GENERALIZED MINIMUM PHASE PROPERTY FOR FINITE-DIMENSIONAL CONTINUOUS-TIME SISO LTI SYSTEMS WITH ADDITIVE DISTURBANCES. PART II: FURTHER PROPERTIES

ZIGANG PAN * AND TAMER BAŞAR †

January 29, 2013

Abstract. In Part I of the paper, we introduced the generalized concept of minimum phase for finite-dimensional continuous-time single-input and single-output linear time-invariant systems. In this paper, we further investigate the properties of minimum phase systems using the definition of Part I. We first establish two technical lemmas that lead to the minimum phase property for linear systems with specific structures in their state space representations. Based on these results, we study the minimum phase property of interconnected systems. With two minimum phase systems in sequential interconnection with additional output feedback, we prove that the composite system is again minimum phase. Another result is that a minimum phase system in feedback connection with another linear system satisfying certain boundedness condition yields a minimum phase composite system. It is well known that a minimum phase system may be inverted, that is tracking of an arbitrary reference signal with bounded derivatives up to certain order, without leaving the internal states unbounded. We establish the following results in this regard. When a minimum phase linear system admits relative degree 0 from the control input to the output, a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a minimum phase linear system admits a positive and finite relative degree, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output is bounded, then the output of a "stable" system sharing the same inputs as the minimum phase system is bounded if the relative degree of the "stable" system is greater than or equal to the relative degree of the minimum phase system. When a minimum phase linear system admits a positive and finite relative degree, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to certain order are bounded, then, the output of a "stable" system sharing the same inputs as the minimum phase system is bounded if the relative degree of the "stable" system is greater than or equal to the relative degree of the highest ordered noiseless derivative of the output for the minimum phase system. These results will have significant impact on model reference control design and analysis.

Key words. continuous-time systems, extended zero dynamics canonical form, minimum phase, extended zero dynamics.

AMS subject classifications. 34A30, 34H05, 93C05, 93C15, 93B10, 93B99, 93D99, ???

1. Introduction.

The minimum phase property is of paramont importance in model reference control theory. The classical definition of the minimum phase property for a finite-dimensional continuous-time SISO LTI system is that the numerator of the transfer function is a Hurwitz polynomial ([1]). When a system is minimum phase, then, it can be safely inverted without leaving the system states going unbounded, that is, the system output may track any desired sufficiently smooth reference trajectory.

Because of its importance in control theory, significant efforts have been devoted to the generalization of the minimum phase property to nonlinear systems. In [2], the concept of zero dynamics is introduced for finite-dimensional continuous-time nonlinear systems that are linear with respect to the inputs. The zero dynamics is the internal dynamics of the system when the output of the system is kept to be identically

*Present address: 4797 Bordeaux Lane, Mason, OH 45040, USA. Tel: 513-398-7825; Fax: 513-398-7825; Email: zigangpan2002@mac.com.

†Coordinated Science Laboratory, University of Illinois, 1308 West Main Street, Urbana, IL 61801, USA. Tel: 217-333-3607; Fax: 217-244-1653; Email: basar1@illinois.edu.

zero. It is shown that, for a finite-dimensional continuous-time SISO LTI system, the system is minimum phase if, and only if, its zero dynamics is asymptotically stable. The zero dynamics defined in [2] is an autonomous system without any inputs. This concept has its limitations in stability analysis due to the nonlinear nature of the problem. Recently, the minimum phase property for nonlinear systems is revisited in [3], which is based on the concept of weakly uniform 0-detectability. The idea in [3] is in line of the development of input-output-to-state stability and output-to-state stability ([6]). It demonstrates the importance of bounding the internal states given the bound of the output of the system.

In Part I of the paper ([5]), the minimum phase concept is generalized for finite-dimensional continuous-time SISO LTI systems with additional distubance inputs in an attempt to make it necessary for the solvability of the output feedback model reference control problem. When the system admits a finite relative degree, then it may be transformed into the extended zero dynamics canonical form representation. Based on this canonical form representation, the extended zero dynamics for the system is defined which is simply the zero dynamics of the system as defined in [2] together with driving terms including the noiseless output and the disturbance input of the system. It is shown that the extended zero dynamics thus defined is invariant under smooth nonlinear coordinate transformations that transform the system into certain canonical forms. The original system is said to be minimum phase with respect to the given set of admissible initial conditions and the given set of admissible disturbance waveforms if the extended zero dynamics is absent or satisfies that the zero dynamics state is bounded for any bounded admissible initial condition (for the extended zero dynamics), any bounded noiseless output waveform, and any bounded admissible disturbance waveform. The relationship of the generalized concept of minimum phase with that introduced in [2] and the classical definition [1] has been investigated. Furthermore, the generalized minimum phase property is proved to be necessary for the achievement of perfect tracking of any bounded reference trajectory with bound derivatives up to rth order without any disturbances and the existence of bounded state trajectory for any bounded admissible initial condition, any bounded admissible disturbance waveform, and any bounded reference trajectory with bounded derivatives up to rth order in model reference control of the system, where r is relative degree of the system.

In this paper, we further investigate the properties of minimum phase systems using the definition of Part I. Two technical lemmas that lead to the minimum phase property for linear systems with specific structures in their state space representations are proved. Based on these results, we prove that the composite system consisting of two minimum phase systems in sequential interconnection with additional output feedback is again minimum phase. Another result is that a minimum phase system in feedback connection with another linear system satisfying certain boundedness condition yields a minimum phase composite system. It is well known that a minimum phase system may be inverted, that is tracking of an arbitrary reference signal with bounded derivatives up to certain order, without leaving the internal states unbounded. We establish the following results in this regard. When a minimum phase linear system admits the relative degree 0 from the control input to the output, a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a minimum phase linear system admits relative degree $r_1 \in \mathbb{N}$, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output is bounded, then, the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the relative degree of the "stable" system is greater than or equal to r_1. When a minimum phase linear system admits relative degree $r_1 \in \mathbb{N}$, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to k_0th order are bounded, then, the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the relative degree of the "stable" system is greater than or equal to $r_1 - k_0$. These results will have significant impact on model reference control design and analysis.

The balance of the paper is as follows. In the next section, we list the notations to be used in the paper. Then, in § 3, we present two technical lemmas that establish the minimum phase property for finite-dimensional continuous-time SISO LTI systems with specific structures in their state space representations. In § 4, the minimum phase property of composite systems are proved for two types of interconnected systems. The boundedness of the inverse of minimum phase systems is presented in § 5. The paper ends with some concluding remarks in § 6 and an appendix containing some useful technical lemmas necessary for the derivation in the main body of the paper.

2. Notations. We let $\mathbb{R}$ denote the real line; let $\mathbb{R}_e$ denote the extended real line, $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ to be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; and $\mathbb{C}$ to be the set of complex numbers. Unless specified, all signals, constants, and matrices are real. For a function f, we say that it belongs to $\mathcal{C}$ (or $\mathcal{C}_0$) if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously differentiable (Fréchet differentiability), which is equivalent to that all partial derivatives up to kth order are continuous when $\mathrm{dom}\,(f)$ is open, $k \in \mathbb{N}$. We say that a function is $\mathcal{L}_2$ if it is square integrable; and that it is $\mathcal{L}_\infty$ if it is bounded. We will write $\mathcal{C}_k(A, B)$ and $\mathcal{L}_p(A, B)$ to denote set of functions of A to B which are k-times continuously differentiable and set of functions of A to B which have a finite $\mathcal{L}_p$ norm, respectively. For any matrix A, A' denotes its transpose. For any $m, n \in \mathbb{Z}_+$ and any $m \times n$-dimensional matrix M, $\mathcal{R}\,(M)$ denotes the range space of M and $\mathcal{N}\,(M)$ denotes the null space of M. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, $|z|$ denotes $\sqrt{z'z}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, and any $n \times n$-dimensional symmetric matrix M, $|z|_M^2 := z'Mz$. For $n \times n$-dimensional symmetric matrices M_1 and M_2, where $n \in \mathbb{Z}_+$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{Z}_+$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. For $n \in \mathbb{Z}_+$, I_n denotes the $n \times n$-dimensional identity matrix. For $n \in \mathbb{Z}_+$ and $n \times n$-dimensional matrix A, we set $A^0 = I_n$. For any matrix M, $\|M\|_{p,p}$ denotes its p-induced norm, $1 \leq p \leq \infty$. For any $m, n \in \mathbb{Z}_+$, $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are zeros. For any $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$, $e_{n,k}$ denotes the kth n-dimensional unit vector, i. e., $\begin{bmatrix} \mathbf{0}_{1\times(k-1)} & 1 & \mathbf{0}_{1\times(n-k)} \end{bmatrix}'$. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0, t_f), \mathbb{R}^p)$, where $t_f \in (0, \infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$ and $\|u_{[0,t_f)}\|_q = \left(\int_0^{t_f} |u(t)|^q \,\mathrm{d}t\right)^{1/q}$, $q \in [1, \infty)$. For a sufficiently smooth signal v, $v^{(i)}$ denotes the ith order derivative of v, $v^{[i]}$ denotes $\begin{bmatrix} v' & (v^{(1)})' & \cdots & (v^{(i)})' \end{bmatrix}'$, $i \in \mathbb{Z}_+$. For a $\mathcal{C}_\infty$ vector field f and a $\mathcal{C}_\infty$ function h, $L_f h$ denotes the derivative of h along f, which equals to $\frac{\partial h}{\partial x}(x) f(x)$ in local coordinates. $L_f^{k+1} h = L_f(L_f^k h)$, $k \in \mathbb{N}$; $L_f^0 h = h$. $\forall \lambda \in \mathbb{C}$, $\mathrm{Re}\,(\lambda)$ and $\mathrm{Im}\,(\lambda)$ denote the real part and the imaginary part of λ, respectively. We will denote constants or matrices of no specific interest or relevance to the analysis by $\star$. We will denote $m \times n$-dimensional matrices of no specific interest or relevance to the analysis by $\star_{m\times n}$.

3. Two lemmas to establish minimum phase property. We present two lemmas that establishes the minimum phase property for finite-dimensional continuous-time SISO LTI systems in more general canonical forms than those introduced in [5], which arises in interconnected systems.

LEMMA 3.1. *Consider a finite-dimensional continuous-time SISO LTI system given in the following canonical form*

$$\dot{x}_z = A_o x_z + A_{o1} x_1 + \cdots + A_{or} x_r + A_{o\,r+1} b_0 u + D_o w \tag{3.1a}$$
$$\dot{x}_i = A_{io} x_z + a_{i1} x_1 + \cdots + a_{ii} x_i + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \tag{3.1b}$$
$$\dot{x}_r = A_{ro} x_z + a_{r1} x_1 + \cdots + a_{rr} x_r + b_0 u + D_r w \tag{3.1c}$$
$$y = x_1 + Ew \tag{3.1d}$$

where x_z is $(n-r)$-dimensional, $n \in \mathbb{N}$, $r \in \mathbb{N}$, $n-r \in \mathbb{N}$; x_i, $i = 1, \ldots, r$, are scalars; $b_0 \neq 0$; y is the scalar output; u is the scalar control input; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; all matrices are of appropriate dimensions and constant. Let $x = \begin{bmatrix} x_z' & x_1 & \cdots & x_r \end{bmatrix}'$, $x(0) = x_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$. Assume that $\exists m_o \in \{1, \ldots, r+1\}$ such that $A_{oj} = \mathbf{0}_{(n-r)\times 1}$, $\forall j \in \{m_o+1, \ldots, r+1\}$; and $A_{jo} = \mathbf{0}_{1\times(n-r)}$, $\forall j \in \{1, \ldots, m_o-1\}$. Then, system (3.1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits relative degree r from u to y if the dynamics (3.1a) satisfies $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_{z0} \in \mathcal{D}_{z0} = P_z(\mathcal{D}_0)$ with $|x_{z0}| \leq c_w$, where $P_z : \mathbb{R}^n \to \mathbb{R}^{n-r}$ is the projection of $\mathbb{R}^n$ onto its first $n-r$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\| x_{i[0,\infty)} \right\|_\infty \leq c_w$, $i = 1, \ldots, m_o$, ($x_{r+1} := b_0 u$ for notational consistency) we have $\left\| x_{z[0,\infty)} \right\|_\infty \leq c_c$.

Proof. We will prove the lemma using mathematical induction on m_o.

1° $m_o = 1$. Clearly $A_{o2} = \cdots = A_{or+1} = \mathbf{0}_{(n-r)\times 1}$. Then, clearly, the system (3.1) is in the form (3.3) of [5]. Then, (3.1a) is the extended zero dynamics of the system (3.1). By Lemma 3.9 of [5], (3.1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits relative degree r. This completes the initialization step. If $r = 1$, then the lemma is proved. If $r \geq 2$, we continue to the next step.

2° Assume the lemma holds for $m_o = 1, \ldots, k$, where $k \in \{1, \ldots, r\}$.

3° Consider the case $m_o = k+1 \in \{2, \ldots, r+1\}$. Consider the state transformation

$$\bar{x} := \begin{bmatrix} \bar{x}_z' & x_1 & \cdots & x_r \end{bmatrix}' = T_1^{-1} x = \begin{bmatrix} x_z' - A_{o\,k+1}' x_k & x_1 & \cdots & x_r \end{bmatrix}'$$

Then, in $\bar{x}$ coordinates, the system (3.1) admits the state space representation

$$\begin{aligned} \dot{\bar{x}}_z &= A_o x_z + A_{o1} x_1 + \cdots + A_{o\,k+1} x_{k+1} + D_o w - A_{o\,k+1}(a_{k1} x_1 + \cdots \\ &\quad + a_{kk} x_k + x_{k+1} + D_k w) = A_o \bar{x}_z + A_o A_{o\,k+1} x_k + A_{o1} x_1 + \cdots + A_{ok} x_k \\ &\quad + D_o w - A_{o\,k+1}(a_{k1} x_1 + \cdots + a_{kk} x_k + D_k w) \\ &=: A_o \bar{x}_z + \bar{A}_{o1} x_1 + \cdots + \bar{A}_{ok} x_k + \bar{D}_o w \end{aligned} \tag{3.2a}$$
$$\dot{x}_i = a_{i1} x_1 + \cdots + a_{ii} x_i + x_{i+1} + D_i w; \quad i = 1, \ldots, k \tag{3.2b}$$
$$\dot{x}_i = A_{io} \bar{x}_z + A_{io} A_{o\,k+1} x_k + a_{i1} x_1 + \cdots + a_{ii} x_i + x_{i+1} + D_i w; \quad i = k+1, \ldots, r \tag{3.2c}$$
$$y = x_1 + Ew \tag{3.2d}$$

with $\bar{x}(0) = \bar{x}_0 := \left[\begin{array}{cccc} \bar{x}'_{z0} & x_{1,0} & \cdots & x_{r,0} \end{array} \right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_0) \neq \emptyset$.

Clearly, the representation (3.2) is in the form of (3.1) with $\bar{m}_o = k$. We will apply the inductive assumption to show that (3.2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$, which will complete the induction proof. Toward this end, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \bar{x}_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|\bar{x}_{z0}| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\| x_{i[0,\infty)} \right\|_\infty \leq c_w$, $i = 1, \ldots, k$. Since $\bar{x}_{z0} \in \bar{\mathcal{D}}_{z0} = P_z(\bar{\mathcal{D}}_0)$, by Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_0$, and $\exists \acute{x}_{1,0}, \ldots, \acute{x}_{r,0} \in \mathbb{R}$ such that $\acute{x}_0 := \left[\begin{array}{cccc} \bar{x}'_{z0} & \acute{x}_{1,0} & \cdots & \acute{x}_{r,0} \end{array} \right]' \in \bar{\mathcal{D}}_0$ and $|\acute{x}_0| \leq c_1 c_w$. Then, $T_1 \acute{x}_0 = \left[\begin{array}{cccc} \acute{x}'_{z0} & \acute{x}_{1,0} & \cdots & \acute{x}_{r,0} \end{array} \right]' \in \mathcal{D}_0$ where $\acute{x}_{z0} := \bar{x}_{z0} + A_{o\,k+1}\acute{x}_{k,0} \in P_z(\mathcal{D}_0)$ with $|\acute{x}_{z0}| \leq \| T_1 \|_{2,2}\, c_1 c_w =: c_1 c_2 c_w$. The solution to (3.2a) may be decomposed by linearity as the sum of the following three systems.

$$\begin{aligned}
&\text{(3.3a)} && \dot{\eta}_1 = A_o \eta_1 + A_{o1} x_1 + \cdots + A_{ok} x_k + A_{o\,k+1} (-a_{k1} x_1 - \cdots - a_{kk} x_k \\
& && \qquad -D_k w) + D_o w; \qquad \eta_1(0) = \bar{x}_{z0} + A_{o\,k+1}\acute{x}_{k,0} \in P_z(\mathcal{D}_0) \\
&\text{(3.3b)} && \dot{\eta}_2 = A_o \eta_2; \qquad \eta_2(0) = -A_{o\,k+1}\acute{x}_{k,0} \\
&\text{(3.3c)} && \dot{\eta}_3 = A_o \eta_3 + A_o A_{o\,k+1} x_k; \qquad \eta_3(0) = \mathbf{0}_{(n-r)\times 1} \\
&\text{(3.3d)} && \bar{x}_{z[0,\infty)} = \eta_{1[0,\infty)} + \eta_{2[0,\infty)} + \eta_{3[0,\infty)}
\end{aligned}$$

Let $\bar{x}_{k+1} = -a_{k1} x_1 - \cdots - a_{kk} x_k - D_k w$. Then, $\bar{x}_{k+1[0,\infty)} \in \mathcal{C}$ and $\left\| \bar{x}_{k+1[0,\infty)} \right\|_\infty \leq (\sum_{i=1}^k |a_{ki}|) c_w + \| D_k \|_{2,2}\, c_w =: \bar{c}_{w1} c_w$. For (3.3a), by the assumption of the lemma, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, $\bar{c}_{w1} c_w$, and $c_1 c_2 c_w$, such that $\left\| \eta_{1[0,\infty)} \right\|_\infty \leq c_{c1}$.

Under the assumption of the lemma, by Lemma A.11 of [5], the following system is bounded input and bounded state stable.

$$\dot{\xi} = A_o \xi + A_{o\,k+1} v; \qquad \xi(0) = \mathbf{0}_{(n-r)\times 1}$$

For (3.3b), by Lemma A.1, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which does not depend on any thing, such that $\left\| \eta_{2[0,\infty)} \right\|_\infty \leq c_{c2}\, |\acute{x}_{k,0}| \leq c_{c2} c_1 c_w$.

Again by Lemma A.1, (3.3c) is bounded input and bounded state stable. Then, by Lemma A.3 of [5], $\exists c_{c3} \in [0,\infty) \subset \mathbb{R}$, which does not depend on any thing, such that $\left\| \eta_{3[0,\infty)} \right\|_\infty \leq c_{c3} c_w$.

Then, we have $\left\| \bar{x}_{z[0,\infty)} \right\|_\infty \leq c_{c1} + c_{c2} c_1 c_w + c_{c3} c_w$. This shows that (3.2) satisfies the inductive assumption. Hence, (3.2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree r, which is equivalent to that (3.1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits relative degree r from u to y.

This completes the induction process.

This completes the proof of the lemma. □

LEMMA 3.2. *Consider a finite-dimensional continuous-time SISO LTI system:*

$$\begin{aligned}
&\text{(3.4a)} && \dot{x}_{za} = A_{oa} x_{za} + A_{oa1} x_1 + \cdots + A_{oar} x_r + A_{oa\,r+1} b_0 u + D_{oa} w \\
&\text{(3.4b)} && \dot{x}_{zb} = A_{oba} x_{za} + A_{ob} x_{zb} + A_{ob1} x_1 + \cdots + A_{obr} x_r + A_{ob\,r+1} b_0 u + D_{ob} w \\
&\text{(3.4c)} && \dot{x}_i = A_{ioa} x_{za} + A_{iob} x_{zb} + a_{i1} x_1 + \cdots + a_{ii} x_i + x_{i+1} + D_i w;\ i = 1, \ldots, r-1 \\
&\text{(3.4d)} && \dot{x}_r = A_{roa} x_{za} + A_{rob} x_{zb} + a_{r1} x_1 + \cdots + a_{rr} x_r + b_0 u + D_r w \\
&\text{(3.4e)} && y = x_1 + E w
\end{aligned}$$

where x_{za} is m_a-dimensional, $m_a \in \mathbb{N}$; x_{zb} is m_b-dimensional, $m_b \in \mathbb{N}$; x_i, $i = 1, \ldots, r$, are scalars, with $m_a + m_b + r = n \in \mathbb{N}$ and $r \in \mathbb{N}$; $b_0 \neq 0$; y is the *scalar output; u is the scalar control input; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; and all matrices are of appropriate dimensions and constant. Let $x = \begin{bmatrix} x'_{za} & x'_{zb} & x_1 & \cdots & x_r \end{bmatrix}'$, $x(0) = x_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ be a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$.*

Assume that $\exists m_{oa}, m_{ob} \in \{1, \ldots, r+1\}$ with $m_{oa} \leq m_{ob}$, such that, $A_{oaj} = \mathbf{0}_{m_a \times 1}$, $j = m_{oa}+1, \ldots, r+1$; $A_{joa} = \mathbf{0}_{1 \times m_a}$, $j = 1, \ldots, m_{oa}-1$; $A_{obj} = \mathbf{0}_{m_b \times 1}$, $j = m_{ob}+1, \ldots, r+1$; and $A_{job} = \mathbf{0}_{1 \times m_b}$, $j = 1, \ldots, m_{ob}-1$.

