

An Extended Zero Dynamics Canonical Form for Finite-Dimensional Continuous-Time Parallel Interconnected Square MIMO LTI Systems

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Abstract: In this paper, we have obtained the extended zero-dynamics canonical form for a class of square MIMO LTI systems comprised of multiple square MIMO LTI subsystems in parallel interconnection satisfying an interconnection property. We assume that each subsystem has already been analyzed and extended to admit uniform vector relative degree (and has uniform observability indices), and thus is suitable for the design of robust adaptive controllers. We prescribe an interconnection property under which the composite system (without any further modification or extension) admits the extended zero-dynamics canonical form even though it does not have uniform vector relative degree. This will allow a centralized robust adaptive controller design for the composite square MIMO LTI system if the composite system can be shown to be minimum phase.

Keywords: Minimum phase, zero dynamics canonical form, extended zero dynamics, extended zero dynamics canonical form, observability indices.

1. INTRODUCTION

The minimum phase concept for linear systems is crucial for the generalization of robust adaptive control system design for finite-dimensional continuous-time SISO LTI systems to that for finite-dimensional continuous-time MIMO LTI systems (see Pan and Başar (2000, 2018); Başar and Pan (2020); Pan and Başar (2021)). In robust adaptive control for SISO systems (Pan and Başar, 2000) it has been observed that key canonical forms for the underlying system are the observer canonical form and the extended zero-dynamics canonical form. For square MIMO LTI systems, the zero-dynamics canonical form exists if there exists a vector relative degree for the system. This zero-dynamic canonical form then reveals the extended zero dynamics for the system. However, the extended zero-dynamics canonical form exists for general square MIMO LTI systems under a more restrictive assumption: the system must admit uniform vector relative degree (Başar and Pan, 2020). This means that one must extend the square MIMO LTI system to admit uniform vector relative degree before attempting to design a robust adaptive controller. These extra steps of extension lead to a larger system order and therefore a more complicated adaptive controller, and it does not allow for an easy expansion of the system when additional subsystems are incorporated into the composite system.

In this paper, we have obtained the extended zero-dynamics canonical form for a class of square MIMO LTI systems that is comprised of multiple square MIMO LTI subsystems in parallel interconnection satisfying an inter-

connection property. We assume that each subsystem has already been analyzed and extended to admit uniform vector relative degree (and has uniform observability indices). We have multiple such subsystems parallel-interconnected to form a composite system, where the composite system admits vector relative degree but not uniform vector relative degree. We prescribe an interconnection property, under which the composite system (without any further modification or extension) admits the extended zero-dynamics canonical form even though it does not have uniform vector relative degree. Thus, the composite system would be in a form suitable for robust adaptive control design with nonuniform vector relative degree if it is further minimum phase according to Başar and Pan (2020). The interconnection property we prescribe is that for each subsystem i , the connections from subsystem j , $j \neq i$, satisfy the properties that the relative degree from each component of \mathbf{y}_j to each component of \mathbf{y}_i is greater than $\max\{0, r_i - r_j\}$, where r_i and r_j are the uniform vector relative degrees for the i th subsystem and the j th subsystem, respectively, and the relative degree from each component of \mathbf{u}_j to each component of \mathbf{y}_i is greater than $\max\{r_i, r_j\}$. Thus, when a number of subsystems are to be incorporated into a robust adaptive control system, we just need to make sure that these subsystems are themselves with uniform vector relative degree (and have uniform observability indices), and the interconnections of these subsystems and those of the original system are compatible, i.e., they satisfy the interconnection property. Then, the (centralized) robust adaptive controller can be redesigned and applied to the larger system without requiring any changes in the subsys-

tems if the composite system is minimum phase according to Başar and Pan (2020).

The balance of the paper is as follows. In the next section, we introduce the notations used in the paper. In Section 3, we introduce the definition of the extended zero dynamics canonical form of a class of square MIMO LTI systems which is the composite system of multiple square MIMO LTI systems in parallel interconnection further satisfying the interconnection property. The availability of the extended zero dynamics canonical form for this class of systems then leads to the bounding result that is stated in Section 4, with the composite system serving as the reference system. The paper ends with some concluding remarks in Section 5.

