Robust Adaptive Controller Design and Disturbance Attenuation for Minimum Phase Square MIMO LTI Systems with Uniform Vector Relative Degree of Zero *

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Abstract

In this paper, we present a systematic procedure for robust adaptive control design for minimum phase uncertain square multiple-input multiple-output (MIMO) linear time-invariant (LTI) systems that admit uniform vector relative degree of zero, under the assumption that the upper bounds for the observability indices of all measurement channels are known. For square MIMO LTI systems, it is always possible to pad dummy state variables to arrive at a system model that admits uniform vector relative degree of zero, which is further minimum phase and has the observability indices the same as the known upper bounds. We assume that the system with such padded dummy state variables admits the strict observer canonical form. We also assume that the unknown parameter vector lies in a convex compact set such that the high frequency gain matrix remains invertible for any parameter vector value in the set. These are the assumptions that allow for a successful design of a robust adaptive controller. A numerical example is included to fully illustrate the controller design and the effectiveness of the controller. As compared with the recent paper Pan and Başar (2023), the problem with uniform vector relative degree of zero allows us to relieve the block diagonally identical backbone structure for the measurement channels and choose a general quadratic cost structure that weighs the tracking errors arbitrarily.

Keywords: nonlinear H^∞ control based robust adaptive control; multiple-input multiple-output linear uncertain systems; minimum phase; extended zero dynamics canonical form; strict observer canonical form.

1 INTRODUCTION

Robust adaptive control design for uncertain linear systems has attracted a lot of research attention since the 1980s, (Morse, 1980; Goodwin and Sin, 1984; Ioannou and Sun, 1996; Pan and Başar, 2000; Zhao *et al.*, 2008, 2009; Zeng and Pan, 2009; Zeng *et al.*, 2010; Tezcan and Başar, 1999; Zeng, 2012; Pan and Başar, 2023). A satisfactory solution to single-input single-output (SISO) linear systems was obtained in Pan and Başar (2000) using the game theoretic approach (Başar and Bernhard, 1995). See Pan and Başar (2000) for a complete literature review of the game-theoretic approach to robust adaptive control and nonlinear adaptive control at the onset of the millenium. There, one can further find extensive simulation results comparing the performance of the robust adaptive control strategy with those of nonadaptive H^∞-control. The solution to the SISO problem has further been refined in Zhao *et al.* (2008), and generalized in follow-up works in multiple

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directions, namely zero relative degree case (Zeng *et al.*, 2010), the three degrees of freedom problem (Zeng and Pan, 2009), and a class of multiple-input multiple-output (MIMO) linear systems that consists of parallel interconnected SISO linear systems with limited output feedback (Zeng, 2012). The solution methodology has also been successfully generalized to SISO uncertain nonlinear systems in Tezcan and Başar (1999). The design of Zhao *et al.* (2008) has also been studied in detail on the convergence properties of the closed-loop system in Zhao *et al.* (2009). It is observed that the minimum phase assumption is the key to the success of robust adpative control design for SISO uncertain linear systems. The generalization of the robust adaptive control design to MIMO systems depends critically on the generalization of the minimum phase assumption to MIMO linear systems under additive disturbances. In Pan and Başar (2018), a generalized minimum phase assumption has been introduced for SISO systems, which is necessary for a successful design of a model reference controller for SISO linear systems. It has been proved that for SISO systems the generalized minimum phase condition is equivalent to all zeros of the transfer function from the control input to the output to have negative real parts if the system is controllable from the control input and is observable from the output (Proposition 3 of Pan and Başar (2018)). More relationships between the generalized minimum phase assumption and its classical counterpart have been obtained in Pan and Başar (2018). This generalized minimum phase assumption has been extended to MIMO linear systems with additive disturbances in Başar and Pan (2020). It has been observed that the generalized minimum phase assumption is necessary for a successful design of model reference controller for MIMO linear systems. It has also been noted in Başar and Pan (2020) that the generalized minimum phase assumption is invariant under finite steps of dynamic extensions (Isidori, 1995). We observe that the key canonical forms of uncertain linear systems are the extended zero dynamics canonical form and the strict observer canonical form. In Başar and Pan (2019), we established methodologies to extend (dynamically) a given minimum phase uncertain MIMO linear system model to achieve an extended system that admits the extended zero dynamics canonical form and the strict observer canonical form without rendering the system non-minimum phase. In Pan and Başar (2023), robust adaptive control for MIMO uncertain linear systems was solved for the case where the system admits positive uniform vector relative degree.

In this paper, we present a systematic procedure for robust adaptive control design for uncertain minimum phase square MIMO linear systems that admit uniform vector relative degree of **zero**. We assume that the MIMO linear system has m output terminals, and a set of upper bounds $n_1, \ldots, n_m \in \mathbb{Z}_+$ for the observability indices (Chen, 1984) $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ of the system is known. For this class of systems, it is always possible to pad dummy state variables (Başar and Pan, 2019) to arrive at a system model that admits the observability indices $n_1, \ldots, n_m$, remains minimum phase according to Başar and Pan (2020), and admits uniform vector relative degree of zero. We assume that this extended system admits strict observer canonical form. Thus, this extended system admits the extended zero dynamics canonical form and the strict observer canonical form. The assumption that the system admits the strict observer canonical form is not restrictive at all, since when $n = n_1 = \cdots = n_m$, i.e., the upper bounds for the observability indices are uniform, then the extended system admits the strict observer canonical form (Başar and Pan, 2020). This assumption is introduced to allow flexibility in the robust adaptive controller design. The observable part of the extended system is then the design model for the system. The design procedure resembles that for the SISO case Zeng *et al.* (2010), but we allow for the additional flexibility of some measured disturbance inputs. The general objective of the control design is to attenuate the effect of the disturbance input on the system tracking error. Using a game theoretic approach, we formulate the robust adaptive control problem as a nonlinear H^∞ optimal control problem with a single cost function. By making use of the *cost-to-come* function methodology for affine nonlinear H^∞ optimal control, we obtain a closed-form expression for an upper bound of the value function of the identifier for the unknown system, which provides a finite-dimensional estimator structure for the uncertain linear system. Assuming the existence of a known convex compact set for the true values of the system parameters such that the high frequency gain matrix will remain invertible for any parameter values in that set, we introduce a smooth parameter projection scheme for the identifier. With this projection algorithm, the adaptive control system becomes robust with or without persistently exciting input signals. Using the explicit form of the value function for the identifier, the nonlinear H^∞ adaptive control problem is then transformed into a full-information nonlinear robust control problem, and the control law is obtained by setting the estimated output to the desired reference trajectory as in Zeng *et al.* (2010). Due to robustness concerns not related to the objective of the current paper, we propose an alternative proof than that of Zeng *et al.* (2010), where the state of a first order filter for the difference between measurement output and the reference trajectory was shown to be bounded. The adaptive controller achieves the desired disturbance attenuation level for all admissible continuous exogenous input waveforms and all continuous reference trajectories on the infinite horizon with any admissible initial conditions. Furthermore, it is proved rigorously that the control law also leads to boundedness of all closed-loop signals under bounded admissible initial condition, bounded admissible disturbance inputs, and bounded reference trajectory without the need for any persistency of excitation condition or any stochastic noise assumptions. The tracking error is proved to converge to zero when, in addition, the unmeasured disturbance input is of finite energy, while the measured disturbance and the reference trajectory are bounded and uniformly continuous.

The balance of the paper is organized as follows. In the next section, we list the notations used in the paper. In Section 3, we provide a precise formulation of the problem to be solved, delineate the basic assumptions regarding the underlying system, as well as the input signals, and include a brief discussion of the solution methodology adopted. In Section 4, we present the identification design and control design for the nonlinear H^∞ adaptive control problem, with detailed discussions on the projection algorithm used in the construction. This identifier then becomes the system to be controlled in a worst-case sense, under an equivalent expression for the cost function that depends only on the identifier states. The control design is simply to set the estimated output to the desired reference trajectory (certainty equivalence). In Section 5, we present the precise statement and complete proofs of the properties of the closed-loop adaptive system. The theoretical results are also illustrated on a numerical example in Section 6, which clearly illustrates the effectiveness of the design methodology. The paper ends with the concluding remarks of Section 7.

2 NOTATIONS

Let $\mathbb{R}$ denote the real line; $\mathbb{R}_+ := (0,\infty) \subset \mathbb{R}$; $\mathbb{R}_- := (-\infty,0) \subset \mathbb{R}$; $\overline{\mathbb{R}_+} := [0,\infty) \subset \mathbb{R}$; $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; $\mathbb{C}$ be the set of complex numbers, where i is the complex unit. For any number $a \in \mathbb{C}$, $\overline{a}$ denotes its complex conjugate and $\mathrm{Re}\,(a)$ denotes its real part. Unless specified, all signals, constants, and matrices are real. For a continuous function f, we say that it belongs to $\mathcal{C}$; if it is k-times continuously differentiable, we say that it belongs to $\mathcal{C}_k$; its lth order derivative is denoted by $\mathrm{D}^l f$ or $f^{(l)}$; its partial derivative with respect to some variable x is denoted by $\frac{\partial f}{\partial x}$. For a $\mathcal{B}_{\mathrm{B}}(\mathbb{R})$-measurable function $f : I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is an interval, we say f is $\bar{\mathrm{L}}_p$, where $p \in [1,\infty) \subset \mathbb{R}$, if $(\int_I |f(\tau)|^p \,\mathrm{d}\tau)^{1/p} < \infty$; the class of all functions g that $g = f$ a.e. in I is denoted by $[f] \in \mathrm{L}_p$; when f is continuous, we say that f is L_∞ if $\max\{\sup_{t\in I} |f(t)|, 0\} < \infty$. We let $\mathbb{R}^n$ denote the Euclidean space, with norm $|z| := \sqrt{z'z}$, unless specified otherwise. For any matrix A, A' denotes its transpose. We will denote $n \times n$-dimensional real symmetric, positive-semidefinite, and positive-definite matrices by $\mathcal{S}_n$, $\mathcal{S}_{\mathrm{psd}\,n}$, and $\mathcal{S}_{+\,n}$, respectively, and say $Q_1 \leq Q_2$, if $Q_2 - Q_1 \in \mathcal{S}_{\mathrm{psd}\,n}$, and $Q_1 < Q_2$, if $Q_2 - Q_1 \in \mathcal{S}_{+\,n}$, $\forall Q_1, Q_2 \in \mathcal{S}_n$; $\mathrm{Tr}\,(Q_1)$ denotes the trace of Q_1. For any tensor $A \in \mathrm{B}(\mathbb{R}^{m_1}, \mathrm{B}(\mathbb{R}^{m_2}, \mathcal{Y}))$, $A^{T_{2,1}}$ denotes the transpose of tensor A between the last two indices, and thus $A(x)(y) = A^{T_{2,1}}(y)(x) \in \mathcal{Y}$, $\forall x \in \mathbb{R}^{m_1}$, $\forall y \in \mathbb{R}^{m_2}$. For any $z \in \mathbb{R}^n$ and any $Q \in \mathcal{S}_{\mathrm{psd}\,n}$, $|z|_Q^2$ denotes $z'Qz$. I_n denotes the $n \times n$-dimensional identity matrix. For any matrix A, $A^0 = I$. For any matrix M, $\|M\|_p$ denotes its p-induced norm, $1 \leq p \leq \infty$; for $p = 2$, we simply write it as $\|M\|$. $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f), \mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$; when this quantity is bounded, we say that $u_{[0,t_f)} \in \mathcal{C}_{\mathrm{b}}([0,t_f), \mathbb{R}^p)$. For any real (complex) Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, we will write $\mathrm{B}(\mathcal{X}_1, \mathcal{X}_2)$ to denote the set of all bounded linear operators from $\mathcal{X}_1$ to $\mathcal{X}_2$, and write $\mathcal{B}_{\mathcal{X}_1}(x, r)$ to denote the open ball centered at x with radius $r \in \mathbb{R}_+$ in $\mathcal{X}_1$. On $\mathbb{R}$, we will denote by $\bar{\mathrm{r}}_{a,b}$ the compact interval $[a,b] \subset \mathbb{R}$, where $a \leq b$ and $a, b \in \mathbb{R}$. For any sets A, B with $A \subseteq B$, $\chi_{A,B}$ denote the indicator function of the set A on B, i.e., $\chi_{A,B}(x) := \begin{cases} 1 & x \in A \\ 0 & x \in B \setminus A \end{cases}$, $\forall x \in B$.

Any signal with a hat accent (like $\hat{x}, \hat{\theta}, \hat{\xi}$) is the worst-case estimate of the corresponding signal without the accent, which is something we design like the control signal. Any signal with a check accent (like $\check{x}$, $\check{\theta}$, $\check{w}$) is some signal we can measure, or the estimate of the corresponding signal without the accent that is produced by the cost-to-come function analysis. Any signal with a grave accent (like $\grave{x}$) is some signal that is unknown in general and is associated with the given unknown MIMO linear system. Any signal without any accent is a signal in the design model. Any signal with tilde accent (like $\tilde{x}$, $\tilde{\theta}$, $\tilde{\xi}$) is the estimation error of the signal without the accent, which equals to the signal without the accent minus the signal with the check accent.

3 PROBLEM FORMULATION

We consider the adaptive control problem for continuous-time finite-dimensional minimum phase square MIMO LTI systems with uniform vector relative degree of zero.