Assume that the system (3.4a) and (3.4b) satisfy $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_{z0} := \begin{bmatrix} x'_{za0} & x'_{zb0} \end{bmatrix}' \in P_z(\mathcal{D}_0)$ with $|x_{z0}| \leq c_w$, where $P_z : \mathbb{R}^n \to \mathbb{R}^{m_a+m_b}$ is the projection of $\mathbb{R}^n$ onto the first m_a+m_b coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\| x_{i[0,\infty)} \right\|_\infty \leq c_w$, $i = 1, \ldots, m_{ob}$, ($x_{r+1} := b_0 u$ for notational consistency) we have $\left\| x_{z[0,\infty)} \right\|_\infty \leq c_c$, where $x_z := \begin{bmatrix} x'_{za} & x'_{zb} \end{bmatrix}'$. Then, the system (3.4) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits relative degree r from u to y.

Proof. When $m_{oa} = m_{ob} =: m_o$, by Lemma 3.1, the result holds. We will prove the lemma by mathematical induction on $m_{ob} - m_{oa} \in \{0, \ldots, r\}$.

1° $m_{ob} - m_{oa} = 0$. The result holds. This completes the initialization step.

2° Assume that the result holds for $m_{ob} - m_{oa} = k \in \{0, \ldots, r-1\}$.

3° Consider $m_{ob} - m_{oa} = k+1 \in \{1, \ldots, r\}$. Then, $m_{oa} \in \{1, \ldots, r\}$ and $m_{ob} \in \{2, \ldots, r+1\}$. Introduce the coordinate transformation

$$\bar{x} = T_1^{-1} x = \begin{bmatrix} x'_{za} & x'_{zb} - A'_{obm_{ob}} x_{m_{ob}-1} & x_1 & \cdots & x_r \end{bmatrix}'$$

Then, in $\bar{x} := \begin{bmatrix} x'_{za} & \bar{x}'_{zb} & x_1 & \cdots & x_r \end{bmatrix}'$ coordinates, the system (3.4) admits the following state space representation

$$\begin{aligned}
&\text{(3.5a)} & \dot{x}_{za} &= A_{oa} x_{za} + A_{oa1} x_1 + \cdots + A_{oam_{oa}} x_{m_{oa}} + D_{oa} w \\
&\text{(3.5b)} & \dot{\bar{x}}_{zb} &= A_{oba} x_{za} + A_{ob} \bar{x}_{zb} + A_{ob} A_{obm_{ob}} x_{m_{ob}-1} + A_{ob1} x_1 + \cdots + A_{obm_{ob}} x_{m_{ob}} \\
&&& \quad + D_{ob} w - A_{obm_{ob}} (A_{m_{ob}-1\,oa} x_{za} + a_{m_{ob}-1\,1} x_1 + \cdots + a_{m_{ob}-1\,m_{ob}-1} \\
&&& \quad \cdot x_{m_{ob}-1} + x_{m_{ob}} + D_{m_{ob}-1} w) = A_{oba} x_{za} + A_{ob} \bar{x}_{zb} + A_{ob} A_{obm_{ob}} x_{m_{ob}-1} \\
&&& \quad + A_{ob1} x_1 + \cdots + A_{ob\,m_{ob}-1} x_{m_{ob}-1} + D_{ob} w - A_{obm_{ob}} (A_{m_{ob}-1\,oa} x_{za} \\
&&& \quad + a_{m_{ob}-1\,1} x_1 + \cdots + a_{m_{ob}-1\,m_{ob}-1} x_{m_{ob}-1} + D_{m_{ob}-1} w) \\
&&& =: \bar{A}_{oba} x_{za} + A_{ob} \bar{x}_{zb} + \bar{A}_{ob1} x_1 + \cdots + \bar{A}_{ob\,m_{ob}-1} x_{m_{ob}-1} + \bar{D}_{ob} w \\
&\text{(3.5c)} & \dot{x}_i &= A_{ioa} x_{za} + a_{i1} x_1 + \cdots + a_{ii} x_i + x_{i+1} + D_i w; \quad i = 1, \ldots, m_{ob}-1 \\
&\text{(3.5d)} & \dot{x}_i &= A_{ioa} x_{za} + A_{iob} \bar{x}_{zb} + A_{iob} A_{obm_{ob}} x_{m_{ob}-1} + a_{i1} x_1 + \cdots + a_{ii} x_i + x_{i+1} \\
&&& \quad + D_i w \\
&&& =: A_{ioa} x_{za} + A_{iob} \bar{x}_{zb} + \bar{a}_{i1} x_1 + \cdots + \bar{a}_{ii} x_i + x_{i+1} + D_i w \quad i = m_{ob}, \ldots, r \\
&\text{(3.5e)} & y &= x_1 + E w
\end{aligned}$$

with $\bar{x}(0) = \bar{x}_0 = \begin{bmatrix} x'_{za0} & \bar{x}'_{zb0} & x_{1,0} & \cdots & x_{r,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_0) \neq \emptyset$. Clearly, the representation (3.5) is in the form of (3.4) with $\bar{m}_{oa} = m_{oa}$ and $\bar{m}_{ob} = m_{ob} - 1$ such that $\bar{m}_{ob} - \bar{m}_{oa} = k$. We will apply the inductive assumption to show that (3.5) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree r, which will complete the induction process. Toward this end, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \bar{x}_{z0} := \begin{bmatrix} x'_{za0} & \bar{x}'_{zb0} \end{bmatrix}' \in P_z(\bar{\mathcal{D}}_0)$ with $|\bar{x}_{z0}| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\| x_{i[0,\infty)} \right\|_\infty \leq c_w$, $i = 1, \ldots, \bar{m}_{ob}$. By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_0$, and $\exists \acute{x}_{1,0}, \ldots, \acute{x}_{r,0} \in \mathbb{R}$ such that $\acute{x}_0 :=$

$\begin{bmatrix} \bar{x}'_{z0} & \acute{x}_{1,0} & \cdots & \acute{x}_{r,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ and $|\acute{x}_0| \leq c_1 c_w$. Then, $\begin{bmatrix} \acute{x}'_{z0} & \acute{x}_{1,0} & \cdots & \acute{x}_{r,0} \end{bmatrix}' = T_1 \acute{x}_0 \in \mathcal{D}_0$ where $\acute{x}_{z0} := \begin{bmatrix} x'_{za0} & \bar{x}'_{zb0} + A'_{ob\, m_{ob}} \acute{x}_{m_{ob}-1,0} \end{bmatrix}' \in P_z(\mathcal{D}_0)$ with $|\acute{x}_{z0}| \leq |T_1 \acute{x}_0| \leq \|T_1\|_{2,2} |\acute{x}_0| \leq \|T_1\|_{2,2} c_1 c_w =: c_1 c_2 c_w$. By the structure of (3.5a) and (3.5b), which exhibits that the dynamics of x_{za} is identical to (3.4a) and is not affected by x_{zb} or $x_{m_{ob}}$, we may apply the assumption of the lemma to conclude that $\exists c_{c1} \in [0, \infty) \subset \mathbb{R}$, which depends only on c_w, such that $\|x_{za[0,\infty)}\|_\infty \leq c_{c1}$.

Let $\bar{x}_{m_{ob}} = -A_{m_{ob}-1\, oa} x_{za} - a_{m_{ob}-1\,1} x_1 - \cdots - a_{m_{ob}-1\, m_{ob}-1} x_{m_{ob}-1} - D_{m_{ob}-1} w$. Let $\bar{x}_z = \begin{bmatrix} x'_{za} & \bar{x}'_{zb} \end{bmatrix}'$. Then, $\bar{x}_{z[0,\infty)}$ may be generated by

$$\begin{aligned}
\dot{\eta}_1 &= \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix} \eta_1 + \begin{bmatrix} A_{oa1} \\ A_{ob1} \end{bmatrix} x_1 + \cdots + \begin{bmatrix} A_{oa\bar{m}_{ob}} \\ A_{ob\bar{m}_{ob}} \end{bmatrix} x_{\bar{m}_{ob}} \\
&\quad + \begin{bmatrix} \mathbf{0}_{m_a \times 1} \\ A_{obm_{ob}} \end{bmatrix} \bar{x}_{m_{ob}} + \begin{bmatrix} D_{oa} \\ D_{ob} \end{bmatrix} w; \qquad \eta_1(0) = \acute{x}_{z0} = \begin{bmatrix} x_{za0} \\ \bar{x}_{zb0} + A_{obm_{ob}} \acute{x}_{\bar{m}_{ob},0} \end{bmatrix} \\
\dot{\eta}_2 &= \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix} \eta_2; \qquad \eta_2(0) = \begin{bmatrix} \mathbf{0}_{m_a \times 1} \\ -A_{obm_{ob}} \end{bmatrix} \acute{x}_{\bar{m}_{ob},0} \\
\dot{\eta}_3 &= \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix} \eta_3 + \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{m_a \times 1} \\ A_{obm_{ob}} \end{bmatrix} x_{m_{ob}-1}; \;\; \eta_3(0) = \mathbf{0}_{(m_a+m_b)\times 1} \\
\bar{x}_{z[0,\infty)} &= \eta_{1[0,\infty)} + \eta_{2[0,\infty)} + \eta_{3[0,\infty)}
\end{aligned}$$

Note that $\bar{x}_{m_{ob}[0,\infty)} \in \mathcal{C}$ and $\|\bar{x}_{m_{ob}[0,\infty)}\|_\infty \leq \|A_{\bar{m}_{ob}\, oa}\|_{2,2} c_{c1} + |a_{\bar{m}_{ob}\,1}| c_w + \cdots + |a_{\bar{m}_{ob}\, \bar{m}_{ob}}| c_w + \|D_{\bar{m}_{ob}}\|_{2,2} c_w =: c_{w1}$. By the assumption of the lemma, $\exists c_{c2} \in [0, \infty) \subset \mathbb{R}$, which depends only c_{w1}, $c_1 c_2 c_w$, and c_w, such that $\|\eta_{1[0,\infty)}\|_\infty \leq c_{c2}$.

By the assumption of the lemma and Lemma A.11 of [5], we have the following system

$$\dot{\xi} = \begin{bmatrix} A_{oa} & \mathbf{0} \\ A_{oba} & A_{ob} \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0}_{m_a \times 1} \\ A_{obm_{ob}} \end{bmatrix} v; \qquad \xi(0) = \mathbf{0}_{(m_a+m_b)\times 1}$$

is bounded input and bounded state stable. For the dynamics of η_2, by Lemma A.1, $\exists c_{c3} \in [0, \infty) \subset \mathbb{R}$, which does not depend on any thing, such that $\|\eta_{2[0,\infty)}\|_\infty \leq c_{c3} c_1 c_w$.

Again by Lemma A.1, the dynamics for η_3 is bounded input and bounded state stable. Then, by Lemma A.3 of [5], $\exists c_{c4} \in [0, \infty) \subset \mathbb{R}$, which does not depend on any thing, such that $\|\eta_{3[0,\infty)}\|_\infty \leq c_{c4} c_w$. Hence, we have $\|\bar{x}_{z[0,\infty)}\|_\infty \leq c_{c2} + c_{c3} c_1 c_w + c_{c4} c_w$. This proves that (3.5) satisfies the inductive assumption. Hence, (3.5) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree r from u to y.

This completes the induction process and hence the proof of the lemma. ∎

4. Minimum phase property for interconnected systems. We first present a result that shows that the composite system consisting of two sequencially interconnected minimum phase finite-dimensional continuous-time SISO LTI systems with additional output feedback is again a minimum phase system. The block diagram of the system is shown in Figure 4.1.

LEMMA 4.1. *Consider two finite-dimensional continuous-time SISO LTI systems*

$$S_1 : \begin{cases} \dot{x} = A_1 x + B_1 y_2 + D_1 w; \qquad x(0) = x_0 \\ y_1 = C_1 x + K_1 y_2 + E_1 w \end{cases} \tag{4.1}$$

$$S_2 : \begin{cases} \dot{\eta} = A_2 \eta + B_2 u + D_2 w + A_{21} y_1; \qquad \eta(0) = \eta_0 \\ y_2 = C_2 \eta + K_2 u + E_2 w + K_{21} y_1 \end{cases} \tag{4.2}$$

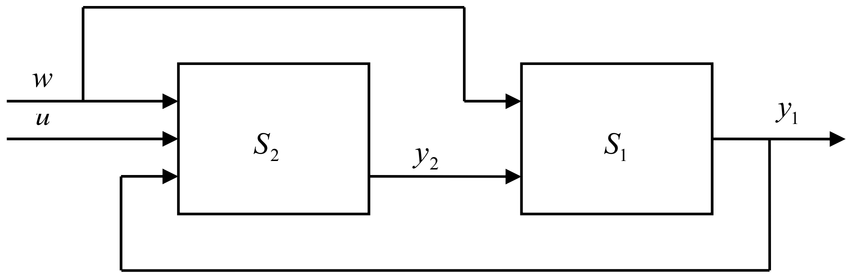

FIG. 4.1. *Block diagram of two sequentially interconnected systems with additional output feedback.*

where x is the n_1-dimensional state for S_1, $n_1 \in \mathbb{Z}_+$; y_2 is the scalar control input of S_1, and is also the output of S_2; y_1 is the scalar output of S_1, and is also an input to S_2; u is the scalar control input for S_2; η is the n_2-dimensional state for S_2, $n_2 \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; A_i, B_i, D_i, C_i, K_i, E_i, $i = 1, 2$, A_{21}, and K_{21} are constant matrices of appropriate dimensions; $x_0 \in \mathcal{D}_{x0} \neq \emptyset$, $\mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$ is a subspace; $\eta_0 \in \mathcal{D}_{\eta 0} \neq \emptyset$, $\mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$ is a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$.

Assume that S_1 admits relative degree $r_1 \in \{0, 1, \ldots, n_1\}$ from y_2 to y_1 and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$. Assume that S_2 admits relative degree $r_2 \in \{0, 1, \ldots, n_2\}$ from u to y_2 and is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty), \mathbb{R})$, where w and y_1 are viewed as disturbance inputs.

Let S be the composite system with output y_1, control input u, disturbance input w, and state $\begin{bmatrix} x' & \eta' \end{bmatrix}'$. Assume that the composite system is well posed, that is $\hat{K} := 1 - K_1 K_{21} \neq 0$.

Then, the composite system S admits relative degree $r_1 + r_2$ from u to y_1; and is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

Proof. We will distinguish 18 exhaustive and mutually exclusive cases, which are listed in Table 4.1.

TABLE 4.1
18 exhaustive and mutually exclusive cases for Lemma 4.1.

Case 1:	$0 = r_1 < n_1$, $0 = r_2 < n_2$	Case 2:	$0 = r_1 < n_1$, $1 = r_2 < n_2$
Case 3:	$0 = r_1 < n_1$, $2 \leq r_2 < n_2$	Case 4:	$0 = r_1 < n_1$, $1 = r_2 = n_2$
Case 5:	$0 = r_1 < n_1$, $2 \leq r_2 = n_2$	Case 6:	$1 \leq r_1 < n_1$, $0 = r_2 < n_2$
Case 7:	$1 \leq r_1 = n_1$, $0 = r_2 < n_2$	Case 8:	$1 \leq r_1 = n_1$, $1 \leq r_2 = n_2$
Case 9:	$1 \leq r_1 < n_1$, $1 \leq r_2 = n_2$	Case 10:	$1 \leq r_1 = n_1$, $1 \leq r_2 < n_2$
Case 11:	$1 \leq r_1 < n_1$, $1 \leq r_2 < n_2$	Case 12:	$0 = r_1 < n_1$, $0 = r_2 = n_2$
Case 13:	$1 \leq r_1 = n_1$, $0 = r_2 = n_2$	Case 14:	$1 \leq r_1 < n_1$, $0 = r_2 = n_2$
Case 15:	$0 = r_1 = n_1$, $0 = r_2 < n_2$	Case 16:	$0 = r_1 = n_1$, $1 \leq r_2 = n_2$
Case 17:	$0 = r_1 = n_1$, $1 \leq r_2 < n_2$	Case 18:	$0 = r_1 = n_1 = r_2 = n_2$

Case 1: $0 = r_1 < n_1$ and $0 = r_2 < n_2$. By Lemma 3.3 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} = \bar{A}_1 x + \bar{B}_1 (y_1 - E_1 w) + D_1 w =: \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0 \\ y_1 = C_1 x + K_1 y_2 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta} &= \bar{A}_2\eta + \bar{B}_2(y_2 - E_2w - K_{21}y_1) + D_2w + A_{21}y_1 \\ &=: \bar{A}_2\eta + \bar{B}_2y_2 + \bar{D}_2w + \bar{A}_{21}y_1; \qquad \eta(0) = \eta_0 \\ y_2 &= C_2\eta + K_2u + E_2w + K_{21}y_1 \end{cases}$$

Note that

$$\begin{aligned} &y_1 = C_1x + K_1\left(C_2\eta + K_2u + E_2w + K_{21}y_1\right) + E_1w \qquad \Rightarrow \\ &y_1 = \hat{K}^{-1}\left(C_1x + K_1C_2\eta + K_1K_2u + (K_1E_2 + E_1)w\right) \\ &y_2 = K_1^{-1}\left(y_1 - C_1x - E_1w\right) \end{aligned}$$

Then, S admits the following representation:

(4.3a) $\dot{x} = \bar{A}_1x + \bar{B}_1y_1 + \bar{D}_1w; \qquad x(0) = x_0$

(4.3b) $\dot{\eta} = \bar{A}_2\eta + \bar{B}_2K_1^{-1}\left(y_1 - C_1x - E_1w\right) + \bar{D}_2w + \bar{A}_{21}y_1$
$= -\bar{B}_2K_1^{-1}C_1x + \bar{A}_2\eta + (\bar{A}_{21} + \bar{B}_2K_1^{-1})y_1 + (\bar{D}_2 - \bar{B}_2K_1^{-1}E_1)w; \ \eta(0) = \eta_0$

(4.3c) $y_1 = \hat{K}^{-1}\left(C_1x + K_1C_2\eta + K_1K_2u + (K_1E_2 + E_1)w\right)$

Clearly, (4.3) admits relative degree $0 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1K_2 \neq 0$ and is in the extended zero dynamics canonical form (3.9) of [5] with (4.3a) and (4.3b) defining the extended zero dynamics of S.

Next, we show that S is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall \left[\begin{array}{cc} x_0' & \eta_0' \end{array}\right]' \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ with $\left|\left[\begin{array}{cc} x_0' & \eta_0' \end{array}\right]'\right| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\|w_{[0,\infty)}\right\|_\infty \leq c_w$, $\forall y_{1[0,\infty)} \in \mathcal{C}$ with $\left\|y_{1[0,\infty)}\right\|_\infty \leq c_w$, let $x_{[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (4.3a) and (4.3b). Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0, \infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\|x_{[0,\infty)}\right\|_\infty \leq c_{c1}$. Note that $y_{2[0,\infty)} \in \mathcal{C}$ with $\left\|y_{2[0,\infty)}\right\|_\infty \leq \left|K_1^{-1}\right| c_w + \left\|K_1^{-1}C_1\right\|_{2,2} c_{c1} + \left\|K_1^{-1}E_1\right\|_{2,2} c_w =: \bar{c}_w \in [0, \infty) \subset \mathbb{R}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0, \infty), \mathbb{R})$, then, $\exists c_{c2} \in [0, \infty) \subset \mathbb{R}$, which depends only on $\bar{c}_w$ and c_w, such that $\left\|\eta_{[0,\infty)}\right\|_\infty \leq c_{c2}$. Therefore, $\left\|\xi_{[0,\infty)}\right\|_\infty \leq \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi = \left[\begin{array}{cc} x' & \eta' \end{array}\right]'$. This proves that (4.3) is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This completes the proof for Case 1.

