# Properties of the Generalized Minimum Phase Concept for SISO LTI Systems with Additive Disturbances

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***Abstract*— In [1], we had introduced a generalized concept of minimum phase for finite-dimensional continuous-time single-input and single-output linear time-invariant systems with additive disturbances. In this paper, we investigate further properties of minimum phase systems using this concept. We prove that a minimum phase system in feedback connection with another linear system satisfying a certain boundedness condition yields a minimum phase composite system. We establish that a minimum phase system may be inverted, that is an arbitrary reference signal with bounded derivatives up to certain order can be tracked, without making the internal states unbounded. When a minimum phase linear system has relative degree (RD) *zero* from the control input to the output, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a minimum phase linear system has RD $r_1 \in \mathbb{N}$, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to $k_0$th ($0 \leq k_0 < r_1$) order are bounded, then the output of a "stable" system sharing the same inputs as the minimum phase system is bounded if the relative degree of the "stable" system satisfies $r_2 \geq r_1 - k_0$. These results have significant implications on model reference control theory.**

***Index Terms*— continuous-time systems, extended zero dynamics canonical form, minimum phase, extended zero dynamics.**

## I. Introduction

The minimum phase property is of paramount importance in model reference control theory, attracting sustained research attention [2], [3], [4]. In an earlier paper [1], we generalized the minimum phase concept for SISO LTI systems with additive disturbance inputs in an attempt to make it necessary for solvability of the output feedback model reference control problem. When the system has a finite relative degree (RD) $r$, then it may be transformed into the extended zero dynamics canonical form (EZDCF) representation. Based on this canonical form representation, the extended zero dynamics (EZD) for the system was defined, which is simply the zero dynamics as defined in [3] together with the driving terms, including the (noiseless) output and the disturbance input of the system. The original system is said to be minimum phase with respect to the given set of admissible initial conditions and the given set of admissible disturbance waveforms if the EZD is absent or satisfies the properties that the zero dynamics state is bounded for any bounded admissible initial condition (for the EZD), any bounded noiseless output waveform, and any bounded admissible disturbance waveform. In our earlier paper, the relationship of the generalized concept of minimum phase with that introduced in [3] and [2] was investigated. Furthermore, the generalized minimum phase property was proved to be necessary in model reference control of the system.

In this paper, we investigate further properties of minimum phase systems using the definition in [1]. We prove that a composite system consisting of a minimum phase system in feedback connection with another linear system satisfying a certain boundedness condition is itself a minimum phase system. We further establish the inversion result for minimum phase systems as defined in [1]. When a minimum phase linear system has the RD $r_1 = 0$ from the control input to the output, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a minimum phase linear system has RD $r_1 \in \mathbb{N}$, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to $k_0$th ($0 \leq k_0 < r_1$) order are bounded, then the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the RD of the "stable" system satisfies $r_2 \geq r_1 - k_0$. These results have significant implications on model reference control theory. In an accompanying paper [5], which is technically dependent on this paper, submitted also to this conference, we proved that a composite system comprised of two minimum phase systems in series interconnection with additional output feedback is itself a minimum phase system under the generalized concept.

The balance of the paper is as follows. In the next section, we list the notations used in the paper. Then, in Section III, the minimum phase property of the composite system is proved for feedback interconnected systems. The boundedness of the inverse of minimum phase systems is presented in Section IV. The paper ends with some concluding remarks in Section V, and an appendix containing some useful technical lemmas necessary for the derivations in the main body of the paper.

## II. Notations

Let $\mathbb{R}$ denote the real line; $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Unless specified, all signals, constants, and matrices are real. For a continuous function $f$, we say that it belongs to $\mathcal{C}$. We say that a function is $\mathrm{L}_\infty$ if it is bounded. For any matrix $A$, $A'$ denotes its transpose. For any $z \in \mathbb{R}^n$, $|z|$ denotes $\sqrt{z'z}$. $I_n$ denotes the $n \times n$-dimensional identity matrix. For any

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matrix $A$, $A^0 = I$. For any matrix $M$, $\|M\|_{p,p}$ denotes its $p$-induced norm, $1 \le p \le \infty$. $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f),\mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$.

## III. Minimum phase property for feedback interconnected systems

We first present a lemma which establishes the minimum phase property for SISO LTI systems in a general canonical form, which arises in interconnected systems.

*Lemma 1:* Consider a SISO LTI system

$$\dot{x}_z = A_o x_z + A_{o1}x_1 + \cdots + A_{or}x_r + A_{o\,r+1}b_0 u + D_o w \quad (1a)$$
$$\dot{x}_i = A_{io}x_z + a_{i1}x_1 + \cdots + a_{ii}x_i + x_{i+1} + D_i w;\ 1 \le i < r \quad (1b)$$
$$\dot{x}_r = A_{ro}x_z + a_{r1}x_1 + \cdots + a_{rr}x_r + b_0 u + D_r w \quad (1c)$$
$$y = x_1 + Ew \quad (1d)$$

where $x_z \in \mathbb{R}^{n-r}$, $0 < r < n$; $x_i \in \mathbb{R}$, $i = 1,\ldots,r$; $b_0 \neq 0$; $y \in \mathbb{R}$ is the output; $u \in \mathbb{R}$ is the control input; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; all matrices are of appropriate dimensions and constant. Let $x = [x_z'\ x_1 \cdots x_r]'$, $x(0) = x_0 \in \mathcal{D}_0 \neq \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$ (See Definition 2 of [1]). Assume that $\exists m_o \in \{1,\ldots,r+1\}$ such that $A_{oj} = \mathbf{0}_{(n-r)\times 1}$, $\forall j \in \{m_o+1,\ldots,r+1\}$; and $A_{jo} = \mathbf{0}_{1\times(n-r)}$, $\forall j \in \{1,\ldots,m_o-1\}$. Then, system (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits RD $r$ from $u$ to $y$ if the dynamics (1a) satisfies $\forall c_w \ge 0$, $\exists c_c \ge 0$, $\forall x_{z0} \in \mathcal{D}_{z0} = P_z(\mathcal{D}_0)$ with $|x_{z0}| \le c_w$, where $P_z : \mathbb{R}^n \to \mathbb{R}^{n-r}$ is the projection of $\mathbb{R}^n$ onto its first $n-r$ coordinates, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \le c_w$, $i = 1,\ldots,m_o$, ($x_{r+1} := b_0 u$ for notational consistency) we have $\|x_{z[0,\infty)}\|_\infty \le c_c$.

*Proof:* We will prove the lemma using mathematical induction on $m_o$.

$1^\circ$ $m_o = 1$. Clearly $A_{o2} = \cdots = A_{or+1} = \mathbf{0}_{(n-r)\times 1}$. Then, (1) is in the form (3) of [1]. Then, (1a) is the EZD of (1). By Proposition 1 of [1], (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits RD $r$. This proves $1^\circ$.

$2^\circ$ Suppose that the lemma holds for $m_o \le k (\le r)$.

$3^\circ$ Consider the case $m_o = k+1 \in \{2,\ldots,r+1\}$. Consider the state transformation $\bar{x} := [\bar{x}_z'\ x_1 \cdots x_r]' = T_1^{-1}x = \left[x_z' - A_{o\,k+1}'x_k\ x_1 \cdots x_r\right]'$. Then, in $\bar{x}$ coordinates, system (1) admits the representation with $\bar{x}(0) = \bar{x}_0 := [\bar{x}_{z0}'\ x_{1,0} \cdots x_{r,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_0) \neq \emptyset$.

$$\begin{aligned}\dot{\bar{x}}_z &= A_o\bar{x}_z + A_oA_{o\,k+1}x_k + A_{o1}x_1 + \cdots + A_{ok}x_k \\ &\quad + D_o w - A_{o\,k+1}\left(a_{k1}x_1 + \cdots + a_{kk}x_k + D_k w\right) \\ &=: A_o\bar{x}_z + \bar{A}_{o1}x_1 + \cdots + \bar{A}_{ok}x_k + \bar{D}_o w \end{aligned} \quad (2a)$$
$$\dot{x}_i = a_{i1}x_1 + \cdots + a_{ii}x_i + x_{i+1} + D_i w;\ i = 1,\ldots,k \quad (2b)$$
$$\begin{aligned}\dot{x}_i &= A_{io}\bar{x}_z + A_{io}A_{o\,k+1}x_k + a_{i1}x_1 + \cdots + a_{ii}x_i + x_{i+1} \\ &\quad + D_i w; \quad i = k+1,\ldots,r\end{aligned} \quad (2c)$$
$$y = x_1 + Ew \quad (2d)$$