2. NOTATIONS

We let \mathbb{R} denote the real line; $\mathbb{R}_+ := (0, \infty) \subset \mathbb{R}$; $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; \mathbb{N} be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; and \mathbb{C} be the set of complex numbers. Unless otherwise specified, all signals, constants, and matrices are real. For a function f , we say that it belongs to \mathcal{C} if it is continuous; we say that it belongs to \mathcal{C}_k if it is k -times continuously differentiable (Fréchet differentiability), which is equivalent to all partial derivatives up to the k th order being continuous when the domain of f is open, $k \in \mathbb{N} \cup \{\infty\}$. We say that a function is L_∞ if it is bounded. For any matrix A , A' denotes its transpose. For any vector $\mathbf{z} \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, $|\mathbf{z}|$ denotes the Euclidean norm $\sqrt{\mathbf{z}'\mathbf{z}}$. For $n \in \mathbb{Z}_+$, I_n denotes the $n \times n$ -dimensional identity matrix. For $n \in \mathbb{Z}_+$ and $n \times n$ -dimensional matrix A , we set $A^0 = I_n$. For any matrix M , $\|M\|_{p,p}$ denotes its p -induced norm, $1 \leq p \leq \infty$. For any waveform $\mathbf{u}_{[0,t_f]} \in \mathcal{C}([0, t_f], \mathbb{R}^p)$, where $t_f \in (0, \infty) \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|\mathbf{u}_{[0,t_f]}\|_\infty = \sup_{t \in [0, t_f]} |\mathbf{u}(t)|$. For any $m, n \in \mathbb{Z}_+$, $\mathbf{0}_{m \times n}$ denotes the $m \times n$ -dimensional matrix whose elements are all zeros. We will denote constants or matrices of no specific interest or relevance to the analysis by \star . We will denote $m \times n$ -dimensional matrices of no specific interest or relevance to the analysis by $\star_{m \times n}$.

3. THE EXTENDED ZERO DYNAMICS CANONICAL FORM

In this section, we first recall the definition of the extended zero dynamics for a MIMO LTI system (Başar and Pan (2020)).

Consider a general MIMO LTI system (not necessarily square)

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + D\mathbf{w}; \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{D}_0 \quad (1a)$$

$$\mathbf{y} = C\mathbf{x} + F\mathbf{u} + E\mathbf{w} \quad (1b)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $\mathbf{u} \in \mathbb{R}^p$ is the control input, $p \in \mathbb{Z}_+$; $\mathbf{y} \in \mathbb{R}^m$ is the output, $m \in \mathbb{Z}_+$; $\mathbf{w} \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $\mathbf{x}_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $\mathbf{w}_{[0,\infty)} \in \mathcal{W}_d$ of class \mathcal{B}_q , A , B , D , C , F , and E are constant matrices of appropriate dimensions. The extended zero dynamics of (1) is defined by the maximal solution $K \in \mathbb{R}^{s \times n}$, $A_z \in \mathbb{R}^{s \times s}$, $A_{z1} \in \mathbb{R}^{s \times m}$ to the following matrix equations.

$$KA = A_zK + A_{z1}C \quad (2a)$$

$$A_{z1}F = KB \quad (2b)$$

where K is of full row rank such that s is maximal. Then, defining $\mathbf{x}_z := K\mathbf{x}$, it evolves according to

$$\dot{\mathbf{x}}_z = A_z\mathbf{x}_z + A_{z1}\mathbf{y} + (KD - A_{z1}E)\mathbf{w}; \quad (3)$$

$$\mathbf{x}_z(0) = K\mathbf{x}_0 \in K(\mathcal{D}_0)$$

This is said to be the *extended zero dynamics* of (1). (Note that $s = 0$ is also a possible solution, which corresponding to the case when the extended zero dynamics is absent)

Then, we recall the canonical form that reveals the extended zero dynamics for square MIMO LTI systems with vector relative degree (See Isidori (1995) or Başar and Pan (2020)).

Lemma 1. Consider a square MIMO LTI system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + D\mathbf{w}; \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{D}_0 \quad (4a)$$

$$\mathbf{y} = C\mathbf{x} + F\mathbf{u} + E\mathbf{w} \quad (4b)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $\mathbf{u} \in \mathbb{R}^m$ is the control input, $m \in \mathbb{Z}_+$; $\mathbf{y} \in \mathbb{R}^m$ is the output; $\mathbf{w} \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $\mathbf{x}_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $\mathbf{w}_{[0,\infty)} \in \mathcal{W}_d$ of class \mathcal{B}_q (Pan and Başar, 2018), A , B , D , C , F , and E are constant matrices of appropriate dimensions.

Let the system admit vector relative degree $r_1, \dots, r_m \in \{0, \dots, n\}$ from \mathbf{u} to \mathbf{y} , that is, $i = 1, \dots, m$,

$$F_{i,:} = C_{i,:}B = \dots = C_{i,:}A^{r_i-1}B = \mathbf{0}_{1 \times m}$$

where $F_{i,:}$ and $C_{i,:}$ are the i th row vectors of the matrices F and C , respectively, and

$$\begin{bmatrix} C_{1,:}A^{r_1-1}B \\ \vdots \\ C_{m,:}A^{r_m-1}B \end{bmatrix} =: B_0$$

is an invertible matrix (for those $i = 1, \dots, m$ with $r_i = 0$, the corresponding row in B_0 is replaced by $F_{i,:}$). The matrix B_0 is said to be the *high frequency gain matrix*. Then, there exists an invertible matrix T_o such that, in

