

$$\dot{\eta} = \bar{A}_2\eta + \bar{B}_2y_2 + \bar{D}_2w + \bar{A}_{21}y_1 = \bar{A}_2\eta + (\bar{A}_{21} + \bar{B}_2K_1^{-1})y_1 + (\bar{D}_2 - \bar{B}_2K_1^{-1}E_1)w; \quad (19a)$$
$$y_1 = \hat{K}^{-1}\left(K_1C_2\eta + K_1K_2u + (K_1E_2+E_1)w\right) \quad (19b)$$

Clearly, (19) is in EZDCF (6) of [1]; and admits RD $0 = r_1 + r_2$ from u to y_1, since $\hat{K}^{-1}K_1K_2 \neq 0$.

$\forall c_w \geq 0$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty \leq c_w$, let $\eta_{[0,\infty)}$ be the solution to (19a). Then, $y_{2[0,\infty)} := K_1^{-1}(y_{1[0,\infty)} - E_1w_{[0,\infty)}) \in \mathcal{C}$ with $\|y_{2[0,\infty)}\|_\infty \leq |K_1^{-1}| \cdot c_w + \|K_1^{-1}E_1\|_{2,2}c_w =: c_{w1}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then $\exists c_c \geq 0$, which depends only on c_w and c_{w1}, such that $\|\eta_{[0,\infty)}\|_\infty \leq c_c$. Hence, (19) is minimum phase with respect to $\mathcal{D}_{\eta 0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This proves Case 15.

Case 16: $0 = r_1 = n_1$ and $1 \leq r_2 = n_2$. By Definition 1 and Lemma 2 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \neq 0$, $b_2 \neq 0$, with $\eta = [\eta_1 \cdots \eta_{r_2}]'$; $\eta_i \in \mathbb{R}$, $1 \leq i \leq r_2$; $\eta(0) = \eta_0 := [\eta_{1,0} \cdots \eta_{r_2,0}]' \in \mathcal{D}_{\eta 0}$; and $\mathcal{D}_{x0} = \mathbb{R}^0$,

$$S_1: \; y_1 = K_1y_2 + E_1w$$
$$S_2: \begin{cases} \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1; \; 1 \leq i < r_2 \\ \dot{\eta}_{r_2} = a_{2,r_21}\eta_1 + b_2u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2w + K_{21}y_1 \end{cases}$$

Note that

$$y_1 = K_1(\eta_1 + E_2w + K_{21}y_1) + E_1w \quad \Rightarrow$$
$$y_1 = \hat{K}^{-1}(K_1\eta_1 + (K_1E_2 + E_1)w) =: \bar{\eta}_1 + \bar{E}_1w$$

Define $\bar{\eta}_i = \hat{K}^{-1}K_1\eta_i$, $1 \leq i \leq r_2$. Then, S admits the following representation in $\xi = [\bar{\eta}_1 \cdots \bar{\eta}_{r_2}]' = T_1^{-1}\eta$ coordinates:

$$\dot{\bar{\eta}}_i = \hat{K}^{-1}K_1(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}(\bar{\eta}_1 + \bar{E}_1w)) =: \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w; \; 1 \leq i < r_2 \quad (20a)$$
$$\dot{\bar{\eta}}_{r_2} = \hat{K}^{-1}K_1(a_{2,r_21}\eta_1 + b_2u + D_{2,r_2}w + A_{21,r_2}(\bar{\eta}_1 + \bar{E}_1w)) =: \bar{a}_{2,r_21}\bar{\eta}_1 + \hat{K}^{-1}K_1b_2u + \bar{D}_{2,r_2}w \quad (20b)$$
$$y_1 = \bar{\eta}_1 + \bar{E}_1w \quad (20c)$$

Clearly, (20) is in the EZDCF (5) of [1], which is minimum phase with respect to $T_1(\mathcal{D}_{\eta 0}) = T_1(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$ and $\mathcal{W}_d$ since the EZD is absent, and admits RD $r_2 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1b_2 \neq 0$. This proves Case 16.

Case 17: $0 = r_1 = n_1$ and $1 \leq r_2 < n_2$. By Definition 1 and Lemma 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \neq 0$, $b_2 \neq 0$, with $\eta = [\eta_z' \; \eta_1 \cdots \eta_{r_2}]'$; $\eta_z \in \mathbb{R}^{n_2 - r_2}$; $\eta_i \in \mathbb{R}$, $1 \leq i \leq r_2$; $\eta(0) = \eta_0 := [\eta_{z0}' \; \eta_{1,0} \cdots \eta_{r_2,0}]' \in \mathcal{D}_{\eta 0}$; and $\mathcal{D}_{x0} = \mathbb{R}^0$,

$$S_1: \; y_1 = K_1y_2 + E_1w$$
$$S_2: \begin{cases} \dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\ \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1; \; 1 \leq i < r_2 \\ \dot{\eta}_{r_2} = A_{2,r_2z}\eta_z + a_{2,r_21}\eta_1 + b_2u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2w + K_{21}y_1 \end{cases}$$