We are given system $\grave{S}$ with state space representation:

$$\dot{\grave{x}} = \grave{A}\grave{x} + \grave{B}u + \grave{D}\grave{w}; \quad \grave{x}(0) = \grave{x}_0 \in \grave{\mathcal{D}}_0 \tag{1a}$$

$$y = \grave{C}\grave{x} + \grave{F}u + \grave{E}\grave{w} \tag{1b}$$

where $\grave{x} \in \mathbb{R}^{\grave{n}}$ is the state vector, $\grave{n} \in \mathbb{Z}_+$; $\grave{x}_0 \in \grave{\mathcal{D}}_0$ is the initial condition, where $\grave{\mathcal{D}}_0 \subseteq \mathbb{R}^{\grave{n}}$ is a subspace (we generally have $\grave{\mathcal{D}}_0 = \mathbb{R}^{\grave{n}}$); $u \in \mathbb{R}^m$ is the control input, $m \in \mathbb{N}$; $\grave{w} \in \mathbb{R}^{\grave{q}}$ is the disturbance input, $\grave{q} \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the measurement output; and the matrices $\grave{A}$, $\grave{B}$, $\grave{D}$, $\grave{C}$, $\grave{F}$, and $\grave{E}$ are constant matrices of appropriate dimensions and generally unknown. It is assumed that the disturbance input is partitioned into $\grave{w} := (\check{w}, \grave{w}_b)$, where $\check{w} \in \mathbb{R}^{\check{q}}$ are measured disturbance inputs (in addition to the measurements y), $\check{q} \in \mathbb{Z}_+$; and the waveform of $\grave{w}_{[0,\infty)}$ is assumed to belong to $\grave{\mathcal{W}}_d$ (generally $= \mathcal{C}(\overline{\mathbb{R}_+}, \mathbb{R}^{\grave{q}})$), which is of class $\mathcal{B}_{\grave{q}}$ (Pan and Başar, 2018). Thus, we are only considering $\grave{w}_{[0,\infty)}$ that is continuous. We now state a number of assumptions, which are quite natural in this context.

Assumption 1 *The system (1) (with control input u, output y, and disturbance input $\grave{w}$) is minimum phase with respect to $\grave{\mathcal{D}}_0$ and $\grave{\mathcal{W}}_d$ as defined in Başar and Pan (2020).*

Assumption 2 *The system (1) admits uniform vector relative degree of* **zero***, that is the matrix $\grave{F}$ is invertible.*

Assumption 3 *A set of upper bounds $n_1, \ldots, n_m \in \mathbb{Z}_+$ for the observability indices $\nu_1, \ldots, \nu_m \in \mathbb{Z}_+$ of system $\grave{S}$ is known. ($\nu_1, \ldots, \nu_m$ are the observability indices of the pair $(\grave{A}, \grave{C})$; and $0 \leq \nu_i \leq n_i$, $i = 1, \ldots, m$.)*

It is always possible to pad dummy state variables to the system (1) to arrive at a system that admits the observability indices $n_1, \ldots, n_m$:

$$\dot{\acute{x}} = \acute{A}\acute{x} + \acute{B}u + \acute{D}\grave{w}; \quad \acute{x}(0) = \acute{x}_0 \in \acute{\mathcal{D}}_0 \tag{2a}$$

$$y = \acute{C}\acute{x} + \grave{F}u + \grave{E}\grave{w} \tag{2b}$$

where $\acute{x} \in \mathbb{R}^{\grave{n}+\sum_{i=1}^m n_i - \sum_{i=1}^m \nu_i}$ is the state vector; $\acute{x}_0 \in \acute{\mathcal{D}}_0$ is the initial state, where $\acute{\mathcal{D}}_0 := \grave{\mathcal{D}}_0 \times \left\{ \mathbf{0}_{\sum_{i=1}^m n_i - \sum_{i=1}^m \nu_i} \right\} \subseteq \mathbb{R}^{\grave{n}+\sum_{i=1}^m n_i - \sum_{i=1}^m \nu_i}$ is a subspace; the matrices $\acute{A}$, $\acute{B}$, $\acute{D}$, and $\acute{C}$ are constant matrices of appropriate dimensions and generally unknown; and the system (2) admits uniform vector relative degree of zero and is minimum phase with respect to $\acute{\mathcal{D}}_0$ and $\grave{\mathcal{W}}_d$. We will refer to the system (2) as $\acute{S}$.

By Lemma 4 of Başar and Pan (2020), the system (2) admits the extended zero-dynamics canonical form:

$$\dot{\acute{x}} = \bar{\acute{A}}\acute{x} + \bar{\acute{B}}y + \bar{\acute{D}}\grave{w}; \quad \acute{x}(0) = \acute{x}_0 \in \acute{\mathcal{D}}_0 \tag{3a}$$

$$y = \acute{C}\acute{x} + \grave{F}u + \grave{E}\grave{w} \tag{3b}$$

Assumption 4 *The system (2) admits strict observer canonical form.*

This assumption is made to allow flexibility in the robust adaptive controller design. When $n_1 = n_2 = \cdots = n_m$, Assumption 4 is always true.

By Corollary 1 of Başar and Pan (2019), there exists an invertible matrix $\grave{T}$ such that in $(x_{\bar{o}}, x) := (x_{\bar{o}}, x_1, \ldots, x_n) = \grave{T}^{-1}\acute{x}$ coordinates, we have that x_i is m_i-dimensional, $i = 1, \ldots, n$, $n := \max\{n_1, \ldots, n_m\}$, $m_i := \sum_{l=1}^{m} \chi_{\{\cdot \geq i\},\mathbb{Z}}(n_l)$; $m \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, and $m_n > 0$ if $n > 0$; $\sum_{i=1}^{m} n_i = \sum_{i=1}^{n} m_i =: n_O$; and the system (2) admits the strict observer canonical form representation

$$\dot{x}_{\bar{o}} = A_{\bar{o}} x_{\bar{o}} + A_{\bar{o},1} x_1 + B_{\bar{o}} u + D_{\bar{o}} \grave{w} \tag{4a}$$

$$\dot{x}_i = A_{i,1} x_1 + A_{i,i+1} x_{i+1} + B_i u + \acute{D}_i \grave{w}; \quad i = 1, \ldots, n-1 \tag{4b}$$

$$\dot{x}_n = A_{n,1} x_1 + B_n u + \acute{D}_n \grave{w} \tag{4c}$$

$$y = C_1 x_1 + \grave{F} u + \grave{E} \grave{w} \tag{4d}$$

where all matrices are constant and of appropriate dimensions, $C_1 \in \mathbb{R}^{m \times m_1}$ is known, whose column vectors are a subset of the column vectors of I_m; $A_{i,i+1} \in \mathbb{R}^{m_i \times m_{i+1}}$ is known, whose column vectors are a subset of the column vectors of I_{m_i}, $i = 1, \ldots, n-1$.

By further taking only the observable part of the system (4) and introducing a disturbance transformation $w_b = \grave{M} \grave{w}_b$, where w_b is q_b-dimensional, $q_b \in \mathbb{N}$, and $\grave{M}$ is an unknown constant matrix, we may obtain the following design model for the dynamics of $x = (x_1, \ldots, x_n)$ in (4):

$$\dot{x} = Ax + Bu + \check{D}\check{w} + Dw_b + (A_{211,1} y + A_{211,3} \check{w} + A_{212} u)\theta \tag{5a}$$

$$y = Cx + B_0 u + (C_{1,3} \check{w} + C_{1,2} u)\theta + E w_b \tag{5b}$$

where the matrices A, B, $\check{D}$, D, C, B_0, and E are known matrices of appropriate dimensions; $\theta \in \Theta \subseteq \mathbb{R}^{\sigma}$ is the unknown parameter vector of the system; $A_{211,1}$, $A_{211,3}$, and A_{212}, are known second-order $\mathbb{R}^{n_O}$-valued tensors of appropriate dimensions; $C_{1,3}$ and $C_{1,2}$ are known second-order $\mathbb{R}^m$-valued tensors of appropriate dimensions; and further we have the pair (A, C) being observable.

Assumption 5 *There exists a known smooth nonnegative proper convex function $P(\bar{\theta})$, such that the true value of θ lies in the convex compact set $\Theta := \left\{ \bar{\theta} \in \mathbb{R}^{\sigma} \;\middle|\; P(\bar{\theta}) \leq 1 \right\}$. Furthermore, $\forall \bar{\theta} \in \Theta$, the matrix $B_0 + C_{1,2}^{T_{2,1}} \bar{\theta} =: B_{p0}(\bar{\theta})$ is invertible.*

$B_{p0}(\bar{\theta})$ being invertible follows from the fact that system (4) admits uniform vector relative degree of zero from u to y.

Assumption 6 *Associated with system (4), we are given an m-dimensional reference trajectory $y_d(t)$ that y is to track. The reference trajectory y_d is assumed to be continuous on the interval $\overline{\mathbb{R}_+}$ and is available for feedback.*

The objective of the control design is to achieve asymptotic tracking of the reference trajectory while rejecting the uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \acute{\mathcal{D}}_0 \times \Theta \times \grave{\mathcal{W}}_d \times \mathcal{C}(\overline{\mathbb{R}_+}, \mathbb{R}^m) =: \grave{\mathcal{W}}$, which comprises the initial state of the system $\acute{S}$, the true values of the unknown parameters, the disturbance input waveforms, and the reference trajectories. We will obtain a class of causal robust adaptive controllers,

$$u(t) = \mu(t, y_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}) \tag{6}$$

$\forall t \in \overline{\mathbb{R}_+}$ to achieve the desired tracking and disturbance attenuation objectives (to be delineated shortly). Let us denote the class of these causal admissible controllers by $\mathcal{M}$.

The control design objective is now made precise in the following.

Definition 1 *A controller μ is said to achieve disturbance attenuation level* 0 *with respect to $\check{w}$ and disturbance attenuation level $\gamma \in \mathbb{R}_+$ with respect to w_b, if there exist nonnegative functions $l(t, \theta, x_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]})$ such that for all $t_f \geq 0$ the following dissipation inequality holds :*

$$\sup_{(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}) \in \grave{\mathcal{W}}} J_{\gamma t_f} \leq 0 \tag{7}$$

where

$$\begin{aligned} J_{\gamma t_f} &:= \int_0^{t_f} \left(\left| Cx(\tau) + B_0 u(\tau) + (C_{1,2}u(\tau))\theta + (C_{1,3}\check{w}(\tau))\theta - y_d(\tau) \right|_Q^2 + l(\tau, \theta, x_{[0,\tau]}, y_{[0,\tau]}, \check{w}_{[0,\tau]}, y_{d[0,\tau]}) \right. \\ &\quad \left. -\gamma^2 \left| w_b(\tau) \right|^2 \right) \mathrm{d}\tau - \gamma^2 \left| (\theta - \check{\theta}_0, x(0) - \check{x}_0) \right|_{\bar{Q}_0}^2 \end{aligned} \tag{8}$$

Here, $Q \in \mathcal{S}_{+\,m}$ is the positive-definite weighting matrix for the quadratic norm of the tracking error; $\check{\theta}_0$ is the initial guess of the unknown parameters; $\check{x}_0$ is the initial guess of the unknown initial state $x(0)$; and $\bar{Q}_0 \in \mathcal{S}_{+\,(\sigma+n_O)}$ is the positive-definite weighting matrix for the quadratic norm of the initial estimation error, quantifying our level of confidence in the a priori estimates of θ and $x(0)$; and $\bar{Q}_0^{-1}$ admits the structure $\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi_0' \end{bmatrix}$, where $Q_0 \in \mathcal{S}_{+\,\sigma}$ and $\Pi_0 \in \mathcal{S}_{+\,n_O}$, respectively.[1]

Note that, in the above definition, the negative weighting on the disturbance input $\grave{w}$ is through the negative weightings on the transformed disturbance inputs w_b. The motivation behind the above definition is to guarantee that, for each time instant $t_f \geq 0$, the weighted squared L_2 norm of the output tracking error $Cx + B_{p0}(\theta)u + (C_{1,3}\check{w})\theta - y_d$ on $[0, t_f]$ is bounded by γ^2 times the squared L_2 norm of the transformed disturbance input $w_{b[0,t_f]}$ plus some constant that depends only on the initial condition of the system. When the disturbance inputs $\grave{w}_b$ have finite L_2 norms on $[0, \infty)$, then the L_2 norm of the tracking error $Cx + B_{p0}(\theta)u + (C_{1,3}\check{w})\theta - y_d$ is also finite, which further implies that $\lim_{t\to\infty}(Cx(t) + B_{p0}(\theta)u(t) + (C_{1,3}\check{w}(t))\theta - y_d(t)) = \mathbf{0}_m$, under additional stability conditions of the closed-loop system. On the other hand, for nonvanishing disturbance inputs $\grave{w}_b$, whose truncated squared L_2 norms increase linearly with t_f, the rate of increase for an upper bound of the truncated squared L_2 norm of the tracking error $Cx + B_{p0}(\theta)u + (C_{1,3}\check{w})\theta - y_d$ is also linear, and is bounded by γ^2 times the rate for the disturbance w_b. Clearly, when such an objective is achieved, the closed-loop system will be robust with respect to the disturbance $\grave{w}$, but the exact attenuation level with respect to $\grave{w}_b$ will in general depend on the unknown transformation matrix $\grave{M}$. Under Assumption 5, $\grave{M}$ can be selected to have a known bound for its norm, which then guarantees a known bound for the attenuation level from $\grave{w}_b$ to the tracking error.