Clearly, the representation (2) is in the form of (1) with $\bar{m}_o = k$. We will apply the inductive argument to show that (2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. Toward this end, $\forall c_w \ge 0$, $\forall \bar{x}_{z0} \in P_z(\bar{\mathcal{D}}_0)$ with $|\bar{x}_{z0}| \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \le c_w$, $i = 1,\ldots,k$. Since $\bar{x}_{z0} \in \bar{\mathcal{D}}_{z0} = P_z(\bar{\mathcal{D}}_0)$, by Lemma 5, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_0$, and $\exists \acute{x}_{1,0},\ldots,\acute{x}_{r,0} \in \mathbb{R}$ such that $\acute{x}_0 := [\bar{x}_{z0}'\ \acute{x}_{1,0} \cdots \acute{x}_{r,0}]' \in \bar{\mathcal{D}}_0$ and $|\acute{x}_0| \le c_1 c_w$. Then, $T_1\acute{x}_0 = [\acute{x}_{z0}'\ \acute{x}_{1,0} \cdots \acute{x}_{r,0}]' \in \mathcal{D}_0$ where $\acute{x}_{z0} := \bar{x}_{z0} + A_{o\,k+1}\acute{x}_{k,0} \in P_z(\mathcal{D}_0)$ with $|\acute{x}_{z0}| \le \|T_1\|_{2,2}c_1c_w =: c_1c_2c_w$. The solution to (2a) may be decomposed by linearity as the sum of the following three systems.

$$\begin{aligned}\dot{\eta}_1 &= A_o\eta_1 + A_{o1}x_1 + \cdots + A_{ok}x_k + A_{o\,k+1}(-a_{k1}x_1 - \cdots \\ &\quad - a_{kk}x_k - D_k w) + D_o w; \\ &\quad \eta_1(0) = \bar{x}_{z0} + A_{o\,k+1}\acute{x}_{k,0} \in P_z(\mathcal{D}_0)\end{aligned} \quad (3a)$$
$$\dot{\eta}_2 = A_o\eta_2; \qquad \eta_2(0) = -A_{o\,k+1}\acute{x}_{k,0} \quad (3b)$$
$$\dot{\eta}_3 = A_o\eta_3 + A_oA_{o\,k+1}x_k; \qquad \eta_3(0) = \mathbf{0}_{(n-r)\times 1} \quad (3c)$$
$$\bar{x}_{z[0,\infty)} = \eta_{1[0,\infty)} + \eta_{2[0,\infty)} + \eta_{3[0,\infty)} \quad (3d)$$

Let $\bar{x}_{k+1} = -a_{k1}x_1 - \cdots - a_{kk}x_k - D_k w$. Then, $\bar{x}_{k+1[0,\infty)} \in \mathcal{C}$ and $\|\bar{x}_{k+1[0,\infty)}\|_\infty \le (\sum_{i=1}^k |a_{ki}|)c_w + \|D_k\|_{2,2}c_w =: \bar{c}_{w1}c_w$. For (3a), by the assumption of the lemma, $\exists c_{c1} \ge 0$, which depends only on $c_w$, $\bar{c}_{w1}c_w$, and $c_1c_2c_w$, such that $\|\eta_{1[0,\infty)}\|_\infty \le c_{c1}$.

Under the assumption of the lemma, by Lemma 9 of [1], the following system is bounded input bounded state (BIBS) stable: $\dot{\xi} = A_o\xi + A_{o\,k+1}v$, $\xi(0) = \mathbf{0}_{(n-r)\times 1}$. For (3b), by Lemma 3, $\exists c_{c2} \ge 0$, which does not depend on any other constant, such that $\|\eta_{2[0,\infty)}\|_\infty \le c_{c2}|\acute{x}_{k,0}| \le c_{c2}c_1c_w$.

Again by Lemma 3, (3c) is BIBS stable. Then, by Lemma 6 of [1], $\exists c_{c3} \ge 0$, which does not depend on any other constant, such that $\|\eta_{3[0,\infty)}\|_\infty \le c_{c3}c_w$.

Then, we have $\|\bar{x}_{z[0,\infty)}\|_\infty \le c_{c1} + c_{c2}c_1c_w + c_{c3}c_w$. This shows that (2) satisfies the inductive assumption. Hence, (2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits RD $r$, which is equivalent to that (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits RD $r$ from $u$ to $y$.

This completes the induction process and the proof. ■

Next, we present a theorem that establishes the minimum phase property for the composite system consisting of two subsystems in feedback configuration. The block diagram of the system is shown in Figure 1.

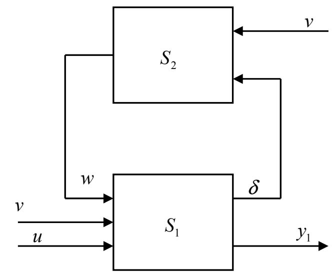


Fig. 1. Block diagram of two feedback interconnected systems.

*Theorem 1:* Consider two LTI systems in feedback:

$$S_1 : \begin{cases} \dot{x} = A_1x + B_1u + D_{1,w}w + D_{1,v}v;\ x(0) = x_0 \\ y = C_1x + K_1u + E_{1,w}w + E_{1,v}v \\ \delta = C_1x + K_1u \end{cases} \quad (4a)$$
$$S_2 : \begin{cases} \dot{\eta} = A_2\eta + B_2\delta + D_2v; \qquad \eta(0) = \eta_0 \\ w = C_2\eta + K_2\delta + E_2v \end{cases} \quad (4b)$$

where $x \in \mathbb{R}^{n_1}$ is the state for $S_1$, $n_1 \in \mathbb{Z}_+$; $y \in \mathbb{R}$ is the output of $S_1$; $u \in \mathbb{R}$ is the input for $S_1$; $\delta \in \mathbb{R}$ is the noiseless output of $S_1$, which is also the input to $S_2$; $w \in \mathbb{R}^{q_w}$ is the disturbance input for $S_1$, $q_w \in \mathbb{Z}_+$, which is also the output of $S_2$; $v \in \mathbb{R}^{q_v}$ is the disturbance input for both $S_1$ and $S_2$, $q_v \in \mathbb{Z}_+$; $\eta \in \mathbb{R}^{n_2}$ is the state for $S_2$, $n_2 \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_{x0}$, $\mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$ is a subspace; $\eta_0 \in \mathcal{D}_{\eta 0}$, $\mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$ is a subspace; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant and of appropriate dimensions.

The composite system $S$ has control input $u$, output $y$, and disturbance $v$. Assume that

TABLE I
6 EXHAUSTIVE AND MUTUALLY EXCLUSIVE CASES FOR THEOREM 1.

| | |
|---|---|
| Case 1: | $0 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$ |
| Case 2: | $1 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$ |
| Case 3: | $2 \leq r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$ |
| Case 4: | $1 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$ |
| Case 5: | $2 \leq r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$ |
| Case 6: | $0 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$ |

1) $S_1$ admits RD $r_1 \in \{0, \ldots, n_1\}$ from $u$ to $y$ and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, where $w$ and $v$ are considered as disturbances;
2) $\hat{E}_{1,w} := 1 + E_{1,w}K_2 \neq 0$, such that $S$ admits a well defined RD; and
3) the following associated system of $S_2$, $\bar{S}_2$,
$$\begin{cases} \dot{\psi} = (A_2 - B_2\hat{E}_{1,w}^{-1}E_{1,w}C_2)\psi + B_2\tau + D_2 v \\ \quad =: \bar{A}_2\psi + B_2\tau + D_2 v; \quad \psi(0) = \eta_0 \in \mathcal{D}_{\eta 0} \\ \phi = \tau \end{cases} \tag{5}$$
with scalar control input $\tau$ and disturbance $v$ and scalar output $\phi$ is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

Then, the composite system $S$ admits RD $r_1$ from $u$ to $y$ and is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

*Proof:* Note that
$$\begin{aligned} y &= \delta + E_{1,w}\left(C_2\eta + K_2\delta + E_2 v\right) + E_{1,v}v \\ &= (1 + E_{1,w}K_2)\delta + E_{1,w}C_2\eta + (E_{1,v} + E_{1,w}E_2)v \\ &=: \hat{E}_{1,w}\delta + E_{1,w}C_2\eta + \hat{E}_{1,v}v \end{aligned} \tag{6}$$

We will distinguish 6 exhaustive and mutually exclusive cases, which are listed in Table I.