Case 2: $0 = r_1 < n_1$ and $1 = r_2 < n_2$. By Lemmas 3.3 and 3.1 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} &= \bar{A}_1x + \bar{B}_1(y_1 - E_1w) + D_1w =: \bar{A}_1x + \bar{B}_1y_1 + \bar{D}_1w; \qquad x(0) = x_0 \\ y_1 &= C_1x + K_1y_2 + E_1w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_z &= A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\ \dot{\eta}_1 &= A_{2,1z}\eta_z + a_{2,11}\eta_1 + b_2u + D_{2,1}w + A_{21,1}y_1 \\ y_2 &= \eta_1 + E_2w + K_{21}y_1 \end{cases}$$

where $\eta = \left[\begin{array}{cc} \eta_z' & \eta_1 \end{array}\right]'$; η_z is $(n_2 - r_2)$-dimensional; η_1 is a scalar, $\eta(0) = \eta_0 := \left[\begin{array}{cc} \eta_{x0}' & \eta_{1,0} \end{array}\right]' \in \mathcal{D}_{\eta 0}$. Note that

$$\begin{aligned} &y_1 = C_1x + K_1\left(\eta_1 + E_2w + K_{21}y_1\right) + E_1w \qquad \Rightarrow \\ &y_1 = \hat{K}^{-1}\left(C_1x + K_1\eta_1 + (E_1 + K_1E_2)w\right) =: \bar{\eta}_1 + \bar{E}_1w \end{aligned}$$

Then, $\eta_1 = K_1^{-1}\left(\hat{K}\bar{\eta}_1 - C_1x\right)$, and the composite system S admits the following state space representation, in $\xi = \left[\begin{array}{ccc} x' & \eta_z' & \bar{\eta}_1 \end{array}\right]' = T_1^{-1}\left[\begin{array}{ccc} x' & \eta_z' & \eta_1 \end{array}\right]'$ coordinates,

(4.4a) $\quad \dot{x} = \bar{A}_1x + \bar{B}_1y_1 + \bar{D}_1w = \bar{A}_1x + \bar{B}_1\bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1\bar{E}_1)w$

$$\text{(4.4b)}\quad \begin{aligned}\dot{\eta}_z &= A_{2,z}\eta_z + A_{2,z1}K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1x) + D_{2,z}w + A_{21,z}(\bar{\eta}_1 + \bar{E}_1w)\\ &= -A_{2,z1}K_1^{-1}C_1x + A_{2,z}\eta_z + (A_{2,z1}K_1^{-1}\hat{K} + A_{21,z})\bar{\eta}_1 + (D_{2,z}\\ &\quad + A_{21,z}\bar{E}_1)w\end{aligned}$$

$$\text{(4.4c)}\quad \begin{aligned}\dot{\bar{\eta}}_1 &= \hat{K}^{-1}\Big(C_1(\bar{A}_1x + \bar{B}_1\bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1\bar{E}_1)w) + K_1\Big(A_{2,1z}\eta_z + a_{2,11}K_1^{-1}\\ &\quad \cdot(\hat{K}\bar{\eta}_1 - C_1x) + b_2u + D_{2,1}w + A_{21,1}(\bar{\eta}_1 + \bar{E}_1w)\Big)\Big)\\ &=: \bar{A}_{2,1x}x + \bar{A}_{2,1z}\eta_z + \bar{a}_{2,11}\bar{\eta}_1 + \hat{K}^{-1}K_1b_2u + \bar{D}_{2,1}w\end{aligned}$$

$$\text{(4.4d)}\quad y_1 = \bar{\eta}_1 + \bar{E}_1w$$

where $\xi(0) = \xi_0 := \begin{bmatrix} x_0' & \eta_{z0}' & \bar{\eta}_{1,0}\end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. Clearly, (4.4) admits relative degree $1 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1b_2 \neq 0$; and is in the extended zero dynamics canonical form (3.2) of [5] with (4.4a) and (4.4b) defining the extended zero dynamics for S.

Now, we will show that (4.4) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \xi_{z0} := \begin{bmatrix} x_0' & \eta_{z0}'\end{bmatrix}' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_1+n_2-1 coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{\eta}_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{[0,\infty)}$ and $\eta_{z[0,\infty)}$ be the solution to (4.4a) and (4.4b). By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} \xi_{z0}' & \acute{\eta}_{1,0}\end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left|\acute{\xi}_0\right| \leq c_1c_w$. Then, $\xi_{z0} = P_zT_1\acute{\xi}_0 \in P_z(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) = \mathcal{D}_{x0} \times P_{\eta z}(\mathcal{D}_{\eta 0})$ and $|\xi_{z0}| \leq c_w$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - 1$ coordinates. Hence, $x_0 \in \mathcal{D}_{x0}$ and $\eta_{z0} \in P_{\eta z}(\mathcal{D}_{\eta 0})$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\|y_{1[0,\infty)}\|_\infty \leq c_w + \|\bar{E}_1\|_{2,2}c_w =: c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\|x_{[0,\infty)}\|_\infty \leq c_{c1}$. Note that $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\|\eta_{1[0,\infty)}\|_\infty \leq \left|K_1^{-1}\hat{K}\right|c_w + \|K_1^{-1}C_1\|_{2,2}c_{c1} =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\|\eta_{z[0,\infty)}\|_\infty \leq c_{c2}$. Therefore, $\|\xi_{z[0,\infty)}\|_\infty \leq \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = \begin{bmatrix} x' & \eta_z'\end{bmatrix}'$. This proves that (4.4) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 3.9 of [5]. This completes the proof for Case 2.

Case 3: $0 = r_1 < n_1$ and $2 \leq r_2 < n_2$. By Lemmas 3.3 and 3.1 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} = \bar{A}_1x + \bar{B}_1(y_1 - E_1w) + D_1w =: \bar{A}_1x + \bar{B}_1y_1 + \bar{D}_1w; & x(0) = x_0\\ y_1 = C_1x + K_1y_2 + E_1w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1\\ \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1; \qquad i = 1,\ldots,r_2-1\\ \dot{\eta}_{r_2} = A_{2,r_2z}\eta_z + a_{2,r_21}\eta_1 + b_2u + D_{2,r_2}w + A_{21,r_2}y_1\\ y_2 = \eta_1 + E_2w + K_{21}y_1 \end{cases}$$

where $\eta = \begin{bmatrix} \eta_z' & \eta_1 & \cdots & \eta_{r_2}\end{bmatrix}'$; η_z is $(n_2 - r_2)$-dimensional; η_i, $i = 1,\ldots,r_2$, are scalars; $\eta(0) = \eta_0 := \begin{bmatrix} \eta_{z0}' & \eta_{1,0} & \cdots & \eta_{r_2,0}\end{bmatrix}' \in \mathcal{D}_{\eta 0}$. Note that

$$y_1 = C_1x + K_1(\eta_1 + E_2w + K_{21}y_1) + E_1w \qquad \Rightarrow$$

$$y_1 = \hat{K}^{-1}\left(C_1 x + K_1\eta_1 + (E_1 + K_1E_2)w\right) =: \bar{\eta}_1 + \bar{E}_1 w$$

Then, $\eta_1 = K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x)$. Define $\bar{\eta}_i = \hat{K}^{-1}K_1\eta_i$, $i = 2,\ldots,r_2$. Then, the composite system S admits the following state space representation, in $\xi = T_1^{-1}\left[\begin{array}{cc} x' & \eta' \end{array}\right]' = \left[\begin{array}{ccccc} x' & \eta_z' & \bar{\eta}_1 & \cdots & \bar{\eta}_{r_2}\end{array}\right]'$ coordinates,

$$\begin{aligned}
&(4.5a)\quad \dot{x} = \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w = \bar{A}_1 x + \bar{B}_1\bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1\bar{E}_1)w\\
&(4.5b)\quad \dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x) + D_{2,z}w + A_{21,z}(\bar{\eta}_1 + \bar{E}_1 w)\\
&\qquad = -A_{2,z1}K_1^{-1}C_1 x + A_{2,z}\eta_z + (A_{2,z1}K_1^{-1}\hat{K} + A_{21,z})\bar{\eta}_1 + (D_{2,z}\\
&\qquad\quad + A_{21,z}\bar{E}_1)w\\
&(4.5c)\quad \dot{\bar{\eta}}_1 = \hat{K}^{-1}\Bigg(C_1(\bar{A}_1 x + \bar{B}_1\bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1\bar{E}_1)w) + K_1\Big(a_{2,11}K_1^{-1}(\hat{K}\bar{\eta}_1\\
&\qquad\quad - C_1 x) + \eta_2 + D_{2,1}w + A_{21,1}(\bar{\eta}_1 + \bar{E}_1 w)\Big)\Bigg)\\
&\qquad =: \bar{A}_{2,1x}x + \bar{a}_{2,11}\bar{\eta}_1 + \bar{\eta}_2 + \bar{D}_{2,1}w\\
&(4.5d)\quad \dot{\bar{\eta}}_i = \hat{K}^{-1}K_1\left(a_{2,i1}K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x) + \eta_{i+1} + D_{2,i}w + A_{21,i}(\bar{\eta}_1 + \bar{E}_1 w)\right)\\
&\qquad =: \bar{A}_{2,ix}x + \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w; \qquad i = 2,\ldots,r_2-1\\
&(4.5e)\quad \dot{\bar{\eta}}_{r_2} = \hat{K}^{-1}K_1\Bigg(A_{2,r_2z}\eta_z + a_{2,r_21}K_1^{-1}(\hat{K}\bar{\eta}_1 - C_1 x) + b_2 u + D_{2,r_2}w\\
&\qquad\quad + A_{21,r_2}(\bar{\eta}_1 + \bar{E}_1 w)\Bigg)\\
&\qquad =: \bar{A}_{2,r_2x}x + \bar{A}_{2,r_2z}\eta_z + \bar{a}_{2,r_21}\bar{\eta}_1 + \hat{K}^{-1}K_1 b_2 u + \bar{D}_{2,r_2}w\\
&(4.5f)\quad y_1 = \bar{\eta}_1 + \bar{E}_1 w
\end{aligned}$$

where $\xi(0) = \xi_0 := \left[\begin{array}{ccccc} x_0' & \eta_{z0}' & \bar{\eta}_{1,0} & \cdots & \bar{\eta}_{r_2,0}\end{array}\right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0}\times\mathcal{D}_{\eta 0}) \neq \emptyset$. Clearly, (4.5) admits relative degree $r_2 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1 b_2 \neq 0$; and is in the form (3.1) with $m_o = 1$.

Now, we will show that (4.5) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by applying Lemma 3.1. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \xi_{z0} := \left[\begin{array}{cc} x_0' & \eta_{z0}'\end{array}\right]' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1+n_2-r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)}\right\|_\infty \le c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\left\|\bar{\eta}_{1[0,\infty)}\right\|_\infty \le c_w$, let $x_{[0,\infty)}$ and $\eta_{z[0,\infty)}$ be the solution to (4.5a) and (4.5b). By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0}\times\mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \left[\begin{array}{ccc} \xi_{z0}' & \acute{\bar{\eta}}_{1,0} & \cdots\acute{\bar{\eta}}_{r_2,0}\end{array}\right]' \in \bar{\mathcal{D}}_0$ with $\left|\acute{\xi}_0\right| \le c_1 c_w$. Then, $\xi_{z0} = P_z T_1\acute{\xi}_0 \in P_z(\mathcal{D}_{x0}\times\mathcal{D}_{\eta 0}) = \mathcal{D}_{x0}\times P_{\eta z}(\mathcal{D}_{\eta 0})$ and $|\xi_{z0}| \le c_w$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first n_2-r_2 coordinates. Hence, $x_0 \in \mathcal{D}_{x0}$ and $\eta_{z0} \in P_{\eta z}(\mathcal{D}_{\eta 0})$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)}\right\|_\infty \le c_w + \left\|\bar{E}_1\right\|_{2,2} c_w =: c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| x_{[0,\infty)}\right\|_\infty \le c_{c1}$. Note that $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\left\|\eta_{1[0,\infty)}\right\|_\infty \le \left|K_1^{-1}\hat{K}\right| c_w + \left\| K_1^{-1}C_1\right\|_{2,2} c_{c1} =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\left\|\eta_{z[0,\infty)}\right\|_\infty \le c_{c2}$. Therefore, $\left\|\xi_{z[0,\infty)}\right\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = \left[\begin{array}{cc} x' & \eta_z'\end{array}\right]'$. By Lemma 3.1, (4.5) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This completes the proof for Case 3.

Case 4: $0 = r_1 < n_1$ and $1 = r_2 = n_2$. By Lemmas 3.3 and 3.2 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} &= \bar{A}_1 x + \bar{B}_1(y_1 - E_1 w) + D_1 w =: \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0 \\ y_1 &= C_1 x + K_1 y_2 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_1 &= a_{2,11}\eta_1 + b_2 u + D_{2,1} w + A_{21,1} y_1 \\ y_2 &= \eta_1 + E_2 w + K_{21} y_1 \end{cases}$$

where $\eta = \eta_1$ is a scalar. Note that

$$\begin{aligned} y_1 &= C_1 x + K_1 (\eta_1 + E_2 w + K_{21} y_1) + E_1 w \qquad \Rightarrow \\ y_1 &= \hat{K}^{-1} (C_1 x + K_1 \eta_1 + (E_1 + K_1 E_2) w) =: \bar{\eta}_1 + \bar{E}_1 w \end{aligned}$$

Then, $\eta_1 = K_1^{-1} (\hat{K}\bar{\eta}_1 - C_1 x)$, and the composite system S admits the following state space representation, in $\xi = \begin{bmatrix} x' & \bar{\eta}_1 \end{bmatrix}' = T_1^{-1} \begin{bmatrix} x' & \eta_1 \end{bmatrix}'$ coordinates,

$$\text{(4.6a)} \quad \dot{x} = \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w = \bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w$$

$$\begin{aligned} \text{(4.6b)} \quad \dot{\bar{\eta}}_1 &= \hat{K}^{-1} \Big(C_1 (\bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w) + K_1 \Big(a_{2,11} K_1^{-1} (\hat{K}\bar{\eta}_1 \\ &\quad - C_1 x) + b_2 u + D_{2,1} w + A_{21,1} (\bar{\eta}_1 + \bar{E}_1 w) \Big) \Big) \\ &=: \bar{A}_{2,1x} x + \bar{a}_{2,11} \bar{\eta}_1 + \hat{K}^{-1} K_1 b_2 u + \bar{D}_{2,1} w \end{aligned}$$

$$\text{(4.6c)} \quad y_1 = \bar{\eta}_1 + \bar{E}_1 w$$

where $\xi(0) = \xi_0 := \begin{bmatrix} x_0' & \bar{\eta}_{1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta_0}) \neq \emptyset$. Clearly, (4.6) admits relative degree $1 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1} K_1 b_2 \neq 0$; and is in the extended zero dynamics canonical form (3.2) of [5]. Hence, (4.6a) defines the extended zero dynamics for S.

Now, we will show that (4.6) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall x_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|x_0| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_1 coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\| w_{[0,\infty)} \|_\infty \leq c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\| \bar{\eta}_{1[0,\infty)} \|_\infty \leq c_w$, let $x_{[0,\infty)}$ be the solution to (4.6a). By Lemma A.3, $\exists c_1 \in [0, \infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} x_0' & \acute{\bar{\eta}}_{1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left| \acute{\xi}_0 \right| \leq c_1 c_w$. Then, $x_0 = P_z T_1 \acute{\xi}_0 \in P_z(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) = \mathcal{D}_{x0}$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\| y_{1[0,\infty)} \|_\infty \leq c_w + \| \bar{E}_1 \|_{2,2} c_w =: c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0, \infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\| x_{[0,\infty)} \|_\infty \leq c_{c1}$. This proves that (4.6) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 3.9 of [5]. This completes the proof for Case 4.

Case 5: $0 = r_1 < n_1$ and $2 \leq r_2 = n_2$. By Lemmas 3.3 and 3.2 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x} &= \bar{A}_1 x + \bar{B}_1(y_1 - E_1 w) + D_1 w =: \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0 \\ y_1 &= C_1 x + K_1 y_2 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_i &= a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i} w + A_{21,i} y_1; \qquad i = 1, \ldots, r_2 - 1 \\ \dot{\eta}_{r_2} &= a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2} w + A_{21,r_2} y_1 \\ y_2 &= \eta_1 + E_2 w + K_{21} y_1 \end{cases}$$

where $\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_{r_2} \end{bmatrix}'$; η_i, $i = 1, \ldots, r_2$, are scalars; and $\eta(0) = \eta_0 := \begin{bmatrix} \eta_{1,0} & \cdots & \eta_{r_2,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0}$. Note that

$$\begin{aligned} y_1 &= C_1 x + K_1 (\eta_1 + E_2 w + K_{21} y_1) + E_1 w \qquad \Rightarrow \\ y_1 &= \hat{K}^{-1} (C_1 x + K_1 \eta_1 + (E_1 + K_1 E_2) w) =: \bar{\eta}_1 + \bar{E}_1 w \end{aligned}$$

Then, $\eta_1 = K_1^{-1} (\hat{K} \bar{\eta}_1 - C_1 x)$. Define $\bar{\eta}_i = \hat{K}^{-1} K_1 \eta_i$, $i = 2, \ldots, r_2$. Then, the composite system S admits the following state space representation, in $\xi = T_1^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}' = \begin{bmatrix} x' & \bar{\eta}_1 & \cdots & \bar{\eta}_{r_2} \end{bmatrix}'$ coordinates,

$$\begin{aligned}
&\text{(4.7a)} \quad \dot{x} = \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w = \bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w \\
&\text{(4.7b)} \quad \dot{\bar{\eta}}_1 = \hat{K}^{-1} \Big(C_1 (\bar{A}_1 x + \bar{B}_1 \bar{\eta}_1 + (\bar{D}_1 + \bar{B}_1 \bar{E}_1) w) + K_1 \Big(a_{2,11} K_1^{-1} (\hat{K} \bar{\eta}_1 \\
&\qquad - C_1 x) + \eta_2 + D_{2,1} w + A_{21,1} (\bar{\eta}_1 + \bar{E}_1 w) \Big) \Big) \\
&\qquad =: \bar{A}_{2,1x} x + \bar{a}_{2,11} \bar{\eta}_1 + \bar{\eta}_2 + \bar{D}_{2,1} w \\
&\text{(4.7c)} \quad \dot{\bar{\eta}}_i = \hat{K}^{-1} K_1 \Big(a_{2,i1} K_1^{-1} (\hat{K} \bar{\eta}_1 - C_1 x) + \eta_{i+1} + D_{2,i} w + A_{21,i} (\bar{\eta}_1 + \bar{E}_1 w) \Big) \\
&\qquad =: \bar{A}_{2,ix} x + \bar{a}_{2,i1} \bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i} w; \qquad i = 2, \ldots, r_2 - 1 \\
&\text{(4.7d)} \quad \dot{\bar{\eta}}_{r_2} = \hat{K}^{-1} K_1 \left(a_{2,r_2 1} K_1^{-1} (\hat{K} \bar{\eta}_1 - C_1 x) + b_2 u + D_{2,r_2} w + A_{21,r_2} (\bar{\eta}_1 + \bar{E}_1 w) \right) \\
&\qquad =: \bar{A}_{2,r_2 x} x + \bar{a}_{2,r_2 1} \bar{\eta}_1 + \hat{K}^{-1} K_1 b_2 u + \bar{D}_{2,r_2} w \\
&\text{(4.7e)} \quad y_1 = \bar{\eta}_1 + \bar{E}_1 w
\end{aligned}$$

where $\xi(0) = \xi_0 := \begin{bmatrix} x_0' & \bar{\eta}_{1,0} & \cdots & \bar{\eta}_{r_2,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. Clearly, (4.7) admits relative degree $r_2 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1} K_1 b_2 \neq 0$; and is in the form (3.1) with $m_o = 1$.

Now, we will show that (4.7) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by applying Lemma 3.1. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall x_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|x_0| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_1 coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{\eta}_{1[0,\infty)} \right\|_\infty \le c_w$, let $x_{[0,\infty)}$ be the solution to (4.7a). By Lemma A.3, $\exists c_1 \in [0, \infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} x_0' & \acute{\bar{\eta}}_{1,0} & \cdots & \acute{\bar{\eta}}_{r_2,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left| \acute{\xi}_0 \right| \le c_1 c_w$. Then, $x_0 = P_z T_1 \acute{\xi}_0 \in P_z(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) = \mathcal{D}_{x0}$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)} \right\|_\infty \le c_w + \left\| \bar{E}_1 \right\|_{2,2} c_w =: c_{w1}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0, \infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| x_{[0,\infty)} \right\|_\infty \le c_{c1}$. By Lemma 3.1, (4.7) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This completes the proof for Case 5.

Case 6: $1 \le r_1 < n_1$ and $0 = r_2 < n_2$. By Lemmas 3.1 and 3.3 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x}_z &= A_{1,z} x_z + A_{1,z1} x_1 + D_{1,z} w \\ \dot{x}_i &= a_{1,i1} x_1 + x_{i+1} + D_{1,i} w; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &= A_{1,r_1 z} x_z + a_{1,r_1 1} x_1 + b_1 y_2 + D_{1,r_1} w \\ y_1 &= x_1 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta} &= \bar{A}_2\eta + \bar{B}_2(y_2 - E_2 w - K_{21}y_1) + D_2 w + A_{21}y_1 \\ &=: \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1; \qquad \eta(0) = \eta_0 \\ y_2 &= C_2\eta + K_2 u + E_2 w + K_{21}y_1 \end{cases}$$

where $x = \begin{bmatrix} x_z' & x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_z is $(n_1 - r_1)$-dimensional; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 := \begin{bmatrix} x_{z0}' & x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$. The composite system $\bar{S}$ with state vector $\begin{bmatrix} x' & \eta' \end{bmatrix}'$, y_1 as output, and y_2 as input admits the following state space representation, in $\xi := \begin{bmatrix} x_z' & \eta' & x_1 & \cdots & x_{r_1} \end{bmatrix}' = T_1^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \tag{4.8a}$$

$$\begin{aligned} \dot{\eta} &= \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1 \\ &= \bar{A}_2\eta + \bar{A}_{21}x_1 + \bar{B}_2 b_1^{-1}(b_1 y_2) + (\bar{D}_2 + \bar{A}_{21}E_1)w \end{aligned} \tag{4.8b}$$

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i = 1, \ldots, r_1 - 1 \tag{4.8c}$$

$$\dot{x}_{r_1} = A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \tag{4.8d}$$

$$y_1 = x_1 + E_1 w \tag{4.8e}$$

with $\xi(0) = \xi_0 := \begin{bmatrix} x_{z0}' & \eta_0' & x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. Clearly, (4.8) is in the form (3.4) with $m_{oa} = 1$ and $m_{ob} = r_1 + 1$.