$$\bar{\mathbf{x}} := T_o^{-1}\mathbf{x} = [\mathbf{x}'_z \mathbf{x}_{1,1} \dots \mathbf{x}_{1,r_1} \dots \mathbf{x}_{m,1} \dots \mathbf{x}_{m,r_m}]'$$

coordinates, the system (4) admits the representation

$$\dot{\mathbf{x}}_z = A_z\mathbf{x}_z + \sum_{i=1}^m A_{z1,i}\mathbf{y}_i + D_z\mathbf{w} \quad (5a)$$

$$\dot{\mathbf{x}}_{i,j} = \mathbf{x}_{i,j+1} + D_{i,j}\mathbf{w}; \quad (5b)$$

$$1 \leq i \leq m \text{ with } r_i > 0, 1 \leq j < r_i$$

$$\dot{\mathbf{x}}_{i,r_i} = A_i\bar{\mathbf{x}} + C_{i,:}A^{r_i-1}B\mathbf{u} + D_{i,r_i}\mathbf{w}; \quad (5c)$$

$$1 \leq i \leq m \text{ with } r_i > 0$$

$$\mathbf{y}_i = \mathbf{x}_{i,1} + E_{i,:}\mathbf{w}; \quad 1 \leq i \leq m \text{ with } r_i > 0 \quad (5d)$$

$$\mathbf{y}_i = \bar{C}_{i,:}\bar{\mathbf{x}} + F_{i,:}\mathbf{u} + E_{i,:}\mathbf{w}; \quad 1 \leq i \leq m \text{ with } r_i = 0 \quad (5e)$$

where $\mathbf{x}_z \in \mathbb{R}^{n-\sum_{i=1}^m r_i}$; $\mathbf{x}_{i,j} \in \mathbb{R}$, $1 \leq i \leq m$ with $r_i > 0$, $1 \leq j \leq r_i$. (5) is called the *zero dynamics canonical form* of system (4). (Note that here (5) is not the extended zero dynamics canonical form.) The dynamics (5a) is said to be the *extended zero dynamics* of system (4).

When the vector relative degree (r_1, \dots, r_m) is uniform, then Başar and Pan (2020) further defines the extended

zero dynamics canonical form for (4) in four possible cases depending on the value $r = r_1 = \dots = r_m$ and n .

In this paper, we consider a special class of square MIMO LTI systems formed as the parallel interconnected square MIMO LTI systems as depicted in Figure 1. (For brevity, the figure only includes two interconnected subsystems, but we consider here an arbitrary number, p , of such interconnected subsystems.)

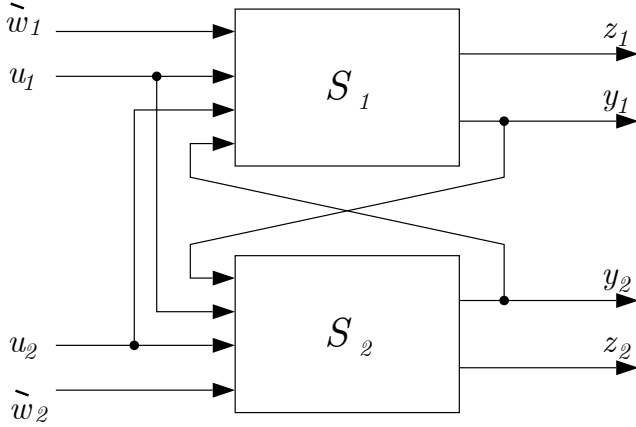


Fig. 1. Two parallel interconnected subsystems.

We assume that

Assumption 1. Each subsystem S_i , $i = 1, \dots, p$, is a finite-dimensional continuous-time square MIMO LTI system of order $n_i \geq 0$ and with uniform vector relative degree $0 \leq r_i \leq \frac{n_i}{m_i}$ from \mathbf{u}_i to \mathbf{y}_i , where \mathbf{u}_i and \mathbf{y}_i are $m_i \in \mathbb{N}$ dimensional.

We assume the following interconnection properties:

Assumption 2. (Interconnection Property). Fix any $i = 1, \dots, p$, for subsystem S_i , the relative degree from each component of \mathbf{y}_j , $j \neq i$ to each component of \mathbf{y}_i is greater than $0 \vee (r_i - r_j)$; and the relative degree from each component of \mathbf{u}_j , $j \neq i$, to each component of \mathbf{y}_i is greater than $r_i \vee r_j$, $j = 1, \dots, p$.

It is straightforward to verify that the composite system S with input $\mathbf{u} := (\mathbf{u}_1, \dots, \mathbf{u}_p)$ and output $\mathbf{y} := (\mathbf{y}_1, \dots, \mathbf{y}_p)$ admits vector relative degree $(\underbrace{r_1, \dots, r_1}_{m_1\text{-times}}, \dots, \underbrace{r_p, \dots, r_p}_{m_p\text{-times}})$.