Case 1: $0 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3 of [1], without loss of generality, assume that $S_1$ is given in the following EZDCF representation, $K_1 \neq 0$,
$$\begin{aligned} \dot{x} &= \bar{A}_1 x + \bar{B}_1\left(y - E_{1,w}w - E_{1,v}v\right) + D_{1,w}w + D_{1,v}v \\ &=: \bar{A}_1 x + \bar{B}_1 y + \bar{D}_{1,w}w + \bar{D}_{1,v}v; \qquad x(0) = x_0 \\ y &= \delta + E_{1,w}w + E_{1,v}v \\ \delta &= C_1 x + K_1 u \end{aligned}$$
Note that, by (6),
$$\begin{aligned} \delta &= \hat{E}_{1,w}^{-1}\left(y - E_{1,w}C_2\eta - \hat{E}_{1,v}v\right) \\ w &= C_2\eta + K_2\delta + E_2 v =: \bar{C}_2\eta + K_2\hat{E}_{1,w}^{-1}y + \bar{E}_2 v \end{aligned}$$
Then, the composite system $S$ admits the following representation, in $\xi = [x'\ \eta']'$ coordinates,
$$\begin{aligned} \dot{x} &= \bar{A}_1 x + \bar{B}_1 y + \bar{D}_{1,w}\left(\bar{C}_2\eta + K_2\hat{E}_{1,w}^{-1}y + \bar{E}_2 v\right) + \bar{D}_{1,v}v \\ &=: \bar{A}_1 x + \hat{A}_{12}\eta + \hat{B}_1\left(y - \hat{E}_{1,v}v\right) + \hat{D}_{1,v}v & \text{(7a)} \\ \dot{\eta} &= A_2\eta + B_2\hat{E}_{1,w}^{-1}\left(y - E_{1,w}C_2\eta + \hat{E}_{1,v}v\right) + D_2 v \\ &= \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v) + D_2 v & \text{(7b)} \\ y &= \hat{E}_{1,w}C_1 x + E_{1,w}C_2\eta + \hat{E}_{1,w}K_1 u + \hat{E}_{1,v}v & \text{(7c)} \end{aligned}$$
Clearly, (7) is in the EZDCF (6) of [1] with (7a) and (7b) as its EZD, since $\hat{E}_{1,w}K_1 \neq 0$. Hence, $S$ admits RD $0 = r_1$ from $u$ to $y$. $\forall c_w \geq 0$, $\forall\, [x_0'\ \eta_0']' \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ with $\left|[x_0'\ \eta_0']'\right| \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty \leq c_w$, let $x_{[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (7a) and (7b). Let $\tau = \hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v)$. Then, $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \leq (\|\hat{E}_{1,w}^{-1}\|_{2,2} + \|\hat{E}_{1,w}^{-1}\hat{E}_{1,v}\|_{2,2}) \cdot c_w =: c_{w1}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \geq 0$, which depends only on $c_w$ and $c_{w1}$, such that $\|\eta_{[0,\infty)}\|_\infty \leq c_{c1}$. Note that $w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty \leq \|\bar{C}_2\|_{2,2}c_{c1} + \|K_2\hat{E}_{1,w}^{-1}\|_{2,2}c_w + \|\bar{E}_2\|_{2,2}c_w =: c_{w2}$. Since $S_1$ is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, $\exists c_{c2} \geq 0$, which depends only of $c_w$ and $c_{w2}$, such that $\|x_{[0,\infty)}\|_\infty \leq c_{c2}$. Then, $\|\xi_{[0,\infty)}\|_\infty \leq \sqrt{c_{c1}^2 + c_{c2}^2}$. Hence, (7) is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This proves Case 1.

Case 2: $1 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 2 of [1], without loss of generality, assume that $S_1$ is given in the following EZDCF representation, $b_1 \neq 0$,
$$\begin{aligned} \dot{x}_1 &= a_{1,11}x_1 + b_1 u + D_{1,1w}w + D_{1,1v}v \\ \delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v \end{aligned}$$
where $x = x_1 \in \mathbb{R}$; $x(0) = x_{1,0} \in \mathcal{D}_{x0}$. Note that, by (6),
$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$
The system $S$ admits the following representation, in $\xi = [\eta'\ \bar{x}_1]' = T_1^{-1}[x\ \eta']'$ coordinates with $\xi(0) = \xi_0 := [\eta_0'\ \bar{x}_{1,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$. ,
$$\begin{aligned} \dot{\eta} &= A_2\eta + B_2 x_1 + D_2 v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v \\ &= \bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v & \text{(8a)} \\ \dot{\bar{x}}_1 &= E_{1,w}C_2\left(\bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v\right) + \hat{E}_{1,w}\left(a_{1,11}x_1\right. \\ &\quad \left. + b_1 u + D_{1,1w}\left(C_2\eta + K_2 x_1 + E_2 v\right) + D_{1,1v}v\right) \\ &=: \bar{A}_{1,1\eta}\eta + \bar{a}_{1,11}\bar{x}_1 + \hat{E}_{1,w}b_1 u + \bar{D}_{1,1v}v & \text{(8b)} \\ y &= \bar{x}_1 + \hat{E}_{1,v}v & \text{(8c)} \end{aligned}$$

We will further distinguish two exhaustive and mutually exclusive subcases: Case 2a: $n_2 \in \mathbb{N}$; Case 2b: $n_2 = 0$.

Case 2a: $n_2 \in \mathbb{N}$. Clearly, (8) is in EZDCF (2) of [1], and admits RD $1 = r_1$ from $u$ to $y$, since $\hat{E}_{1,w}b_1 \neq 0$.

$\forall c_w \geq 0$, $\forall \eta_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_0| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_2$ coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \leq c_w$, let $\eta_{[0,\infty)}$ be the solution to (8a). Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Then, $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \leq \|\hat{E}_{1,w}^{-1}\|_{2,2}c_w =: c_{w1}$. By Lemma 5, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, such that $\exists \acute{\xi}_0 := \left[\eta_0'\ \acute{\bar{x}}_{1,0}\right]' \in \bar{\mathcal{D}}_0$ with $|\acute{\xi}_0| \leq c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\eta_0 \in \mathcal{D}_{\eta 0}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \geq 0$, which depends only on $c_w$ and $c_{w1}$, such that $\|\eta_{[0,\infty)}\|_\infty \leq c_{c1}$. Hence, (8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Proposition 1 of [1]. This completes the proof for Case 2a.

Case 2b: $n_2 = 0$. (8) is in the EZDCF (5) of [1]. Then, (8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ since its EZD is absent. This proves Case 2b and Case 2.

Case 3: $2 \leq r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 2 of [1], without loss of generality, assume that $S_1$ is given in the following EZDCF representation, $b_1 \neq 0$,
$$\begin{aligned} \dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v; \ 1 \leq i < r_1 \\ \dot{x}_{r_1} &= a_{1,r_1 1}x_1 + b_1 u + D_{1,r_1 w}w + D_{1,r_1 v}v \\ \delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v \end{aligned}$$
where $x = [x_1 \cdots x_{r_1}]'$; $x_i \in \mathbb{R}$, $i = 1, \ldots, r_1$; $x(0) = x_0 := [x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$. Note that, by (6),
$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$
Define $\bar{x}_i = \hat{E}_{1,w}x_i$, $i = 2, \ldots, r_1$. Then, $S$ admits the following representation, in $\xi = [\eta'\ \bar{x}_1 \cdots \bar{x}_{r_1}]' = T_1^{-1}[x'\ \eta']'$ coordinates with $\xi(0) = \xi_0 = [\eta_0'\ \bar{x}_{1,0} \cdots \bar{x}_{r_1,0}]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,
$$\dot{\eta} = A_2\eta + B_2 x_1 + D_2 v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v$$

$$
\begin{aligned}
&= \bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v && \text{(9a)}\\
\dot{\bar{x}}_1 &= E_{1,w}C_2\left(\bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v\right) + \hat{E}_{1,w}\left(a_{1,11}x_1 \right.\\
&\quad \left. + x_2 + D_{1,1w}\left(C_2\eta + K_2x_1 + E_2v\right) + D_{1,1v}v\right)\\
&=: \bar{A}_{1,1\eta}\eta + \bar{a}_{1,11}\bar{x}_1 + \bar{x}_2 + \bar{D}_{1,1v}v && \text{(9b)}\\
\dot{\bar{x}}_i &= \hat{E}_{1,w}\left(a_{1,i1}x_1 + x_{i+1} + D_{1,iw}\left(C_2\eta + K_2x_1 \right.\right.\\
&\quad \left.\left. + E_2v\right) + D_{1,iv}v\right); \quad i = 2,\ldots,r_1-1\\
&=: \bar{A}_{1,i\eta}\eta + \bar{a}_{1,i1}\bar{x}_1 + \bar{x}_{i+1} + \bar{D}_{1,iv}v; && \text{(9c)}\\
\dot{\bar{x}}_{r_1} &= \hat{E}_{1,w}\left(a_{1,r_11}x_1 + b_1u + D_{1,r_1w}\left(C_2\eta + K_2x_1 \right.\right.\\
&\quad \left.\left. + E_2v\right) + D_{1,r_1v}v\right)\\
&=: \bar{A}_{1,r_1\eta}\eta + \bar{a}_{1,r_11}\bar{x}_1 + \hat{E}_{1,w}b_1u + \bar{D}_{1,r_1v}v && \text{(9d)}\\
y &= \bar{x}_1 + \hat{E}_{1,v}v && \text{(9e)}
\end{aligned}
$$

We will further distinguish two exhaustive and mutually exclusive subcases: Case 3a: $n_2 \in \mathbb{N}$; Case 3b: $n_2 = 0$.