We will apply Lemma 3.2 to prove that (4.8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall \xi_{z0} := \begin{bmatrix} x_{z0}' & \eta_0' \end{bmatrix}' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1 + n_2 - r_1$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\| x_{i[0,\infty)} \right\|_\infty \le c_w$, $i = 1, \ldots, r_1 + 1$ ($x_{r_1+1} := b_1 y_2$ for notational consistency), let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (4.8a) and (4.8b). By Lemma A.3, $\exists c_1 \in [0, \infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} \xi_{z0}' & \acute{x}_{1,0} & \cdots & \acute{x}_{r_1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left| \acute{\xi}_0 \right| \le c_1 c_w$. Then, $T_1 \acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $\begin{bmatrix} x_{z0}' & \acute{x}_{1,0} & \cdots & \acute{x}_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$, and $\eta_0 \in \mathcal{D}_{\eta 0}$. Hence, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1 - r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0, \infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\| x_{z[0,\infty)} \right\|_\infty \le c_{c1}$. Note that $y_{2[0,\infty)} \in \mathcal{C}$ with $\left\| y_{2[0,\infty)} \right\|_\infty \le \left| b_1^{-1} \right| c_w =: c_{w1}$; $y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)} \right\|_\infty \le c_w + \| E_1 \|_{2,2} c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0, \infty), \mathbb{R})$, then, $\exists c_{c2} \in [0, \infty) \subset \mathbb{R}$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \le c_{c2}$. Then, $\left\| \xi_{z[0,\infty)} \right\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = \begin{bmatrix} x_z' & \eta' \end{bmatrix}'$. By Lemma 3.2, (4.8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree $r_1 = r_1 + r_2$ from y_2 to y_1.

By Lemma 3.1 of [5], $\bar{S}$ admits the following extended zero dynamics canonical form representation, in $\hat{\xi} = \begin{bmatrix} \hat{\xi}_z' & \hat{\xi}_1 & \cdots & \hat{\xi}_{r_1} \end{bmatrix}' = T_2^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\dot{\hat{\xi}}_z = A_z\hat{\xi}_z + A_{z1}\hat{\xi}_1 + D_z w \tag{4.9a}$$

$$\dot{\hat{\xi}}_i = a_{i1}\hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w; \qquad i = 1, \ldots, r_1 - 1 \tag{4.9b}$$

$$\dot{\hat{\xi}}_{r_1} = A_{r_1 z}\hat{\xi}_z + a_{r_1 1}\hat{\xi}_1 + b y_2 + D_{r_1}w \tag{4.9c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{4.9d}$$

where $\hat{\xi}_z$ is $(n_2 + n_1 - r_1)$-dimensional; $\hat{\xi}_i$, $i = 1, \ldots, r_1$, are scalars; $b \neq 0$; $\hat{\xi}(0) = \hat{\xi}_0 := \begin{bmatrix} \hat{\xi}_{z0}' & \hat{\xi}_{1,0} & \cdots & \hat{\xi}_{r_1,0} \end{bmatrix}' \in \hat{\mathcal{D}}_0 := T_2^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. In $\hat{\xi}$ coordinates, we have

$y_2 = \hat{C}_z\hat{\xi}_z + \hat{C}_1\hat{\xi}_1 + \cdots + \hat{C}_{r_1}\hat{\xi}_{r_1} + K_2 u + (E_2 + K_{21}E_1)w$. Then, the composite system S with state vector $\begin{bmatrix} x' & \eta' \end{bmatrix}'$, y_1 as output, and u as input admits the following state space representation, in $\hat{\xi}$ coordinates,

$$\dot{\hat{\xi}}_z = A_z\hat{\xi}_z + A_{z1}\hat{\xi}_1 + D_z w \tag{4.10a}$$

$$\dot{\hat{\xi}}_i = a_{i1}\hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w; \qquad i = 1, \ldots, r_1 - 1 \tag{4.10b}$$

$$\dot{\hat{\xi}}_{r_1} = \hat{A}_{r_1 z}\hat{\xi}_z + \hat{a}_{r_1 1}\hat{\xi}_1 + \cdots + \hat{a}_{r_1 r_1}\hat{\xi}_{r_1} + bK_2 u + \hat{D}_{r_1} w \tag{4.10c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{4.10d}$$

with $bK_2 \neq 0$. Clearly, (4.10) is in the form of (3.1) with $m_o = 1$. By the fact that (4.8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$, then (4.9) is minimum phase with respect to $\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$. By Lemma 3.9 of [5], $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall \hat{\xi}_{z0} \in P_z(\hat{\mathcal{D}}_0)$ with $\left|\hat{\xi}_{z0}\right| \leq c_w$, $\forall \hat{\xi}_{1[0,\infty)} \in \mathcal{C}$ with $\left\|\hat{\xi}_{1[0,\infty)}\right\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\|w_{[0,\infty)}\right\|_\infty \leq c_w$, we have $\left\|\hat{\xi}_{z[0,\infty)}\right\|_\infty \leq c_c$, where $\hat{\xi}_{z[0,\infty)}$ is the solution to (4.9a) with initial condition $\hat{\xi}_{z0}$ and inputs $\hat{\xi}_{1[0,\infty)}$ and $w_{[0,\infty)}$. Then, the assumption for Lemma 3.1 is satisfied for (4.10). Therefore, (4.10) is minimum phase with respect to $\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree $r_1 = r_1 + r_2$ from u to y_1. This completes the proof for Case 6.

Case 7: $1 \leq r_1 = n_1$ and $0 = r_2 < n_2$. By Lemmas 3.2 and 3.3 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &= a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 &= x_1 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta} &= \bar{A}_2\eta + \bar{B}_2(y_2 - E_2 w - K_{21}y_1) + D_2 w + A_{21}y_1 \\ &=: \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1; \qquad \eta(0) = \eta_0 \\ y_2 &= C_2\eta + K_2 u + E_2 w + K_{21}y_1 \end{cases}$$

where $x = \begin{bmatrix} x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 := \begin{bmatrix} x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$. The composite system $\bar{S}$ with state vector $\begin{bmatrix} x' & \eta' \end{bmatrix}'$, y_1 as output, and y_2 as input admits the following state space representation, in $\xi := \begin{bmatrix} \eta' & x_1 & \cdots & x_{r_1} \end{bmatrix}' = T_1^{-1}\begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\begin{aligned} \dot{\eta} &= \bar{A}_2\eta + \bar{B}_2 y_2 + \bar{D}_2 w + \bar{A}_{21}y_1 \\ &= \bar{A}_2\eta + \bar{A}_{21}x_1 + \bar{B}_2 b_1^{-1}(b_1 y_2) + (\bar{D}_2 + \bar{A}_{21}E_1)w \end{aligned} \tag{4.11a}$$

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i = 1, \ldots, r_1 - 1 \tag{4.11b}$$

$$\dot{x}_{r_1} = a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \tag{4.11c}$$

$$y_1 = x_1 + E_1 w \tag{4.11d}$$

with $\xi(0) = \xi_0 := \begin{bmatrix} \eta_0' & x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0} \times \mathcal{D}_{x0} =: \bar{\mathcal{D}}_0 \neq \emptyset$. Clearly, (4.11) is in the form (3.1) with $m_o = r_1 + 1$.

We will apply Lemma 3.1 to prove that (4.11) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree r_1. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_0 \in P_z(\bar{\mathcal{D}}_0) = \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_2 coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\|w_{[0,\infty)}\right\|_\infty \leq c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\|x_{i[0,\infty)}\right\|_\infty \leq$

c_w, $i = 1, \ldots, r_1 + 1$ ($x_{r_1+1} := b_1 y_2$ for notational consistency), let $\eta_{[0,\infty)}$ be the solution to (4.11a). Note that $y_{2[0,\infty)} \in \mathcal{C}$ with $\left\| y_{2[0,\infty)} \right\|_\infty \le \left| b_1^{-1} \right| c_w =: c_{w1}$; $y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)} \right\|_\infty \le c_w + \| E_1 \|_{2,2} c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty), \mathbb{R})$, then, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \le c_{c2}$. By Lemma 3.1, (4.11) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree $r_1 = r_1 + r_2$ from y_2 to y_1.

By Lemma 3.1 of [5], $\bar{S}$ admits the following extended zero dynamics canonical form representation, in $\hat{\xi} = \begin{bmatrix} \hat{\xi}_z' & \hat{\xi}_1 & \cdots & \hat{\xi}_{r_1} \end{bmatrix}' = T_2^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\dot{\hat{\xi}}_z = A_z \hat{\xi}_z + A_{z1} \hat{\xi}_1 + D_z w \tag{4.12a}$$

$$\dot{\hat{\xi}}_i = a_{i1} \hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w; \qquad i = 1, \ldots, r_1 - 1 \tag{4.12b}$$

$$\dot{\hat{\xi}}_{r_1} = A_{r_1 z} \hat{\xi}_z + a_{r_1 1} \hat{\xi}_1 + b y_2 + D_{r_1} w \tag{4.12c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{4.12d}$$

where $\hat{\xi}_z$ is n_2-dimensional; $\hat{\xi}_i$, $i = 1, \ldots, r_1$, are scalars; $b \neq 0$; $\hat{\xi}(0) = \hat{\xi}_0 := \begin{bmatrix} \hat{\xi}_{z0}' & \hat{\xi}_{1,0} & \cdots & \hat{\xi}_{r_1,0} \end{bmatrix}' \in \hat{\mathcal{D}}_0 := T_2^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. In $\hat{\xi}$ coordinates, we have $y_2 = \hat{C}_z \hat{\xi}_z + \hat{C}_1 \hat{\xi}_1 + \cdots + \hat{C}_{r_1} \hat{\xi}_{r_1} + K_2 u + (E_2 + K_{21} E_1) w$. Then, the composite system S with state vector $\begin{bmatrix} x' & \eta' \end{bmatrix}'$, y_1 as output, and u as input admits the following state space representation, in $\hat{\xi}$ coordinates,

$$\dot{\hat{\xi}}_z = A_z \hat{\xi}_z + A_{z1} \hat{\xi}_1 + D_z w \tag{4.13a}$$

$$\dot{\hat{\xi}}_i = a_{i1} \hat{\xi}_1 + \hat{\xi}_{i+1} + D_i w; \qquad i = 1, \ldots, r_1 - 1 \tag{4.13b}$$

$$\dot{\hat{\xi}}_{r_1} = \hat{A}_{r_1 z} \hat{\xi}_z + \hat{a}_{r_1 1} \hat{\xi}_1 + \cdots + \hat{a}_{r_1 r_1} \hat{\xi}_{r_1} + b K_2 u + \hat{D}_{r_1} w \tag{4.13c}$$

$$y_1 = \hat{\xi}_1 + E_1 w \tag{4.13d}$$

with $bK_2 \neq 0$. Clearly, (4.13) is in the form of (3.1) with $m_o = 1$. By the fact that (4.11) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$, then (4.12) is minimum phase with respect to $\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$. By Lemma 3.9 of [5], $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall \hat{\xi}_{z0} \in P_z(\hat{\mathcal{D}}_0)$ with $\left| \hat{\xi}_{z0} \right| \le c_w$, $\forall \hat{\xi}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \hat{\xi}_{1[0,\infty)} \right\|_\infty \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, we have $\left\| \hat{\xi}_{z[0,\infty)} \right\|_\infty \le c_c$, where $\hat{\xi}_{z[0,\infty)}$ is the solution to (4.12a) with initial condition $\hat{\xi}_{z0}$ and inputs $\hat{\xi}_{1[0,\infty)}$ and $w_{[0,\infty)}$. Then, the assumption for Lemma 3.1 is satisfied for (4.13). Then, (4.13) is minimum phase with respect to $\hat{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree $r_1 = r_1 + r_2$ from u to y_1. This completes the proof for Case 7.

Case 8: $1 \le r_1 = n_1$ and $1 \le r_2 = n_2$. By Lemma 3.2 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x}_i &= a_{1,i1} x_1 + x_{i+1} + D_{1,i} w; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &= a_{1,r_1 1} x_1 + b_1 y_2 + D_{1,r_1} w \\ y_1 &= x_1 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_i &= a_{2,i1} \eta_1 + \eta_{i+1} + D_{2,i} w + A_{21,i} y_1; \qquad i = 1, \ldots, r_2 - 1 \\ \dot{\eta}_{r_2} &= a_{2,r_2 1} \eta_1 + b_2 u + D_{2,r_2} w + A_{21,r_2} y_1 \\ y_2 &= \eta_1 + E_2 w + K_{21} y_1 \end{cases}$$

where $x = \begin{bmatrix} x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_i, $i = 1, \ldots, r_1$, are scalars; $\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_{r_2} \end{bmatrix}'$; η_i, $i = 1, \ldots, r_2$ are scalars.

It is straightforward to check that the composite system S admits relative degree $r_1 + r_2$ from u to y_1 since $b_1 b_2 \neq 0$. Since the relative degree equals to the dimension of system, which implies that the extended zero dynamics is absent by Lemma 3.2 of [5], then, S is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This completes the proof for Case 8.

Case 9: $1 \le r_1 < n_1$ and $1 \le r_2 = n_2$. By Lemmas 3.1 and 3.2 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \neq 0$, $b_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x}_z &= A_{1,z} x_z + A_{1,z1} x_1 + D_{1,z} w \\ \dot{x}_i &= a_{1,i1} x_1 + x_{i+1} + D_{1,i} w; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &= A_{1,r_1 z} x_z + a_{1,r_1 1} x_1 + b_1 y_2 + D_{1,r_1} w \\ y_1 &= x_1 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_i &= a_{2,i1} \eta_1 + \eta_{i+1} + D_{2,i} w + A_{21,i} y_1; \qquad i = 1, \ldots, r_2 - 1 \\ \dot{\eta}_{r_2} &= a_{2,r_2 1} \eta_1 + b_2 u + D_{2,r_2} w + A_{21,r_2} y_1 \\ y_2 &= \eta_1 + E_2 w + K_{21} y_1 \end{cases}$$

where $x = \begin{bmatrix} x_z' & x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_z is $(n_1 - r_1)$-dimensional; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 := \begin{bmatrix} x_{z0}' & x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$; $\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_{r_2} \end{bmatrix}'$; η_i, $i = 1, \ldots, r_2$, are scalars; $\eta(0) = \eta_0 := \begin{bmatrix} \eta_{1,0} & \cdots & \eta_{r_2,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0}$.

Define $\bar{\eta}_i = b_1 \eta_i$, $i = 1, \ldots, r_2$. Then, the composite system S admits the following state space representation, in $\xi = \begin{bmatrix} x_z' & x_1 & \cdots & x_{r_1} & \bar{\eta}_1 & \cdots & \bar{\eta}_{r_2} \end{bmatrix}' = T_1^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\begin{aligned}
&(4.14\text{a}) & \dot{x}_z &= A_{1,z} x_z + A_{1,z1} x_1 + D_{1,z} w \\
&(4.14\text{b}) & \dot{x}_i &= a_{1,i1} x_1 + x_{i+1} + D_{1,i} w; \qquad i = 1, \ldots, r_1 - 1 \\
&(4.14\text{c}) & \dot{x}_{r_1} &= A_{1,r_1 z} x_z + a_{1,r_1 1} x_1 + b_1 \left(b_1^{-1} \bar{\eta}_1 + E_2 w + K_{21} \left(x_1 + E_1 w \right) \right) + D_{1,r_1} w \\
&& &=: A_{1,r_1 z} x_z + \bar{a}_{1,r_1 1} x_1 + \bar{\eta}_1 + \bar{D}_{1,r_1} w \\
&(4.14\text{d}) & \dot{\bar{\eta}}_i &= b_1 \left(a_{2,i1} \eta_1 + \eta_{i+1} + D_{2,i} w + A_{21,i} \left(x_1 + E_1 w \right) \right) \\
&& &=: \bar{A}_{21,i} x_1 + \bar{a}_{2,i1} \bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i} w; \qquad i = 1, \ldots, r_2 - 1 \\
&(4.14\text{e}) & \dot{\bar{\eta}}_{r_2} &= b_1 \left(a_{2,r_2 1} \eta_1 + b_2 u + D_{2,r_2} w + A_{21,r_2} \left(x_1 + E_1 w \right) \right) \\
&& &=: \bar{A}_{21,r_2} x_1 + \bar{a}_{2,r_2 1} \bar{\eta}_1 + b_1 b_2 u + \bar{D}_{2,r_2} w \\
&(4.14\text{f}) & y_1 &= x_1 + E_1 w
\end{aligned}$$

where $\xi(0) = \xi_0 := \begin{bmatrix} x_{z0}' & x_{1,0} & \cdots & x_{r_1,0} & \bar{\eta}_{1,0} & \cdots & \bar{\eta}_{r_2,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. Clearly, (4.14) is in the form (3.1) with $m_o = 1$.

We will apply Lemma 3.1 to complete the proof for this case. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall x_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|x_{z0}| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1 - r_1$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall x_{1[0,\infty)} \in \mathcal{C}$ with $\left\| x_{1[0,\infty)} \right\|_\infty \le c_w$, let $x_{z[0,\infty)}$ be the solution to (4.14a). By Lemma A.3, $\exists c_1 \in [0, \infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} x_{z0}' & \acute{\xi}_{1,0} & \cdots & \acute{\xi}_{n_2+r_1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left| \acute{\xi}_0 \right| \le c_1 c_w$. Then, $T_1 \acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\begin{bmatrix} x_{z0}' & \acute{\xi}_{1,0} & \cdots & \acute{\xi}_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$. Hence, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\| x_{z[0,\infty)} \right\|_\infty \le c_{c1}$. By Lemma 3.1, (4.14) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree $r_1 + r_2$ from u to y_1. This completes the proof for Case 9.

Case 10: $1 \le r_1 = n_1$ and $1 \le r_2 < n_2$. By Lemmas 3.2 and 3.1 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \ne 0$, $b_2 \ne 0$,

$$S_1 : \begin{cases} \dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i = 1,\ldots,r_1 - 1 \\ \dot{x}_{r_1} &= a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 &= x_1 + E_1 w \end{cases}$$

$$S_2 : \begin{cases} \dot{\eta}_z &= A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\ \dot{\eta}_i &= a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1; \qquad i = 1,\ldots,r_2 - 1 \\ \dot{\eta}_{r_2} &= A_{2,r_2 z}\eta_z + a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 &= \eta_1 + E_2 w + K_{21}y_1 \end{cases}$$

where $x = \begin{bmatrix} x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_i, $i = 1,\ldots,r_1$, are scalars; $x(0) = x_0 := \begin{bmatrix} x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$; $\eta = \begin{bmatrix} \eta_z' & \eta_1 & \cdots & \eta_{r_2} \end{bmatrix}'$; η_z is $(n_2 - r_2)$-dimensional; η_i, $i = 1,\ldots,r_2$, are scalars; $\eta(0) = \eta_0 := \begin{bmatrix} \eta_{z0}' & \eta_{1,0} & \cdots & \eta_{r_2,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0}$.

Define $\bar{\eta}_i = b_1\eta_i$, $i = 1,\ldots,r_2$. Then, the composite system S admits the following state space representation, in $\xi = \begin{bmatrix} \eta_z' & x_1 & \cdots & x_{r_1} & \bar{\eta}_1 & \cdots & \bar{\eta}_{r_2} \end{bmatrix}' = T_1^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\begin{aligned}
&\text{(4.15a)} \quad & \dot{\eta}_z &= A_{2,z}\eta_z + A_{21,z}x_1 + A_{2,z1}b_1^{-1}\bar{\eta}_1 + (D_{2,z} + A_{21,z}E_1)w \\
&\text{(4.15b)} \quad & \dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i = 1,\ldots,r_1 - 1 \\
&\text{(4.15c)} \quad & \dot{x}_{r_1} &= (a_{1,r_1 1} + b_1 K_{21})x_1 + \bar{\eta}_1 + (D_{1,r_1} + b_1 E_2 + b_1 K_{21}E_1)w \\
&& &=: \bar{a}_{1,r_1 1}x_1 + \bar{\eta}_1 + \bar{D}_{1,r_1}w \\
&\text{(4.15d)} \quad & \dot{\bar{\eta}}_i &= b_1\left(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}\left(x_1 + E_1 w\right)\right) \\
&& &=: b_1 A_{21,i}x_1 + \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w; \qquad i = 1,\ldots,r_2 - 1 \\
&\text{(4.15e)} \quad & \dot{\bar{\eta}}_{r_2} &= b_1\left(A_{2,r_2 z}\eta_z + a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}\left(x_1 + E_1 w\right)\right) \\
&& &=: b_1 A_{2,r_2 z}\eta_z + b_1 A_{21,r_2}x_1 + \bar{a}_{2,r_2 1}\bar{\eta}_1 + b_1 b_2 u + \bar{D}_{2,r_2}w \\
&\text{(4.15f)} \quad & y_1 &= x_1 + E_1 w
\end{aligned}$$

where $\xi(0) = \xi_0 = \begin{bmatrix} \eta_{z0}' & x_{1,0} & \cdots & x_{r_1,0} & \bar{\eta}_{1,0} & \cdots & \bar{\eta}_{r_2,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \ne \emptyset$. Clearly, (4.15) is in the form of (3.1) with $m_o = r_1 + 1$.