The problem formulated above can be brought into the framework of H^∞ optimal control for affine-quadratic nonlinear systems with imperfect state measurements. Toward that end, we expand the system dynamics (5) by adjoining the simple dynamics of θ: $\dot{\theta} = \mathbf{0}_\sigma$. Let ξ denote the expanded state $\xi = (\theta, x)$, which satisfies the following dynamics:

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n_O} \\ A_{211,1}y + A_{211,3}\check{w} + A_{212}u & A \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0}_{\sigma\times m} \\ B \end{bmatrix} u + \begin{bmatrix} \mathbf{0}_{\sigma\times\check{q}} \\ \check{D} \end{bmatrix} \check{w} + \begin{bmatrix} \mathbf{0}_{\sigma\times q_b} \\ D \end{bmatrix} w_b \\ &=: \bar{A}\xi + \bar{B}u + \bar{\check{D}}\check{w} + \bar{D}w_b \end{aligned} \tag{9a}$$

$$\bar{y} := y - B_0 u = \begin{bmatrix} C_{1,3}\check{w} + C_{1,2}u & C \end{bmatrix} \xi + Ew_b =: \bar{C}\xi + Ew_b \tag{9b}$$

The worst-case optimization of the cost function (8) can be carried out in two steps: first a maximization over $\acute{x}_0$, θ, and w_b, given all measurements available to the controller, and then maximization over $\check{w}$, y, and y_d. The idea is that the controller can observe the underlying system only through the measurements, and hence once the measurement waveform is fixed, the control input is an open-loop time function with respect

[1] At this point, Π_0 is quite arbitrary. Later, to simplify the structure of the adaptive controller to be derived, we will choose it to be the solution of an algebraic Riccati equation.

to the underlying dynamics. This is precisely the idea that underpins the *cost-to-come* function methodology (Didinsky *et al.*, 1993), leading to the following identity for each fixed $t_f \in \overline{\mathbb{R}_+}$:

$$
\begin{aligned}
\sup_{(\acute{x}_0,\theta,\grave{w}_{[0,\infty)},y_{d\,[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} &= \sup_{y_{[0,\infty)}\in\mathcal{C},y_{d\,[0,\infty)}\in\mathcal{C},\breve{w}_{[0,\infty)}\in\mathcal{C}} \; \sup_{(\acute{x}_0,\theta,\grave{w}_{[0,\infty)},y_{d\,[0,\infty)})\in\grave{\mathcal{W}}|y_{[0,\infty)},y_{d\,[0,\infty)},\breve{w}_{[0,\infty)}} J_{\gamma t_f} \\
&\le \sup_{y_{[0,\infty)}\in\mathcal{C},y_{d\,[0,\infty)}\in\mathcal{C},\breve{w}_{[0,\infty)}\in\mathcal{C}} \; \sup_{(\acute{x}_0,\theta,w_{b[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},y_{d\,[0,\infty)},\breve{w}_{[0,\infty)}} J_{\gamma t_f} \qquad (10)
\end{aligned}
$$

where the right-hand sup operator

$$
\sup_{(\acute{x}_0,\theta,w_{b[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},y_{d\,[0,\infty)},\breve{w}_{[0,\infty)}}
$$

is over all initial conditions $\acute{x}_0 \in \mathbb{R}^{\grave{n}+n_O-\sum_{i=1}^{m}\nu_i}$, parameter value $\theta \in \Theta$, and disturbance waveforms $w_{b[0,\infty)} \in \mathcal{C}$ that generate the output waveform $y_{[0,\infty)}$ with $\breve{w}_{[0,\infty)}$ and $y_{d\,[0,\infty)}$ fixed and known. In the above, we have elected to be conservative that we supremize with respect to $w_{b[0,\infty)}$, instead of $\grave{w}_{b[0,\infty)}$. This is done solely for the consideration of the existence of a finite-dimensional solution for the problem.

The right-hand supremization, which will be carried out first, corresponds to the evaluation of the worst-case performance for any set of known measurement waveforms, which renders the control input waveform independent of the actual disturbance input waveform, since the control input is generated as a function of the output waveform, the measured disturbance waveform, and the reference trajectory. This is the identification design step. Because of the special structure of the problem under consideration, an upper bound of the value function for this step of the optimization, which is related to the *cost-to-come* function for this problem, can be obtained explicitly by utilizing the results of Appendix B of Pan and Başar (2000).

The left-hand supremization, which will be carried out second, corresponds to the computation of the worst-case measurement waveform against a given control law. Since the control law is restricted to be a causal function of the measurements and the reference trajectory, it plays a critical role in the determination of achievability of the objective (7). This is the control design step. Both of these design steps are discussed in Section 4.

The design function $l(t,\theta,x_{[0,t]},y_{[0,t]},\breve{w}_{[0,t]},y_{d[0,t]})$ is selected based on two considerations: the existence of a solution to the problem; and the ease of analysis of stability and robustness of the resulting closed-loop system. It is built up in the identifier design step. In the identifier design step, the weighting functions are selected to provide necessary stability properties, and to yield a desirable structure for the identifier that is amenable to the later control design. In particular, they are selected to maintain a predetermined positive definite lower bound for the worst-case covariance matrix of the parameter estimates, which is necessary for the robustness of the closed-loop system.

In the controller design step, we will simply set the control law according to the certainty equivalence principle since the system admits uniform vector relative degree of zero. This completes the formulation of the robust adaptive control problem and the general solution method to be adopted. We now turn to the controller design in the next section.

4 CONTROLLER DESIGN

In this section, we first present the identification design for the adaptive control problem formulated. In this step, the measurement waveforms $y_{[0,\infty)}$, $\breve{w}_{[0,\infty)}$, $y_{d[0,\infty)}$, and therefore the control waveform $u_{[0,\infty)}$, are assumed to be fixed and known. We consider the cost function:

$$
\begin{aligned}
J_{i\gamma t_f} = & \int_0^{t_f} \left(\left| Cx(\tau) + (C_{1,3}\breve{w}(\tau) + C_{1,2}u(\tau))\theta + B_0u(\tau) - y_d(\tau) \right|_Q^2 + \left| \xi(\tau) - \hat{\xi}(\tau) \right|_{\bar{Q}}^2 - \gamma^2 \left| w_b(\tau) \right|^2 \right) \mathrm{d}\tau \\
& -\gamma^2 \left| (\theta - \breve{\theta}_0, x(0) - \breve{x}_0) \right|_{\bar{Q}_0}^2 \qquad (11)
\end{aligned}
$$

where the first positive definite term is required by the objective of the adaptive control design (7); the second nonnegative definite term is introduced for robustness considerations of the complete adaptive system, where

$\hat{\xi}$ is the worst-case estimate for the expanded state ξ, which is like a control signal yet to be determined; the two negative-definite weighting terms involving the disturbance w_b and the initial conditions are again required by the objective of the adaptive control design (7). The nonnegative-definite weighting function $\bar{Q}$ will exhibit a special structure to be delineated shortly.

To avoid singularity in estimation, we assume that

Assumption 7 *The matrix E is of full row rank, or equivalently, $EE' =: N \in \mathcal{S}_{+m}$.*

By expressing the above cost function completely in the ξ state variables, we can apply Lemma 10 of Pan and Başar (2000) to obtain an equivalent, more transparent, expression for $J_{i\gamma t_f}$.

Let $\bar{\Sigma}$ and $\check{\xi}$ be defined by

$$\begin{aligned}\dot{\bar{\Sigma}} &= (\bar{A} - \bar{L}N^{-1}\bar{C})\bar{\Sigma} + \bar{\Sigma}(\bar{A} - \bar{L}N^{-1}\bar{C})' + \frac{1}{\gamma^2}\bar{D}\bar{D}' - \frac{1}{\gamma^2}\bar{L}N^{-1}\bar{L}' - \bar{\Sigma}(\gamma^2\bar{C}'N^{-1}\bar{C} - \bar{C}'Q\bar{C} - \bar{Q})\bar{\Sigma}; \\ & \qquad \bar{\Sigma}(0) = \frac{1}{\gamma^2}\bar{Q}_0^{-1} = \frac{1}{\gamma^2}\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi_0' \end{bmatrix} \end{aligned} \tag{12a}$$

$$\begin{aligned}\dot{\check{\xi}} &= (\bar{A} + \bar{\Sigma}(\bar{C}'Q\bar{C} + \bar{Q}))\check{\xi} - \bar{\Sigma}(\bar{C}'Q(y_d - B_0 u) + \bar{Q}\hat{\xi}) + \bar{B}u + \bar{\check{D}}\check{w} + (\gamma^2\bar{\Sigma}\bar{C}' + \bar{L})N^{-1}(y - B_0 u - \bar{C}\check{\xi}); \\ & \qquad \check{\xi}(0) = \begin{bmatrix} \check{\theta}_0 \\ \check{x}_0 \end{bmatrix} \end{aligned} \tag{12b}$$

where $\bar{L} := \bar{D}E'$ is given by $\bar{L} = \begin{bmatrix} \mathbf{0}_{\sigma\times m} \\ L \end{bmatrix}$ with $L := DE'$.

Then, the cost function (11) can equivalently be written as (from Lemma 10 of Pan and Başar (2000))

$$\begin{aligned} J_{i\gamma t_f} &= -\left|\xi(t_f) - \check{\xi}(t_f)\right|^2_{(\bar{\Sigma}(t_f))^{-1}} + \int_0^{t_f} \Big(\left|\bar{C}\check{\xi}(\tau) - (y_d(\tau) - B_0 u(\tau))\right|^2_Q - \gamma^2\left|y(\tau) - B_0 u(\tau) - \bar{C}\check{\xi}(\tau)\right|^2_{N^{-1}} \\ &\quad + \left|\check{\xi}(\tau) - \hat{\xi}(\tau)\right|^2_{\bar{Q}} - \gamma^2\left|w_b(\tau) - w_*(\tau, \xi_{[0,\tau]}, y_{[0,\tau]}, u_{[0,\tau]}, \check{w}_{[0,\tau]}, y_{d[0,\tau]}, \hat{\xi}_{[0,\tau]})\right|^2 \Big)\,\mathrm{d}\tau \end{aligned} \tag{13}$$

where

$$\begin{aligned} w_*(\tau, \xi_{[0,\tau]}, y_{[0,\tau]}, u_{[0,\tau]}, \check{w}_{[0,\tau]}, y_{d[0,\tau]}, \hat{\xi}_{[0,\tau]}) &= E'N^{-1}(y(\tau) - B_0 u(\tau) - \bar{C}\xi(\tau)) \\ &\quad + \frac{1}{\gamma^2}(I_{q_b} - E'N^{-1}E)\bar{D}'(\bar{\Sigma}(\tau))^{-1}(\xi(\tau) - \check{\xi}(\tau)) \end{aligned} \tag{14}$$

Furthermore, an upper bound of the value function for this estimation step is $W : \mathcal{S}_{+(\sigma+n_O)} \times \mathbb{R}^{\sigma+n_O} \times \mathbb{R}^{\sigma+n_O} \to \overline{\mathbb{R}_+}$:

$$W(\bar{\Sigma}, \xi, \check{\xi}) := |\xi - \check{\xi}|^2_{\bar{\Sigma}^{-1}} \tag{15}$$

whose time derivative is given by

$$\begin{aligned} \dot{W} &= -\left|\bar{C}\xi - (y_d - B_0 u)\right|^2_Q + \left|y_d - B_0 u - \bar{C}\check{\xi}\right|^2_Q - \left|\xi - \hat{\xi}\right|^2_{\bar{Q}} + \left|\check{\xi} - \hat{\xi}\right|^2_{\bar{Q}} + \gamma^2|w_b|^2 \\ &\quad -\gamma^2\left|y - B_0 u - \bar{C}\check{\xi}\right|^2_{N^{-1}} - \gamma^2\left|w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}, \hat{\xi}_{[0,t]})\right|^2 \end{aligned} \tag{16}$$

$$\forall \bar{\Sigma} \in \mathcal{S}_{+(\sigma+n_O)},\ \forall \xi \in \mathbb{R}^{\sigma+n_O},\ \forall \check{\xi} \in \mathbb{R}^{\sigma+n_O},\ \forall \hat{\xi} \in \mathbb{R}^{\sigma+n_O},\ \forall w_b \in \mathbb{R}^{q_b},\ \forall \check{w} \in \mathbb{R}^{\check{q}},\ \forall u \in \mathbb{R}^m,\ \forall y_d \in \mathbb{R}^m$$

Partition $\check{\xi} := (\check{\theta}, \check{x})$ and $\hat{\xi} := (\hat{\theta}, \hat{x})$ compatible with the partitioning of $\xi = (\theta, x)$. Our intention is to keep $\check{\theta}$ within a vicinity of Θ such that the matrix $B_0 + C_{1,2}^{T_{2,1}}\check{\theta} = B_{p0}(\check{\theta})$ is always invertible with a bounded inverse, by using a smooth projection algorithm. (A straightforward nonsmooth projection function should also work.)

Define $\rho_M := \inf_{\det(B_0+C_{1,2}^{T_{2,1}}\bar{\theta})=0} P(\bar{\theta})$. By Assumption 5, we have $\rho_M \in (1,\infty] \subset \mathbb{R}_e$. Choose $\rho_o \in (1,\rho_M) \subset \mathbb{R}$. We will design the smooth projection algorithm such that the estimate $\check{\theta}$ lies in the open set $\Theta_o := \{ \bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) < \rho_o \}$. It is immediate that this implies that $B_0 + C_{1,2}^{T_{2,1}}\bar{\theta}$ is invertible, $\forall \bar{\theta} \in \Theta_o$, and there exists $c_0 \in \overline{\mathbb{R}_+}$, such that $\| (B_{p0}(\bar{\theta}))^{-1} \| \le c_0$, $\forall \bar{\theta} \in \Theta_o$.