Thus, by Lemma 1, for the composite system S , we have the following zero dynamics canonical form:

$$\dot{\mathbf{x}}_z = \mathbf{A}_z \mathbf{x}_z + \sum_{i=1}^p \mathbf{A}_{z1,i} \mathbf{y}_i + \mathbf{D}_z \mathbf{w} \quad (6a)$$

$$\dot{\mathbf{x}}_{i,j} = \mathbf{x}_{i,j+1} + \mathbf{D}_{i,j} \mathbf{w}; \quad 1 \leq i \leq p \text{ with } r_i > 0, 1 \leq j < r_i \quad (6b)$$

$$\dot{\mathbf{x}}_{i,r_i} = \mathbf{A}_i \mathbf{x} + \mathbf{B}_{0,i} \mathbf{u}_i + \mathbf{D}_{i,r_i} \mathbf{w}; \quad 1 \leq i \leq p \text{ with } r_i > 0 \quad (6c)$$

$$\mathbf{y}_i = \mathbf{x}_{i,1} + \mathbf{E}_i \mathbf{w}; \quad 1 \leq i \leq p \text{ with } r_i > 0 \quad (6d)$$

$$\mathbf{y}_i = \mathbf{C}_i \mathbf{x} + \mathbf{F}_i \mathbf{u}_i + \mathbf{E}_i \mathbf{w}; \quad 1 \leq i \leq p \text{ with } r_i = 0 \quad (6e)$$

where $\mathbf{x}_z \in \mathbb{R}^{\sum_{i=1}^p n_i - \sum_{i=1}^p m_i r_i}$; $\mathbf{y}_i \in \mathbb{R}^{m_i}$, $i = 1, \dots, p$; $\mathbf{x}_{i,j} \in \mathbb{R}^{m_i}$, $i = 1, \dots, p$ with $r_i > 0$, $j = 1, \dots, r_i$, $\mathbf{B}_{0,i}$ is invertible, $i = 1, \dots, p$ with $r_i > 0$;

and \mathbf{F}_i is invertible, $i = 1, \dots, p$ with $r_i = 0$; $\mathbf{x} = (\mathbf{x}_z, \mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,r_1}, \dots, \mathbf{x}_{p,1}, \dots, \mathbf{x}_{p,r_p})$. (6) is the *zero dynamics canonical form* of system S . (Note that here (6) is not the extended zero dynamics canonical form.) The dynamics (6a) is the *extended zero dynamics* of system S . Without loss of generality, assume that

Assumption 3. The uniform vector relative degrees are ordered in the nondecreasing fashion: $r_1 \leq r_2 \leq \dots \leq r_p$.

By Assumption 2, we have \mathbf{C}_i , $i = 1, \dots, p$ with $r_i = 0$, has all zero elements multiplying $\mathbf{x}_{j,1}, \dots, \mathbf{x}_{j,r_j}$, $j = 1, \dots, p$ with $r_j > 0$. Therefore, \mathbf{C}_i has nonzero elements only multiplying \mathbf{x}_z . By Assumption 2, we have \mathbf{A}_i , $i = 1, \dots, p$ with $r_i > 0$, has all zero elements multiplying $\mathbf{x}_{j,r_i+1}, \dots, \mathbf{x}_{j,r_j}$, $j = 1, \dots, p$ with $r_j > r_i$.

Now, consider the dynamics of

$$(\mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,r_1}, \dots, \mathbf{x}_{p,1}, \dots, \mathbf{x}_{p,r_p})$$

It has the following structure:

$$\dot{\mathbf{x}}_{i,j} = \bar{\mathbf{x}}_{i,j+1} + \bar{\mathbf{D}}_{i,j} \mathbf{w}; \quad (7a)$$

$$i = 1, \dots, \bar{p}, j = 1, \dots, \bar{r}_i - 1$$

$$\dot{\mathbf{x}}_{i,\bar{r}_i} = \bar{\mathbf{A}}_{iz} \mathbf{x}_z + \sum_{j=1}^{\bar{p}} \sum_{l=1}^{\bar{r}_j \wedge \bar{r}_i} \bar{\mathbf{A}}_{i,j,l} \bar{\mathbf{x}}_{j,l} + \bar{\mathbf{B}}_{0,i} \bar{\mathbf{u}}_i + \bar{\mathbf{D}}_{i,\bar{r}_i} \mathbf{w}; \quad (7b)$$

$$\bar{\mathbf{y}}_i = \bar{\mathbf{x}}_{i,1} + \bar{\mathbf{E}}_i \mathbf{w}; \quad i = 1, \dots, \bar{p} \quad (7c)$$