We will apply Lemma 3.1 to complete the proof for this case. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_{z0}| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2 - r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_2 - r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\| x_{i[0,\infty)} \right\|_\infty \le c_w$, $i = 1,\ldots,r_1$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{\eta}_{1[0,\infty)} \right\|_\infty \le c_w$, let $\eta_{z[0,\infty)}$ be the solution to (4.15a). By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} \eta_{z0}' & \acute{\xi}_{1,0} & \cdots & \acute{\xi}_{n_1+r_2,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left| \acute{\xi}_0 \right| \le c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\begin{bmatrix} \eta_{z0}' & b_1^{-1}\acute{\xi}_{n_1+1,0} & \cdots & b_1^{-1}\acute{\xi}_{n_1+r_2,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0}$. Hence, $\eta_{z0} \in P_{\eta z}(\mathcal{D}_{\eta 0})$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 - r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - r_2$ coordinates. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)} \right\|_\infty = \left\| x_{1[0,\infty)} + E_1 w_{[0,\infty)} \right\|_\infty \le c_w + \left\| E_1 \right\|_{2,2} c_w =: c_{w1}$ and $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \eta_{1[0,\infty)} \right\|_\infty = \left\| b_1^{-1}\bar{\eta}_{1[0,\infty)} \right\|_\infty \le \left| b_1^{-1} \right| c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$

and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\left\| \eta_{z[0,\infty)} \right\|_\infty \le c_{c1}$. By Lemma 3.1, (4.15) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree r_1+r_2 from u to y_1. This completes the proof for Case 10.

Case 11: $1 \le r_1 < n_1$ and $1 \le r_2 < n_2$. By Lemma 3.1 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \ne 0$, $b_2 \ne 0$,

$$S_1 : \left\{ \begin{array}{rcl} \dot{x}_z &=& A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \\ \dot{x}_i &=& a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i=1,\dots,r_1-1 \\ \dot{x}_{r_1} &=& A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1 y_2 + D_{1,r_1}w \\ y_1 &=& x_1 + E_1 w \end{array} \right.$$

$$S_2 : \left\{ \begin{array}{rcl} \dot{\eta}_z &=& A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\ \dot{\eta}_i &=& a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1; \qquad i=1,\dots,r_2-1 \\ \dot{\eta}_{r_2} &=& A_{2,r_2 z}\eta_z + a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 &=& \eta_1 + E_2 w + K_{21}y_1 \end{array} \right.$$

where $x = \left[\begin{array}{cccc} x_z' & x_1 & \cdots & x_{r_1} \end{array} \right]'$; x_z is (n_1-r_1)-dimensional; x_i, $i=1,\dots,r_1$, are scalars; $x(0)=x_0 := \left[\begin{array}{cccc} x_{z0}' & x_{1,0} & \cdots & x_{r_1,0} \end{array} \right]' \in \mathcal{D}_{x0}$; $\eta = \left[\begin{array}{cccc} \eta_z' & \eta_1 & \cdots & \eta_{r_2} \end{array} \right]'$; η_z is (n_2-r_2)-dimensional; η_i, $i=1,\dots,r_2$, are scalars; $\eta(0)=\eta_0 := \left[\begin{array}{cccc} \eta_{z0}' & \eta_{1,0} & \cdots & \eta_{r_2,0} \end{array} \right]' \in \mathcal{D}_{\eta 0}$.

Define $\bar{\eta}_i = b_1\eta_i$, $i=1,\dots,r_2$. Then, the composite system S admits the following state space representation, in $\xi = \left[\begin{array}{ccccccc} x_z' & \eta_z' & x_1 & \cdots & x_{r_1} & \bar{\eta}_1 & \cdots & \bar{\eta}_{r_2} \end{array} \right]' = T_1^{-1} \left[\begin{array}{cc} x' & \eta' \end{array} \right]'$ coordinates,

$$\begin{aligned}
&(4.16\text{a}) \quad \dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \\
&(4.16\text{b}) \quad \dot{\eta}_z = A_{2,z}\eta_z + A_{21,z}x_1 + A_{2,z1}b_1^{-1}\bar{\eta}_1 + (D_{2,z} + A_{21,z}E_1)w \\
&(4.16\text{c}) \quad \dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i=1,\dots,r_1-1 \\
&(4.16\text{d}) \quad \dot{x}_{r_1} = A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1\left(\eta_1 + K_{21}x_1 + (E_2 + K_{21}E_1)w\right) + D_{1,r_1}w \\
&\qquad\qquad =: A_{1,r_1 z}x_z + \bar{a}_{1,r_1 1}x_1 + \bar{\eta}_1 + \bar{D}_{1,r_1}w \\
&(4.16\text{e}) \quad \dot{\bar{\eta}}_i = b_1\left(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}\left(x_1 + E_1 w\right)\right) \\
&\qquad\qquad =: b_1 A_{21,i}x_1 + \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w; \qquad i=1,\dots,r_2-1 \\
&(4.16\text{f}) \quad \dot{\bar{\eta}}_{r_2} = b_1\left(A_{2,r_2 z}\eta_z + a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}\left(x_1 + E_1 w\right)\right) \\
&\qquad\qquad =: b_1 A_{2,r_2 z}\eta_z + b_1 A_{21,r_2}x_1 + \bar{a}_{2,r_2 1}\bar{\eta}_1 + b_1 b_2 u + \bar{D}_{2,r_2}w \\
&(4.16\text{g}) \quad y_1 = x_1 + E_1 w
\end{aligned}$$

where $\xi(0) = \xi_0 := \left[\begin{array}{ccccccc} x_{z0}' & \eta_{z0}' & x_{1,0} & \cdots & x_{r_1,0} & \bar{\eta}_{1,0} & \cdots & \bar{\eta}_{r_2,0} \end{array} \right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \ne \emptyset$. Clearly, (4.16) is in the form of (3.4) with $m_{oa}=1$ and $m_{ob}=r_1+1$.

We will apply Lemma 3.2 to complete the proof for this case. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \xi_{z0} := \left[\begin{array}{cc} x_{z0}' & \eta_{z0}' \end{array} \right]' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_1-r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1-r_1+n_2-r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\left\| x_{i[0,\infty)} \right\|_\infty \le c_w$, $i=1,\dots,r_1$, and $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{\eta}_{1[0,\infty)} \right\|_\infty \le c_w$, let $x_{z[0,\infty)}$ and $\eta_{z[0,\infty)}$ be the solution to (4.16a) and (4.16b). By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \left[\begin{array}{cccc} \acute{\xi}_{z0}' & \acute{\xi}_{1,0} & \cdots & \acute{\xi}_{r_1+r_2,0} \end{array} \right]' \in \bar{\mathcal{D}}_0$ with $\left| \acute{\xi}_0 \right| \le c_1 c_w$.

Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $\begin{bmatrix} x'_{z0} & \acute{\xi}_{1,0} & \cdots & \acute{\xi}_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates, $\begin{bmatrix} \eta'_{z0} & b_1^{-1}\acute{\xi}_{r_1+1,0} & \cdots & b_1^{-1}\acute{\xi}_{r_1+r_2,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0}$, and $\eta_{z0} \in P_{\eta z}(\mathcal{D}_{\eta 0})$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - r_2$ coordinates. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\| x_{z[0,\infty)} \right\|_\infty \le c_{c1}$. Note that $y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)} \right\|_\infty = \left\| x_{1[0,\infty)} + E_1 w_{[0,\infty)} \right\|_\infty \le c_w + \left\| E_1 \right\|_{2,2} c_w =: c_{w1}$. Note also that $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \eta_{1[0,\infty)} \right\|_\infty = \left\| b_1^{-1}\bar{\eta}_{1[0,\infty)} \right\|_\infty \le \left| b_1^{-1} \right| c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\left\| \eta_{z[0,\infty)} \right\|_\infty \le c_{c2}$. Therefore, $\left\| \xi_{z[0,\infty)} \right\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = \begin{bmatrix} x'_z & \eta'_z \end{bmatrix}'$. By Lemma 3.2, (4.16) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits relative degree $r_1 + r_2$ from u to y_1. This completes the proof for Case 11.

Case 12: $0 = r_1 < n_1$ and $0 = r_2 = n_2$. By Lemma 3.3 and Definition 3.4 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \left\{ \begin{array}{rcl} \dot{x} &=& \bar{A}_1 x + \bar{B}_1(y_1 - E_1 w) + D_1 w =: \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0 \\ y_1 &=& C_1 x + K_1 y_2 + E_1 w \end{array} \right.$$
$$S_2 : \; y_2 \;=\; K_2 u + E_2 w + K_{21} y_1$$

Then, $\mathcal{D}_{\eta 0} = \mathbb{R}^0$. Note that

$$\begin{aligned} & y_1 = C_1 x + K_1 (K_2 u + E_2 w + K_{21} y_1) + E_1 w \qquad \Rightarrow \\ & y_1 = \hat{K}^{-1} (C_1 x + K_1 K_2 u + (E_1 + K_1 E_2) w) \end{aligned}$$

Then, the composite system S admits the following representation

$$\begin{aligned} \dot{x} &= \bar{A}_1 x + \bar{B}_1 y_1 + \bar{D}_1 w; \qquad x(0) = x_0 \\ y_1 &= \hat{K}^{-1} C_1 x + \hat{K}^{-1} K_1 K_2 u + \hat{K}^{-1} (E_1 + K_1 E_2) w \end{aligned}$$

Clearly, the above representation is in the extended zero dynamics canonical form (3.9) of [5] since $\hat{K}^{-1} K_1 K_2 \neq 0$. Hence, S admits relative degree $0 = r_1 + r_2$ from u to y_1 and is minimum phase with respect to $\mathcal{D}_{x0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$. This completes the proof for Case 12.

Case 13: $1 \le r_1 = n_1$ and $0 = r_2 = n_2$. By Lemma 3.2 and Definition 3.4 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \left\{ \begin{array}{rcl} \dot{x}_i &=& a_{1,i1} x_1 + x_{i+1} + D_{1,i} w; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &=& a_{1,r_1 1} x_1 + b_1 y_2 + D_{1,r_1} w \\ y_1 &=& x_1 + E_1 w \end{array} \right.$$
$$S_2 : \; y_2 \;=\; K_2 u + E_2 w + K_{21} y_1$$

where $x = \begin{bmatrix} x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 := \begin{bmatrix} x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$; and $\mathcal{D}_{\eta 0} = \mathbb{R}^0$. Then, the composite system S admits the following state space representation,

$$\dot{x}_i = a_{1,i1} x_1 + x_{i+1} + D_{1,i} w; \qquad i = 1, \ldots, r_1 - 1$$

$$\begin{aligned}\dot{x}_{r_1} &= a_{1,r_11}x_1 + b_1\left(K_2u + E_2w + K_{21}\left(x_1 + E_1w\right)\right) + D_{1,r_1}w \\ &=: \bar{a}_{1,r_11}x_1 + b_1K_2u + \bar{D}_{1,r_1}w \\ y_1 &= x_1 + E_1w\end{aligned}$$

Clearly, S admits relative degree $r_1 = r_1 + r_2 = n_1 + n_2$ from u to y_1 since $b_1K_2 \neq 0$. Then, S is minimum phase with respect to $\mathcal{D}_{x0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof for Case 13.

Case 14: $1 \leq r_1 < n_1$ and $0 = r_2 = n_2$. By Lemma 3.1 and Definition 3.4 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $b_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \begin{cases} \dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \\ \dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &= A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1y_2 + D_{1,r_1}w \\ y_1 &= x_1 + E_1w \end{cases}$$
$$S_2 : \; y_2 = K_2u + E_2w + K_{21}y_1$$

where $x = \left[\begin{array}{cccc} x_z' & x_1 & \cdots & x_{r_1}\end{array}\right]'$; x_z is $(n_1 - r_1)$-dimensional; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 := \left[\begin{array}{cccc} x_{z0}' & x_{1,0} & \cdots & x_{r_1,0}\end{array}\right]' \in \mathcal{D}_{x0}$; and $\mathcal{D}_{\eta 0} = \mathbb{R}^0$. Then, the composite system S admits the following state space representation,

$$\begin{aligned}\dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,z}w \\ \dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,i}w; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &= A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1\left(K_2u + E_2w + K_{21}\left(x_1 + E_1w\right)\right) + D_{1,r_1}w \\ &=: A_{1,r_1z}x_z + \bar{a}_{1,r_11}x_1 + b_1K_2u + \bar{D}_{1,r_1}w \\ y_1 &= x_1 + E_1w\end{aligned}$$

Clearly, S admits relative degree $r_1 = r_1 + r_2$ from u to y_1 since $b_1K_2 \neq 0$. Then, S is minimum phase with respect to $\mathcal{D}_{x0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{W}_d$. This completes the proof for Case 14.

Case 15: $0 = r_1 = n_1$ and $0 = r_2 < n_2$. By Definition 3.4 and Lemma 3.3 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $K_2 \neq 0$,

$$S_1 : \; y_1 = K_1y_2 + E_1w$$
$$S_2 : \begin{cases} \dot{\eta} &= \bar{A}_2\eta + \bar{B}_2(y_2 - E_2w - K_{21}y_1) + D_2w + A_{21}y_1 \\ &=: \bar{A}_2\eta + \bar{B}_2y_2 + \bar{D}_2w + \bar{A}_{21}y_1; \qquad \eta(0) = \eta_0 \\ y_2 &= C_2\eta + K_2u + E_2w + K_{21}y_1 \end{cases}$$

where $\mathcal{D}_{x0} = \mathbb{R}^0$; and $\eta_0 \in \mathcal{D}_{\eta 0}$. Note that

$$\begin{aligned}&y_1 = K_1\left(C_2\eta + K_2u + E_2w + K_{21}y_1\right) + E_1w \qquad \Rightarrow \\ &y_1 = \hat{K}^{-1}\left(K_1C_2\eta + K_1K_2u + (K_1E_2 + E_1)w\right) \\ &y_2 = K_1^{-1}\left(y_1 - E_1w\right)\end{aligned}$$

Hence, the composite system S admits the following representation:

$$\begin{aligned}\text{(4.17a)} \quad \dot{\eta} &= \bar{A}_2\eta + \bar{B}_2y_2 + \bar{D}_2w + \bar{A}_{21}y_1 \\ &= \bar{A}_2\eta + (\bar{A}_{21} + \bar{B}_2K_1^{-1})y_1 + (\bar{D}_2 - \bar{B}_2K_1^{-1}E_1)w; \qquad \eta(0) = \eta_0\end{aligned}$$
$$\text{(4.17b)} \quad y_1 = \hat{K}^{-1}\left(K_1C_2\eta + K_1K_2u + (K_1E_2 + E_1)w\right)$$

Clearly, the above representation is in the extended zero dynamics canonical form (3.9) of [5]; and admits relative degree $0 = r_1 + r_2$ from u to y_1, since $\hat{K}^{-1}K_1K_2 \neq 0$.

$\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)} \right\|_\infty \leq c_w$, let $\eta_{[0,\infty)}$ be the solution to (4.17a). Then, $y_{2[0,\infty)} := K_1^{-1}(y_{1[0,\infty)} - E_1 w_{[0,\infty)}) \in \mathcal{C}$ with $\left\| y_{2[0,\infty)} \right\|_\infty \leq \left| K_1^{-1} \right| c_w + \left\| K_1^{-1} E_1 \right\|_{2,2} c_w =: c_{w1}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty), \mathbb{R})$, then, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \leq c_c$. Hence, the composite system (4.17) is minimum phase with respect to $\mathcal{D}_{\eta 0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This completes the proof for Case 15.

Case 16: $0 = r_1 = n_1$ and $1 \leq r_2 = n_2$. By Definition 3.4 and Lemma 3.2 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$S_1: \; y_1 = K_1 y_2 + E_1 w$$

$$S_2: \begin{cases} \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1; \qquad i = 1,\ldots,r_2-1 \\ \dot{\eta}_{r_2} = a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2 w + K_{21}y_1 \end{cases}$$

where $\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_{r_2} \end{bmatrix}'$; η_i, $i = 1,\ldots,r_2$, are scalars; $\eta(0) = \eta_0 := \begin{bmatrix} \eta_{1,0} & \cdots & \eta_{r_2,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0}$; and $\mathcal{D}_{x0} = \mathbb{R}^0$. Note that

$$y_1 = K_1(\eta_1 + E_2 w + K_{21}y_1) + E_1 w \qquad \Rightarrow$$
$$y_1 = \hat{K}^{-1}(K_1\eta_1 + (K_1E_2 + E_1)w) =: \bar{\eta}_1 + \bar{E}_1 w$$

Define $\bar{\eta}_i = \hat{K}^{-1}K_1\eta_i$, $i = 1,\ldots,r_2$. Then, the composite system S admits the following state space representation in $\xi = \begin{bmatrix} \bar{\eta}_1 & \cdots & \bar{\eta}_{r_2} \end{bmatrix}' = T_1^{-1}\eta$ coordinates:

$$\text{(4.18a)} \qquad \dot{\bar{\eta}}_i = \hat{K}^{-1}K_1(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}(\bar{\eta}_1 + \bar{E}_1 w)) =: \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w; \qquad i = 1,\ldots,r_2-1$$

$$\text{(4.18b)} \qquad \dot{\bar{\eta}}_{r_2} = \hat{K}^{-1}K_1(a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}(\bar{\eta}_1 + \bar{E}_1 w)) =: \bar{a}_{2,r_2 1}\bar{\eta}_1 + \hat{K}^{-1}K_1 b_2 u + \bar{D}_{2,r_2}w$$

$$\text{(4.18c)} \qquad y_1 = \bar{\eta}_1 + \bar{E}_1 w$$

Clearly, the above representation is in the extended zero dynamics canonical form (3.7) of [5], which is minimum phase with respect to $T_1(\mathcal{D}_{\eta 0}) = T_1(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$ and $\mathcal{W}_d$ since the extended zero dynamics is absent, and admits relative degree $r_2 = r_1 + r_2 = n_1 + n_2$ from u to y_1 since $\hat{K}^{-1}K_1 b_2 \neq 0$. This completes the proof for Case 16.

Case 17: $0 = r_1 = n_1$ and $1 \leq r_2 < n_2$. By Definition 3.4 and Lemma 3.1 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $b_2 \neq 0$,

$$\text{(4.19a)} \quad S_1: \; y_1 = K_1 y_2 + E_1 w$$

$$\text{(4.19b)} \quad S_2: \begin{cases} \dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\ \dot{\eta}_i = a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}y_1; \qquad i = 1,\ldots,r_2-1 \\ \dot{\eta}_{r_2} = A_{2,r_2 z}\eta_z + a_{2,r_2 1}\eta_1 + b_2 u + D_{2,r_2}w + A_{21,r_2}y_1 \\ y_2 = \eta_1 + E_2 w + K_{21}y_1 \end{cases}$$

where $\eta = \begin{bmatrix} \eta_z' & \eta_1 & \cdots & \eta_{r_2} \end{bmatrix}'$; η_z is $(n_2 - r_2)$-dimensional; η_i, $i = 1,\ldots,r_2$, are scalars; $\eta(0) = \eta_0 := \begin{bmatrix} \eta_{z0}' & \eta_{1,0} & \cdots & \eta_{r_2,0} \end{bmatrix}' \in \mathcal{D}_{\eta 0}$; and $\mathcal{D}_{x0} = \mathbb{R}^0$. Note that

$$y_1 = K_1(\eta_1 + E_2 w + K_{21}y_1) + E_1 w \qquad \Rightarrow$$

$$y_1 = \hat{K}^{-1}\left(K_1\eta_1 + (K_1E_2 + E_1)w\right) =: \bar{\eta}_1 + \bar{E}_1 w$$

Define $\bar{\eta}_i = \hat{K}^{-1}K_1\eta_i$, $i = 1,\ldots,r_2$. Then, the composite system S admits the following state space representation, in $\xi = \begin{bmatrix} \eta_z' & \bar{\eta}_1 & \cdots & \bar{\eta}_{r_2} \end{bmatrix}' = T_1^{-1}\eta$ coordinates,

$$\begin{aligned}
&\text{(4.20a)} \quad \dot{\eta}_z = A_{2,z}\eta_z + A_{2,z1}\eta_1 + D_{2,z}w + A_{21,z}y_1 \\
&\qquad =: A_{2,z}\eta_z + \bar{A}_{2,z1}\bar{\eta}_1 + \bar{D}_{2,z}w \\
&\text{(4.20b)} \quad \dot{\bar{\eta}}_i = \hat{K}^{-1}K_1\left(a_{2,i1}\eta_1 + \eta_{i+1} + D_{2,i}w + A_{21,i}\left(\bar{\eta}_1 + \bar{E}_1 w\right)\right) \\
&\qquad =: \bar{a}_{2,i1}\bar{\eta}_1 + \bar{\eta}_{i+1} + \bar{D}_{2,i}w; \qquad i = 1,\ldots,r_2-1 \\
&\text{(4.20c)} \quad \dot{\bar{\eta}}_{r_2} = \hat{K}^{-1}K_1\left(A_{2,r_2z}\eta_z + a_{2,r_21}\eta_1 + b_2u + D_{2,r_2}w + A_{21,r_2}\left(\bar{\eta}_1 + \bar{E}_1 w\right)\right) \\
&\qquad =: \bar{A}_{2,r_2z}\eta_z + \bar{a}_{2,r_21}\bar{\eta}_1 + \hat{K}^{-1}K_1b_2u + \bar{D}_{2,r_2}w \\
&\text{(4.20d)} \quad y_1 = \bar{\eta}_1 + \bar{E}_1 w
\end{aligned}$$

with $\xi(0) = \xi_0 := \begin{bmatrix} \eta_{z0}' & \bar{\eta}_{1,0} & \cdots & \bar{\eta}_{r_2,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$.