Case 3a: $n_2 \in \mathbb{N}$. Clearly, (9) is in form (1) with $m_o = 1$, and admits RD $r_1$ from $u$ to $y$, since $\hat{E}_{1,w}b_1 \neq 0$. We will apply Lemma 1 to prove this subcase. $\forall c_w \geq 0$, $\forall \eta_0 \in P_z(\bar{\mathcal{D}}_0)$ with $|\eta_0| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ is the projection from $\mathbb{R}^{n_1+n_2}$ to the first $n_2$ coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \leq c_w$, let $\eta_{[0,\infty)}$ be the solution to (9a). By Lemma 5, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, and $\exists \acute{\xi}_0 = \left[\eta_0' \, \acute{\bar{x}}_{1,0} \cdots \acute{\bar{x}}_{r_1,0}\right]' \in \bar{\mathcal{D}}_0$ such that $|\acute{\xi}_0| \leq c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\eta_0 \in \mathcal{D}_{\eta 0}$. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \leq \|\hat{E}_{1,w}^{-1}\|_{2,2}c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_{c1} \geq 0$, which depends only on $c_w$ and $c_{w1}$, such that $\|\eta_{[0,\infty)}\|_\infty \leq c_{c1}$. Hence, by Lemma 1, (9) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This proves Case 3a.

Case 3b: $n_2 = 0$. (9) is in the EZDCF (5) of [1]. Then, (9) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ since its EZD is absent. This proves Case 3b and Case 3.

Case 4: $1 = r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 1 of [1], without loss of generality, assume that $S_1$ is given in the following EZDCF representation, $b_1 \neq 0$,

$$
\begin{aligned}
\dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v\\
\dot{x}_1 &= A_{1,1z}x_z + a_{1,11}x_1 + b_1u + D_{1,1w}w + D_{1,1v}v\\
\delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v
\end{aligned}
$$

where $x = \left[x_z' \, x_1\right]'$; $x_z \in \mathbb{R}^{n_1-r_1}$; $x_1 \in \mathbb{R}$; and $x(0) = x_0 := \left[x_{z0}' \, x_{1,0}\right]' \in \mathcal{D}_{x0}$. Note that, by (6),

$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$

The composite system $S$ admits the following representation, in $\xi = \left[x_z' \, \eta' \, \bar{x}_1\right]' =: T_1^{-1}\left[x' \, \eta'\right]'$ coordinates with $\xi(0) = \xi_0 = \left[x_{z0}' \, \eta_0' \, \bar{x}_{1,0}\right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$
\begin{aligned}
\dot{x}_z &= A_{1,z}x_z + A_{1,z1}\hat{E}_{1,w}^{-1}\left(\bar{x}_1 - E_{1,w}C_2\eta\right) + D_{1,zw}\left(C_2\eta \right.\\
&\quad \left. + K_2x_1 + E_2v\right) + D_{1,zv}v\\
&=: \bar{A}_{1,\eta}\eta + A_{1,z}x_z + \bar{A}_{1,z1}\bar{x}_1 + \bar{D}_{1,zv}v && \text{(10a)}\\
\dot{\eta} &= A_2\eta + B_2x_1 + D_2v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v && \text{(10b)}\\
\dot{\bar{x}}_1 &= E_{1,w}C_2(\bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v) + \hat{E}_{1,w}(A_{1,1z}x_z\\
&\quad + a_{1,11}x_1 + b_1u + D_{1,1w}(C_2\eta + K_2x_1 + E_2v) + D_{1,1v}v)\\
&=: \bar{A}_{1,1z}x_z + \bar{A}_{1,1\eta}\eta + \bar{a}_{1,11}\bar{x}_1 + \hat{E}_{1,w}b_1u + \bar{D}_{1,1v}v && \text{(10c)}\\
y &= \bar{x}_1 + \hat{E}_{1,v}v && \text{(10d)}
\end{aligned}
$$

Clearly, (10) is in the EZDCF (2) of [1], and admits RD $1 = r_1$ from $u$ to $y$, since $\hat{E}_{1,w}b_1 \neq 0$.

$\forall c_w \geq 0$, $\forall \xi_{z0} := \left[x_{z0}' \, \eta_0'\right]' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1-1+n_2}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1+n_2-1$ coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (10a) and (10b). By Lemma 5, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, and $\exists \acute{\xi}_0 = \left[\xi_{z0}' \, \acute{\bar{x}}_{1,0}\right]' \in \bar{\mathcal{D}}_0$ such that $|\acute{\xi}_0| \leq c_1 c_w$. Then, $T_1\acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, and $\eta_0 \in \mathcal{D}_{\eta 0}$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1-r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \leq \|\hat{E}_{1,w}^{-1}\|_{2,2}c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_{c1} \geq 0$, which depends only on $c_w$ and $c_{w1}$, such that $\|\eta_{[0,\infty)}\|_\infty \leq c_{c1}$. Note that $x_{1[0,\infty)} \in \mathcal{C}$ with $\|x_{1[0,\infty)}\|_\infty = \|\hat{E}_{1,w}^{-1}\left(\bar{x}_{1[0,\infty)} - E_{1,w}C_2\eta_{[0,\infty)}\right)\|_\infty \leq \|\hat{E}_{1,w}^{-1}\|_{2,2}c_w + \|\hat{E}_{1,w}^{-1}E_{1,w}C_2\|_{2,2}c_{c1} =: c_{w2}$; and $w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty = \|C_2\eta_{[0,\infty)} + K_2x_{1[0,\infty)} + E_2v_{[0,\infty)}\|_\infty \leq \|C_2\|_{2,2}c_{c1} + \|K_2\|_{2,2}c_{w2} + \|E_2\|_{2,2}c_w =: c_{w3}$. Since $S_1$ is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$, then, by Proposition 1 of [1], $\exists c_{c2} \geq 0$, which depends only on $c_w$, $c_{w2}$, and $c_{w3}$, such that $\|x_{z[0,\infty)}\|_\infty \leq c_{c2}$. Then, $\|\xi_{z[0,\infty)}\|_\infty \leq \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = \left[x_z' \, \eta'\right]'$. Hence, (10) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Proposition 1 of [1]. This proves Case 4.

Case 5: $2 \leq r_1 < n_1$ and $n_2 \in \mathbb{Z}_+$. By Lemma 1 of [1], without loss of generality, assume that $S_1$ is given in the following EZDCF representation, $b_1 \neq 0$,

$$
\begin{aligned}
\dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v\\
\dot{x}_i &= a_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v; \ 1 \leq i < r_1\\
\dot{x}_{r_1} &= A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1u + D_{1,r_1w}w + D_{1,r_1v}v\\
\delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v
\end{aligned}
$$

where $x = \left[x_z' \, x_1 \cdots x_{r_1}\right]'$; $x_z \in \mathbb{R}^{n_1-r_1}$; $x_i \in \mathbb{R}$, $i = 1,\ldots,r_1$; and $x(0) = x_0 = \left[x_{z0}' \, x_{1,0} \cdots x_{r_1,0}\right]' \in \mathcal{D}_{x0}$. Note that, by (6),