where \bar{p} is equal to the number of distinct r_i , $i = 1, \dots, p$, that are not zeros, which forms the set $\{\bar{r}_1, \dots, \bar{r}_{\bar{p}}\}$; $l_1, \dots, l_{\bar{p}}$ is defined by $r_{l_1} = 0 < \bar{r}_1 := r_{l_1+1} = \dots = r_{l_2} < \bar{r}_2 := r_{l_2+1} = \dots = r_{l_3} < \dots < \bar{r}_{\bar{p}} := r_{l_{\bar{p}}+1} = \dots = r_p$, $l_{\bar{p}+1} = p$; (for notational consistency, we define $r_0 := 0$); $\bar{\mathbf{x}}_{i,j} := (\mathbf{x}_{l_i+1,j}, \dots, \mathbf{x}_{l_{i+1},j})$, $i = 1, \dots, \bar{p}$, $j = 1, \dots, \bar{r}_i$, $\bar{\mathbf{u}}_i := (\mathbf{u}_{l_i+1}, \dots, \mathbf{u}_{l_{i+1}})$, and $\bar{\mathbf{B}}_{0,i} :=$ block diagonal $(\mathbf{B}_{0,l_i+1}, \dots, \mathbf{B}_{0,l_{i+1}})$, $\bar{\mathbf{y}}_i := (\mathbf{y}_{l_i+1}, \dots, \mathbf{y}_{l_{i+1}})$, $i = 1, \dots, \bar{p}$.

For the system (7), \mathbf{x}_z and \mathbf{w} are considered inputs into the system. By Lemma 1 of Başar and Pan (2019), this system (7) is observable with observability indices

$$(\underbrace{\bar{r}_1, \dots, \bar{r}_1}_{\bar{m}_1\text{-times}}, \dots, \underbrace{\bar{r}_{\bar{p}}, \dots, \bar{r}_{\bar{p}}}_{\bar{m}_{\bar{p}}\text{-times}})$$

where $\bar{m}_i := \sum_{j=l_i+1}^{l_{i+1}+1} m_j$, $i = 1, \dots, \bar{p}$. As we had done in Lemma 2 of Başar and Pan (2020), we will transform the system (7) into observer canonical form. By the proof of Lemma 1 of Başar and Pan (2019), we note that the noninterweaved version of the matrix $\mathbf{Q} = l_{\sum_{i=1}^{\bar{p}} \bar{r}_i \bar{m}_i}$. Thus, we may form the matrix \mathbf{S} as in the proof of Lemma 1 of Başar and Pan (2019) (noninterweaved version):

$$\mathbf{S} = \begin{bmatrix} \bar{\mathbf{M}}_{11} & \dots & \bar{\mathbf{M}}_{1\bar{p}} \\ \vdots & \ddots & \vdots \\ \bar{\mathbf{M}}_{\bar{p}1} & \dots & \bar{\mathbf{M}}_{\bar{p}\bar{p}} \end{bmatrix}$$

where $\bar{\mathbf{M}}_{ii}$, $i = 1, \dots, \bar{p}$, is a $\bar{r}_i \bar{m}_i \times \bar{r}_i \bar{m}_i$ -dimensional matrix of the form:

$$\bar{\mathbf{M}}_{ii} = \begin{bmatrix} l_{\bar{m}_i} & \mathbf{0} & \dots & \mathbf{0} \\ \star & l_{\bar{m}_i} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \star & \dots & \star & l_{\bar{m}_i} \end{bmatrix}$$

and \bar{M}_{ij} , $i = 1, \dots, \bar{p} - 1$, $j = i + 1, \dots, \bar{p}$, is a $\bar{r}_i \bar{m}_i \times \bar{r}_j \bar{m}_j$ -dimensional matrix of the form:

$$\bar{M}_{ij} = \begin{bmatrix} \mathbf{0}_{\bar{m}_i \times \bar{m}_j} & \cdots & \cdots & \cdots & \cdots & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} \\ \star_{\bar{m}_i \times \bar{m}_j} & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} & \cdots & \cdots & \cdots & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \star_{\bar{m}_i \times \bar{m}_j} & \cdots & \star_{\bar{m}_i \times \bar{m}_j} & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} & \cdots & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} \end{bmatrix}$$

and \bar{M}_{ij} , $i = 2, \dots, \bar{p}$, $j = 1, \dots, i - 1$, is a $\bar{r}_i \bar{m}_i \times \bar{r}_j \bar{m}_j$ -dimensional matrix of the form:

$$\bar{M}_{ij} = \begin{bmatrix} \mathbf{0}_{\bar{m}_i \times \bar{m}_j} & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} & \cdots & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{\bar{m}_i \times \bar{m}_j} & \vdots & \vdots & \vdots \\ \star_{\bar{m}_i \times \bar{m}_j} & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \star_{\bar{m}_i \times \bar{m}_j} & \cdots & \star_{\bar{m}_i \times \bar{m}_j} & \mathbf{0}_{\bar{m}_i \times \bar{m}_j} \end{bmatrix}$$

We have the following result.