Clearly, (4.20) is in the extended zero dynamics canonical form (3.2) of [5]; and admits relative degree $r_2 = r_1 + r_2$ from u to y_1 since $\hat{K}^{-1}K_1b_2 \neq 0$. We will apply Lemma 3.9 of [5] to complete the proof for this case. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_2 - r_2$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall \bar{\eta}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{\eta}_{1[0,\infty)} \right\|_\infty \leq c_w$, let $\eta_{z[0,\infty)}$ be the solution to (4.20a). By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} \eta_{z0}' & \acute{\xi}_{1,0} & \cdots & \acute{\xi}_{r_2,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left|\acute{\xi}_0\right| \leq c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0} = \mathcal{D}_{\eta 0}$ and $\eta_{z0} = P_{\eta z}T_1\acute{\xi}_0 \in P_{\eta z}(\mathcal{D}_{\eta 0})$, where $P_{\eta z} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2-r_2}$ is the projection of $\mathbb{R}^{n_2}$ onto the first $n_2 - r_2$ coordinates. Note that $\eta_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \eta_{1[0,\infty)} \right\|_\infty = \left\| K_1^{-1}\hat{K}\bar{\eta}_{1[0,\infty)} \right\|_\infty \leq \left| K_1^{-1}\hat{K} \right| c_w =: c_{w1}$ and $y_{1[0,\infty)} \in \mathcal{C}$ with $\left\| y_{1[0,\infty)} \right\|_\infty = \left\| \bar{\eta}_{1[0,\infty)} + \bar{E}_1 w_{[0,\infty)} \right\|_\infty \leq (1 + \left\| \bar{E}_1 \right\|_{2,2})c_w =: c_{w2}$. Since S_2 is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d \times \mathcal{C}([0,\infty),\mathbb{R})$, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w1}, and c_{w2}, such that $\left\| \eta_{z[0,\infty)} \right\|_\infty \leq c_{c1}$. Hence, the composite system (4.20) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 3.9 of [5]. This completes the proof for Case 17.

Case 18: $0 = r_1 = n_1 = r_2 = n_2$. By Definition 3.4 of [5], without loss of generality, assume that S_1 and S_2 are given in the following extended zero dynamics canonical form representations, respectively, $K_1 \neq 0$, $K_2 \neq 0$,

$$\begin{aligned}
S_1 : y_1 &= K_1 y_2 + E_1 w \\
S_2 : y_2 &= K_2 u + E_2 w + K_{21} y_1
\end{aligned}$$

Then, we have

$$\begin{aligned}
&y_1 = K_1\left(K_2 u + E_2 w + K_{21}y_1\right) + E_1 w \qquad \Rightarrow \\
&y_1 = \hat{K}^{-1}\left(K_1K_2u + (E_1 + K_1E_2)w\right)
\end{aligned}$$

Since $\hat{K}^{-1}K_1K_2 \neq 0$, then, the composite system S admits relative degree $0 = r_1 + r_2$ from u to y_1. It is minimum phase with respect to $\mathbb{R}^0 = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof of Case 18.

This completes the proof of the lemma. □

Next, we present a lemma that establishes the minimum phase property for the composite system consisting of two subsystems in feedback configuration. The block diagram of the system is shown in Figure 4.2.

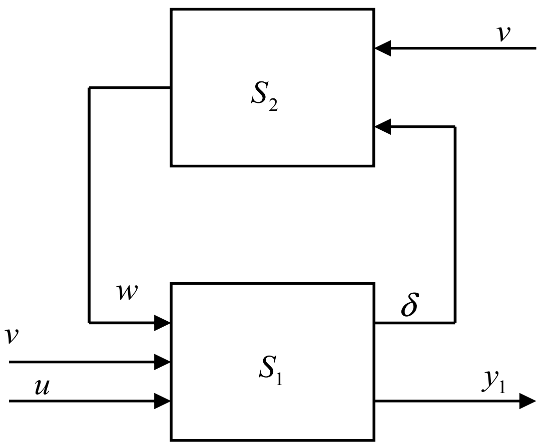

FIG. 4.2. *Block diagram of two feedback interconnected systems.*

LEMMA 4.2. *Consider two finite-dimensional continuous-time LTI systems in feedback configuration:*

$$S_1 : \begin{cases} \dot{x} &= A_1 x + B_1 u + D_{1,w} w + D_{1,v} v; \qquad x(0) = x_0 \\ y &= C_1 x + K_1 u + E_{1,w} w + E_{1,v} v \\ \delta &= C_1 x + K_1 u \end{cases} \tag{4.21a}$$

$$S_2 : \begin{cases} \dot{\eta} &= A_2 \eta + B_2 \delta + D_2 v; \qquad \eta(0) = \eta_0 \\ w &= C_2 \eta + K_2 \delta + E_2 v \end{cases} \tag{4.21b}$$

where x is the n_1-dimensional state for S_1, $n_1 \in \mathbb{Z}_+$; y is the scalar output of S_1; u is the scalar input for S_1; δ is the scalar noiseless output of S_1, which is also the input to S_2; w is the q_w-dimensional disturbance input for S_1, $q_w \in \mathbb{Z}_+$, which is also the output of S_2; v is the q_v-dimensional disturbance input for both S_1 and S_2, $q_v \in \mathbb{Z}_+$; η is the n_2-dimensional state for S_2, $n_2 \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_{x0} \neq \emptyset$, $\mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$ is a subspace; $\eta_0 \in \mathcal{D}_{\eta 0} \neq \emptyset$, $\mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$ is a subspace; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant and of appropriate dimensions.

Let the composite system S with control input u and output y be composed of S_1 in feedback configuration with S_2. Assume that

1. *S_1 admits relative degree $r_1 \in \{0, \ldots, n_1\}$ from u to y and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, where w and v are considered as disturbances;*
2. *$\hat{E}_{1,w} := 1 + E_{1,w} K_2 \neq 0$, such that S admits a well defined relative degree; and*
3. *the following associated system of S_2*

$$\bar{S}_2 : \begin{cases} \dot{\psi} &= (A_2 - B_2 \hat{E}_{1,w}^{-1} E_{1,w} C_2)\psi + B_2 \tau + D_2 v \\ &=: \bar{A}_2 \psi + B_2 \tau + D_2 v; \quad \psi(0) = \eta_0 \in \mathcal{D}_{\eta 0} \\ \phi &= \tau \end{cases} \tag{4.22}$$

with scalar control input τ and disturbance v and scalar output ϕ is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

Then, the composite system S admits relative degree r_1 from u to y and is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

Proof. Note that

$$\begin{aligned} y &= \delta + E_{1,w}\left(C_2\eta + K_2\delta + E_2 v\right) + E_{1,v}v \\ &= (1 + E_{1,w}K_2)\delta + E_{1,w}C_2\eta + (E_{1,v} + E_{1,w}E_2)v \\ &=: \hat{E}_{1,w}\delta + E_{1,w}C_2\eta + \hat{E}_{1,v}v \end{aligned} \tag{4.23}$$

We will distinguish 6 exhaustive and mutually exclusive cases, which are listed in Table 4.2.

TABLE 4.2
6 exhaustive and mutually exclusive cases for Lemma 4.2.

Case 1: $0 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$	Case 2: $1 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$
Case 3: $2 \le r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$	Case 4: $1 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$
Case 5: $2 \le r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$	Case 6: $0 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$

Case 1: $0 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3.3 of [5], without loss of generality, assume that S_1 is given in the following extended zero dynamics canonical form representation, $K_1 \neq 0$,

$$\begin{aligned} \dot{x} &= \bar{A}_1 x + \bar{B}_1\left(y - E_{1,w}w - E_{1,v}v\right) + D_{1,w}w + D_{1,v}v \\ &=: \bar{A}_1 x + \bar{B}_1 y + \bar{D}_{1,w}w + \bar{D}_{1,v}v; \qquad x(0) = x_0 \\ y &= \delta + E_{1,w}w + E_{1,v}v \\ \delta &= C_1 x + K_1 u \end{aligned}$$

Note that, by (4.23),

$$\begin{aligned} \delta &= \hat{E}_{1,w}^{-1}\left(y - E_{1,w}C_2\eta - \hat{E}_{1,v}v\right) \\ w &= C_2\eta + K_2\delta + E_2 v =: \bar{C}_2\eta + K_2\hat{E}_{1,w}^{-1}y + \bar{E}_2 v \end{aligned}$$

Then, the composite system S admits the following representation, in $\xi = \left[\begin{array}{cc} x' & \eta' \end{array}\right]'$ coordinates,

$$\dot{x} = \bar{A}_1 x + \bar{B}_1 y + \bar{D}_{1,w}\left(\bar{C}_2\eta + K_2\hat{E}_{1,w}^{-1}y + \bar{E}_2 v\right) + \bar{D}_{1,v}v \tag{4.24a}$$
$$=: \bar{A}_1 x + \hat{A}_{12}\eta + \hat{B}_1\left(y - \hat{E}_{1,v}v\right) + \hat{D}_{1,v}v$$
$$\dot{\eta} = A_2\eta + B_2\hat{E}_{1,w}^{-1}\left(y - E_{1,w}C_2\eta + \hat{E}_{1,v}v\right) + D_2 v \tag{4.24b}$$
$$= \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v) + D_2 v$$
$$y = \hat{E}_{1,w}C_1 x + E_{1,w}C_2\eta + \hat{E}_{1,w}K_1 u + \hat{E}_{1,v}v \tag{4.24c}$$

Clearly, (4.24) is in the extended zero dynamics canonical form (3.9) of [5] with (4.24a) and (4.24b) as its extended zero dynamics, since $\hat{E}_{1,w}K_1 \neq 0$. Hence, S admits relative degree $0 = r_1$ from u to y. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \left[\begin{array}{cc} x_0' & \eta_0' \end{array}\right]' \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ with $|\left[\begin{array}{cc} x_0' & \eta_0' \end{array}\right]'| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\left\| y_{[0,\infty)} \right\|_\infty \le c_w$, let $x_{[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (4.24a) and (4.24b). Let $\tau = \hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v)$. Then, $\tau_{[0,\infty)} \in \mathcal{C}$ with $\left\| \tau_{[0,\infty)} \right\|_\infty \le (\left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} + \left\| \hat{E}_{1,w}^{-1}\hat{E}_{1,v} \right\|_{2,2}) \cdot$

$c_w =: c_{w1}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \le c_{c1}$. Note that $w_{[0,\infty)} \in \mathcal{C}$ with $\left\| w_{[0,\infty)} \right\|_\infty \le \left\| \bar{C}_2 \right\|_{2,2} c_{c1} + \left\| K_2 \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w + \left\| \bar{E}_2 \right\|_{2,2} c_w =: c_{w2}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only of c_w and c_{w2}, such that $\left\| x_{[0,\infty)} \right\|_\infty \le c_{c2}$. Then, $\left\| \xi_{[0,\infty)} \right\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$. Hence, (4.24) is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This completes the proof for Case 1.

Case 2: $1 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3.2 of [5], without loss of generality, assume that S_1 is given in the following extended zero dynamics canonical form representation, $b_1 \neq 0$,

$$\begin{aligned} \dot{x}_1 &= a_{1,11} x_1 + b_1 u + D_{1,1w} w + D_{1,1v} v \\ \delta &= x_1 \\ y &= x_1 + E_{1,w} w + E_{1,v} v \end{aligned}$$

where $x = x_1$ is a scalar; $x(0) = x_{1,0} \in \mathcal{D}_{x0}$. Note that, by (4.23),

$$y = \hat{E}_{1,w} x_1 + E_{1,w} C_2 \eta + \hat{E}_{1,v} v =: \bar{x}_1 + \hat{E}_{1,v} v$$

The composite system S admits the following state space representation, in $\xi = \begin{bmatrix} \eta' & \bar{x}_1 \end{bmatrix}' = T_1^{-1} \begin{bmatrix} x & \eta' \end{bmatrix}'$ coordinates,

$$\dot{\eta} = A_2 \eta + B_2 x_1 + D_2 v = \bar{A}_2 \eta + B_2 \hat{E}_{1,w}^{-1} \bar{x}_1 + D_2 v =: \bar{A}_2 \eta + \bar{B}_2 \bar{x}_1 + D_2 v \tag{4.25a}$$

$$\begin{aligned} \dot{\bar{x}}_1 &= E_{1,w} C_2 \left(\bar{A}_2 \eta + \bar{B}_2 \bar{x}_1 + D_2 v \right) + \hat{E}_{1,w} \left(a_{1,11} x_1 + b_1 u + D_{1,1w} \left(C_2 \eta \right.\right. \\ &\quad \left.\left. + K_2 x_1 + E_2 v \right) + D_{1,1v} v \right) \\ &=: \bar{A}_{1,1\eta} \eta + \bar{a}_{1,11} \bar{x}_1 + \hat{E}_{1,w} b_1 u + \bar{D}_{1,1v} v \end{aligned} \tag{4.25b}$$

$$y = \bar{x}_1 + \hat{E}_{1,v} v \tag{4.25c}$$

where $\xi(0) = \xi_0 := \begin{bmatrix} \eta_0' & \bar{x}_{1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$.

We will further distinguish two exhaustive and mutually exclusive subcases: Case 2a: $n_2 \in \mathbb{N}$; Case 2b: $n_2 = 0$.

Case 2a: $n_2 \in \mathbb{N}$. Clearly, (4.25) is in extended zero dynamics canonical form (3.2) of [5], and admits relative degree $1 = r_1$ from u to y, since $\hat{E}_{1,w} b_1 \neq 0$.

$\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_0| \le c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_2 coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{x}_{1[0,\infty)} \right\|_\infty \le c_w$, let $\eta_{[0,\infty)}$ be the solution to (4.25a). Let $\tau = \hat{E}_{1,w}^{-1} \bar{x}_1$. Then, $\tau_{[0,\infty)} \in \mathcal{C}$ with $\left\| \tau_{[0,\infty)} \right\|_\infty \le \left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w =: c_{w1}$. By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \begin{bmatrix} \eta_0' & \acute{\bar{x}}_{1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ with $\left| \acute{\xi}_0 \right| \le c_1 c_w$. Then, $T_1 \acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\eta_0 \in \mathcal{D}_{\eta 0}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \le c_{c1}$. Hence, (4.25) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 3.9 of [5]. This completes the proof for Case 2a.

Case 2b: $n_2 = 0$. Clearly, (4.25) is in the extended zero dynamics canonical form (3.7) of [5]. Then, (4.25) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof for Case 2b.

This completes the proof for Case 2.

Case 3: $2 \leq r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3.2 of [5], without loss of generality, assume that S_1 is given in the following extended zero dynamics canonical form representation, $b_1 \neq 0$,

$$\begin{aligned} \dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v; \qquad i = 1, \ldots, r_1 - 1 \\ \dot{x}_{r_1} &= a_{1,r_1 1}x_1 + b_1 u + D_{1,r_1 w}w + D_{1,r_1 v}v \\ \delta &= x_1 \\ y &= x_1 + E_{1,w}w + E_{1,v}v \end{aligned}$$

where $x = \begin{bmatrix} x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 := \begin{bmatrix} x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$. Note that, by (4.23),

$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$

Define $\bar{x}_i = \hat{E}_{1,w}x_i$, $i = 2, \ldots, r_1$. Then, the composite system S admits the following state space representation, in $\xi = \begin{bmatrix} \eta' & \bar{x}_1 & \cdots & \bar{x}_{r_1} \end{bmatrix}' = T_1^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\begin{aligned} &(4.26a) \quad \dot{\eta} = A_2\eta + B_2x_1 + D_2v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v =: \bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2v \\ &(4.26b) \quad \dot{\bar{x}}_1 = E_{1,w}C_2\left(\bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2v\right) + \hat{E}_{1,w}\left(a_{1,11}x_1 + x_2 + D_{1,1w}\left(C_2\eta \right.\right. \\ &\qquad\qquad \left.\left. + K_2x_1 + E_2v\right) + D_{1,1v}v\right) =: \bar{A}_{1,1\eta}\eta + \bar{a}_{1,11}\bar{x}_1 + \bar{x}_2 + \bar{D}_{1,1v}v \\ &(4.26c) \quad \dot{\bar{x}}_i = \hat{E}_{1,w}\left(a_{1,i1}x_1 + x_{i+1} + D_{1,iw}\left(C_2\eta + K_2x_1 + E_2v\right) + D_{1,iv}v\right) \\ &\qquad\qquad =: \bar{A}_{1,i\eta}\eta + \bar{a}_{1,i1}\bar{x}_1 + \bar{x}_{i+1} + \bar{D}_{1,iv}v; \qquad i = 2, \ldots, r_1 - 1 \\ &(4.26d) \quad \dot{\bar{x}}_{r_1} = \hat{E}_{1,w}\left(a_{1,r_1 1}x_1 + b_1u + D_{1,r_1 w}\left(C_2\eta + K_2x_1 + E_2v\right) + D_{1,r_1 v}v\right) \\ &\qquad\qquad =: \bar{A}_{1,r_1\eta}\eta + \bar{a}_{1,r_1 1}\bar{x}_1 + \hat{E}_{1,w}b_1u + \bar{D}_{1,r_1 v}v \\ &(4.26e) \quad y = \bar{x}_1 + \hat{E}_{1,v}v \end{aligned}$$

where $\xi(0) = \xi_0 = \begin{bmatrix} \eta_0' & \bar{x}_{1,0} & \cdots & \bar{x}_{r_1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$.

We will further distinguish two exhaustive and mutually exclusive subcases: Case 3a: $n_2 \in \mathbb{N}$; Case 3b: $n_2 = 0$.

Case 3a: $n_2 \in \mathbb{N}$. Clearly, (4.26) is in form (3.1) with $m_o = 1$, and admits relative degree r_1 from u to y, since $\hat{E}_{1,w}b_1 \neq 0$. We will apply Lemma 3.1 to prove this subcase. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall \eta_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_0| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ is the projection from $\mathbb{R}^{n_1+n_2}$ to the first n_2 coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{x}_{1[0,\infty)} \right\|_\infty \leq c_w$, let $\eta_{[0,\infty)}$ be the solution to (4.26a). By Lemma A.3, $\exists c_1 \in [0, \infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, and $\exists \acute{\xi}_0 = \begin{bmatrix} \eta_0' & \acute{\bar{x}}_{1,0} & \cdots & \acute{\bar{x}}_{r_1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ such that $\left| \acute{\xi}_0 \right| \leq c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\eta_0 \in \mathcal{D}_{\eta 0}$. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\left\| \tau_{[0,\infty)} \right\|_\infty \leq \left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w =: c_{w1}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \in [0, \infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \leq c_{c1}$. Hence, by Lemma 3.1, (4.26) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This completes the proof for Case 3a.

Case 3b: $n_2 = 0$. Clearly, (4.26) is in the extended zero dynamics canonical form (3.7) of [5]. Then, (4.26) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. This completes the proof for Case 3b.

This completes the proof for Case 3.

Case 4: $1 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3.1 of [5], without loss of generality, assume that S_1 is given in the following extended zero dynamics canonical form representation, $b_1 \neq 0$,

$$\begin{aligned}\dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v\\ \dot{x}_1 &= A_{1,1z}x_z + a_{1,11}x_1 + b_1u + D_{1,1w}w + D_{1,1v}v\\ \delta &= x_1\\ y &= x_1 + E_{1,w}w + E_{1,v}v\end{aligned}$$

where $x = \begin{bmatrix} x_z' & x_1 \end{bmatrix}'$; x_z is $(n_1 - r_1)$-dimensional; x_1 is a scalar; and $x(0) = x_0 := \begin{bmatrix} x_{z0}' & x_{1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$. Note that, by (4.23),

$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$

The composite system S admits the following state space representation, in $\xi = \begin{bmatrix} x_z' & \eta' & \bar{x}_1 \end{bmatrix}' =: T_1^{-1}\begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\begin{aligned}
&(4.27\text{a}) & \dot{x}_z &= A_{1,z}x_z + A_{1,z1}\hat{E}_{1,w}^{-1}(\bar{x}_1 - E_{1,w}C_2\eta) + D_{1,zw}(C_2\eta + K_2x_1 + E_2v)\\
&&&\quad + D_{1,zv}v =: \bar{A}_{1,\eta}\eta + A_{1,z}x_z + \bar{A}_{1,z1}\bar{x}_1 + \bar{D}_{1,zv}v\\
&(4.27\text{b}) & \dot{\eta} &= A_2\eta + B_2x_1 + D_2v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v\\
&(4.27\text{c}) & \dot{\bar{x}}_1 &= E_{1,w}C_2(\bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v) + \hat{E}_{1,w}(A_{1,1z}x_z + a_{1,11}x_1 + b_1u\\
&&&\quad + D_{1,1w}(C_2\eta + K_2x_1 + E_2v) + D_{1,1v}v)\\
&&&=: \bar{A}_{1,1z}x_z + \bar{A}_{1,1\eta}\eta + \bar{a}_{1,11}\bar{x}_1 + \hat{E}_{1,w}b_1u + \bar{D}_{1,1v}v\\
&(4.27\text{d}) & y &= \bar{x}_1 + \hat{E}_{1,v}v
\end{aligned}$$

where $\xi(0) = \xi_0 = \begin{bmatrix} x_{z0}' & \eta_0' & \bar{x}_{1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. Clearly, (4.27) is in the extended zero dynamics canonical form (3.2) of [5], and admits relative degree $1 = r_1$ from u to y, since $\hat{E}_{1,w}b_1 \neq 0$.