$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$

Define $\bar{x}_i = \hat{E}_{1,w}x_i$, $i = 2,\ldots,r_1$. The composite system $S$ admits the following representation, in $\xi = \left[x_z' \, \eta' \, \bar{x}_1 \cdots \bar{x}_{r_1}\right]' = T_1^{-1}\left[x' \, \eta'\right]'$ coordinates with $\xi(0) = \xi_0 := \left[x_{z0}' \, \eta_0' \, \bar{x}_{1,0} \cdots \bar{x}_{r_1,0}\right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$
\begin{aligned}
\dot{x}_z &= A_{1,z}x_z + A_{1,z1}\hat{E}_{1,w}^{-1}\left(\bar{x}_1 - E_{1,w}C_2\eta\right) + D_{1,zw}\left(C_2\eta \right.\\
&\quad \left. + K_2x_1 + E_2v\right) + D_{1,zv}v\\
&=: A_{1,z}x_z + \bar{A}_{1,\eta}\eta + \bar{A}_{1,z1}\bar{x}_1 + \bar{D}_{1,zv}v && \text{(11a)}\\
\dot{\eta} &= A_2\eta + B_2x_1 + D_2v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v && \text{(11b)}\\
\dot{\bar{x}}_1 &= E_{1,w}C_2\left(\bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2v\right) + \hat{E}_{1,w}\left(a_{1,11}x_1 \right.\\
&\quad \left. + x_2 + D_{1,1w}\left(C_2\eta + K_2x_1 + E_2v\right) + D_{1,1v}v\right)\\
&=: \bar{A}_{1,1\eta}\eta + \bar{a}_{1,11}\bar{x}_1 + \bar{x}_2 + \bar{D}_{1,1v}v && \text{(11c)}\\
\dot{\bar{x}}_i &= \hat{E}_{1,w}(a_{1,i1}x_1 + x_{i+1} + D_{1,iv}v + D_{1,iw}(C_2\eta + K_2x_1 +\\
&\quad E_2v)) =: \bar{A}_{1,i\eta}\eta + \bar{a}_{1,i1}\bar{x}_1 + \bar{x}_{i+1} + \bar{D}_{1,iv}v; \ 2 \leq i < r_1 && \text{(11d)}\\
\dot{\bar{x}}_{r_1} &= \hat{E}_{1,w}\left(A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1u + D_{1,r_1w}\left(C_2\eta \right.\right.\\
&\quad \left.\left. + K_2x_1 + E_2v\right) + D_{1,r_1v}v\right) =: \bar{A}_{1,r_1z}x_z + \bar{A}_{1,r_1\eta}\eta\\
&\quad + \bar{a}_{1,r_11}\bar{x}_1 + \hat{E}_{1,w}b_1u + \bar{D}_{1,r_1v}v && \text{(11e)}\\
y &= \bar{x}_1 + \hat{E}_{1,v}v && \text{(11f)}
\end{aligned}
$$

Clearly, (11) is in the form (1) with $m_o = 1$, and admits RD $r_1$ from $u$ to $y$, since $\hat{E}_{1,w}b_1 \neq 0$.

We will apply Lemma 1 to prove that (11) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \geq 0$, $\forall \xi_{z0} := \left[x_{z0}' \, \eta_0'\right]' \in P_z(\bar{\mathcal{D}}_0)$ with $|\xi_{z0}| \leq c_w$, where $P_z : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2-r_1}$ is the projection of $\mathbb{R}^{n_1+n_2}$ onto the first $n_1 + n_2 - r_1$ coordinates, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq$

$c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \le c_w$, let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (11a) and (11b). By Lemma 5, $\exists c_1 \in [0,\infty) \subset \mathbb{R}$, which depends only on $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $\exists \acute{\xi}_0 = \left[ \xi'_{z0} \, \acute{\bar{x}}_{1,0} \cdots \acute{\bar{x}}_{r_1,0} \right]' \in \bar{\mathcal{D}}_0$ such that $|\acute{\xi}_0| \le c_1 c_w$. Then, $T_1 \acute{\xi}_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $\eta_0 \in \mathcal{D}_{\eta 0}$, and $x_{z0} \in P_{xz}(\mathcal{D}_{x0})$, where $P_{xz} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1 - r_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the first $n_1 - r_1$ coordinates. Let $\tau = \hat{E}_{1,w}^{-1} \bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \le \|\hat{E}_{1,w}^{-1}\|_{2,2} c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_{c1} \ge 0$, which depends only on $c_w$ and $c_{w1}$, such that $\|\eta_{[0,\infty)}\|_\infty \le c_{c1}$. Note that $x_{1[0,\infty)} \in \mathcal{C}$ with $\|x_{1[0,\infty)}\|_\infty = \|\hat{E}_{1,w}^{-1} (\bar{x}_{1[0,\infty)} - E_{1,w} C_2 \eta_{[0,\infty)})\|_\infty \le \|\hat{E}_{1,w}^{-1}\|_{2,2} c_w + \|\hat{E}_{1,w}^{-1} E_{1,w} C_2\|_{2,2} \cdot c_{c1} =: c_{w2}$; and $w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty = \|C_2 \cdot \eta_{[0,\infty)} + K_2 x_{1[0,\infty)} + E_2 v_{[0,\infty)}\|_\infty \le \|C_2\|_{2,2} c_{c1} + \|K_2\|_{2,2} \cdot c_{w2} + \|E_2\|_{2,2} c_w =: c_{w3}$. Since $S_1$ is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, then, by Proposition 1 of [1], $\exists c_{c2} \ge 0$, which depends only on $c_w$, $c_{w2}$, and $c_{w3}$, such that $\|x_{z[0,\infty)}\|_\infty \le c_{c2}$. Then, $\|\xi_{z[0,\infty)}\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = [x'_z \, \eta']'$. Hence, (11) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This proves Case 5.

Case 6: $0 = r_1 = n_1$ and $n_2 \in \mathbb{Z}_+$. By Definition 1 of [1], without loss of generality, assume that the system $S_1$ is given in the following EZDCF representation, $K_1 \ne 0$,

$$S_1 : y = K_1 u + E_{1,w} w + E_{1,v} v; \quad \delta = K_1 u$$

By (6), we have

$$y = E_{1,w} C_2 \eta + \hat{E}_{1,w} K_1 u + \hat{E}_{1,v} v$$
$$\delta = K_1 u = \hat{E}_{1,w}^{-1} (y - E_{1,w} C_2 \eta - \hat{E}_{1,v} v)$$

Then, the composite system $S$ admits the EZDCF:

$$\begin{aligned} \dot{\eta} &= A_2 \eta + B_2 \hat{E}_{1,w}^{-1} (y - E_{1,w} C_2 \eta - \hat{E}_{1,v} v) + D_2 v \\ &= \bar{A}_2 \eta + B_2 \hat{E}_{1,w}^{-1} (y - \hat{E}_{1,v} v) + D_2 v \end{aligned} \tag{12a}$$

$$y = E_{1,w} C_2 \eta + \hat{E}_{1,w} K_1 u + \hat{E}_{1,v} v \tag{12b}$$

Clearly, (12) admits RD $0 = r_1$ from $u$ to $y$, since $\hat{E}_{1,w} K_1 \ne 0$. If $n_2 = 0$, then (12) is minimum phase with respect to $\mathbb{R}^0 = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since its EZD is absent. If $n_2 \in \mathbb{N}$, then (12) is in the form of (6) of [1] with (12a) defining its EZD. $\forall c_w \ge 0$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty \le c_w$, let $\eta_{[0,\infty)}$ be the solution to (12a). Let $\tau = \hat{E}_{1,w}^{-1} (y - \hat{E}_{1,v} v)$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \le \|\hat{E}_{1,w}^{-1}\|_{2,2} c_w + \|\hat{E}_{1,w}^{-1} \hat{E}_{1,v}\|_{2,2} c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_c \ge 0$, which depends only on $c_w$ and $c_{w1}$, such that $\|\eta_{[0,\infty)}\|_\infty \le c_c$. Hence, $S$ is minimum phase with respect to $\mathcal{D}_{\eta 0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ by Lemma 3 of [1]. This proves Case 6.

This completes the proof of the theorem. ■

## IV. Bounding lemmas on the inversion of minimum phase systems

In this section, we present results on boundedness of the inverse of minimum phase systems. First, we present a result for a SISO LTI system with RD *zero*.

*Proposition 1:* Consider a SISO LTI system:

$$\dot{x} = Ax + Bu + D_w w + D_v v; \qquad x(0) = x_0 \tag{13a}$$
$$y = Cx + Ku + E_w w + E_v v \tag{13b}$$

where $x \in \mathbb{R}^n$ is the state vector, $n \in \mathbb{Z}_+$; $u \in \mathbb{R}$ is the control input; $y \in \mathbb{R}$ is the output; $w \in \mathbb{R}^{q_w}$ is the disturbance input, $q_w \in \mathbb{Z}_+$; $v \in \mathbb{R}^{q_v}$ is the disturbance input, $q_v \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0 \ne \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant.

Assume (13) admits RD 0 from $u$ to $y$ and is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, where $w$ and $v$ are viewed as disturbances. Then, $\forall c_w \ge 0$, $\exists c_c \ge 0$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$; $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$; let $x_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solution to (13); if $\|y_{[0,t_f)}\|_\infty \le c_w$, we have $\|x_{[0,t_f)}\|_\infty \le c_c$; and $\|u_{[0,t_f)}\|_\infty \le c_c$.

*Proof:* We will distinguish 2 exhaustive and mutually exclusive cases: Case 1: $n = 0$; Case 2: $n \in \mathbb{N}$.