By Proposition 4 on Page 178 of Luenberger (1984), we have

$$\frac{\partial P}{\partial \theta}(\check{\theta})(\theta - \check{\theta}) \le P(\theta) - P(\check{\theta}) \le 1 - P(\check{\theta}); \qquad \forall \check{\theta} \in \mathbb{R}^\sigma \tag{17}$$

We now add to the right-hand side of the dynamics (12b) for $\check{\xi}$ the following term when $1 < P(\check{\theta}) < \rho_o$:

$$-\frac{\exp(-\frac{1}{P(\check{\theta})-1})}{(\rho_o - P(\check{\theta}))^3} \bar{\Sigma} \left[\frac{\partial P}{\partial \theta}(\check{\theta}) \ \mathbf{0}_{1\times n_O} \right]'$$

Hence, we have

$$\begin{aligned} \dot{\check{\xi}} \;=\; & -(1-\chi_{\Theta,\mathbb{R}^\sigma}(\check{\theta}))\frac{\exp(-\frac{1}{P(\check{\theta})-1})}{(\rho_o - P(\check{\theta}))^3} \bar{\Sigma} \left[\frac{\partial P}{\partial \theta}(\check{\theta}) \ \mathbf{0}_{1\times n_O} \right]' + (\bar{A} + \bar{\Sigma}(\bar{C}'Q\bar{C} + \bar{Q}))\check{\xi} - \bar{\Sigma}(\bar{C}'Q(y_d - B_0 u) + \bar{Q}\hat{\xi}) \\ & + \bar{B}u + \bar{\check{D}}\check{w} + (\gamma^2\bar{\Sigma}\bar{C}' + \bar{L})N^{-1}(y - B_0 u - \bar{C}\check{\xi}); \quad \check{\xi}(0) = \begin{bmatrix} \check{\theta}_0 \\ \check{x}_0 \end{bmatrix} \end{aligned} \tag{18}$$

It is easy to verify that the following nonlinear functions P_r and p_r

$$P_r(\check{\theta}) := (1-\chi_{\Theta,\mathbb{R}^\sigma}(\check{\theta}))\frac{\exp(-\frac{1}{P(\check{\theta})-1})}{(\rho_o - P(\check{\theta}))^3}(\frac{\partial P}{\partial \theta}(\check{\theta}))' =: p_r(\check{\theta})(\frac{\partial P}{\partial \theta}(\check{\theta}))' = \frac{\kappa_1(P(\check{\theta})-1)}{(\rho_o - P(\check{\theta}))^3}(\frac{\partial P}{\partial \theta}(\check{\theta}))' \tag{19}$$

are $\mathcal{C}_\infty$ on the set Θ_o, where κ_1 is as defined in Pan and Başar (2023). In view of this, the derivative of the value function W given by (15) is equal to

$$\begin{aligned} \dot{W} \;=\; & -\left|\bar{C}\xi - (y_d - B_0 u)\right|_Q^2 + \left|y_d - B_0 u - \bar{C}\check{\xi}\right|_Q^2 - \left|\xi - \hat{\xi}\right|_{\bar{Q}}^2 + \left|\check{\xi} - \hat{\xi}\right|_{\bar{Q}}^2 + \gamma^2|w_b|^2 \\ & -\gamma^2\left|y - B_0 u - \bar{C}\check{\xi}\right|_{N^{-1}}^2 - \gamma^2\left|w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \check{w}_{[0,t]}, y_{d[0,t]}, \hat{\xi}_{[0,t]})\right|^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) \\ & \forall \bar{\Sigma} \in \mathcal{S}_{+(\sigma+n_O)},\ \forall \xi \in \mathbb{R}^{\sigma+n_O},\ \forall \check{\xi} \in \Theta_o \times \mathbb{R}^{n_O},\ \forall \hat{\xi} \in \mathbb{R}^{\sigma+n_O},\ \forall w_b \in \mathbb{R}^{q_b},\ \forall \check{w} \in \mathbb{R}^{\check{q}},\ \forall u \in \mathbb{R}^m,\ \forall y_d \in \mathbb{R}^m \end{aligned}$$

The last term above appears because of the modification in the dynamics of $\check{\xi}$. We now have the following inequality: $\forall \theta \in \Theta$, $\forall \check{\theta} \in \Theta_o$,

$$2(\theta - \check{\theta})'P_r(\check{\theta}) \;=\; 2\frac{\kappa_1(P(\check{\theta})-1)}{(\rho_o - P(\check{\theta}))^3}\frac{\partial P}{\partial \theta}(\check{\theta})(\theta - \check{\theta}) \le 2\frac{\kappa_1(P(\check{\theta})-1)}{(\rho_o - P(\check{\theta}))^3}(1 - P(\check{\theta})) = 2p_r(\check{\theta})(1 - P(\check{\theta})) \le 0$$

which shows that the last term in the expression for $\dot{W}$ is nonpositive, is zero on the set Θ, and approaches $-\infty$ as $\check{\theta}$ approaches the boundary of the set Θ_o (i. e., $P(\check{\theta})$ approaches ρ_o).

To further deduce the existence of the covariance matrix $\bar{\Sigma}$ and the structure of the identifier, we pursue the following line of detailed analysis. First, partition the worst-case covariance matrix $\bar{\Sigma}$ (compatible with the partition of ξ) as

$$\bar{\Sigma} = \begin{bmatrix} \Sigma & \bar{\Sigma}_{12} \\ \bar{\Sigma}_{21} & \bar{\Sigma}_{22} \end{bmatrix} \tag{20}$$

and introduce the quantities:

$$\Phi := \bar{\Sigma}_{21}\Sigma^{-1} \tag{21a}$$

$$\Pi := \gamma^2(\bar{\Sigma}_{22} - \bar{\Sigma}_{21}\Sigma^{-1}\bar{\Sigma}_{12}) \tag{21b}$$

Next, choose the following structure for the weighting matrix $\bar{Q}$:

$$\bar{Q} = \bar{\Sigma}^{-1}\begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n_O} \\ \mathbf{0}_{n_O\times\sigma} & \Delta \end{bmatrix}\bar{\Sigma}^{-1} + \begin{bmatrix} \epsilon(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)'(\gamma^2 N^{-1} - Q)(C_{1,3}\breve{w} + C_{1,2}u + C\Phi) & \mathbf{0}_{\sigma\times n_O} \\ \mathbf{0}_{n_O\times\sigma} & \mathbf{0}_{n_O\times n_O} \end{bmatrix}$$

where $\Delta := \gamma^{-2}\beta_\Delta\Pi + \gamma^{-2}\Delta_1$ with $\beta_\Delta \in \overline{\mathbb{R}_+}$ being a constant and $\Delta_1 \in \mathcal{S}_{+\,n_O}$ being an $n_O\times n_O$-dimensional positive-definite matrix; and ϵ is a scalar function defined by

$$\epsilon(t) := \frac{\operatorname{Tr}\left((\Sigma(t))^{-1}\right)}{K_c} \tag{22}$$

with $K_c \in [\gamma^2 \operatorname{Tr}(Q_0), \infty) \subset \mathbb{R}$ being a constant corresponding to the preselected maximum level for the quantity $\operatorname{Tr}\left((\Sigma(t))^{-1}\right)$. The Riccati differential equation (RDE) for $\bar{\Sigma}$ is expressed as

$$\begin{aligned}\dot{\bar{\Sigma}} &= (\bar{A} - \bar{L}N^{-1}\bar{C})\bar{\Sigma} + \bar{\Sigma}(\bar{A} - \bar{L}N^{-1}\bar{C})' + \frac{1}{\gamma^2}\bar{D}\bar{D}' - \frac{1}{\gamma^2}\bar{L}N^{-1}\bar{L}' - \bar{\Sigma}(\gamma^2\bar{C}'N^{-1}\bar{C} - \bar{C}'Q\bar{C} \\ &\quad - \begin{bmatrix} \epsilon(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)'(\gamma^2 N^{-1} - Q)(C_{1,3}\breve{w} + C_{1,2}u + C\Phi) & \mathbf{0}_{\sigma\times n_O} \\ \mathbf{0}_{n_O\times\sigma} & \mathbf{0}_{n_O\times n_O} \end{bmatrix})\bar{\Sigma} + \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n_O} \\ \mathbf{0}_{n_O\times\sigma} & \Delta \end{bmatrix}; \\ \bar{\Sigma}(0) &= \gamma^{-2}\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi_0' \end{bmatrix}\end{aligned}$$

By Lemma 6 of Pan and Başar (2000), we obtain the following differential equations for Σ, Φ, and Π:

$$\dot{\Sigma} = -(1-\epsilon(t))\Sigma(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)'(\gamma^2 N^{-1} - Q)(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)\Sigma; \quad \Sigma(0) = \gamma^{-2}Q_0^{-1} \tag{23a}$$

$$\begin{aligned}\dot{\Phi} &= (A - LN^{-1}C - \Pi C'(N^{-1} - \frac{1}{\gamma^2}Q)C)\Phi + A_{211,1}y + (A_{211,3} - (LN^{-1} + \Pi C'(N^{-1} - \gamma^{-2}Q))C_{1,3})\breve{w} \\ &\quad + (A_{212} - (LN^{-1} + \Pi C'(N^{-1} - \gamma^{-2}Q))C_{1,2})u; \quad \Phi(0) = \Phi_0\end{aligned} \tag{23b}$$

$$\dot{\Pi} = (A - LN^{-1}C)\Pi + \Pi(A - LN^{-1}C)' - \Pi C'(N^{-1} - \frac{1}{\gamma^2}Q)C\Pi + DD' - LN^{-1}L' + \gamma^2\Delta; \; \Pi(0) = \Pi_0 \tag{23c}$$

We note that in order to guarantee the boundedness of the matrix Σ, we can pick γ such that $\gamma^2 N^{-1} \geq Q$. For the RDE (23c), we note that the pairs (A, C) and $(A, DD' - LN^{-1}L' + \Delta_1)$ are both observable. Then, the RDE (23c) admits a unique positive-definite solution on $[0,\infty)$, and the solution converges, as $t \to \infty$, to the unique positive-definite solution of the corresponding algebraic Riccati equation (24) below, if (24) admits a stabilizing positive-definite solution.

$$\begin{aligned}&(A - LN^{-1}C + \frac{\beta_\Delta}{2}I_{n_O})\Pi + \Pi(A - LN^{-1}C + \frac{\beta_\Delta}{2}I_{n_O})' - \Pi C'(N^{-1} - \frac{1}{\gamma^2}Q)C\Pi + DD' - LN^{-1}L' \\ &\qquad + \Delta_1 = \mathbf{0}_{n_O\times n_O}\end{aligned} \tag{24}$$

Clearly, if $N^{-1} - \gamma^{-2}Q \in \mathcal{S}_{+\,m}$, then (24) admits a unique positive-definite stabilizing solution. We next invoke Assumption 8 — an assumption to clarify the possible choices of γ, which is a very natural condition to impose.

Assumption 8 *The desired disturbance attenuation level γ satisfies $N^{-1} - \gamma^{-2}Q \in \mathcal{S}_{\mathrm{psd}\,m}$ and is such that the algebraic Riccati equation (24) admits a stabilizing solution, that is, the matrix $A - LN^{-1}C + \frac{\beta_\Delta}{2}I_{n_O} - \Pi C'(N^{-1} - \frac{1}{\gamma^2}Q)C$ is Hurwitz.*

Under Assumption 8, the RDE (23c) admits a positive-definite solution on the infinite horizon $[0,\infty)$. To further simplify the controller structure and enable a proof of closed-loop robustness, we assume that $\Pi_0 = \Pi$, where Π is the positive-definite solution to (24). This implies that $\Pi(t) \equiv \Pi = \Pi_0$, where $\Pi(t)$ is the solution to (23c). Then, the matrix $A_f := A - LN^{-1}C - \Pi C'(N^{-1} - \frac{1}{\gamma^2}Q)C$ is Hurwitz.

From its definition, the function $\epsilon(t)$ can be shown to be less than or equal to 1 for any $t \geq 0$. Therefore, the covariance matrix Σ is nonincreasing. This result is summarized in the following lemma.

Lemma 1 *Consider the matrix differential equation (23a) for the covariance matrix Σ. Let Assumption 8 hold. Then, the matrix Σ is uniformly upper and lower bounded as follows:*

$$K_c^{-1} I_{\sigma\times\sigma} \leq \Sigma(t) \leq \Sigma(0) = \gamma^{-2} Q_0^{-1}$$

Proof Let $[0, t_f]$ denote the maximum-length interval on which $\mathrm{Tr}((\Sigma(t))^{-1}) \leq K_c$. Then, on this interval we have: $\dot{\Sigma} \leq \mathbf{0}_{\sigma\times\sigma}$. If t_f is finite, then, we have $\mathrm{Tr}\left((\Sigma(t_f))^{-1}\right) = K_c$, and $\dot{\Sigma} = \mathbf{0}_{\sigma\times\sigma}$ on the interval $[t_f, \infty)$. This shows that t_f cannot be finite. Hence, the matrix Σ is nonincreasing on $[0, \infty)$, and this verifies the upper bound.

Since $t_f = \infty$, we have $\mathrm{Tr}\left((\Sigma(t))^{-1}\right) \leq K_c$ on the interval $[0, \infty)$. Next, we observe the following inequality:

$$\mathrm{Tr}\left((\Sigma(t))^{-1}\right) \geq \lambda_{\max}((\Sigma(t))^{-1}) = \frac{1}{\lambda_{\min}(\Sigma(t))}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote, respectively, the minimum and maximum eigenvalues of a symmetric matrix. Therefore, we have

$$\lambda_{\min}(\Sigma(t)) \geq K_c^{-1}$$

which yields the desired lower bound. □

In actual implementation, it is preferred not to invert the matrix Σ on line. Computation of such an inverse for the purpose of evaluating ϵ can in fact be avoided (see Pan and Başar (2000)). Let $s_\Sigma(t) := \mathrm{Tr}\left((\Sigma(t))^{-1}\right)$; thus, we have $\epsilon(t) = K_c^{-1} s_\Sigma(t)$ and

$$\dot{s}_\Sigma = (1-\epsilon)\,\mathrm{Tr}\left((C_{1,3}\breve{w} + C_{1,2}u + C\Phi)'(\gamma^2 N^{-1} - Q)(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)\right); \quad s_\Sigma(0) = \gamma^2\,\mathrm{Tr}\,(Q_0) \tag{25}$$

For ease of reference, we now summarize collectively the equations describing the identifier derived heretofore.