$\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \xi_{z0} := \begin{bmatrix} x_{z0}' & \eta_0' \end{bmatrix}' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1-1+n_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first n_1+n_2-1 coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\| v_{[0,\infty)} \|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\| \bar{x}_{1[0,\infty)} \|_\infty \leq c_w$, let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (4.27a) and (4.27b). By Lemma A.3, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, and $\exists \acute{\xi}_0 = \begin{bmatrix} \xi_{z0}' & \acute{\bar{x}}_{1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ such that $\left|\acute{\xi}_0\right| \leq c_1c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, and $\eta_0 \in \mathcal{D}_{\eta 0}$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\| \tau_{[0,\infty)} \|_\infty \leq \left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w =: c_{w1}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\| \eta_{[0,\infty)} \|_\infty \leq c_{c1}$. Note that $x_{1[0,\infty)} \in \mathcal{C}$ with $\| x_{1[0,\infty)} \|_\infty = \left\| \hat{E}_{1,w}^{-1}(\bar{x}_{1[0,\infty)} - E_{1,w}C_2\eta_{[0,\infty)}) \right\|_\infty \leq \left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w + \left\| \hat{E}_{1,w}^{-1}E_{1,w}C_2 \right\|_{2,2} c_{c1} =: c_{w2}$; and $w_{[0,\infty)} \in \mathcal{C}$ with $\| w_{[0,\infty)} \|_\infty = \| C_2\eta_{[0,\infty)} + K_2x_{1[0,\infty)} + E_2v_{[0,\infty)} \|_\infty \leq \| C_2 \|_{2,2} c_{c1} + \| K_2 \|_{2,2} c_{w2} + \| E_2 \|_{2,2} c_w =: c_{w3}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, then, by Lemma 3.9 of [5], $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w2}, and c_{w3}, such that $\| x_{z[0,\infty)} \|_\infty \leq c_{c2}$. Then, $\| \xi_{z[0,\infty)} \|_\infty \leq$

$\sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = \begin{bmatrix} x_z' & \eta' \end{bmatrix}'$. Hence, (4.27) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 3.9 of [5]. This completes the proof for Case 4.

Case 5: $2 \leq r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3.1 of [5], without loss of generality, assume that S_1 is given in the following extended zero dynamics canonical form representation, $b_1 \neq 0$,

$$\begin{aligned}
\dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v \\
\dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v; \qquad i = 1, \ldots, r_1 - 1 \\
\dot{x}_{r_1} &= A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1 u + D_{1,r_1 w}w + D_{1,r_1 v}v \\
\delta &= x_1 \\
y &= x_1 + E_{1,w}w + E_{1,v}v
\end{aligned}$$

where $x = \begin{bmatrix} x_z' & x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_z is $(n_1 - r_1)$-dimensional; x_i, $i = 1, \ldots, r_1$, are scalars; and $x(0) = x_0 = \begin{bmatrix} x_{z0}' & x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$. Note that, by (4.23),

$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$

Define $\bar{x}_i = \hat{E}_{1,w}x_i$, $i = 2, \ldots, r_1$. The composite system S admits the following state space representation, in $\xi = \begin{bmatrix} x_z' & \eta' & \bar{x}_1 & \cdots & \bar{x}_{r_1} \end{bmatrix}' = T_1^{-1} \begin{bmatrix} x' & \eta' \end{bmatrix}'$ coordinates,

$$\begin{aligned}
&\text{(4.28a)} \quad \dot{x}_z = A_{1,z}x_z + A_{1,z1}\hat{E}_{1,w}^{-1}(\bar{x}_1 - E_{1,w}C_2\eta) + D_{1,zw}(C_2\eta + K_2x_1 + E_2v) \\
&\qquad\qquad + D_{1,zv}v =: A_{1,z}x_z + \bar{A}_{1,\eta}\eta + \bar{A}_{1,z1}\bar{x}_1 + \bar{D}_{1,zv}v \\
&\text{(4.28b)} \quad \dot{\eta} = A_2\eta + B_2x_1 + D_2v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v \\
&\text{(4.28c)} \quad \dot{\bar{x}}_1 = E_{1,w}C_2(\bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v) + \hat{E}_{1,w}(a_{1,11}x_1 + x_2 \\
&\qquad\qquad + D_{1,1w}(C_2\eta + K_2x_1 + E_2v) + D_{1,1v}v) \\
&\qquad\qquad =: \bar{A}_{1,1\eta}\eta + \bar{a}_{1,11}\bar{x}_1 + \bar{x}_2 + \bar{D}_{1,1v}v \\
&\text{(4.28d)} \quad \dot{\bar{x}}_i = \hat{E}_{1,w}(a_{1,i1}x_1 + x_{i+1} + D_{1,iw}(C_2\eta + K_2x_1 + E_2v) + D_{1,iv}v) \\
&\qquad\qquad =: \bar{A}_{1,i\eta}\eta + \bar{a}_{1,i1}\bar{x}_1 + \bar{x}_{i+1} + \bar{D}_{1,iv}v; \qquad i = 2, \ldots, r_1 - 1 \\
&\text{(4.28e)} \quad \dot{\bar{x}}_{r_1} = \hat{E}_{1,w}(A_{1,r_1 z}x_z + a_{1,r_1 1}x_1 + b_1u + D_{1,r_1 w}(C_2\eta + K_2x_1 + E_2v) \\
&\qquad\qquad + D_{1,r_1 v}v) =: \bar{A}_{1,r_1 z}x_z + \bar{A}_{1,r_1\eta}\eta + \bar{a}_{1,r_1 1}\bar{x}_1 + \hat{E}_{1,w}b_1u + \bar{D}_{1,r_1 v}v \\
&\text{(4.28f)} \quad y = \bar{x}_1 + \hat{E}_{1,v}v
\end{aligned}$$

where $\xi(0) = \xi_0 := \begin{bmatrix} x_{z0}' & \eta_0' & \bar{x}_{1,0} & \cdots & \bar{x}_{r_1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}) \neq \emptyset$. Clearly, (4.28) is in the form (3.1) with $m_o = 1$, and admits relative degree r_1 from u to y, since $\hat{E}_{1,w}b_1 \neq 0$.

We will apply Lemma 3.1 to prove that (4.28) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall \xi_{z0} := \begin{bmatrix} x_{z0}' & \eta_0' \end{bmatrix}' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1 + n_2 - r_1$ coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\left\| \bar{x}_{1[0,\infty)} \right\|_\infty \leq c_w$, let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (4.28a) and (4.28b). By Lemma A.3, $\exists c_1 \in [0, \infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $\exists \acute{\xi}_0 = \begin{bmatrix} \xi_{z0}' & \acute{\bar{x}}_{1,0} & \cdots & \acute{\bar{x}}_{r_1,0} \end{bmatrix}' \in \bar{\mathcal{D}}_0$ such that $\left| \acute{\xi}_0 \right| \leq c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $\eta_0 \in \mathcal{D}_{\eta 0}$, and $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates.

Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\left\| \tau_{[0,\infty)} \right\|_\infty \leq \left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w =: c_{w1}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \leq c_{c1}$. Note that $x_{1[0,\infty)} \in \mathcal{C}$ with $\left\| x_{1[0,\infty)} \right\|_\infty = \left\| \hat{E}_{1,w}^{-1} (\bar{x}_{1[0,\infty)} - E_{1,w}C_2\eta_{[0,\infty)}) \right\|_\infty \leq \left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w + \left\| \hat{E}_{1,w}^{-1} E_{1,w}C_2 \right\|_{2,2} c_{c1} =: c_{w2}$; and $w_{[0,\infty)} \in \mathcal{C}$ with $\left\| w_{[0,\infty)} \right\|_\infty = \left\| C_2\eta_{[0,\infty)} + K_2 x_{1[0,\infty)} + E_2 v_{[0,\infty)} \right\|_\infty \leq \| C_2 \|_{2,2} c_{c1} + \| K_2 \|_{2,2} c_{w2} + \| E_2 \|_{2,2} c_w =: c_{w3}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, then, by Lemma 3.9 of [5], $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, c_{w2}, and c_{w3}, such that $\left\| x_{z[0,\infty)} \right\|_\infty \leq c_{c2}$. Then, $\left\| \xi_{z[0,\infty)} \right\|_\infty \leq \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = \left[\begin{array}{cc} x_z' & \eta' \end{array} \right]'$. Hence, (4.28) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This completes the proof for Case 5.

Case 6: $0 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$. By Definition 3.4 of [5], without loss of generality, assume that the system S_1 is given in the following extended zero dynamics canonical form representation, $K_1 \neq 0$,

$$S_1 : \left\{ \begin{array}{rcl} y & = & K_1 u + E_{1,w} w + E_{1,v} v \\ \delta & = & K_1 u \end{array} \right.$$

By (4.23), we have

$$\begin{aligned} y &= E_{1,w}C_2\eta + \hat{E}_{1,w}K_1 u + \hat{E}_{1,v} v \\ \delta &= K_1 u = \hat{E}_{1,w}^{-1} (y - E_{1,w}C_2\eta - \hat{E}_{1,v} v) \end{aligned}$$

Then, the composite system S admits the following extended zero dynamics canonical form representation:

$$\begin{aligned} \dot{\eta} &= A_2\eta + B_2\hat{E}_{1,w}^{-1} (y - E_{1,w}C_2\eta - \hat{E}_{1,v} v) + D_2 v \\ &= \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v} v) + D_2 v \end{aligned} \tag{4.29a}$$

$$y = E_{1,w}C_2\eta + \hat{E}_{1,w}K_1 u + \hat{E}_{1,v} v \tag{4.29b}$$

Clearly, (4.29) admits relative degree $0 = r_1$ from u to y, since $\hat{E}_{1,w}K_1 \neq 0$. If $n_2 = 0$, then (4.29) is minimum phase with respect to $\mathbb{R}^0 = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since its extended zero dynamics is absent. If $n_2 \in \mathbb{N}$, then (4.29) is in the form of (3.9) of [5] with (4.29a) defining its extended zero dynamics. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\left\| y_{[0,\infty)} \right\|_\infty \leq c_w$, let $\eta_{[0,\infty)}$ be the solution to (4.29a). Let $\tau = \hat{E}_{1,w}^{-1} (y - \hat{E}_{1,v} v)$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\left\| \tau_{[0,\infty)} \right\|_\infty \leq \left\| \hat{E}_{1,w}^{-1} \right\|_{2,2} c_w + \left\| \hat{E}_{1,w}^{-1} \hat{E}_{1,v} \right\|_{2,2} c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\left\| \eta_{[0,\infty)} \right\|_\infty \leq c_c$. Hence, S is minimum phase with respect to $\mathcal{D}_{\eta 0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This completes the proof for Case 6.

This completes the proof of the lemma. □

5. Bounding lemmas on the inversion of minimum phase systems. In this section, we will present results on boundedness of the inverse of minimum phase systems. First, we present a result for a SISO linear system with relative degree 0.

LEMMA 5.1. *Consider a finite-dimensional continuous-time SISO LTI system:*

$$\dot{x} = Ax + Bu + D_w w + D_v v; \qquad x(0) = x_0 \tag{5.1a}$$

$$y = Cx + Ku + E_w w + E_v v \tag{5.1b}$$

where x is the n-dimensional state vector, $n \in \mathbb{Z}_+$; u is the scalar control input; y is the scalar output; w is the q_w-dimensional disturbance input, $q_w \in \mathbb{Z}_+$; v is the q_v-dimensional disturbance input, $q_v \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant and of appropriate dimensions.

Assume (5.1) admits relative degree 0 from u to y and is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w})\times\mathcal{W}_d$, where w and v are viewed as disturbances. Then, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$; $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\left\| w_{[0,t_f)} \right\|_\infty \le c_w$; let $x_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solution to (5.1); if $\left\| y_{[0,t_f)} \right\|_\infty \le c_w$, we have $\left\| x_{[0,t_f)} \right\|_\infty \le c_c$; and $\left\| u_{[0,t_f)} \right\|_\infty \le c_c$.

Proof. We will distinguish 2 exhaustive and mutually exclusive cases: Case 1: $n = 0$; Case 2: $n \in \mathbb{N}$.

Case 1: $n = 0$. By Definition 3.4 of [5], the system (5.1) admits the following extended zero dynamics canonical form, $K \neq 0$,

$$y = Ku + E_w w + E_v v$$

Then, we have $u = K^{-1}(y - E_w w - E_v v)$. Clearly, the desired result holds with $c_c := \left| K^{-1} \right| c_w + \left\| K^{-1} E_w \right\|_{2,2} c_w + \left\| K^{-1} E_v \right\|_{2,2} c_w$. This completes the proof for Case 1.

Case 2: $n \in \mathbb{N}$. By Lemma 3.3 of [5], without loss of generality, assume that the system (5.1) is given in the following extended zero dynamics canonical form representation, $b_0 \neq 0$,

$$\dot{x} = \bar{A}x + \bar{B}(y - E_w w - E_v v) + D_w w + D_v v \tag{5.2a}$$

$$y = Cx + b_0 u + E_w w + E_v v \tag{5.2b}$$

$\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\left\| w_{[0,t_f)} \right\|_\infty \le c_w$; let $x_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solution to (5.2); and let $\left\| y_{[0,t_f)} \right\|_\infty \le c_w$. Since the system (5.1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$, by Lemma A.2, then, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, we have $\left\| x_{[0,t_f)} \right\|_\infty \le c_{c1}$. Note that $u = b_0^{-1}(y - Cx - E_w w - E_v v)$. Then, $\left\| u_{[0,t_f)} \right\|_\infty \le \left| b_0^{-1} \right| c_w + \left\| b_0^{-1} C \right\|_{2,2} c_{c1} + \left\| b_0^{-1} E_w \right\|_{2,2} c_w + \left\| b_0^{-1} E_v \right\|_{2,2} c_w =: c_{c2}$. This completes the proof for Case 2.

This completes the proof of the lemma. ∎

Next, we will present the bounding results for systems with positive relative degrees. First, we present a technical lemma that serves as a stepping stone for the next lemma.

LEMMA 5.2. *Consider the following chain of integrators:*

$$\dot{x}_i = x_{i+1} + D_{iw} w + D_{iv} v; \qquad i = 1,\ldots,r_1 - 1 \tag{5.3a}$$

$$\dot{x}_{r_1} = b_1 u + D_{r_1 w} w + D_{r_1 v} v \tag{5.3b}$$

where x_i, $i = 1,\ldots,r_1$, are scalars, $r_1 \in \mathbb{N}$; w is the q_w-dimensional disturbance input, $q_w \in \mathbb{Z}_+$; v is the q_v-dimensional disturbance input, $q_v \in \mathbb{Z}_+$; u is the scalar control input; $0 \neq b_1 \in \mathbb{R}$; $x = \begin{bmatrix} x_1 & \cdots & x_{r_1} \end{bmatrix}'$; $x(0) = x_0 \in \mathbb{R}^{r_1}$; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and D_{iw} and D_{iv}, $i = 1,\ldots,r_1$, are constant matrices of appropriate dimensions. Consider another finite-dimensional continuous-time SISO LTI system,

S_η, sharing the same set of inputs as (5.3):

$$\dot{\eta} = A\eta + Bu + D_w w + D_v v; \qquad \eta(0) = \eta_0 \tag{5.4a}$$

$$y = C\eta \tag{5.4b}$$

where η is the n-dimensional state, $n \in \mathbb{N}$; y is the scalar output; u, w, and v are the same as (5.3); $\eta_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$. Assume that S_η admits relative degree r_2 ($\geq r_1$) from u to y and satisfies: $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \leq c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $\left\| u_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{C}$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \leq c_w$, we have $\left\| \eta_{[0,\infty)} \right\|_\infty \leq c_{c1}$.

Then, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_{c2} \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathbb{R}^{r_1}$ with $|x_0| \leq c_w$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \leq c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\left\| w_{[0,t_f)} \right\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \leq c_w$. Let $\eta_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solutions to (5.4), and $x_{[0,t_f)}$ be the solution to (5.3). If $\left\| x_{1[0,t_f)} \right\|_\infty \leq c_w$, then $\left\| y_{[0,t_f)} \right\|_\infty \leq c_{c2}$.

Proof. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathbb{R}^{r_1}$ with $|x_0| \leq c_w$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \leq c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\left\| w_{[0,t_f)} \right\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \leq c_w$. Let $\eta_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solutions to (5.4), and $x_{[0,t_f)}$ be the solution to (5.3). Let $\left\| x_{1[0,t_f)} \right\|_\infty \leq c_w$.

Let $z := \eta - \sum_{k=1}^{r_1} A^{r_1-k} B b_1^{-1} x_k : [0,t_f) \to \mathbb{R}^n$. Then, z satisfies the following state space equation:

$$\begin{aligned}
\dot{z} &= Az + \sum_{k=1}^{r_1} A^{r_1-k+1} B b_1^{-1} x_k + Bu + D_w w + D_v v - \sum_{k=1}^{r_1-1} A^{r_1-k} B b_1^{-1} (x_{k+1} \\
&\quad + D_{kw} w + D_{kv} v) - B b_1^{-1} (b_1 u + D_{r_1 w} w + D_{r_1 v} v) \\
&= Az + \sum_{k=0}^{r_1-1} A^{r_1-k} B b_1^{-1} x_{k+1} + D_w w + D_v v - \sum_{k=1}^{r_1-1} A^{r_1-k} B b_1^{-1} x_{k+1} \\
&\quad - \sum_{k=1}^{r_1} A^{r_1-k} B b_1^{-1} (D_{kw} w + D_{kv} v) \\
&= Az + A^{r_1} B b_1^{-1} x_1 + D_w w + D_v v - \sum_{k=1}^{r_1} A^{r_1-k} B b_1^{-1} (D_{kw} w + D_{kv} v) \\
&= Az + B\left(-b_1^{-1}(D_{r_1 w} w + D_{r_1 v} v)\right) + D_w w + D_v v + A^{r_1} B b_1^{-1} x_1 \\
&\quad + \sum_{k=1}^{r_1-1} A^{r_1-k} B \left(-b_1^{-1}(D_{kw} w + D_{kv} v)\right) \\
z(0) &= \eta_0 - \sum_{k=1}^{r_1} A^{r_1-k} B b_1^{-1} x_{k,0}
\end{aligned}$$

where $x_0 = \begin{bmatrix} x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}'$; $x_{i,0}$, $i = 1,\ldots,r_1$, are scalars. Then, by the linearity and uniqueness of solution to linear differential equations, $z_{[0,t_f)}$ may be generated by

$$\begin{aligned}
\dot{\xi}_1 &= A\xi_1 + B\left(-b_1^{-1}(D_{r_1 w} w + D_{r_1 v} v)\right) + D_w w + D_v v; \qquad \xi_1(0) = \eta_0 \\
\dot{\delta}_i &= A\delta_i; \qquad \delta_i(0) = -A^i B b_1^{-1} x_{r_1-i,0}; \qquad i = 0,\ldots,r_1-1 \\
\dot{\zeta}_i &= A\zeta_i + A^i B\left(-b_1^{-1}(D_{r_1-i\,w} w + D_{r_1-i\,v} v)\right); \quad \zeta_i(0) = \mathbf{0}_{n\times 1}; \quad i = 1,\ldots,r_1-1 \\
\dot{\zeta}_{r_1} &= A\zeta_{r_1} + A^{r_1} B b_1^{-1} x_1; \qquad \zeta_{r_1}(0) = \mathbf{0}_{n\times 1}
\end{aligned}$$

$$z_{[0,t_f)} = \xi_{1[0,t_f)} + \sum_{i=0}^{r_1-1} \delta_{i[0,t_f)} + \sum_{i=1}^{r_1} \zeta_{i[0,t_f)}$$

Let $\bar{u}_{[0,t_f)} := -b_1^{-1}(D_{r_1 w} w_{[0,t_f)} + D_{r_1 v} v_{[0,t_f)})$. Note that $\eta_0 \in \mathcal{D}_0$ and $|\eta_0| \leq c_w$; $\bar{u}_{[0,t_f)} \in \mathcal{C}$ with $\| \bar{u}_{[0,t_f)} \|_\infty \leq \| b_1^{-1} D_{r_1 w} \|_{2,2} c_w + \| b_1^{-1} D_{r_1 v} \|_{2,2} c_w =: c_{w1}$; $w_{[0,t_f)} \in \mathcal{C}$ with $\| w_{[0,t_f)} \|_\infty \leq c_w$; and $v_{[0,\infty)} \in \mathcal{W}_d$ with $\| v_{[0,\infty)} \|_\infty \leq c_w$. By the assumption on S_η and Lemma A.2, $\exists c_{ca} \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and c_{w1}, such that $\| \xi_{1[0,t_f)} \|_\infty \leq c_{ca}$.