Note that

$$y_1 = K_1(\eta_1 + E_2w + K_{21}y_1) + E_1w \quad \Rightarrow$$
$$y_1 = \hat{K}^{-1}(K_1\eta_1 + (K_1E_2 + E_1)w) =: \bar{\eta}_1 + \bar{E}_1w$$

Define $\bar{\eta}_i = \hat{K}^{-1}K_1\eta_i$, $1 \leq i \leq r_2$. Then, S admits the following representation, in $\xi = [\eta_z' \; \bar{\eta}_1 \cdots \bar{\eta}_{r_2}]' = T_1^{-1}\eta$ coordinates, with $\xi(0) = \xi_0 := [\eta_{z0}' \; \bar{\eta}_{1,0} \cdots \bar{\eta}_{r_2,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 =: A_{2,z}\eta_z + \bar{A}_{2,z1}\bar{\eta}_1 + \bar{D}_{2,z}w \quad (21a)$$
$$\dot{\bar{\eta}}_i = \hat{K}^{-1}K_1(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}(\bar{\eta}_1 + \bar{E}_1w)) =: \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w; \; 1 \leq i < r_2 \quad (21b)$$
$$\dot{\bar{\eta}}_{r_2} = \hat{K}^{-1}K_1(A_{2,r_2z}\eta_z + a_{2,r_21}\eta_1 + b_2u + D_{2,r_2}w + A_{21,r_2} \cdot (\bar{\eta}_1 + \bar{E}_1w)) =: \bar{A}_{2,r_2z}\eta_z + \bar{a}_{2,r_21}\bar{\eta}_1 + \hat{K}^{-1}K_1b_2u + \bar{D}_{2,r_2}w \quad (21c)$$
$$y_1 = \bar{\eta}_1 + \bar{E}_1w \quad (21d)$$

Clearly, (21) is in the EZDCF (2) of [1]; and admits RD $r_2 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1b_2 \neq 0$. We will apply Proposition 1 of [1] to prove this case. $\forall c_w \geq 0$, $\forall \eta_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_2 - r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty \leq c_w$, let $\eta_{z[0,\infty)}$ be the solution to (21a). By Lemma 5 of [5], $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := [\eta_{z0}' \; \acute{\xi}_{1,0} \cdots \acute{\xi}_{r_2,0}]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \leq c_1c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0} = \mathcal{D}_{\eta 0}$ and $\eta_{z0} = P_{\eta z}T_1\acute{\xi}_0 \in P_{\eta z}(\mathcal{D}_{\eta 0})$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 - r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - r_2$ coordinates. Note that $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\|\eta_{1[0,\infty)}\|_\infty = \|K_1^{-1}\hat{K}\bar{\eta}_{1[0,\infty)}\|_\infty \leq |K_1^{-1}\hat{K}|c_w =: c_{w1}$ and $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty = \|\bar{\eta}_{1[0,\infty)} + \bar{E}_1w_{[0,\infty)}\|_\infty \leq (1 + \|\bar{E}_1\|_{2,2})c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then $\exists c_{c1} \geq 0$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{z[0,\infty)}\|_\infty \leq c_{c1}$. Hence, (21) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This proves Case 17.

Case 18: $0 = r_1 = n_1 = r_2 = n_2$. By Definition 1 of [1], without loss of generality, assume that S_1 and S_2 are given in EZDCFs, respectively, $K_1 \neq 0$, $K_2 \neq 0$,

$$S_1: \; y_1 = K_1y_2 + E_1w$$
$$S_2: \; y_2 = K_2u + E_2w + K_{21}y_1$$

Then, we have

$$y_1 = K_1(K_2u + E_2w + K_{21}y_1) + E_1w \quad \Rightarrow$$
$$y_1 = \hat{K}^{-1}(K_1K_2u + (E_1 + K_1E_2)w)$$

Since $\hat{K}^{-1}K_1K_2 \neq 0$, then, the composite system S admits RD $0 = r_1 + r_2$ from u to y_1. It is minimum phase with respect to $\mathbb{R}^0 = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since its EZD is absent. This proves Case 18.

This completes the proof of the theorem. ∎

IV. Conclusions

In this paper, we have further investigated properties of minimum phase systems using the generalized definition introduced in [1]. We proved that a composite system consisting of two minimum phase systems in series interconnection with additional output feedback is itself minimum phase.

Future research along this direction lies in the generalization of the minimum phase concept for finite-dimensional continuous-time multiple-input and multiple-output linear time-invariant systems under some specific structural assumptions. Another fruitful research topic lies in model reference robust adaptive control using the new definition of minimum phase. Both of these topics are currently under investigation.

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