Case 1: $n = 0$. By Definition 1 of [1], (13) admits the following EZDCF, $K \ne 0$, $y = Ku + E_w w + E_v v$. Then, we have $u = K^{-1} (y - E_w w - E_v v)$. Clearly, the desired result holds with $c_c := |K^{-1}| c_w + \|K^{-1} E_w\|_{2,2} c_w + \|K^{-1} E_v\|_{2,2} c_w$. This proves Case 1.

Case 2: $n \in \mathbb{N}$. By Lemma 3 of [1], without loss of generality, assume that (13) is given in the following EZDCF, $b_0 \ne 0$,

$$\dot{x} = \bar{A} x + \bar{B}(y - E_w w - E_v v) + D_w w + D_v v \tag{14a}$$
$$y = Cx + b_0 u + E_w w + E_v v \tag{14b}$$

$\forall c_w \ge 0$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$; let $x_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solution to (14); and let $\|y_{[0,t_f)}\|_\infty \le c_w$. Since (13) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, by Lemma 3 of [1] and Lemma 4, then, $\exists c_{c1} \ge 0$, which depends only on $c_w$, we have $\|x_{[0,t_f)}\|_\infty \le c_{c1}$. Note that $u = b_0^{-1} (y - Cx - E_w w - E_v v)$. Then, $\|u_{[0,t_f)}\|_\infty \le |b_0^{-1}| c_w + \|b_0^{-1} C\|_{2,2} c_{c1} + \|b_0^{-1} E_w\|_{2,2} c_w + \|b_0^{-1} E_v\|_{2,2} c_w =: c_{c2}$. This proves Case 2.

This completes the proof of the proposition. ■

Next, we present results for systems with positive RDs. First, we present a technical lemma.

*Lemma 2:* Consider the following chain of integrators:

$$\dot{x}_i = x_{i+1} + D_{iw} w + D_{iv} v; \; i = 1, \ldots, r_1 - 1 \tag{15a}$$
$$\dot{x}_{r_1} = b_1 u + D_{r_1 w} w + D_{r_1 v} v \tag{15b}$$

where $x_i \in \mathbb{R}$, $i = 1, \ldots, r_1$, $r_1 \in \mathbb{N}$; $w \in \mathbb{R}^{q_w}$ is the disturbance input, $q_w \in \mathbb{Z}_+$; $v \in \mathbb{R}^{q_v}$ is the disturbance input, $q_v \in \mathbb{Z}_+$; $u \in \mathbb{R}$ is the control input; $0 \ne b_1 \in \mathbb{R}$; $x = [x_1 \cdots x_{r_1}]'$; $x(0) = x_0 \in \mathbb{R}^{r_1}$; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and $D_{iw}$ and $D_{iv}$, $i = 1, \ldots, r_1$, are constant matrices. Consider another SISO LTI system, $S_\eta$, sharing the same set of inputs as (15):

$$\dot{\eta} = A\eta + Bu + D_w w + D_v v; \qquad \eta(0) = \eta_0 \tag{16a}$$
$$y = C\eta \tag{16b}$$

where $\eta \in \mathbb{R}^n$ is the state, $n \in \mathbb{N}$; $y \in \mathbb{R}$ is the output; $u$, $w$, and $v$ are as (15); $\eta_0 \in \mathcal{D}_0 \ne \emptyset$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$. Assume $S_\eta$ admits RD $r_2$ $(\ge r_1)$ from $u$ to $y$ and satisfies: $\forall c_w \ge 0$, $\exists c_{c1} \ge 0$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \le c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $c_w \ge \|u_{[0,\infty)}\|_\infty$, $\forall w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, we have $\|\eta_{[0,\infty)}\|_\infty \le c_{c1}$.

Then, $\forall c_w \ge 0$, $\exists c_{c2} \ge 0$, $\forall x_0 \in \mathbb{R}^{r_1}$ with $|x_0| \le c_w$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$. Let $\eta_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solutions to (16), and $x_{[0,t_f)}$ be the solution to (15). If $\|x_{1[0,t_f)}\|_\infty \le c_w$, then $\|y_{[0,t_f)}\|_\infty \le c_{c2}$.

*Proof:* $\forall c_w \geq 0$, $\forall x_0 \in \mathbb{R}^{r_1}$ with $|x_0| \leq c_w$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \leq c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $c_w \geq \|v_{[0,\infty)}\|_\infty$. Let $\eta_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solutions to (16), and $x_{[0,t_f)}$ be the solution to (15). Let $\|x_{1[0,t_f)}\|_\infty \leq c_w$.

Let $z := \eta - \sum_{k=1}^{r_1} A^{r_1-k}Bb_1^{-1}x_k : [0,t_f) \to \mathbb{R}^n$. Then, $z$ satisfies the following state space equation:

$$\begin{aligned}\dot{z} &= Az + B\left(-b_1^{-1}\left(D_{r_1w}w + D_{r_1v}v\right)\right) + A^{r_1}Bb_1^{-1}x_1 + D_v v \\ &\quad + D_w w + \sum_{k=1}^{r_1-1} A^{r_1-k}B\left(-b_1^{-1}\left(D_{kw}w + D_{kv}v\right)\right) \\ z(0) &= \eta_0 - \sum_{k=1}^{r_1} A^{r_1-k}Bb_1^{-1}x_{k,0}\end{aligned}$$

where $x_0 = [x_{1,0} \cdots x_{r_1,0}]'$; $x_{i,0} \in \mathbb{R}$, $i = 1,\ldots,r_1$. Then, by the linearity and uniqueness of solution to linear differential equations, $z_{[0,t_f)}$ may be generated by

$$\begin{aligned}&\dot{\xi}_1 = A\xi_1 + B\left(-b_1^{-1}\left(D_{r_1w}w + D_{r_1v}v\right)\right) + D_w w + D_v v; \\ &\qquad \xi_1(0) = \eta_0 \\ &\dot{\delta}_i = A\delta_i;\ \delta_i(0) = -A^iBb_1^{-1}x_{r_1-i,0};\ i = 0,\ldots,r_1-1 \\ &\dot{\zeta}_i = A\zeta_i + A^iB\left(-b_1^{-1}\left(D_{r_1-i\,w}w + D_{r_1-i\,v}v\right)\right); \\ &\qquad \zeta_i(0) = \mathbf{0}_{n\times 1};\ i = 1,\ldots,r_1-1 \\ &\dot{\zeta}_{r_1} = A\zeta_{r_1} + A^{r_1}Bb_1^{-1}x_1;\ \zeta_{r_1}(0) = \mathbf{0}_{n\times 1} \\ &z_{[0,t_f)} = \xi_{1[0,t_f)} + \sum_{i=0}^{r_1-1}\delta_{i[0,t_f)} + \sum_{i=1}^{r_1}\zeta_{i[0,t_f)}\end{aligned}$$

Let $\bar{u}_{[0,t_f)} := -b_1^{-1}\left(D_{r_1w}w_{[0,t_f)} + D_{r_1v}v_{[0,t_f)}\right)$. Note that $\eta_0 \in \mathcal{D}_0$ and $|\eta_0| \leq c_w$; $\bar{u}_{[0,t_f)} \in \mathcal{C}$ with $\|\bar{u}_{[0,t_f)}\|_\infty \leq \|b_1^{-1} \cdot D_{r_1w}\|_{2,2}c_w + \|b_1^{-1}D_{r_1v}\|_{2,2}c_w =: c_{w1}$; $w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \leq c_w$; and $v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$. By the assumption on $S_\eta$ and Lemma 4, $\exists c_{ca} \geq 0$, which depends only on $c_w$ and $c_{w1}$, such that $\|\xi_{1[0,t_f)}\|_\infty \leq c_{ca}$.