$$\begin{aligned}
&(A - LN^{-1}C + \frac{\beta_\Delta}{2} I_{n_O})\Pi + \Pi(A - LN^{-1}C + \frac{\beta_\Delta}{2} I_{n_O})' - \Pi C'(N^{-1} - \frac{1}{\gamma^2}Q)C\Pi + DD' - LN^{-1}L' \\
&\qquad + \Delta_1 = \mathbf{0}_{n_O\times n_O} && \text{(26a)}\\
\dot{\Sigma} &= -(1-\epsilon(t))\Sigma(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)'(\gamma^2 N^{-1} - Q)(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)\Sigma; \quad \Sigma(0) = \gamma^{-2}Q_0^{-1} && \text{(26b)}\\
\dot{s}_\Sigma &= (1-\epsilon)\,\mathrm{Tr}\left((C_{1,3}\breve{w} + C_{1,2}u + C\Phi)'(\gamma^2 N^{-1} - Q)(C_{1,3}\breve{w} + C_{1,2}u + C\Phi)\right); \quad s_\Sigma(0) = \gamma^2\,\mathrm{Tr}\,(Q_0); && \text{(26c)}\\
\epsilon &= K_c^{-1} s_\Sigma && \text{(26d)}\\
A_f &= A - \check{L}C; \qquad \check{L} := LN^{-1} + \Pi C'(N^{-1} - \gamma^{-2}Q) && \text{(26e)}\\
\dot{\Phi} &= A_f\Phi + A_{211,1}y + (A_{211,3} - \check{L}C_{1,3})\breve{w} + (A_{212} - \check{L}C_{1,2})u; \quad \Phi(0) = \Phi_0 && \text{(26f)}\\
\dot{\check{\theta}} &= -\Sigma P_r(\check{\theta}) - \Sigma(C\Phi + C_{1,3}\breve{w} + C_{1,2}u)'Q(y_d - B_0u - C\check{x} - (C_{1,3}\breve{w} + C_{1,2}u)\check{\theta}) - \left[\,\Sigma\ \ \Sigma\Phi'\,\right]\bar{Q}\xi_c \\
&\quad + \gamma^2\Sigma(C\Phi + C_{1,3}\breve{w} + C_{1,2}u)'N^{-1}(y - B_0u - C\check{x} - (C_{1,3}\breve{w} + C_{1,2}u)\check{\theta}); \quad \check{\theta}(0) = \check{\theta}_0 && \text{(26g)}\\
\dot{\check{x}} &= -\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\Pi C'\gamma^{-2} + \Phi\Sigma(C\Phi + C_{1,3}\breve{w} + C_{1,2}u)')Q(y_d - B_0u - C\check{x} - (C_{1,3}\breve{w} + C_{1,2}u)\check{\theta}) \\
&\quad + (A_{211,1}y + A_{211,3}\breve{w} + A_{212}u)\check{\theta} + Bu + \check{D}\breve{w} - \left[\,\Phi\Sigma\ \ \gamma^{-2}\Pi + \Phi\Sigma\Phi'\,\right]\bar{Q}\xi_c \\
&\quad + (\Pi C' + \gamma^2\Phi\Sigma(C\Phi + C_{1,3}\breve{w} + C_{1,2}u)' + L)N^{-1}(y - B_0u - C\check{x} - (C_{1,3}\breve{w} + C_{1,2}u)\check{\theta}); \ \check{x}(0) = \check{x}_0 && \text{(26h)}
\end{aligned}$$

where $\xi_c := \hat{\xi} - \check{\xi}$. Associated with this identifier, we have the upper bound of the value function $W : \mathcal{S}_{+\,\sigma} \times \mathbb{R}^{n_O\times\sigma} \times \mathbb{R}^\sigma \times \mathbb{R}^{n_O} \times \mathbb{R}^\sigma \times \mathbb{R}^{n_O} \to \overline{\mathbb{R}_+}$:

$$W(\Sigma, \Phi, \theta, x, \check{\theta}, \check{x}) = |\xi - \check{\xi}|^2_{\Sigma^{-1}} = |\theta - \check{\theta}|^2_{\Sigma^{-1}} + \gamma^2|x - \check{x} - \Phi(\theta - \check{\theta})|^2_{\Pi^{-1}} \tag{27}$$

whose time derivative is given by

$$\dot W = -\left| Cx + (C_{1,3}\breve w + C_{1,2}u)\theta + B_0 u - y_d \right|_Q^2 + \left| C\breve x + (C_{1,3}\breve w + C_{1,2}u)\breve\theta + B_0 u - y_d \right|_Q^2 - \left| \xi - \hat\xi \right|_{\bar Q}^2 \tag{28}$$
$$+ \left| \xi_c \right|_{\bar Q}^2 + \gamma^2 \left| w_b \right|^2 + 2(\theta - \breve\theta)' P_r(\breve\theta) - \gamma^2 \left| y - B_0 u - C\breve x - (C_{1,3}\breve w + C_{1,2}u)\breve\theta \right|_{N^{-1}}^2$$
$$-\gamma^2 \left| w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \breve w_{[0,t]}, y_{d[0,t]}, \hat\xi_{[0,t]}) \right|^2$$
$$\forall \Sigma \in \mathcal{S}_{+\sigma},\ \forall \Phi \in \mathbb{R}^{n_O \times \sigma},\ \forall \theta \in \mathbb{R}^\sigma,\ \forall x \in \mathbb{R}^{n_O},\ \forall \breve\theta \in \Theta_o,\ \forall \breve x \in \mathbb{R}^{n_O},\ \forall \hat\xi \in \mathbb{R}^{\sigma + n_O},\ \forall w_b \in \mathbb{R}^{q_b},$$
$$\forall \breve w \in \mathbb{R}^{\breve q},\ \forall u \in \mathbb{R}^m,\ \forall y_d \in \mathbb{R}^m$$

Also, the cost function (11) can equivalently be written as:

$$J_{i\gamma t_f} = -\left| \xi(t_f) - \breve\xi(t_f) \right|_{(\bar\Sigma(t_f))^{-1}}^2 + \int_0^{t_f} \Big(\left| C\breve x(\tau) + (C_{1,3}\breve w(\tau) + C_{1,2}u(\tau))\breve\theta(\tau) + B_0 u(\tau) - y_d(\tau) \right|_Q^2$$
$$+ \left| \xi_c(\tau) \right|_{\bar Q(\tau, y_{[0,\tau]}, \breve w_{[0,\tau]}, y_{d[0,\tau]}, u_{[0,\tau]}, \hat\xi_{[0,\tau]})}^2 - \gamma^2 \left| w_b(\tau) - w_*(\tau, \xi_{[0,\tau]}, y_{[0,\tau]}, u_{[0,\tau]}, \breve w_{[0,\tau]}, y_{d[0,\tau]}, \hat\xi_{[0,\tau]}) \right|^2$$
$$+2(\theta - \breve\theta(\tau))' P_r(\breve\theta(\tau)) - \gamma^2 \left| y(\tau) - B_0 u(\tau) - C\breve x(\tau) - (C_{1,3}\breve w(\tau) + C_{1,2}u(\tau))\breve\theta(\tau) \right|_{N^{-1}}^2 \Big)\, \mathrm{d}\tau \tag{29}$$

Note that the matrix Φ may be suitably generated by prefilters of signals of y, $\breve w$, and u, to replace dynamics (26f) to further simplify the identifier structure.

This completes the identification design step. We now turn to the control design for the uncertain system, with the identifier above in place.

Based on (29), the optimal choice for ξ_c is $\xi_c = \mathbf{0}_{\sigma + n_O}$, i. e.,

$$\hat\xi = \breve\xi \tag{30}$$

and the optimal choice for u is simply $\mu : \Theta_o \times \mathbb{R}^{n_O} \times \mathbb{R}^{\breve q} \times \mathbb{R}^m \to \mathbb{R}^m$

$$u = \mu(\breve\theta, \breve x, \breve w, y_d) = (B_{p0}(\breve\theta))^{-1}(y_d - C\breve x - (C_{1,3}\breve w)\breve\theta) \tag{31}$$

This completes the adaptive controller design step. With the optimal choices of the (30) and (31), we have the controller expressed as (26a)–(26f) and the following dynamics for $\breve\theta$ and $\breve x$:

$$\dot{\breve\theta} = -\Sigma P_r(\breve\theta) + \gamma^2 \Sigma (C\Phi + C_{1,3}\breve w + C_{1,2}u)' N^{-1}(y - y_d); \quad \breve\theta(0) = \breve\theta_0 \tag{32a}$$
$$\dot{\breve x} = -\Phi\Sigma P_r(\breve\theta) + A\breve x + (A_{211,1}y + A_{211,3}\breve w + A_{212}u)\breve\theta + Bu + \breve D\breve w$$
$$+(\Pi C' + \gamma^2 \Phi\Sigma(C\Phi + C_{1,3}\breve w + C_{1,2}u)' + L)N^{-1}(y - y_d); \quad \breve x(0) = \breve x_0 \tag{32b}$$

where u is given by (31). Furthermore,

$$\dot W = -\left| Cx + (C_{1,3}\breve w + C_{1,2}u)\theta + B_0 u - y_d \right|_Q^2 - \left| \xi - \breve\xi \right|_{\bar Q}^2 + \gamma^2 \left| w_b \right|^2 + 2(\theta - \breve\theta)' P_r(\breve\theta) - \gamma^2 \left| y - y_d \right|_{N^{-1}}^2$$
$$-\gamma^2 \left| w_b - w_*(t, \xi_{[0,t]}, y_{[0,t]}, u_{[0,t]}, \breve w_{[0,t]}, y_{d[0,t]}, \hat\xi_{[0,t]}) \right|^2 \tag{33}$$
$$= -\left| Cx + (C_{1,3}\breve w + C_{1,2}u)\theta + B_0 u - y_d \right|_Q^2 - \left| \xi - \breve\xi \right|_{\bar Q}^2 + \gamma^2 \left| w_b \right|^2 + 2(\theta - \breve\theta)' P_r(\breve\theta) - \gamma^2 \left| w_b - w_{b\mathrm{opt}} \right|^2$$
$$\forall \Sigma \in \mathcal{S}_{+\sigma},\ \forall \Phi \in \mathbb{R}^{n_O \times \sigma},\ \forall \theta \in \mathbb{R}^\sigma,\ \forall x \in \mathbb{R}^{n_O},\ \forall \breve\theta \in \Theta_o,\ \forall \breve x \in \mathbb{R}^{n_O},\ \forall w_b \in \mathbb{R}^{q_b},\ \forall \breve w \in \mathbb{R}^{\breve q},\ \forall y_d \in \mathbb{R}^m$$

where the worst-case disturbance with respect to the value function W is given by

$$w_{b\mathrm{opt}}(t, \xi_{[0,t]}, u_{[0,t]}, \breve w_{[0,t]}, y_{d[0,t]}, \breve\xi_{[0,t]}) = E' N^{-1}(y_d(t) - B_0 u(t) - \bar C\xi(t))$$
$$+\frac{1}{\gamma^2}(I_{q_b} - E' N^{-1} E)\bar D'(\bar\Sigma(t))^{-1}(\xi(t) - \breve\xi(t)) \tag{34}$$

and

$$J_{i\gamma t_f} = -\left|\xi(t_f) - \check{\xi}(t_f)\right|^2_{(\bar{\Sigma}(t_f))^{-1}} + \int_0^{t_f} (2(\theta - \check{\theta}(\tau))' P_r(\check{\theta}(\tau))$$
$$-\gamma^2 \left| w_b(\tau) - w_{b\mathrm{opt}}(\tau, \xi_{[0,\tau]}, u_{[0,\tau]}, \breve{w}_{[0,\tau]}, y_{d[0,\tau]}, \check{\xi}_{[0,\tau]}) \right|^2) \,\mathrm{d}\tau \le 0 \tag{35}$$

Next, we turn to study the robustness and tracking properties of the proposed adaptive control law.

5 MAIN RESULT

In this section, we present the main result of this paper by stating a theorem on the robustness and tracking properties of the proposed adaptive control law.

The closed-loop system dynamics are

$$\dot{X} = F(X, y_d, \breve{w}) + G(X, y_d, \breve{w}) w_b = F(X, y_d, \breve{w}) + G(X, y_d, \breve{w}) \grave{M} \grave{w}_b; \qquad X(0) = X_0 \tag{36}$$

where $X := (x_{\bar{o}}, \theta, x, \Sigma, s_\Sigma, \Phi, \check{\theta}, \check{x})$; F, and G are smooth mappings on $\mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^{\breve{q}}$; and

$$X_0 \in \mathcal{D}_0 := \big\{ X_0 \in \mathcal{D} \;\big|\; \theta \in \Theta,\ \Sigma(0) = \gamma^{-2} Q_0^{-1} \in \mathcal{S}_{+\sigma},\ \mathrm{Tr}\left((\Sigma(0))^{-1}\right) \le K_c,$$
$$s_\Sigma(0) = \gamma^2 \,\mathrm{Tr}\,(Q_0)\,,\ \grave{T}(x_{\bar{o}}(0), x_1(0), \cdots, x_n(0)) = \grave{T}(x_{\bar{o}}(0), x(0)) \in \acute{\mathcal{D}}_0,\ \check{\theta}_0 \in \Theta \big\}$$
$$\mathcal{D} := \big\{ X \in \mathbb{R}^{\grave{n} - \sum_{i=1}^m \nu_i} \times \mathbb{R}^\sigma \times \mathbb{R}^{n_O} \times \mathcal{S}_\sigma \times \mathbb{R} \times \mathbb{R}^{n_O \times \sigma} \times \mathbb{R}^\sigma \times \mathbb{R}^{n_O} \;\big|\; \Sigma \in \mathcal{S}_{+\sigma}, s_\Sigma \in \mathbb{R}_+, \check{\theta} \in \Theta_o \big\}$$

Since (33) holds, the value function W satisfies the following Hamilton-Jacobi-Isaacs equation by Lemma 8 of Pan and Başar (2023):

$$\frac{\partial W}{\partial X}(X) F(X, y_d, \breve{w}) + \frac{1}{4\gamma^2} \left\| \frac{\partial W}{\partial X}(X) G(X, y_d, \breve{w}) \right\|^2_{\mathbb{R}^{q_b}} + \hat{Q}(X, y_d, \breve{w}) = 0;\ \forall X \in \mathcal{D},\ \forall y_d \in \mathbb{R}^m, \forall \breve{w} \in \mathbb{R}^{\breve{q}} \tag{37}$$

where $\hat{Q} : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^{\breve{q}} \to \mathbb{R}$ is smooth and given by

$$\hat{Q}(X, y_d, \breve{w}) = \left| Cx + B_0 u + (C_{1,3}\breve{w} + C_{1,2}u)\theta - y_d \right|^2_Q + \gamma^4 \left| x - \check{x} - \Phi(\theta - \check{\theta}) \right|^2_{\Pi^{-1}\Delta\Pi^{-1}}$$
$$+\epsilon \left| (C\Phi + C_{1,3}\breve{w} + C_{1,2}u)(\theta - \check{\theta}) \right|^2_{\gamma^2 N^{-1} - Q} - 2(\theta - \check{\theta})' P_r(\check{\theta})$$

Clearly, $\hat{Q}$ is nonnegative as long as $X \in \mathcal{D}$ and $\theta \in \Theta$.

Since the value function W is not a positive-definite function for the entire closed-loop system state X, we cannot deduce stability properties of the closed-loop system directly from the value function W. As it turns out, the closed-loop adaptive system possesses a strong stability property: all closed-loop signals remain bounded with respect to bounded disturbance $\grave{w}_{[0,\infty)} \in \grave{\mathcal{W}}_d$ the initial condition $\acute{x}_0 \in \acute{\mathcal{D}}_0$, and bounded reference trajectory $y_{d[0,\infty)}$, in addition to the above stated attenuation (dissipation) property. This is made precise in the following theorem.