By the assumption on S_η and Lemma A.11 of [5], we have that the following dynamics

$$\dot{\kappa}_1 = A\kappa_1 + B\rho; \qquad \kappa_1(0) = \mathbf{0}_{n\times 1}$$

is bounded input and bounded state stable. By repeated application of Lemma A.1, we have that the following dynamics

$$\dot{\kappa}_i = A\kappa_i + A^{i-1}B\rho; \qquad \kappa_i(0) = \mathbf{0}_{n\times 1}; \qquad i = 2, \ldots, r_1 + 1$$

are bounded input and bounded state stable. $\forall i = 1, \ldots, r_1 - 1$. Let $\rho_i := -b_1^{-1} \cdot (D_{r_1 - i\, w} w + D_{r_1 - i\, v} v) : [0, t_f) \to \mathbb{R}$. Then, $\rho_{i[0,t_f)} \in \mathcal{C}$ and $\| \rho_{i[0,t_f)} \|_\infty \leq \| b_1^{-1} D_{r_1 - i\, w} \|_{2,2} c_w + \| b_1^{-1} D_{r_1 - i\, v} \|_{2,2} c_w =: c_{wAi} c_w$. Since κ_{i+1} dynamics is bounded input and bounded state stable, then, ζ_i dynamics is bounded input and bounded state stable. By Lemma A.3 of [5], $\exists c_{Ai} \in [0,\infty) \subset \mathbb{R}$, which depends only on A and $A^i B$, such that $\| \zeta_{i[0,t_f)} \|_\infty \leq c_{Ai} c_{wAi} c_w$. Since κ_{r_1+1} dynamics is bounded input and bounded state stable, then, ζ_{r_1} dynamics is bounded input and bounded state stable. Let $\rho_{r_1} := b_1^{-1} x_1$. Then $\rho_{r_1[0,t_f)} \in \mathcal{C}$ and $\| \rho_{r_1[0,t_f)} \|_\infty \leq | b_1^{-1} | c_w =: c_{wAr_1} c_w$. Then, by Lemma A.3 of [5], $\exists c_{Ar_1} \in [0,\infty) \subset \mathbb{R}$, which depends only on A and $A^{r_1} B$, such that $\| \zeta_{r_1[0,t_f)} \|_\infty \leq c_{Ar_1} c_{wAr_1} c_w$.

$\forall i = 0, \ldots, r_1 - 1$. Since κ_{i+1} system is bounded input and bounded state stable, by Lemma A.1, note that $| -b_1^{-1} x_{r_1 - i, 0} | \leq | b_1^{-1} | c_w =: c_1 c_w$, then, $\exists c_{Bi} \in [0,\infty) \subset \mathbb{R}$, which depends only on A and $A^i B$, such that $\| \delta_{i[0,t_f)} \|_\infty \leq c_{Bi} c_1 c_w$.

Hence, we have $\| z_{[0,t_f)} \|_\infty \leq c_{ca} + \sum_{i=0}^{r_1-1} c_{Bi} c_1 c_w + \sum_{i=1}^{r_1} c_{Ai} c_{wAi} c_w =: c_{c1}$.

Note that $y = Cz + \sum_{k=1}^{r_1} CA^{r_1-k} B b_1^{-1} x_k = Cz + CA^{r_1-1} B b_1^{-1} x_1$ since $r_1 \leq r_2$. Hence, $\| y_{[0,t_f)} \|_\infty \leq \| C \|_{2,2} c_{c1} + \| CA^{r_1-1} B b_1^{-1} \|_{2,2} c_w =: c_{c2}$.

This completes the proof of the lemma. ∎

Next, we present two bounding results for systems with positive relative degrees.

LEMMA 5.3. *Consider two finite-dimensional continuous-time SISO LTI systems sharing the same inputs:*

$$S_1 : \left\{ \begin{array}{rcl} \dot{x} & = & A_1 x + B_1 u + D_{1,w} w + D_{1,v} v; \qquad x(0) = x_0 \\ y_1 & = & C_1 x + E_{1,w} w + E_{1,v} v \end{array} \right. \tag{5.5}$$

$$S_2 : \left\{ \begin{array}{rcl} \dot{\eta} & = & A_2 \eta + B_2 u + D_{2,w} w + D_{2,v} v; \qquad \eta(0) = \eta_0 \\ y_2 & = & C_2 \eta \end{array} \right. \tag{5.6}$$

where x is the n_1-dimensional state of S_1, $n_1 \in \mathbb{N}$; u is the scalar control input; y_1 is the scalar output of S_1; w is the q_w-dimensional disturbance input, $q_w \in \mathbb{Z}_+$; v is the q_v-dimensional disturbance input, $q_v \in \mathbb{Z}_+$; η is the n_2-dimensional state of S_2, $n_2 \in \mathbb{N}$; y_2 is the scalar output of S_2; $x_0 \in \mathcal{D}_{x0} \neq \emptyset$, $\mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$; $\eta_0 \in \mathcal{D}_{\eta 0} \neq \emptyset$, $\mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant and of appropriate dimensions.

Assume that

(i) *S_1 admits relative degree r_1 from u to y_1, $r_1 \in \{1, \ldots, n_1\}$, and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, where w and v are viewed as disturbance inputs;*

(ii) *S_2 satisfies that $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_{c1} \in [0,\infty) \subset \mathbb{R}$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $\left\| u_{[0,\infty)} \right\|_\infty \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{C}$ with $\left\| w_{[0,\infty)} \right\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$, we have $\left\| \eta_{[0,\infty)} \right\|_\infty \le c_{c1}$; and*

(iii) *S_2 admits relative degree $r_2 \in \mathbb{N}$ from u to y_2.*

By Lemmas 3.1 and 3.2 of [5], without loss of generality, assume that S_1 is given in the extended zero dynamics canonical form representation: Case 1: $r_1 = n_1 \in \mathbb{N}$

$$\dot{x}_i = a_{1,i1} x_1 + x_{i+1} + D_{1,iw} w + D_{1,iv} v; \qquad i = 1, \ldots, r_1 - 1 \tag{5.7a}$$

$$\dot{x}_{r_1} = a_{1,r_1 1} x_1 + b_1 u + D_{1,r_1 w} w + D_{1,r_1 v} v \tag{5.7b}$$

$$y_1 = x_1 + E_{1,w} w + E_{1,v} v \tag{5.7c}$$

where $x = \begin{bmatrix} x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 = \begin{bmatrix} x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$; and $b_1 \ne 0$; Case 2: $1 \le r_1 < n_1$

$$\dot{x}_z = A_{1,z} x_z + A_{1,z1} x_1 + D_{1,zw} w + D_{1,zv} v \tag{5.8a}$$

$$\dot{x}_i = a_{1,i1} x_1 + x_{i+1} + D_{1,iw} w + D_{1,iv} v; \qquad i = 1, \ldots, r_1 - 1 \tag{5.8b}$$

$$\dot{x}_{r_1} = A_{1,r_1 z} x_z + a_{1,r_1 1} x_1 + b_1 u + D_{1,r_1 w} w + D_{1,r_1 v} v \tag{5.8c}$$

$$y_1 = x_1 + E_{1,w} w + E_{1,v} v \tag{5.8d}$$

where $x = \begin{bmatrix} x_z' & x_1 & \cdots & x_{r_1} \end{bmatrix}'$; x_z is $(n_1 - r_1)$-dimensional; x_i, $i = 1, \ldots, r_1$, are scalars; $x(0) = x_0 = \begin{bmatrix} x_{z0}' & x_{1,0} & \cdots & x_{r_1,0} \end{bmatrix}' \in \mathcal{D}_{x0}$; and $b_1 \ne 0$.

Then, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \le c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\left\| w_{[0,t_f)} \right\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (5.6), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (5.5). If $\left\| x_{1[0,t_f)} \right\|_\infty \le c_w$, $\ldots$, $\left\| x_{k_0[0,t_f)} \right\|_\infty \le c_w$, for some fixed $k_0 \in \{1, \ldots, r_1\}$, and $r_2 \ge r_1 - k_0 + 1$, then $\left\| y_{2[0,t_f)} \right\|_\infty \le c_c$.

Proof. $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \le c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\left\| w_{[0,t_f)} \right\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \le c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (5.6), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (5.5). Let $\left\| x_{1[0,t_f)} \right\|_\infty \le c_w$, $\ldots$, $\left\| x_{k_0[0,t_f)} \right\|_\infty \le c_w$.

We will distinguish two exhaustive and mutually exclusive cases: Case 1: $r_1 = n_1$; Case 2: $1 \le r_1 < n_1$.

Case 1: $r_1 = n_1 \in \mathbb{N}$. S_1 admits the representation (5.7). We will apply Lemma 5.2 to prove this case. The chain of integrators is

$$\dot{x}_i = x_{i+1} + \begin{bmatrix} a_{1,i1} & D_{1,iw} \end{bmatrix} \begin{bmatrix} x_1 \\ w \end{bmatrix} + D_{1,iv} v; \qquad i = k_0, \ldots, r_1 - 1$$

$$\dot{x}_{r_1} = b_1 u + \begin{bmatrix} a_{1,r_1 1} & D_{1,r_1 w} \end{bmatrix} \begin{bmatrix} x_1 \\ w \end{bmatrix} + D_{1,r_1 v} v$$

Then, by Lemma 5.2, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\| y_{2[0,t_f)} \right\|_\infty \le c_c$. This completes the proof for this case.

Case 2: $1 \le r_1 < n_1$. S_1 admits the representation (5.8).

CLAIM 5.3.1. *$\exists \bar{c}_c \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\| x_{z[0,t_f)} \right\|_\infty \leq \bar{c}_c$.*

Proof. By the fact that S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$ and Lemma 3.9 of [5] and Lemma A.2, $\exists \bar{c}_c \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w, such that $\left\| x_{z[0,t_f)} \right\|_\infty \leq \bar{c}_c$. This completes the proof of the claim. $\square$

Now, we will apply Lemma 5.2 to prove this case. The chain of integrators is

$$\dot{x}_i = x_{i+1} + \begin{bmatrix} \mathbf{0}_{1\times(n_1-r_1)} & a_{1,i1} & D_{1,iw} \end{bmatrix} \begin{bmatrix} x_z \\ x_1 \\ w \end{bmatrix} + D_{1,iv} v; \qquad i = k_0, \ldots, r_1 - 1$$

$$\dot{x}_{r_1} = b_1 u + \begin{bmatrix} A_{1,r_1 z} & a_{1,r_1 1} & D_{1,r_1 w} \end{bmatrix} \begin{bmatrix} x_z \\ x_1 \\ w \end{bmatrix} + D_{1,r_1 v} v$$

Then, by Lemma 5.2, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, which depends only on c_w and $\bar{c}_c$, such that $\left\| y_{2[0,t_f)} \right\|_\infty \leq c_c$. This completes the proof for Case 2.

This completes the proof of the lemma. $\square$

In the application of Lemma 5.3, we will refer S_1 as the reference system. Finally, we present a corollary of Lemma 5.3, which has a stronger assumption on S_2.

COROLLARY 5.4. *Consider two finite-dimensional continuous-time SISO LTI systems sharing the same inputs (5.5) and (5.6) as in Lemma 5.3.*

Assume that

(i) S_1 admits relative degree r_1 from u to y_1, $r_1 \in \{1,\ldots,n_1\}$, and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$, where w and v are viewed as disturbance inputs; and

(ii) the matrix A_2 is Hurwitz and S_2 admits relative degree $r_2 \in \mathbb{N}$ from u to y_2.

Let S_1 be given in the extended zero dynamics canonical form (5.7) or (5.8) depending on $r_1 = n_1$ or $r_1 < n_1$, respectively.

Then, $\forall c_w \in [0,\infty) \subset \mathbb{R}$, $\exists c_c \in [0,\infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \leq c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\left\| w_{[0,t_f)} \right\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| v_{[0,\infty)} \right\|_\infty \leq c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (5.6), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (5.5). If $\left\| x_{1[0,t_f)} \right\|_\infty \leq c_w$, $\ldots$, $\left\| x_{k_0[0,t_f)} \right\|_\infty \leq c_w$, for some fixed $k_0 \in \{1,\ldots,r_1\}$, and $r_2 \geq r_1 - k_0 + 1$, then $\left\| y_{2[0,t_f)} \right\|_\infty \leq c_c$.

Proof. In view of Lemma 5.3, we only need to show that (ii) of Lemma 5.3 holds under the assumptions of this corollary. Since the matrix A_2 is Hurwitz, by Lemmas A.2, A.1, and A.3 of [5], the result holds. This completes the proof of the corollary. $\square$

In the application of Corollary 5.4, we will refer S_1 as the reference system.

6. Conclusions. In this paper, we further investigated the properties of minimum phase systems using the definition of Part I [5]. We proved two technical lemmas that lead to the minimum phase property for linear systems with specific structures in their state space representations. Based on these results, we proved that the composite system consisting of two minimum phase systems in sequential interconnection with additional output feedback is again minimum phase. Another result is that a minimum phase system in feedback connection with another linear system satisfying certain boundedness condition yields a minimum phase composite system. It is well known that a minimum phase system may be inverted, that is tracking of an arbitrary sufficiently smooth bounded reference signal with bounded derivatives up to rth order, without leaving the internal states unbounded, where r is the relative degree of the system. We established the following results in this regard. When a minimum phase linear system admits the relative degree 0 from the control input to the output, a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a minimum phase linear system admits relative degree $r_1 \in \mathbb{N}$, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output is bounded, then, the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the relative degree of the "stable" system is greater than or equal to r_1. When a minimum phase linear system admits relative degree $r_1 \in \mathbb{N}$, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to k_0th order are bounded, then, the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the relative degree of the "stable" system is greater than or equal to $r_1 - k_0$. These results will have significant impact on model reference control design and analysis.

Future research along this direction lies in the generalization of the minimum phase concept for finite-dimensional continuous-time multiple-input and multiple-output linear time-invariant systems, as well as nonlinear systems. Another fruitful research topic lies in the model reference robust adaptive control using the new definition of minimum phase. Both of these topics are currently under investigation.

Appendix A. Some useful results. We present a lemma that is essential for the proof of Lemma 3.1.

LEMMA A.1. *Consider a finite-dimensional continuous-time LTI system:*

$$\dot{z} = Az + Bv; \qquad z(0) = \mathbf{0}_{n\times 1} \tag{A.1}$$

where z is the n-dimensional state, $n \in \mathbb{Z}_+$; and v is the p-dimensional input, $p \in \mathbb{Z}_+$. Assume that the system (A.1) is bounded input and bounded state stable. Then, the following statements hold.

1. *For system $\dot{\eta} = A\eta$, $\eta(0) = B\xi$, where $\xi \in \mathbb{R}^p$, there exist $k \in [0,\infty) \subset \mathbb{R}$ and $\lambda \in (0,\infty) \subset \mathbb{R}$ such that $\forall \xi \in \mathbb{R}^p$ with $|\xi| \le c_w \in [0,\infty) \subset \mathbb{R}$, we have $|\eta(t)| \le c_w k \mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty)$.*
2. *The system $\dot{x} = Ax + ABu$, $x(0) = \mathbf{0}_{n\times 1}$, is bounded input and bounded state stable.*

Proof. By Lemma A.3 of [5], $\exists k \in [0,\infty) \subset \mathbb{R}$ and $\exists \lambda \in (0,\infty) \subset \mathbb{R}$ such that $\left\| \mathrm{e}^{At}B \right\|_{2,2} \le k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty) \subset \mathbb{R}$.

For the first statement, By Lemma A.1 of [5], $|\eta(t)| = \left| \mathrm{e}^{At}B\xi \right| \le k\mathrm{e}^{-\lambda t}c_w$, $\forall t \in [0,\infty) \subset \mathbb{R}$.

For the second statement, we note that $\mathrm{e}^{At}AB = \left(\sum_{i=0}^{\infty}(At)^i\right)AB = A \cdot \left(\sum_{i=0}^{\infty}(At)^i\right)B = A\mathrm{e}^{At}B$, $\forall t \in \mathbb{R}$. Then, $\left\| \mathrm{e}^{At}AB \right\|_{2,2} = \left\| A\mathrm{e}^{At}B \right\|_{2,2} \le \| A \|_{2,2} \cdot \left\| \mathrm{e}^{At}B \right\|_{2,2} \le \| A \|_{2,2} k\mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty) \subset \mathbb{R}$. By Lemma A.3 of [5], the result holds.

This completes the proof of the lemma. ◻

Next, we present a technical lemma that shows that a finite-dimensional continuous-time LTI system satisfies certain bounding property on sub-intervals if it satisfies the property on $[0, \infty)$.

LEMMA A.2. *Consider a finite-dimensional continuous-time LTI system:*

$$\dot{x} = Ax + Bu + Dw; \qquad x(0) = x_0 \tag{A.2}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; u is the p-dimensional input, $p \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$. Assume that (A.2) satisfies that $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\exists c_c \in [0, \infty) \subset \mathbb{R}$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $\left\| u_{[0,\infty)} \right\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, we have $\left\| x_{[0,\infty)} \right\|_\infty \leq c_c$.

Then, $\forall c_w \in [0, \infty) \subset \mathbb{R}$, let c_c be the constant defined previously, $\forall t_f \in (0, \infty] \subset \mathbb{R}_e$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall u_{[0,t_f)} \in \mathcal{C}$ with $\left\| u_{[0,t_f)} \right\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$, we have $\left\| x_{[0,t_f)} \right\|_\infty \leq c_c$.

Proof. $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall t_f \in (0, \infty] \subset \mathbb{R}_e$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall u_{[0,t_f)} \in \mathcal{C}$ with $\left\| u_{[0,t_f)} \right\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\left\| w_{[0,\infty)} \right\|_\infty \leq c_w$. Let $x_{[0,t_f)}$ be the solution to (A.2). $\forall t \in [0, t_f)$, we will show that $|x(t)| \leq c_c$. Definte $\bar{u}_{[0,\infty)} \in \mathcal{C}$ be given by $\bar{u}(s) = \begin{cases} u(s) & s \in [0, t] \\ u(t) & s \in (t, \infty) \end{cases}$. Clearly, we have $\left\| \bar{u}_{[0,\infty)} \right\|_\infty \leq c_w$. Let $\bar{x}_{[0,\infty)}$ be the solution to (A.2) due to $\bar{u}_{[0,\infty)}$, $w_{[0,\infty)}$, and x_0. Then, $\left\| \bar{x}_{[0,\infty)} \right\|_\infty \leq c_c$. By the causality of (A.2), we have $x_{[0,t]} = \bar{x}_{[0,t]}$. Hence, $|x(t)| \leq c_c$. This completes the proof of the lemma. □

LEMMA A.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{K}$ and $\mathcal{D}_0 \subseteq \mathcal{X}$ be a closed subspace, P be a bounded linear operator of $\mathcal{X}$ to $\mathcal{Y}$, and $P(\mathcal{D}_0) \subseteq \mathcal{Y}$ is closed. Then, $\exists c \in [0, \infty) \subset \mathbb{R}$, $\forall c_w \in [0, \infty) \subset \mathbb{R}$, $\forall b \in P(\mathcal{D}_0) \subseteq \mathcal{Y}$ with $\| b \|_{\mathcal{Y}} \leq c_w$, $\exists x \in \mathcal{D}_0 \subseteq \mathcal{X}$ such that $b = Px$ and $\| x \|_{\mathcal{X}} \leq c c_w$.*

Proof. This is immediate from the Open Mapping Theorem [4]. □

REFERENCES

[1] P. A. IOANNOU AND J. SUN, *Robust Adaptive Control*, Prentice Hall, Upper Saddle River, NJ, 1996.

[2] ALBERTO ISIDORI, *Nonlinear Control Systems*, Springer-Verlag, London, 3rd ed., 1995.

[3] D. LIBERZON, A. S. MORSE, AND E. D. SONTAG, *Output-input stability and minimum-phase nonlinear systems*, IEEE Transactions on Automatic Control, 47 (2002), pp. 422–436.

[4] D. G. LUENBERGER, *Optimization by Vector Space Methods*, Wiley, New York, 1969.

[5] ZIGANG PAN AND TAMER BAŞAR, *Generalized minimum phase property for finite-dimensional continuous-time SISO LTI systems with additive disturbances. Part I: Definition and some properties*. Submitted to *SIAM J. Control and Optimization*, May 2007.

[6] E. D. SONTAG AND Y. WANG, *Output-to-state stability and detectability of nonlinear systems*, Systems and Control Letters, 29 (1997), pp. 279–290.