By the assumption on $S_\eta$ and Lemma 9 of [1], we have that the following dynamics $\dot{\kappa}_1 = A\kappa_1 + B\rho$, $\kappa_1(0) = \mathbf{0}_{n\times 1}$, is BIBS stable. By repeated application of Lemma 3, we have that the following dynamics

$$\dot{\kappa}_i = A\kappa_i + A^{i-1}B\rho;\ \kappa_i(0) = \mathbf{0}_{n\times 1};\ i = 2,\ldots,r_1+1$$

are BIBS stable. $\forall i = 1,\ldots,r_1-1$. Let $\rho_i := -b_1^{-1} \cdot (D_{r_1-i\,w}w + D_{r_1-i\,v}v) : [0,t_f) \to \mathbb{R}$. Then, $\rho_{i[0,t_f)} \in \mathcal{C}$ and $\|\rho_{i[0,t_f)}\|_\infty \leq \|b_1^{-1}D_{r_1-i\,w}\|_{2,2}c_w + \|b_1^{-1}D_{r_1-i\,v}\|_{2,2} \cdot c_w =: c_{wAi}c_w$. Since $\kappa_{i+1}$ dynamics is BIBS stable, then, $\zeta_i$ dynamics is BIBS stable. By Lemma 6 of [1], $\exists c_{Ai} \geq 0$, which depends only on $A$ and $A^iB$, such that $\|\zeta_{i[0,t_f)}\|_\infty \leq c_{Ai}c_{wAi}c_w$. Since $\kappa_{r_1+1}$ dynamics is BIBS stable, then, $\zeta_{r_1}$ dynamics is BIBS stable. Let $\rho_{r_1} := b_1^{-1}x_1$. Then $\rho_{r_1[0,t_f)} \in \mathcal{C}$ and $\|\rho_{r_1[0,t_f)}\|_\infty \leq |b_1^{-1}|c_w =: c_{wAr_1}c_w$. Then, by Lemma 6 of [1], $\exists c_{Ar_1} \geq 0$, which depends only on $A$ and $A^{r_1}B$, such that $\|\zeta_{r_1[0,t_f)}\|_\infty \leq c_{Ar_1}c_{wAr_1}c_w$.

$\forall i = 0,\ldots,r_1-1$. Since $\kappa_{i+1}$ system is BIBS stable, by Lemma 3, note that $|-b_1^{-1}x_{r_1-i,0}| \leq |b_1^{-1}|c_w =: c_1c_w$, then, $\exists c_{Bi} \geq 0$, which depends only on $A$ and $A^iB$, such that $\|\delta_{i[0,t_f)}\|_\infty \leq c_{Bi}c_1c_w$.

Hence, we have $\|z_{[0,t_f)}\|_\infty \leq c_{ca} + \sum_{i=0}^{r_1-1}c_{Bi}c_1c_w + \sum_{i=1}^{r_1}c_{Ai}c_{wAi}c_w =: c_{c1}$.

Note that $y = Cz + \sum_{k=1}^{r_1}CA^{r_1-k}Bb_1^{-1}x_k = Cz + CA^{r_1-1}Bb_1^{-1}x_1$ since $r_1 \leq r_2$. Hence, $\|y_{[0,t_f)}\|_\infty \leq \|C\|_{2,2}c_{c1} + \|CA^{r_1-1}Bb_1^{-1}\|_{2,2}c_w =: c_{c2}$.

This completes the proof of the lemma. ■

*Proposition 2:* Consider two SISO LTI systems sharing the same inputs:

$$S_1 : \begin{cases}\dot{x} = A_1x + B_1u + D_{1,w}w + D_{1,v}v;\ x(0) = x_0 \\ y_1 = C_1x + E_{1,w}w + E_{1,v}v\end{cases} \tag{17}$$

$$S_2 : \begin{cases}\dot{\eta} = A_2\eta + B_2u + D_{2,w}w + D_{2,v}v;\ \eta(0) = \eta_0 \\ y_2 = C_2\eta\end{cases} \tag{18}$$

where $x \in \mathbb{R}^{n_1}$ is the state of $S_1$, $n_1 \in \mathbb{N}$; $u \in \mathbb{R}$ is the control input; $y_1 \in \mathbb{R}$ is the output of $S_1$; $w \in \mathbb{R}^{q_w}$ is the disturbance input, $q_w \in \mathbb{Z}_+$; $v \in \mathbb{R}^{q_v}$ is the disturbance input, $q_v \in \mathbb{Z}_+$; $\eta \in \mathbb{R}^{n_2}$ is the state of $S_2$, $n_2 \in \mathbb{N}$; $y_2 \in \mathbb{R}$ is the output of $S_2$; $x_0 \in \mathcal{D}_{x0} \neq \emptyset$, $\mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$; $\eta_0 \in \mathcal{D}_{\eta 0} \neq \emptyset$, $\mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant.

Assume that

(i) $S_1$ admits RD $r_1$ from $u$ to $y_1$, $r_1 \in \{1,\ldots,n_1\}$, and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$, where $w$ and $v$ are viewed as disturbance inputs;

(ii) $S_2$ satisfies that $\forall c_w \geq 0$, $\exists c_{c1} \geq 0$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $\|u_{[0,\infty)}\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, we have $\|\eta_{[0,\infty)}\|_\infty \leq c_{c1}$; and

(iii) $S_2$ admits RD $r_2 \in \mathbb{N}$ from $u$ to $y_2$.

By Lemmas 1 and 2 of [1], without loss of generality, assume that $S_1$ is given in the EZDCF: Case 1: $r_1 = n_1$

$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v;\ 1 \leq i < r_1 \tag{19a}$$
$$\dot{x}_{r_1} = a_{1,r_11}x_1 + b_1u + D_{1,r_1w}w + D_{1,r_1v}v \tag{19b}$$
$$y_1 = x_1 + E_{1,w}w + E_{1,v}v \tag{19c}$$

where $x = [x_1 \cdots x_{r_1}]'$; $x_i \in \mathbb{R}$, $i = 1,\ldots,r_1$; $x(0) = x_0 = [x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$; and $b_1 \neq 0$; Case 2: $1 \leq r_1 < n_1$

$$\dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v \tag{20a}$$
$$\dot{x}_i = a_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v;\ 1 \leq i < r_1 \tag{20b}$$
$$\dot{x}_{r_1} = A_{1,r_1z}x_z + a_{1,r_11}x_1 + b_1u + D_{1,r_1w}w + D_{1,r_1v}v \tag{20c}$$
$$y_1 = x_1 + E_{1,w}w + E_{1,v}v \tag{20d}$$

where $x = [x_z'\, x_1 \cdots x_{r_1}]'$; $x_z \in \mathbb{R}^{n_1-r_1}$; $x_i \in \mathbb{R}$, $i = 1,\ldots,r_1$; $x(0) = x_0 = [x_{z0}'\, x_{1,0} \cdots x_{r_1,0}]' \in \mathcal{D}_{x0}$; and $b_1 \neq 0$.

Then, $\forall c_w \geq 0$, $\exists c_c \geq 0$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \leq c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (18), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (17). If $\|x_{1[0,t_f)}\|_\infty \leq c_w$, $\ldots$, $\|x_{k_0[0,t_f)}\|_\infty \leq c_w$, for some fixed $k_0 \in \{1,\ldots,r_1\}$, and $r_2 \geq r_1 - k_0 + 1$, we have $\|y_{2[0,t_f)}\|_\infty \leq c_c$.

*Proof:* $\forall c_w \geq 0$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \leq c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (18), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (17). Let $\|x_{1[0,t_f)}\|_\infty \leq c_w$, $\ldots$, $\|x_{k_0[0,t_f)}\|_\infty \leq c_w$.

We will distinguish two exhaustive and mutually exclusive cases: Case 1: $r_1 = n_1$; Case 2: $1 \leq r_1 < n_1$.

Case 1: $r_1 = n_1$. $S_1$ admits the representation (19). We will apply Lemma 2 to prove this case. The chain of integrators is

$$\begin{aligned}&\dot{x}_i = x_{i+1} + [a_{1,i1}\ D_{1,iw}]\,[x_1\ w']' + D_{1,iv}v;\ k_0 \leq i < r_1 \\ &\dot{x}_{r_1} = b_1u + [a_{1,r_11}\ D_{1,r_1w}]\,[x_1\ w']' + D_{1,r_1v}v\end{aligned}$$

Then, by Lemma 2, $\exists c_c \geq 0$, which depends only on $c_w$, such that $\|y_{2[0,t_f)}\|_\infty \leq c_c$. This proves Case 1.

Case 2: $1 \leq r_1 < n_1$. $S_1$ admits the representation (20).

*Claim 1:* $\exists \bar{c}_c \geq 0$, which depends only on $c_w$, such that $\|x_{z[0,t_f)}\|_\infty \leq \bar{c}_c$.

*Proof:* By the fact that $S_1$ is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w})\times\mathcal{W}_d$ and Proposition 1 of [1] and Lemma 4, $\exists\bar{c}_c\geq 0$, which depends only on $c_w$, such that $\|x_{z[0,t_f)}\|_\infty\leq\bar{c}_c$. This proves the claim. ■

Now, we will apply Lemma 2 to prove this case. The chain of integrators is

$$\dot{x}_i = x_{i+1} + [\mathbf{0}_{1\times(n_1-r_1)}\; a_{1,i1}\; D_{1,iw}]\,[x_z'\; x_1\; w']' + D_{1,iv}v;\quad i=k_0,\ldots,r_1-1$$

$$\dot{x}_{r_1} = b_1 u + [A_{1,r_1 z}\; a_{1,r_1 1}\; D_{1,r_1 w}]\,[x_z'\; x_1\; w']' + D_{1,r_1 v}v$$

Then, by Lemma 2, $\exists c_c\geq 0$, which depends only on $c_w$ and $\bar{c}_c$, such that $\|y_{2[0,t_f)}\|_\infty\leq c_c$. This proves Case 2 and completes the proof of the proposition. ■

In application of Proposition 2, we will refer to $S_1$ as the reference system. Finally, we present a corollary without proof, which has a stronger assumption on $S_2$.