Lemma 2. The matrix S is invertible and S^{-1} admits the same structure as S .

Proof. We will show that S is always invertible and S^{-1} admits the same structure as S using mathematical induction on \bar{p} .

1° Consider the case $\bar{p} = 1$. The result is obvious.

2° Assume that the result holds when $\bar{p} = k \in \mathbb{N}$.

3° Consider the case when $\bar{p} = k + 1 \in \{2, 3, \dots\}$. Denote

$$\bar{S} := \begin{bmatrix} \bar{M}_{11} & \cdots & \bar{M}_{1k} \\ \vdots & \ddots & \vdots \\ \bar{M}_{k1} & \cdots & \bar{M}_{kk} \end{bmatrix} \text{ and then } \begin{bmatrix} \bar{S} & \bar{M}_{1\bar{p}} \\ \hline \bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1} & \bar{M}_{\bar{p}\bar{p}} \end{bmatrix}$$

$= S$. By 2°, \bar{S} is invertible and \bar{S}^{-1} admits the same form as \bar{S} . By 1°, $\bar{M}_{\bar{p}\bar{p}}^{-1}$ exists and admits the same form as $\bar{M}_{\bar{p}\bar{p}}$.

By Matrix Inversion Lemma, we have $S^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$

when \bar{S} , $\bar{M}_{\bar{p}\bar{p}}$ are invertible, and T_{11} and T_{22} are invertible,

$$\text{where } T_{11} = \bar{S} - \begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix} \bar{M}_{\bar{p}\bar{p}}^{-1} [\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}], \quad T_{21} = -\bar{M}_{\bar{p}\bar{p}}^{-1} [\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}] T_{11}^{-1}, \quad T_{22} = -[\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}] \cdot \bar{S}^{-1} \begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix} + \bar{M}_{\bar{p}\bar{p}}, \text{ and } T_{12} = -\bar{S}^{-1} \begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix} T_{22}^{-1}.$$

Now, based on the structures preceding the lemma, we can easily show that T_{11} admits the same structure as \bar{S} and therefore invertible, by 2°. Then, T_{11}^{-1} admits the same structure as \bar{S}^{-1} , which is the same structure as \bar{S} , by 2°. We can also conclude that T_{22} admits the same structure as $\bar{M}_{\bar{p}\bar{p}}$. By 1°, T_{22}^{-1} exists and admits the same structure as $\bar{M}_{\bar{p}\bar{p}}$. Furthermore, T_{12} admits the same

structure as $\begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix}$ and T_{21} admits the same structure as $[\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}]$. Thus, S^{-1} admits the same structure as S .

This completes the induction process and therefore the proof of the lemma. \square

By Lemma 1 of Başar and Pan (2019), in the coordinates of

$$(\tilde{\mathbf{x}}_{1,1}, \dots, \tilde{\mathbf{x}}_{1,\bar{r}_1}, \dots, \tilde{\mathbf{x}}_{\bar{p},1}, \dots, \tilde{\mathbf{x}}_{\bar{p},\bar{r}_{\bar{p}}}) \\ := S^{-1}(\bar{\mathbf{x}}_{1,1}, \dots, \bar{\mathbf{x}}_{1,\bar{r}_1}, \dots, \bar{\mathbf{x}}_{\bar{p},1}, \dots, \bar{\mathbf{x}}_{\bar{p},\bar{r}_{\bar{p}}})$$

the system (7) admits the observer canonical form representation:

$$\dot{\tilde{\mathbf{x}}}_{i,j} = \sum_{l=1}^{\bar{p}} \tilde{A}_{i,j,l} \tilde{\mathbf{x}}_{l,1} + \tilde{\mathbf{x}}_{i,j+1} + \tilde{D}_{i,j} \mathbf{w}; \quad (8a)$$

$$i = 1, \dots, \bar{p}, j = 1, \dots, \bar{r}_i - 1$$

$$\dot{\tilde{\mathbf{x}}}_{i,\bar{r}_i} = \bar{A}_{iz} \mathbf{z} + \sum_{l=1}^{\bar{p}} \tilde{A}_{i,\bar{r}_i,l} \tilde{\mathbf{x}}_{l,1} + \bar{B}_{0,i} \bar{\mathbf{u}}_i + \tilde{D}_{i,\bar{r}_i} \mathbf{w}; \quad (8b)$$

$$\bar{\mathbf{y}}_i = \tilde{\mathbf{x}}_{i,1} + \bar{E}_i \mathbf{w}; \quad i = 1, \dots, \bar{p} \quad (8c)$$

where $\tilde{\mathbf{x}}_{i,j}$ is of \bar{m}_i -dimensional, $i = 1, \dots, \bar{p}$, $j = 1, \dots, \bar{r}_i$, $\tilde{A}_{i,j,l} = \mathbf{0}_{\bar{m}_i \times \bar{m}_l}$ if $\bar{r}_i - \bar{r}_l - j \geq 0$, $i = 1, \dots, \bar{p}$, $j = 1, \dots, \bar{r}_i$, $l = 1, \dots, \bar{p}$, and we have made use of the structure of S and S^{-1} in the above formulas.