Remark 1 *Assumptions 1 – 8 are standard as in the SISO case Zeng* et al. *(2010).*

Theorem 1 *Consider the robust adaptive control problem formulated in Section 3, with Assumptions 1 – 8 holding. Then, the adaptive controller μ given by (31) with the worst-case estimate $\hat{\xi}$ generated by the optimal policy (30), achieves the following strong robustness properties for the closed-loop system.*

1. *Given $c_w \in \overline{\mathbb{R}_+}$ and $c_d \in \overline{\mathbb{R}_+}$, there exists a constant $c_c \in \overline{\mathbb{R}_+}$ and a compact set $\Theta_c \subset \Theta_o$ such that for any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$ with*

$$|\,\acute{x}_0\,| \le c_w;\ \acute{x}_0 \in \acute{\mathcal{D}}_0; \quad |\,\grave{w}(t)\,| \le c_w;\ \grave{w}_{[0,\infty)} \in \grave{\mathcal{W}}_d; \quad |\,y_d(t)\,| \le c_d; \qquad \forall t \in \overline{\mathbb{R}_+}$$

all closed-loop state variables $x_{\bar{o}}$, x, $\check{x}$, $\check{\theta}$, Σ, s_Σ, and Φ exist and are bounded as follows, $\forall t \in \overline{\mathbb{R}_+}$,

$$|x_{\bar{o}}(t)| \le c_c, \quad |x(t)| \le c_c, \quad |\check{x}(t)| \le c_c, \quad \check{\theta}(t) \in \Theta_c, \quad \|\Phi(t)\| \le c_c,$$
$$K_c^{-1} I_\sigma \le \Sigma(t) \le \gamma^{-2} Q_0^{-1}, \quad \gamma^2 \operatorname{Tr}(Q_0) \le s_\Sigma(t) \le K_c.$$

Therefore, there is a compact set $S \subset \mathcal{D}$ such that $X(t) \in S$, $\forall t \in [0, \infty)$. Hence, there exists a constant $c_u \in \overline{\mathbb{R}_+}$ such that $|u(t)| \le c_u$, $\forall t \in [0,\infty)$.

2. *The controller μ belongs to $\mathcal{M}$ and achieves disturbance attenuation level 0 with respect to $\breve{w}$ and disturbance attenuation level γ with respect to w_b for any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$.*

3. *For any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$ with $\breve{w}_{[0,\infty)} \in \bar{\mathrm{L}}_\infty$, $\grave{w}_{b[0,\infty)} \in \bar{\mathrm{L}}_2 \cap \bar{\mathrm{L}}_\infty$, $y_{d\,[0,\infty)} \in \bar{\mathrm{L}}_\infty$, and $\breve{w}_{[0,\infty)}$ and $y_{d\,[0,\infty)}$ being uniformly continuous on the interval $\overline{\mathbb{R}_+}$, the output of the system $Cx + B_{p0}(\theta)u + (C_{1,3}\breve{w})\theta$ asymptotically tracks the reference trajectory y_d, i. e.,*

$$\lim_{t\to\infty} (Cx(t) + B_{p0}(\theta)u(t) + (C_{1,3}\breve{w}(t))\theta - y_d(t)) = \mathbf{0}_m$$

Proof We consider the first statement. Fix an uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$ with

$$|\acute{x}_0| \le c_w;\ \acute{x}_0 \in \acute{\mathcal{D}}_0; \quad |\grave{w}(t)| \le c_w;\ \grave{w}_{[0,\infty)} \in \grave{\mathcal{W}}_d; \quad |y_d(t)| \le c_d; \qquad \forall t \in [0,\infty)$$

for some $c_w \in \overline{\mathbb{R}_+}$ and $c_d \in \overline{\mathbb{R}_+}$. With the controller μ and $\hat{\xi}$ designed, we have a fixed initial condition $X_0 \in \mathcal{D}_0$ for the closed-loop system (36). Consider the maximal length interval $[0, T_f)$ where the differential equation (36) for the closed-loop system admits a solution that lies in $\mathcal{D}$, which is clearly an open set. Then, by the smoothness of the system, the solution $X(t)$ is unique on $[0, T_f)$. Note that the maximal length of the interval, T_f, may depend on the specific waveform for the disturbance $\grave{w}_{[0,\infty)}$ and the reference $y_{d\,[0,\infty)}$. We will show that the maximal length of the interval, T_f, is always ∞.

By Lemma 1, the covariance matrix Σ and the signal s_Σ are uniformly upper bounded and uniformly bounded away from 0, as depicted in the first statement of the theorem. By Proposition 3 of Pan and Başar (2023), Σ and s_Σ are inside compact subsets of $\mathcal{S}_{+\sigma}$ and $\mathbb{R}_+$, respectively. The reference trajectory is uniformly bounded since $|y_d(t)| \le c_d$, $\forall t \ge 0$.

Introduce the dynamics $\dot{\tilde{\eta}} = \lambda_m \tilde{\eta} + y - y_d$; $\tilde{\eta}(0) = \mathbf{0}_m$, where $\lambda_m \approx \max(\operatorname{Re}(\lambda(A_f))) \in \mathbb{R}_-$ and $\lambda(A_f)$ denotes all eigenvalues of the matrix A_f. There exist positive definite matrices $Z, Y \in \mathcal{S}_{+m}$ such that

$$2\lambda_m Z + \gamma^{-2} Z N Z + Y = \mathbf{0}_{m\times m}$$

Then, taking $V : \mathbb{R}^m \to \overline{\mathbb{R}_+}$, $V(\tilde{\eta}) := |\tilde{\eta}|_Z^2$, we have $\dot{V} = -|\tilde{\eta}|_Y^2 + \gamma^2 |y - y_d|_{N^{-1}}^2 - \gamma^2 \left| y - y_d - \gamma^{-2} N Z \tilde{\eta} \right|_{N^{-1}}^2$, $\forall \tilde{\eta} \in \mathbb{R}^m$, $\forall y \in \mathbb{R}^m$, $\forall y_d \in \mathbb{R}^m$. Define the vector of variables

$$X_e := (\check{\theta}, \tilde{x} - \Phi\tilde{\theta}, \tilde{\eta})$$

Clearly, $X_e : [0, T_f) \to \mathcal{D}_e := \Theta_o \times \mathbb{R}^{n_O} \times \mathbb{R}^m$, and the function $U := V + W$ can be written as $U = \bar{U}(t, X_e(t))$, where $\bar{U} : [0, T_f) \times \mathcal{D}_e \to \overline{\mathbb{R}_+}$. Under the assumption that $\grave{w}$ is uniformly bounded on $[0, \infty)$, we have the following inequality for the derivative of U:

$$\begin{aligned}
\dot{U} &\le -\left|\xi - \check{\xi}\right|_{\bar{Q}}^2 + \gamma^2 \left\| \grave{M} \right\|^2 c_w^2 + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|_Y^2 - \gamma^2 \left| y - y_d - \gamma^{-2} N Z \tilde{\eta} \right|_{N^{-1}}^2 \\
&\le -\gamma^4 \left| \tilde{x} - \Phi\tilde{\theta} \right|_{\Pi^{-1}\Delta\Pi^{-1}}^2 + 2(\theta - \check{\theta})' P_r(\check{\theta}) - |\tilde{\eta}|_Y^2 + \bar{c}_w^2
\end{aligned}$$

where $\bar{c}_w := \gamma c_w \left\| \grave{M} \right\|$. Then, there exists a compact set $\Omega_1(c_w) \subset \mathcal{D}_e$ such that, $\forall t \in [0, T_f)$, if $X_e \in \mathcal{D}_e \setminus \Omega_1(c_w)$ then $\dot{U} < 0$. Let

$$U_M(X_e) := K_c \left| \theta - \check{\theta} \right|^2 + \gamma^2 \left| \tilde{x} - \Phi\tilde{\theta} \right|_{\Pi^{-1}}^2 + |\tilde{\eta}|_Z^2$$

$$U_m(X_e) := \gamma^2 \left|\theta - \check{\theta}\right|^2_{Q_0} + \gamma^2 \left|\tilde{x} - \Phi\tilde{\theta}\right|^2_{\Pi^{-1}} + \left|\tilde{\eta}\right|^2_Z$$

Then, clearly $U_m(X_e) \leq \bar{U}(t, X_e) \leq U_M(X_e)$, $\forall t \in [0, T_f)$, $\forall X_e \in \mathcal{D}_e$. By Lemma 5 of Pan and Başar (2023), there is a constant $c_1 \in \overline{\mathbb{R}_+}$ such that $U_m(X_e(t)) \leq c_1$, $\forall t \in [0, T_f)$.

Then, on the interval $[0, T_f)$, the vector X_e is uniformly bounded. Hence, we have that $\tilde{\theta}$, $\tilde{x} - \Phi\tilde{\theta}$, and $\tilde{\eta}$ are uniformly bounded. ($\tilde{\theta}$ is bounded to begin with since $\theta \in \Theta$ and $\check{\theta} \in \Theta_o$.)

Note that the signal $\tilde{\eta}$ is uniformly bounded, it is minimum phase and has uniform vector relative degree 1 with respect to the input y with respect to $\mathcal{D}_{\tilde{\eta}_0} := \mathbb{R}^m$ and $\mathcal{C}$ according to Başar and Pan (2020), where the signal y_d is regarded as disturbance. Then, this signal $\tilde{\eta}$ is minimum phase with respect to $\mathcal{D}_{\tilde{\eta}_0} \times \acute{\mathcal{D}}_0$ and $\mathcal{C} \times \grave{\mathcal{W}}_d$ and has relative degree 1 with respect to the input u (by a straightforward vectorized version of Theorem 1 of Pan and Başar (2019a)). The signal $\tilde{\eta}$ is uniformly bounded. It is easy to see that the $\tilde{\eta}$ dynamics with input y and output $\tilde{\eta}$ may serve as a reference system in the application of Proposition 2 of Pan and Başar (2019b) (more precisely, a straightforward vectorized version of it). By Theorem 1 of Pan and Başar (2019a), the composite system with control input u, output $\tilde{\eta}$, and disturbance inputs y_d and $\grave{w}$ may serve as a reference system in the application of Proposition 2 of Pan and Başar (2019b).

We conclude the boundedness of the variables Φ as follows. Define

$$\dot{\Phi}_u = A_f \Phi_u + (A_{212} - \check{L}C_{1,2})u; \quad \Phi_u(0) = \mathbf{0}_{n_O \times \sigma} \tag{38a}$$

$$\dot{\Phi}_y = A_f \Phi_y + A_{211,1} y + (A_{211,3} - \check{L}C_{1,3})\breve{w}; \quad \Phi_y(0) = \Phi_0 \tag{38b}$$

Then, we have $\Phi = \Phi_y + \Phi_u$.

The relative degree for each of the elements of Φ_u is at least 1 with respect to the input u, and is the output of a stable linear system. By Proposition 2 of Pan and Başar (2019b), this yields that Φ_u is uniformly bounded, where the reference system has output $\tilde{\eta}$ and inputs u, $\grave{w}$, and y_d.

The relative degree for each of the elements of Φ_y is at least 1 with respect to the input y, and is the output of a stable linear system. By Proposition 2 of Pan and Başar (2019b), this yields that Φ_y is uniformly bounded, where the reference system has output $\tilde{\eta}$ and input y and y_d.

Hence, Φ is uniformly bounded on $[0, T_f)$. Since $\tilde{x} - \Phi\tilde{\theta}$ and $\tilde{\theta}$ are uniformly bounded, we have that the signal $\tilde{x}$ is uniformly bounded.

Note that the dynamics of x is (5a), which can be written as

$$\dot{x} = A_f x + \check{L}(y - (C_{1,3}\breve{w} + C_{1,2}u)\theta - B_0 u - E w_b) + Bu + \check{D}\breve{w} + D w_b + (A_{211,1} y + A_{211,3}\breve{w} + A_{212}u)\theta$$

To apply Proposition 2 of Pan and Başar (2019b), the dynamics are separated into y dependent and u dependent parts using the linearity of the system, $x =: x_u + x_y$. The dynamics of x_u and x_y are given by

$$\dot{x}_u = A_f x_u - \check{L}B_{p0}(\theta)u + Bu + (A_{212}u)\theta; \quad x_u(0) = x(0)$$

$$\dot{x}_y = A_f x_y + \check{L}(y - (C_{1,3}\breve{w})\theta - E w_b) + \check{D}\breve{w} + D w_b + (A_{211,1} y + A_{211,3}\breve{w})\theta; \quad x_y(0) = \mathbf{0}_{n_O}$$

The signal x_u has relative degree at least 1 with respect to u. It is uniformly bounded by Proposition 2 of Pan and Başar (2019b), where the reference system has inputs u, y_d, and $\grave{w}$, and output $\tilde{\eta}$. The signal x_y has relative degree at least 1 with respect to y. It is uniformly bounded by Proposition 2 of Pan and Başar (2019b), where the reference system has inputs y and y_d, and output $\tilde{\eta}$. Hence, x is uniformly bounded.

Then, $\check{x}$ is uniformly bounded since both x and $\tilde{x}$ are uniformly bounded.

It can further be concluded that u is uniformly bounded by the control law (31) and $B_{p0}(\check{\theta})$ is uniformly bounded away from singularity due the $\check{\theta} \in \Theta_o$, $\forall t \in [0, T_f)$, (see the last paragraph preceding (17)). This then yields the boundedness of y by (5b).

By the minimum phase condition on the system (2) and the canonical form (3), we have that the state $\acute{x}$ is uniformly bounded. Therefore, $x_{\bar{o}}$ is bounded as desired.