*Corollary 1:* Consider two SISO LTI systems sharing the same inputs (17) and (18) as in Proposition 2.

Assume that

(i) $S_1$ admits RD $r_1$ from $u$ to $y_1$, $r_1\in\{1,\ldots,n_1\}$, and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w})\times\mathcal{W}_d$, where $w$ and $v$ are viewed as disturbance inputs; and

(ii) the matrix $A_2$ is Hurwitz and $S_2$ admits RD $r_2\in\mathbb{N}$ from $u$ to $y_2$.

Let $S_1$ be given in the EZDCF (19) or (20) depending on whether $r_1=n_1$ or $r_1<n_1$, respectively.

Then, $\forall c_w\geq 0$, $\exists c_c\geq 0$, $\forall x_0\in\mathcal{D}_{x0}$ with $|x_0|\leq c_w$, $\forall\eta_0\in\mathcal{D}_{\eta0}$ with $|\eta_0|\leq c_w$, $\forall t_f\in(0,\infty]\subset\mathbb{R}_e$, $\forall u_{[0,t_f)}\in\mathcal{C}$, $\forall w_{[0,t_f)}\in\mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty\leq c_w$, $\forall v_{[0,\infty)}\in\mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty\leq c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (18), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (17). If $\|x_{1[0,t_f)}\|_\infty\leq c_w$, $\ldots$, $\|x_{k_0[0,t_f)}\|_\infty\leq c_w$, for some fixed $k_0\in\{1,\ldots,r_1\}$, and $r_2\geq r_1-k_0+1$, we have $\|y_{2[0,t_f)}\|_\infty\leq c_c$.

## V. Conclusions

In this paper, we have further investigated properties of minimum phase systems using the generalized definition introduced in [1]. We proved that the composite system consisting of a minimum phase system in feedback connection with another linear system satisfying a certain boundedness condition is itself a minimum phase system. We have established two results (Propositions 1 and 2) on the stable invertibility of minimum phase systems. When a minimum phase linear system has RD *zero* from the control input to the output, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a minimum phase linear system has RD $r_1\in\mathbb{N}$, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to $k_0$th order are bounded, then, the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the RD of the "stable" system satisfies $r_2\geq r_1-k_0$.

Future research along this direction lies in the generalization of the minimum phase concept for MIMO LTI systems under structural assumptions. Another fruitful research topic lies in model reference robust adaptive control using the new definition of minimum phase. Both of these topics are currently under investigation.

## Appendix

*Lemma 3:* Consider the LTI system: $\dot{z}=Az+Bv$, $z(0)=\mathbf{0}_{n\times 1}$, where $z$ is the $n$-dimensional state, $n\in\mathbb{Z}_+$; and $v$ is the $p$-dimensional input, $p\in\mathbb{Z}_+$. Assume that the system is BIBS stable. Then, the following statements hold.

1) For system $\dot{\eta}=A\eta$, $\eta(0)=B\xi$, where $\xi\in\mathbb{R}^p$, there exist $k\geq 0$ and $\lambda>0$ such that $\forall\xi\in\mathbb{R}^p$ with $|\xi|\leq c_w\geq 0$, we have $|\eta(t)|\leq c_w k\mathrm{e}^{-\lambda t}$, $\forall t\in[0,\infty)$.
2) The system $\dot{x}=Ax+ABu$, $x(0)=\mathbf{0}_{n\times 1}$, is BIBS stable.

*Proof:* By Lemma 6 of [1], $\exists k\geq 0$ and $\exists\lambda>0$ such that $\|\mathrm{e}^{At}B\|_{2,2}\leq k\mathrm{e}^{-\lambda t}$, $\forall t\geq 0$. 1) By Theorem 4-4 of [6], $|\eta(t)|=|\mathrm{e}^{At}B\xi|\leq k\mathrm{e}^{-\lambda t}c_w$, $\forall t\geq 0$. 2) we note that $(\sum_{i=0}^\infty\frac{1}{i!}(At)^i)AB=\mathrm{e}^{At}AB=A(\sum_{i=0}^\infty\frac{1}{i!}(At)^i)B=A\mathrm{e}^{At}B$, $\forall t\in\mathbb{R}$. Then, $\|\mathrm{e}^{At}AB\|_{2,2}=\|A\mathrm{e}^{At}B\|_{2,2}\leq\|A\|_{2,2}\|\mathrm{e}^{At}B\|_{2,2}\leq\|A\|_{2,2}\cdot k\mathrm{e}^{-\lambda t}$, $\forall t\geq 0$. By Lemma 6 of [1], the result holds. ■

*Lemma 4:* Consider the LTI system:

$$\dot{x}=Ax+Bu+Dw;\qquad x(0)=x_0\tag{21}$$

where $x$ is the $n$-dimensional state, $n\in\mathbb{Z}_+$; $u$ is the $p$-dimensional input, $p\in\mathbb{Z}_+$; $w$ is the $q$-dimensional disturbance input, $q\in\mathbb{Z}_+$; $x_0\in\mathcal{D}_0\neq\emptyset$, $\mathcal{D}_0\subseteq\mathbb{R}^n$; $w_{[0,\infty)}\in\mathcal{W}_d$ of class $\mathcal{B}_q$. Assume that (21) satisfies: $\forall c_w\geq 0$, $\exists c_c\geq 0$, $\forall x_0\in\mathcal{D}_0$ with $|x_0|\leq c_w$, $\forall u_{[0,\infty)}\in\mathcal{C}$ with $\|u_{[0,\infty)}\|_\infty\leq c_w$, $\forall w_{[0,\infty)}\in\mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty\leq c_w$, we have $\|x_{[0,\infty)}\|_\infty\leq c_c$.

Then, $\forall c_w\geq 0$, if $c_c$ is the constant defined previously, $\forall t_f\in(0,\infty]\subset\mathbb{R}_e$, $\forall x_0\in\mathcal{D}_0$ with $|x_0|\leq c_w$, $\forall u_{[0,t_f)}\in\mathcal{C}$ with $\|u_{[0,t_f)}\|_\infty\leq c_w$, $\forall w_{[0,\infty)}\in\mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty\leq c_w$, we have $\|x_{[0,t_f)}\|_\infty\leq c_c$.

*Proof:* $\forall c_w\geq 0$, $\forall t_f\in(0,\infty]\subset\mathbb{R}_e$, $\forall x_0\in\mathcal{D}_0$ with $|x_0|\leq c_w$, $\forall u_{[0,t_f)}\in\mathcal{C}$ with $\|u_{[0,t_f)}\|_\infty\leq c_w$, $\forall w_{[0,\infty)}\in\mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty\leq c_w$. Let $x_{[0,t_f)}$ be the solution to (21). $\forall t\in[0,t_f)$, we will show that $|x(t)|\leq c_c$. Let $\bar{u}_{[0,\infty)}\in\mathcal{C}$ be given by $\bar{u}(s)=\begin{cases}u(s) & s\in[0,t]\\ u(t) & s\in(t,\infty)\end{cases}$. Clearly, we have $\|\bar{u}_{[0,\infty)}\|_\infty\leq c_w$. Let $\bar{x}_{[0,\infty)}$ be the solution to (21) due to $\bar{u}_{[0,\infty)}$, $w_{[0,\infty)}$, and $x_0$. Then, $\|\bar{x}_{[0,\infty)}\|_\infty\leq c_c$. By the causality of (21), we have $x_{[0,t]}=\bar{x}_{[0,t]}$. Hence, $|x(t)|\leq c_c$. This completes the proof. ■

*Lemma 5:* Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{K}$ and $\mathcal{D}_0\subseteq\mathcal{X}$ be a closed subspace, $P$ be a bounded linear operator of $\mathcal{X}$ to $\mathcal{Y}$, and $P(\mathcal{D}_0)\subseteq\mathcal{Y}$ is closed. Then, $\exists c\geq 0$, $\forall c_w\geq 0$, $\forall b\in P(\mathcal{D}_0)\subseteq\mathcal{Y}$ with $\|b\|_{\mathcal{Y}}\leq c_w$, $\exists x\in\mathcal{D}_0\subseteq\mathcal{X}$ such that $b=Px$ and $\|x\|_{\mathcal{X}}\leq cc_w$.

*Proof:* This result is well known; see [7]. ■

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