Summarizing the preceding, we state the following result.

Proposition 3. Consider the finite-dimensional continuous-time square MIMO LTI system which is composed of $p \in \mathbb{N}$ finite-dimensional continuous-time square MIMO LTI systems in parallel configuration as illustrated in Figure 1 (which is for the special case $p = 2$). We assume that Assumptions 1, 2, and 3 hold for the interconnected systems. Then, the composite system S admits state space representation (6) by Lemma 1. Furthermore, the system admits the following *extended zero dynamic canonical form*:

$$\dot{\mathbf{x}}_z = A_z \mathbf{x}_z + \sum_{i=1}^{l_1} A_{z1,i} \mathbf{y}_i + \sum_{i=1}^{\bar{p}} \tilde{A}_{z1,i} \bar{\mathbf{y}}_i + D_z \mathbf{w} \quad (9a)$$

$$\dot{\tilde{\mathbf{x}}}_{i,j} = \sum_{l=1}^{\bar{p}} \tilde{A}_{i,j,l} \tilde{\mathbf{x}}_{l,1} + \tilde{\mathbf{x}}_{i,j+1} + \tilde{D}_{i,j} \mathbf{w}; \quad (9b)$$

$$i = 1, \dots, \bar{p}, j = 1, \dots, \bar{r}_i - 1$$

$$\dot{\tilde{\mathbf{x}}}_{i,\bar{r}_i} = \bar{A}_{iz} \mathbf{z} + \sum_{l=1}^{\bar{p}} \tilde{A}_{i,\bar{r}_i,l} \tilde{\mathbf{x}}_{l,1} + \bar{B}_{0,i} \bar{\mathbf{u}}_i + \tilde{D}_{i,\bar{r}_i} \mathbf{w}; \quad (9c)$$

$$\bar{\mathbf{y}}_i = \tilde{\mathbf{x}}_{i,1} + \bar{E}_i \mathbf{w}; \quad i = 1, \dots, \bar{p} \quad (9d)$$

$$\mathbf{y}_i = C_{iz} \mathbf{x}_z + F_i \mathbf{u}_i + E_i \mathbf{w}; \quad 1 \leq i \leq l_1 \quad (9e)$$

where \bar{p} is equal to the number of distinct r_i , $i = 1, \dots, p$, that are not zeros, which forms the set $\{\bar{r}_1, \dots, \bar{r}_{\bar{p}}\}$; $l_1, \dots, l_{\bar{p}+1}$ is defined by $r_0 = \dots = r_{l_1} = 0 < \bar{r}_1 := r_{l_1+1} = \dots = r_{l_2} < \bar{r}_2 := r_{l_2+1} = \dots = r_{l_3} < \dots < \bar{r}_{\bar{p}} := r_{l_{\bar{p}+1}} = \dots = r_p$, $l_{\bar{p}+1} := p$; (for notational consistency, we define $r_0 := 0$); $\bar{\mathbf{u}}_i := (\mathbf{u}_{l_i+1}, \dots, \mathbf{u}_{l_{i+1}})$, and $\bar{B}_{0,i} :=$ block diagonal $(B_{0,l_i+1}, \dots, B_{0,l_{i+1}})$, which is invertible; $\bar{\mathbf{y}}_i := (\mathbf{y}_{l_i+1}, \dots, \mathbf{y}_{l_{i+1}})$, $i = 1, \dots, \bar{p}$; $\mathbf{x}_z \in \mathbb{R}^{\sum_{i=1}^p n_i - \sum_{i=1}^p m_i r_i}$; $\tilde{\mathbf{x}}_{i,j} \in \mathbb{R}^{\bar{m}_i} = \mathbb{R}^{\sum_{k=l_i+1}^{l_{i+1}} m_k}$, $i = 1, \dots, \bar{p}$, $j = 1, \dots, \bar{r}_i$; F_i is invertible, $i = 1, \dots, l_1$; and $\tilde{A}_{i,j,l} = \mathbf{0}_{\bar{m}_i \times \bar{m}_l}$ if $\bar{r}_i - \bar{r}_l - j \geq 0$, $i = 1, \dots, \bar{p}$, $j = 1, \dots, \bar{r}_i$, $l = 1, \dots, \bar{p}$.