In order to show the existence of a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$, introduce the function

$$\Upsilon := W + P(\check{\theta})(\rho_o - P(\check{\theta}))^{-1}$$

Clearly, Υ can be written as $\Upsilon(t) = \bar{\Upsilon}(t, \bar{X}_e(t))$, where $\bar{\Upsilon} : [0, T_f) \times \bar{\mathcal{D}}_e \to \overline{\mathbb{R}_+}$ and $\bar{X}_e := (\check{\theta}, \tilde{x} - \Phi\tilde{\theta}) \in \bar{\mathcal{D}}_e := \Theta_o \times \mathbb{R}^{n_O}$. The total time derivative of Υ is given by

$$
\begin{aligned}
\dot{\Upsilon} &= \dot{W} + \rho_o \left(\rho_o - P(\check{\theta})\right)^{-2} \frac{\partial P}{\partial \theta}(\check{\theta})\dot{\check{\theta}} \\
&\le -\gamma^4 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \bar{c}_w^2 - \gamma^2 \left| y - y_d \right|^2_{N^{-1}} + \rho_o \left(\rho_o - P(\check{\theta})\right)^{-2} \Bigg(- \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma P_r(\check{\theta}) \\
&\quad + \frac{\partial P}{\partial \theta}(\check{\theta})\gamma^2 \Sigma (C\Phi + C_{1,3}\check{w} + C_{1,2}u)' N^{-1}(y - y_d) \Bigg) \\
&\le -\gamma^4 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \bar{c}_w^2 - \rho_o p_r(\check{\theta}) \left(\rho_o - P(\check{\theta})\right)^{-2} \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \\
&\quad + \gamma^2 \left| \rho_o \left(\rho_o - P(\check{\theta})\right)^{-2} (C\Phi + C_{1,3}\check{w} + C_{1,2}u)\Sigma \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \right|^2_{N^{-1}} \\
&= -\gamma^4 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \bar{c}_w^2 - \rho_o p_r(\check{\theta}) \left(\rho_o - P(\check{\theta})\right)^{-2} \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \\
&\quad + \gamma^2 \rho_o^2 \left(\rho_o - P(\check{\theta})\right)^{-4} \left| (C\Phi + C_{1,3}\check{w} + C_{1,2}u)\Sigma \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \right|^2_{N^{-1}} \\
&\le -\gamma^4 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \bar{c}_w^2 - \rho_o p_r(\check{\theta}) \left(\rho_o - P(\check{\theta})\right)^{-2} \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \\
&\quad + \gamma^2 \rho_o^2 c_2 \left(\rho_o - P(\check{\theta})\right)^{-4} \frac{\partial P}{\partial \theta}(\check{\theta})\Sigma \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)'
\end{aligned}
$$

for some $c_2 \in \overline{\mathbb{R}_+}$, where the last inequality follows from the uniform boundedness of $\check{w}$, Φ, u, and Σ. Then, there exists a compact set $\Omega_2(c_2, \bar{c}_w) \subset \bar{\mathcal{D}}_e$ such that, $\forall t \in [0, T_f)$, if $\bar{X}_e \in \bar{\mathcal{D}}_e \setminus \Omega_2(c_2, \bar{c}_w)$ then $\dot{\Upsilon} < 0$. Note that, $\forall (t, \bar{X}_e) \in [0, T_f) \times \bar{\mathcal{D}}_e$,

$$
\begin{aligned}
\Upsilon_m(\bar{X}_e) := \gamma^2 \left| \tilde{\theta} \right|^2_{Q_0} + \gamma^2 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}} + P(\check{\theta})(\rho_o - P(\check{\theta}))^{-1} \le \bar{\Upsilon}(t, X_e) \le K_c \left| \tilde{\theta} \right|^2 + \gamma^2 \left| \tilde{x} - \Phi\tilde{\theta} \right|^2_{\Pi^{-1}} \\
+ P(\check{\theta})(\rho_o - P(\check{\theta}))^{-1} =: \Upsilon_M(\bar{X}_e)
\end{aligned}
$$

By Lemma 5 of Pan and Başar (2023), there exists a constant $c_3 \in \overline{\mathbb{R}_+}$ such that $\Upsilon_m(\bar{X}_e(t)) \le c_3$, $\forall t \in [0, T_f)$. Hence, there exists a compact set $\Theta_c \subset \Theta_o$ such that $\check{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$.

Thus, statement 1 holds on the interval $[0, T_f)$. Then, there exists a compact set $S \subseteq \mathcal{D}$, such that $X(t) \in S$, $\forall t \in [0, T_f)$. By a standard result in ordinary differential equations, we have $T_f = \infty$. Thus, statement 1 is proved.

Next, we prove the second statement. Fix any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$. Then, we have $\acute{x}_0 \in \acute{\mathcal{D}}_0$, $\theta \in \Theta$, and $\grave{w}_{[0,\infty)} \in \grave{\mathcal{W}}_d$. For any $t_f \in \overline{\mathbb{R}_+}$, there exist constants $c_w \ge 0$ and $c_d \ge 0$ such that $|\acute{x}_0| \le c_w$, $|\grave{w}(t)| \le c_w$, and $|y_d(t)| \le c_d$, $\forall t \in [0, t_f]$, since $\grave{w}$ and y_d are continuous. By the first statement and the causality of the closed-loop system, there exists a solution $X : [0, t_f] \to \mathcal{D}$ for the closed-loop system. Hence, the closed-loop system (36) admits a unique solution on $[0, \infty)$. This further implies that the proposed adaptive control law belongs to $\mathcal{M}$. Choose

$$
\begin{aligned}
l(t, \theta, x_{[0,t]}, y_{[0,t]}, \check{w}_{[0,t]}, y_{d\,[0,t]}) &= \gamma^4 \left| x - \check{x} - \Phi(\theta - \check{\theta}) \right|^2_{\Pi^{-1}\Delta\Pi^{-1}} + \epsilon \left| (C\Phi + C_{1,3}\check{w} + C_{1,2}u)(\theta - \check{\theta}) \right|^2_{\gamma^2 N^{-1} - Q} \\
&\quad - 2(\theta - \check{\theta})' P_r(\check{\theta})
\end{aligned}
$$

The function l is clearly nonnegative as long as $X(t) \in \mathcal{D}$ with $\theta \in \Theta$, which is guaranteed by the first statement. Then, we have

$$
J_{\gamma t_f} = J_{\gamma t_f} + \int_0^{t_f} \dot{W} \, \mathrm{d}\tau + W(0) - W(t_f) \le -W(t_f) \le 0
$$

This shows that the controller μ given by (31), with the optimal choice $\hat{\xi} = \check{\xi}$ given by (30), achieves the disturbance attenuation level 0 with respect to $\breve{w}$ and disturbance attenuation γ with respect to w_b as prescribed by Definition 1. This establishes the second statement.

Last, we prove the third statement. For any uncertainty quadruple $(\acute{x}_0, \theta, \grave{w}_{[0,\infty)}, y_{d\,[0,\infty)}) \in \grave{\mathcal{W}}$ with $\breve{w}_{[0,\infty)} \in \bar{\mathrm{L}}_\infty$, $\grave{w}_{b[0,\infty)} \in \bar{\mathrm{L}}_2 \cap \bar{\mathrm{L}}_\infty$ and $y_{d\,[0,\infty)} \in \bar{\mathrm{L}}_\infty$, we have that Statements 1 and 2 hold. Then,

$$\int_0^\infty |Cx(t) + B_0u(t) + (C_{1,3}\breve{w}(t) + C_{1,2}u(t))\theta - y_d(t)|_Q^2 \,\mathrm{d}t \le W(0) + \gamma^2 \int_0^\infty \left| \grave{M}\grave{w}_b(t) \right|^2 \mathrm{d}t < +\infty$$

by the dissipation inequality (33) and the second statement. This implies that $Cx + B_0u + (C_{1,3}\breve{w} + C_{1,2}u)\theta - y_d \in \bar{\mathrm{L}}_2$ on the interval $[0, \infty)$. By the uniform continuity assumption on $\breve{w} : \overline{\mathbb{R}_+} \to \mathbb{R}^{\breve{q}}$ and $y_d : \overline{\mathbb{R}_+} \to \mathbb{R}^m$, and the boundedness of $\dot{x}$, we have that $Cx + B_0u + (C_{1,3}\breve{w} + C_{1,2}u)\theta - y_d : \overline{\mathbb{R}_+} \to \mathbb{R}^m$ is uniformly continuous. Therefore, by Lemma 4.3 of Zhao *et al.* (2009), we have

$$\lim_{t\to\infty} (Cx(t) + B_0u(t) + (C_{1,3}\breve{w}(t) + C_{1,2}u(t))\theta - y_d(t)) = \mathbf{0}_m$$

This completes the proof of the theorem. □

6 AN EXAMPLE

In this section, we present a numerical example that serves to illustrate the robust adaptive control design presented in this paper. The designs for the example were carried out using MATLAB.

We consider the following adaptive noise cancellation problem. The uncertain linear system is given as follows, where $\bar{\theta}_1 \in \bar{\mathrm{r}}_{-\frac{3}{2},-\frac{1}{2}}$, $\bar{\theta}_2 \in \bar{\mathrm{r}}_{\frac{1}{2},\frac{3}{2}}$, and $\bar{\theta}_3 \in \bar{\mathrm{r}}_{\frac{1}{2},\frac{3}{2}}$ are unknown parameters,

$$\dot{\grave{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\bar{\theta}_3 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} 2\bar{\theta}_1 & \bar{\theta}_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \grave{w}_b; \qquad \grave{x}_0 = \begin{bmatrix} 0 \\ \frac{1}{10} \\ \frac{1}{10} \end{bmatrix} \tag{39a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} \bar{\theta}_1 & \bar{\theta}_2 \\ -\bar{\theta}_2 & \bar{\theta}_1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \grave{w}_b \tag{39b}$$

$$z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} \bar{\theta}_1 & \bar{\theta}_2 \\ -\bar{\theta}_2 & \bar{\theta}_1 \end{bmatrix} u \tag{39c}$$

This uncertain system admits uniform vector relative degree of zero. The system of (39) has the extended zero dynamics canonical form

$$\dot{\grave{x}} = \begin{bmatrix} -(1 + \frac{\bar{\theta}_1^2}{\bar{\theta}_1^2+\bar{\theta}_2^2}) & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\bar{\theta}_3 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} 1 + \frac{\bar{\theta}_1^2}{\bar{\theta}_1^2+\bar{\theta}_2^2} & -\frac{\bar{\theta}_1\bar{\theta}_2}{\bar{\theta}_1^2+\bar{\theta}_2^2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} y + \begin{bmatrix} -(1 + \frac{\bar{\theta}_1^2}{\bar{\theta}_1^2+\bar{\theta}_2^2}) & \frac{\bar{\theta}_1\bar{\theta}_2}{\bar{\theta}_1^2+\bar{\theta}_2^2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \grave{w}_b$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \grave{x} + \begin{bmatrix} \bar{\theta}_1 & \bar{\theta}_2 \\ -\bar{\theta}_2 & \bar{\theta}_1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \grave{w}_b$$

This implies that the extended zero dynamics is of third order and the system is minimum phase with respect to $\mathbb{R}^3$ and $\mathcal{C}$ if $\bar{\theta}_1^2 + \bar{\theta}_2^2 > 0$ according to Başar and Pan (2020). Assume that we know that the observability indices for the measurement channels are 3 and 0 for channels 1 and 2, respectively. Clearly, the system (39) admits strict observer canonical form. Now, we transform the system into such a form and introduce a disturbance transformation to arrive at the following design model.

Introducing the state transformation $\grave{x} = \grave{T}x$ with

$$\grave{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\bar{\theta}_3 & 0 & 1 \end{bmatrix}$$

we have

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -\bar{\theta}_3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 2\bar{\theta}_1 & \bar{\theta}_2 \\ 0 & 0 \\ 2\bar{\theta}_1\bar{\theta}_3 & \bar{\theta}_2\bar{\theta}_3 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{\theta}_3 \end{bmatrix} \grave{w}_b \tag{40a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} \bar{\theta}_1 & \bar{\theta}_2 \\ -\bar{\theta}_2 & \bar{\theta}_1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \grave{w}_b \tag{40b}$$

We assume that the true value of $\bar{\theta}_1$ is -1, the true value of $\bar{\theta}_2$ is 1, and the true value of $\bar{\theta}_3$ is 1. To normalize the unknown parameters, we set $\theta_1 = 2\bar{\theta}_1 + 2$, $\theta_2 = 2\bar{\theta}_2 - 2$, $\theta_3 = 2\bar{\theta}_3 - 2$, and define $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, where $\theta_4 = \theta_1\theta_3$ and $\theta_5 = \theta_2\theta_3$. Then the true value for parameter vector θ is $\mathbf{0}_5$, and θ_1, θ_2, θ_3, θ_4, θ_5 all belong to $\bar{\mathrm{r}}_{-1,1}$.

We then introduce disturbance transformation $w = \grave{M}\grave{w}_b$

$$\grave{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\theta_3}{2} & 0 & 0 \\ 0 & 0 & \frac{\theta_3+2}{2} \end{bmatrix}$$

and obtain the design model as follows:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ -\frac{\theta_3}{2} & 0 \\ 0 & 0 \end{bmatrix} y + \begin{bmatrix} \theta_1 & \frac{\theta_2}{2} \\ \frac{\theta_4}{4} - \frac{\theta_3}{2} & \frac{\theta_5}{4} + \frac{\theta_3}{2} \\ \frac{\theta_4}{2} + \theta_1 - \theta_3 & \frac{\theta_5}{4} + \frac{\theta_2+\theta_3}{2} \end{bmatrix} u + \begin{bmatrix} -2 & 1 \\ 0 & 0 \\ -2 & 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} w \tag{41a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} u + \begin{bmatrix} \frac{\theta_1}{2} & \frac{\theta_2}{2} \\ -\frac{\theta_2}{2} & \frac{\theta_1}{2} \end{bmatrix} u + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} w \tag{41b}$$

We choose the design parameter as

$$\gamma = \sqrt{5};\ Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix};\ Q_0 = \frac{1}{10} I_5;\ \rho_o = \frac{12}{5};\ \beta_\Delta = 0;\ \Delta_1 = I_3;\ K_c = \frac{5}{2};\ \Phi_0 = \mathbf{0}_{3\times 5};\ \epsilon = K_c^{-1} s_\Sigma$$

and the convex function $P(\theta)$ as

$$P(\theta) = \frac{1}{5}\Big(\exp(2(\theta_1^2 - 1)) + \exp(2(\theta_2^2 - 1)) + \exp(2(\theta_3^2 - 1)) + \exp(2(\theta_4^2 - 1)) + \exp(2(\theta_5^2 - 1))\Big)$$

Finally, we choose the initial conditions and the reference trajectory as

$$\check{x}_0 = \mathbf{0}_3;\ \check{\theta}_0 = (-0.5, 0.5, 0.25, 0.25, 0.25);\ y_{d1} = 0;\ y_{d2} = 0$$

We present two sets of simulation results for this example. The first set of simulations is aimed to demonstrate the asymptotic tracking capability of the adaptive controller. The disturbance input $\grave{w}_b$ is fixed to be identically zero. The simulation results are shown in Figure 1. We see that the tracking errors are converging to zero as predicted and the transient of the system response is well behaved. The state estimates and parameter estimates are well behaved. The parameter estimation errors do not converge to zero since there is no persistent excitation in the system. We also observe that control inputs are bounded in magnitude by 0.1 and the integral performance index grows from zero to a positive constant that is less than 0.005. These simulation results corroborate our theoretical results.