4. FURTHER RESULTS

Based on the extended zero-dynamics canonical form (9), we observe that the system outputs are just a chain of integrators from the corresponding control input, which are perturbed by bounded signals, if the system outputs are bounded and the composite system is minimum phase according to Başar and Pan (2020). This observation is made precise by the following theorem.

Theorem 4. Consider the finite-dimensional continuous-time square MIMO LTI system which is composed of $p \in \mathbb{N}$ finite-dimensional continuous-time square MIMO LTI systems in parallel configuration as illustrated in Figure 1 (which is for the special case $p = 2$). We assume that Assumptions 1, 2, and 3 hold for the interconnected systems, and that the composite system S is minimum phase with respect to \mathcal{D}_0 and \mathcal{W}_d , where $\mathcal{D}_0 \subseteq \mathbb{R}^{\sum_{i=1}^p n_i}$ is a subspace and \mathcal{W}_d is of class \mathcal{B}_q , and $q \in \mathbb{Z}_+$ is the dimension of the exogenous disturbance input \mathbf{w} . Then, the composite system S admits extended zero dynamic canonical form representation (9) by Proposition 3. Assume that the entire output of composite system $\mathbf{y} := (\mathbf{y}_1, \dots, \mathbf{y}_p)$ and the disturbance input \mathbf{w} are bounded on some time interval $[0, t_f]$, where $t_f \in \mathbb{R}_+$, the initial condition of the composite system is bounded and in the subspace \mathcal{D}_0 , the disturbance waveform is in \mathcal{W}_d , and $\exists i_0 \in \{1, \dots, \bar{p}\}$ and $\exists k_0 \in \{0, \dots, \bar{r}_{i_0} - 1\}$, such that $\tilde{\mathbf{x}}_{i_0,1}, \dots, \tilde{\mathbf{x}}_{i_0,1+k_0}$ are bounded on $[0, t_f]$. Then, any stably filtered signal of $\tilde{\mathbf{u}}_{i_0}$:

$$\dot{\boldsymbol{\eta}} = \mathbf{F}\boldsymbol{\eta} + \mathbf{G}\tilde{\mathbf{u}}_{i_0} \quad (10a)$$

$$\boldsymbol{\xi} = \mathbf{H}\boldsymbol{\eta} \quad (10b)$$

where the system (10) is finite-dimensional and LTI and \mathbf{F} is Hurwitz, and each component of the output $\boldsymbol{\xi}$ has relative degree with respect to each component of the input $\tilde{\mathbf{u}}_{i_0}$ to be greater than or equal to $\bar{r}_{i_0} - k_0$, then the signal $\boldsymbol{\xi}$ is bounded on the interval $[0, t_f]$.

Proof. Under the assumption that \mathbf{y} and \mathbf{w} are bounded on $[0, t_f]$, and the composite system S is minimum phase according to Başar and Pan (2020), we have that the state of the extended zero-dynamics \mathbf{x}_z is bounded on $[0, t_f]$. Then, the result follows directly from Lemma 2 of Pan and Başar (2019). \square

In the application of the above theorem, the composite system S is referred to as the reference system.

5. CONCLUSIONS

In this paper, we have obtained the extended zero-dynamics canonical form for a class of square MIMO LTI systems comprised of multiple square MIMO LTI subsystems in parallel interconnection satisfying the interconnection property. Under the assumption that each subsystem has already been analyzed and extended to admit uniform vector relative degree (and has uniform observability indices), we have considered multiple such subsystems, parallel-interconnected to form a composite system which admits vector relative degree but not uniform vector relative degree. We have prescribed an interconnection property under which the composite system (without any further modification or extension) admits the extended zero-dynamics canonical form even though

it does not have uniform vector relative degree. Thus, the composite system is in a form suitable for robust adaptive control design with nonuniform vector relative degree if it is further minimum phase according to Başar and Pan (2020). The interconnection property we have prescribed is one where for each subsystem i , the connections from subsystem j , $j \neq i$, satisfy the properties that the relative degree from each component of \mathbf{y}_j to each component of \mathbf{y}_i is greater than $\max\{0, r_i - r_j\}$, where r_i and r_j are the uniform vector relative degrees for the i th subsystem and the j th subsystem, respectively, and the relative degree from each component of \mathbf{u}_j to each component of \mathbf{y}_i is greater than $\max\{r_i, r_j\}$. Thus, when a number of subsystems are to be incorporated into a robust adaptive control system, we just need to make sure that these subsystems are themselves with uniform vector relative degree (and have uniform observability indices), and the interconnections of these subsystems and those of the original system are compatible, i.e., they satisfy the interconnection property. Then, the (centralized) robust adaptive controller can be redesigned and applied to the larger system without requiring any changes in the subsystems if the composite system is minimum phase according to Başar and Pan (2020).

The remaining question along this line of research is what kind of interconnection property is required such that the composite system will automatically be minimum phase given that the subsystems are minimum phase. This question is currently under investigation.

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