The second set of simulation results is to illustrate the robustness property of the controller. We set $\grave{w}_1 = 0.1\sin(0.025t)$ + band-limited white noise with power 0.0001, $\grave{w}_2 = 0.15\sin(0.02t + \pi)$ + band-limited white noise with power 0.0001, and $\grave{w}_3 = 0.05\sin(0.015t + \pi)$ + band-limited white noise with power 0.0001.

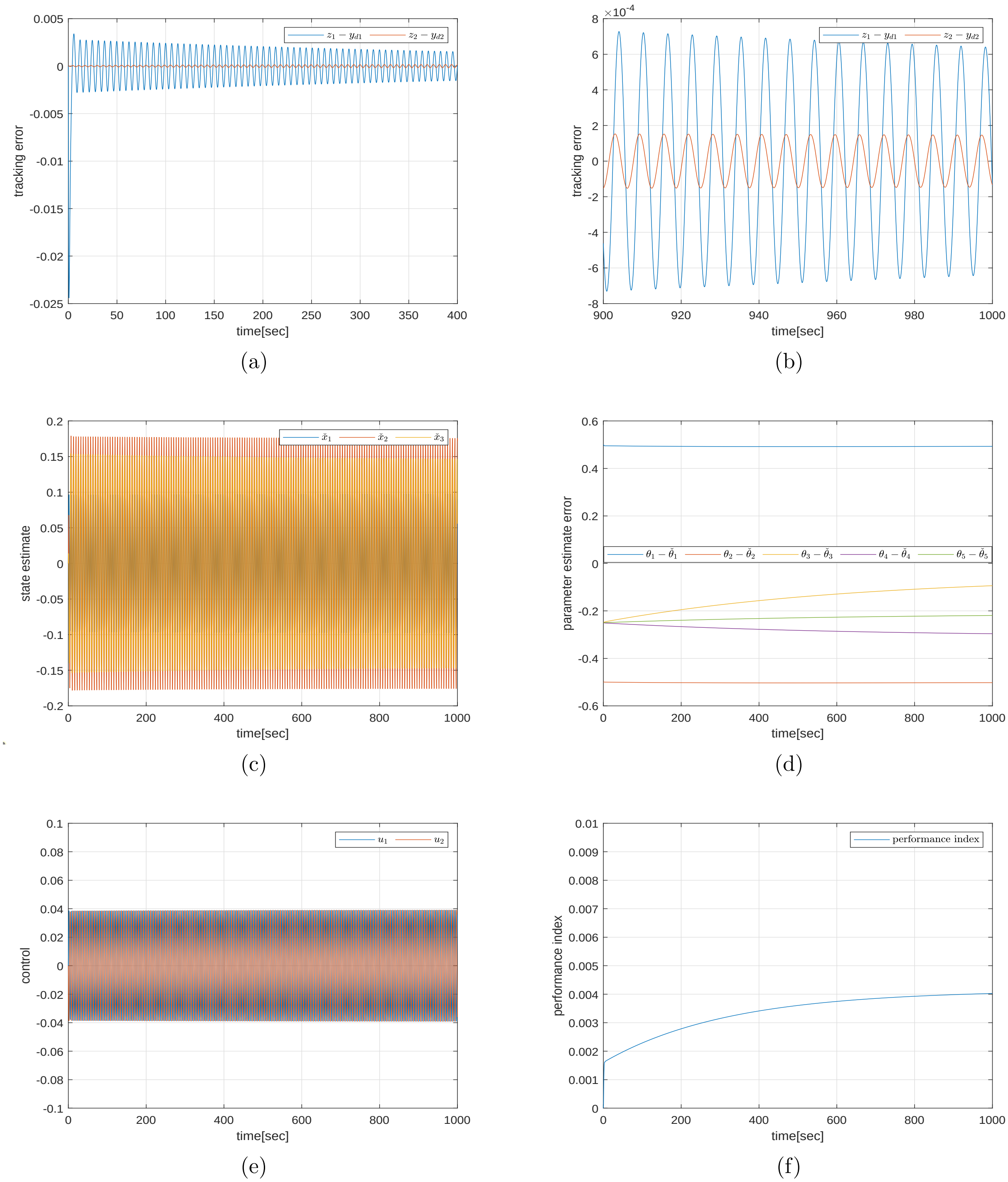

Figure 1: System response in the absence of any exogeneous disturbance.

(a) Tracking errors (short term); (b) Tracking errors (long term); (c) State estimates; (d) Parameter estimation errors; (e) Control inputs; (f) $\int_0^t (|z(\tau) - y_d(\tau)|_Q^2 - \gamma^2 |w(\tau)|^2)\, \mathrm{d}\tau$

The simulation results are shown in Figure 2. We see that the tracking errors are bounded in magnitude by 0.15; the control inputs are bounded in magnitude by 0.12; and the transient of the system response is well behaved. The parameter estimates are well behaved. The integral performance index is upper bounded by 0 and show negative slopes converging to negative infinity. These simulation results corroborate our theoretical results.

7 CONCLUSIONS

In this paper, we have presented a systematic procedure for robust adaptive control design for uncertain minimum phase square MIMO LTI systems that admit uniform vector relative degree of zero. We assumed that the MIMO linear system has m output terminals, and a set of upper bounds $n_1, \ldots, n_m \in \mathbb{Z}_+$ for the observability indices of the system is known. Then, it is always possible to pad dummy state variables to the system to arrive at a model that admits the observability indices $n_1, \ldots, n_m$, remains minimum phase, and admits uniform vector relative degree of zero. We assumed that this extended system admits strict observer canonical form. As mentioned earlier in the paper, this assumption is not restrictive at all, since when $n = n_1 = \cdots = n_m$ then the extended system admits the strict observer canonical form. The design procedure resembles that for the SISO case (Zeng *et al.*, 2010), except that we allow part of the disturbance inputs to be measured in this paper. The objective of the control design is to achieve attenuation level of 0 with respect to measured disturbance inputs and the desired attenuation level of $\gamma \in \mathbb{R}_+$ with respect to unmeasured disturbance inputs. We have formulated the underlying robust adaptive control problem as a nonlinear H^∞ optimal control problem with a single cost function. By making use of the *cost-to-come* function methodology in game theory for affine nonlinear H^∞ optimal control, we have obtained a finite-dimensional closed-form expression for an upper bound of the value function of the identifier for the unknown system. Assuming the existence of a known convex compact set for the true values of the system parameters, on which the high frequency gain matrix will remain invertible, we have introduced a smooth parameter projection scheme for the identifier, such that the adaptive control system is robust with or without persistently exciting input signals. Using the explicit form of the value function for the identifier, the nonlinear H^∞ adaptive control problem becomes a full-information nonlinear robust control problem, and the optimal control law is the certainty equivalent control law (Zeng *et al.*, 2010). An important observation on the worst case disturbance inputs we have made is that it will keep the measurement outputs identically equal to the reference trajectory, and thus deprive of the controller any information into the system. The adaptive controller achieves the desired disturbance attenuation objective and guarantees boundedness of all closed-loop signals under bounded admissible disturbance waveforms, bounded admissible initial states, and bounded reference trajectories without the need for any persistency of excitation condition or any stochastic noise assumptions. We have proved that the tracking error converges to zero when, in addition, the unmeasured disturbance inputs are of finite energy, and the measured disturbance and the reference trajectory are uniformly continuous. A numerical example of a system with two inputs and two outputs has been included and the simulation results have corroborated our theoretical findings.

A number of future research directions stand out as promising. One fruitful direction pertains to the study of the counterpart of the theory developed here to nonlinear systems with noisy output measurements. Another interesting topic would be to study the robustness of the adaptive system with respect to unmodeled fast dynamics. Another interesting direction lies in the study of networked robust adaptive control systems. It has been observed and proved that robust adaptive control systems designed according to Pan and Başar (2000) can be networked in a feedback loop fashion, and under the satisfaction of the small gain condition for the L_2-gains of the closed-loop system, the closed-loop signals will remain bounded for any admissible bounded exogeneous disturbance inputs and any admissible bounded initial conditions that are further convergent (that is, the tracking errors converge to zero) when the exogeneous disturbance inputs are L_2 and vanishing. We believe that this result holds for the adaptive systems addressed in this paper and in Pan and Başar (2023). This will pave the way for the application of the robust adaptive control system theory in practical use. Another fruitful direction lies in the case when the given MIMO LTI system is

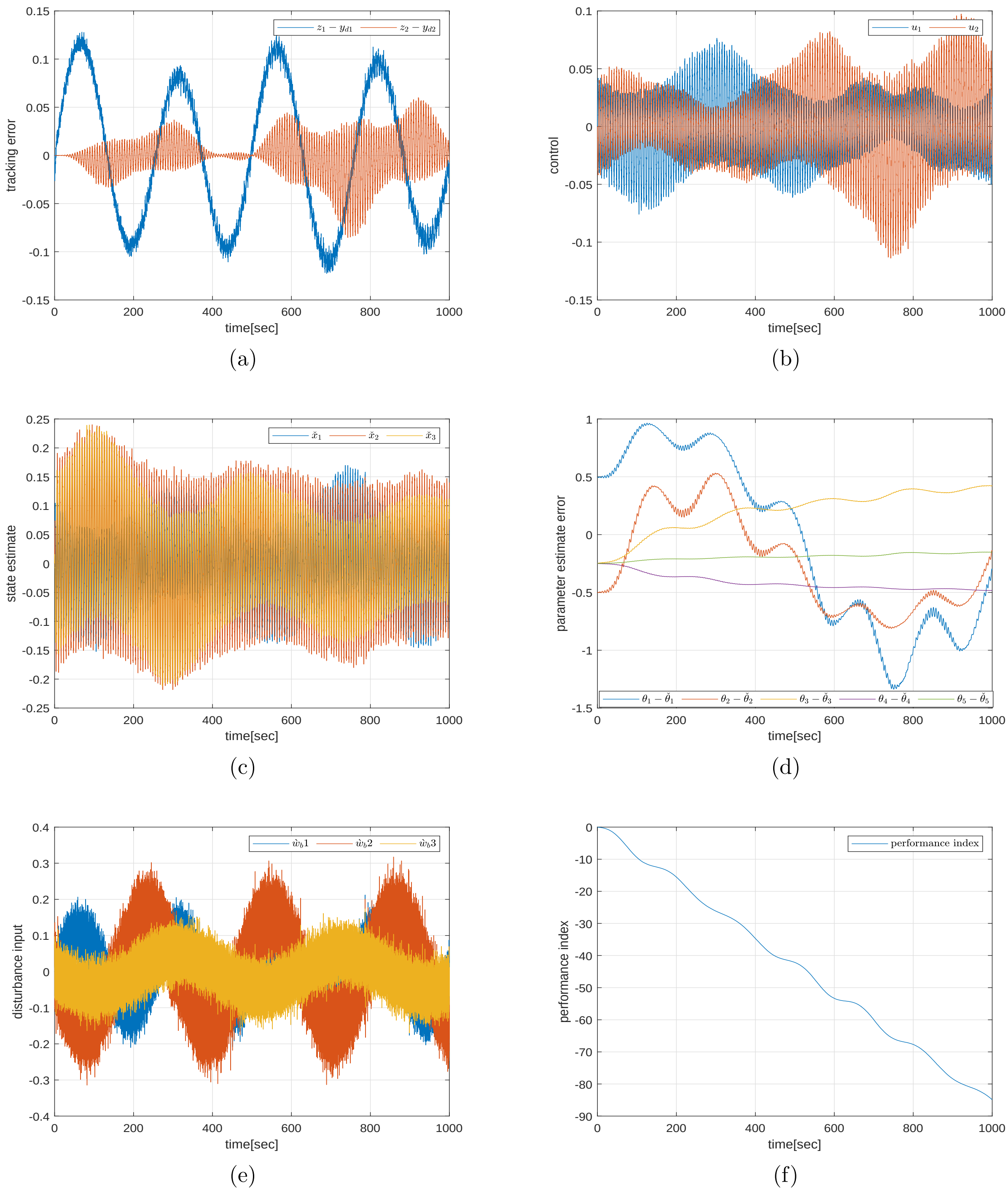

Figure 2: System response under exogeneous disturbances.

(a) Tracking errors; (b) Control inputs; (c) State estimates;
(d) Parameter estimation errors; (e) Disturbance inputs; (f) $\int_0^t (|z(\tau) - y_d(\tau)|_Q^2 - \gamma^2 |w(\tau)|^2)\, \mathrm{d}\tau$

comprised of multiple square MIMO LTI subsystems in parallel interconnection, satisfying an interconnection property, where the subsystems are assumed to be robust adaptive control ready (i. e., with positive uniform vector relative degree and uniform observability indices, or with zero uniform vector relative degree) but the composite system may have nonuniform vector relative degree and/or nonuniform observability indices. We envision that a centralized controller can be designed without requiring any dynamic extension or adding dummy state variables to the design model. We believe that the results of this paper and of Pan and Başar (2023) will serve as building blocks for the solution to the parallel interconnected MIMO LTI system problem.

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